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STABILIZATION OF BILINEAR TIME-VARYING SYSTEMS WITH NORM-BOUNDED CONTROLS

Abstract. In this paper, we consider a class of bilinear time-varying systems. We study the stabilization problem for these systems with norm-bounded controls by using Lyapunov techniques and the solutions of Riccati differential equations. A numerical example is given to illustrate the efficiency of the obtained result.

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1 Introduction

The problem of controllability and stabilizability for linear control systems has received a considerable amount of interest in the last few years [5,9,10]. This problem is an extension of the classical Kalman result [3] on the controllability and stability of linear control systems. Linear nonautonomous control systems are usually represented in the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in \mathbb{R}^+, \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. We assume that $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$ are the matrices, continuously depending on t. The global null-controllability (GNC) problem of the linear system (1.1) concerns the question of finding an admissible control $u(t)$ which leads an arbitrary state x_0 to the origin. The stabilization problem is aimed by means of a linear control to find a control $u(t) = K(t)x(t)$ such that the zero solution of the closed-loop system

$$\dot{x}(t) = \big[A(t) + B(t)K(t)\big]x(t), \quad t \geq 0,$$

is asymptotically stable in the Lyapunov sense. In this case one says that the system is stabilizable with the stabilizing feedback control $u(t) = K(t)x(t)$. For linear time-varying (LTV) systems, the first result on the relationship between GNC problem and Riccati differential equation (RDE) was given in [3] where it was proven that if the LTV control system (1.1) is GNC, then the RDE

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) - P(t)B(t)B^T(t)P(t) + Q(t) = 0, \tag{1.2}$$

where $Q(t) \geq 0$, has a positive semi-definite solution $P(t)$. However, the existence of the positive definite solution $P(t)$ of the above RDE is not sufficient for the GNC. In [2], the authors prove that the system is completely stabilizable if it is uniformly globally null-controllable. In [6], the authors have developed the relationship between the exact controllability and complete stabilizability for linear time-varying control systems in Hilbert spaces. In [7], the authors study the stabilization of linear nonautonomous systems with norm-bounded controls (1.1), where the control $u(t)$ satisfies the following condition:

$$\|u(t)\| \leq r, \quad t \in \mathbb{R}^+.$$

For autonomous systems, where the constant matrix A satisfies some appropriate spectral properties, Slemrod [8] proposed a nonsmooth feedback control of the form

$$u(t) = \begin{cases} \dfrac{-rB^T x(t)}{\|B^T x(t)\|} & \text{if } \|B^T x(t)\| \geq r, \\ -B^T x(t) & \text{if } \|B^T x(t)\| \leq r. \end{cases}$$

In this paper, we consider the following bilinear time-varying (BTV) control system:

$$\dot{x}(t) = A(t)x(t) + u(t)B(t)x(t), \quad t \in \mathbb{R}^+, \tag{1.3}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times n}$.

The purpose of this paper is to discuss the problem of global uniform stabilization of the BTV control system (1.3) with norm-bounded controls by using the Lyapunov techniques.

2 Preliminary results

We start by recalling some classical notation and definitions that will be useful throughout the paper.

- $\mathbb{R}^+$ denotes the set of all real nonnegative numbers.
- $\mathbb{R}^n$ denotes the n-dimensional space.
- $\langle x, y\rangle$ or $x^T y$ denote the scalar inner product of two vectors $x, y \in \mathbb{R}^n$.

- $\|x\|$ denotes the Euclidean vector norm of x.
- $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ matrices.
- I_n denotes the identity matrix.

Let $A \in \mathbb{R}^{n\times n}$:

- A^T denotes the transpose matrix of A; A is symmetric if and only if $A^T = A$.
- $\lambda(A)$ denotes the set of all eigenvalues of A.
- $\lambda_{\max}(A) = \max\{\mathrm{Re}(\lambda) : \lambda \in \lambda(A)\}$, $\lambda_{\min}(A) = \min\{\mathrm{Re}(\lambda) : \lambda \in \lambda(A)\}$.
- $\mu(A)$ denotes the matrix measure of the matrix A defined by

$$\mu(A) = \frac{1}{2}\,\lambda_{\max}(A + A^T).$$

- $\mathbf{L}_2([t,s],\mathbb{R})$ denotes the set of all square integrable $\mathbb{R}$-valued functions on $[t,s]$.
- The matrix A is bounded on $\mathbb{R}^+$ if there exists $M > 0$ such that $\sup\limits_{t\geq 0}\|A(t)\| \leq M$.
- The matrix $A \in \mathbb{R}^{n\times n}$ is positive semi-definite ($A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{R}^n$.
- $\mathbf{M}([0,\infty),\mathbb{R}^n_+)$ is the set of all symmetric positive semi-definite matrix functions, continuous and bounded on $[0,\infty)$.
- The matrix function $A(t)$ is positive definite ($A(t) > 0$) if there exists a constant $c > 0$ such that $\langle A(t)x, x\rangle \geq c\|x\|^2$ for all $x \in \mathbb{R}^n$, $t \geq 0$.

Now, we recall some classical definitions and results.
Let the system is described by the equation

$$\dot{x} = f(t,x), \tag{2.1}$$

where the map $f : \mathbb{R} \times U \to \mathbb{R}^n$ is continuous locally Lipschitz with respect to x, $f(t,0) = 0\ \forall\, t \geq 0$, and U is an open set of $\mathbb{R}^n$ ($0 \in U$). Denote by $x(t,t_0)$ the solution of (2.1) starting at x_0 at time t_0.

Definition 2.1. The equilibrium point $x = 0$ of system (2.1) is said to be

(i) stable if $\forall\, \varepsilon > 0$, $\forall\, t_0 \geq 0$, $\exists\, \delta = \delta(t_0,\varepsilon) > 0$ such that $\forall\, x_0 \in \mathbb{R}^n$ one has

$$\|x_0\| < \delta \Longrightarrow \|x(t,t_0)\| < \varepsilon,\ \ \forall\, t \geq t_0;$$

(ii) uniformly stable if (i) holds where $\delta = \delta(\varepsilon)$ is independent of t_0;

(iii) attractive if there exists a neighborhood $\mathcal{V}$ of 0 such that for any initial condition x_0 belonging to $\mathcal{V}$, the corresponding solution $x(t,t_0)$ is defined for all $t \geq 0$ and $\lim\limits_{t\to+\infty} x(t,t_0) = 0$. If $\mathcal{V} = \mathbb{R}^n$, then $x = 0$ is globally attractive;

(iv) asymptotically stable if it is stable and attractive;

(v) uniformly asymptotically stable if it is uniformly stable and, in addition, there exists $c > 0$ such that for all $\varepsilon > 0$, there exists $\tau > 0$ such that for all $x_0 \in \mathbb{R}^n$

$$\|x_0\| < c \Longrightarrow \|x(t,t_0)\| < \varepsilon,\ \ \forall\, t \geq \tau + t_0;$$

(vi) globally uniformly asymptotically stable if it is uniformly stable, $\delta(\varepsilon)$ can be chosen to satisfy $\lim\limits_{\varepsilon\to+\infty} \delta(\varepsilon) = +\infty$, and for all $c > 0$ and for all $\varepsilon > 0$, there exists $\tau > 0$ such that for all $x_0 \in \mathbb{R}^n$,

$$\|x_0\| < c \Longrightarrow \|x(t,t_0)\| < \varepsilon,\ \ \forall\, t \geq \tau + t_0.$$

Definition 2.2. The pair $(A(t), B(t))$ is said to be GNC if the associated linear control system (1.1) is GNC in the following sense:

for every $x_0 \in \mathbb{R}^n$, there exist a number $\tau > 0$ and an admissible control $u(t)$ such that $x(\tau) = 0$.

We recall the following controllability criterion that will be used later.

Proposition 2.1 ([1,3]). *The pair $(A(t), B(t))$ is GNC if and only if one of the following conditions holds:*

(i) *there exist $t > 0$ and $c > 0$ such that*

$$\int_0^t \left\|B^T(s)U^T(t,s)\right\| ds \geq c\,\|U^T(t,0)\|^2, \quad \forall\, x \in \mathbb{R}^n;$$

(ii) *$A(t)$, $B(t)$ are analytic on $\mathbb{R}_+$ and the rank $M(t_0) = n$ for some $t_0 > 0$, where*

$$M(t) = \big[M_0(t), M_1(t), \ldots, M_{n-1}(t)\big];$$
$$M_0 := B(t), \quad M_{i+1}(t) = -A(t)M_i(t) + \frac{d}{dt}M_i(t), \quad i = 0, 1, \ldots, n-2.$$

Definition 2.3. A scalar continuous function $\alpha(r)$ defined for $r \in [0, a[$ belongs to the class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It belongs to the class $\mathcal{K}_\infty$ if it is defined for all $r \geq 0$ and $\alpha(r) \to \infty$ as $r \to \infty$.

Theorem 2.1 ([4]). *Let $r > 0$ and denote $\mathcal{B}_r = \{x \in \mathbb{R}^n, \ \ \|x\| < r\}$. Let $V : \mathbb{R}^+ \times \mathcal{B}_r \to \mathbb{R}$ be a smooth function. If there exists functions α_1, α_2 and α_3 of the class $\mathcal{K}$ defined on $[0, a[$ and satisfying: $\forall\, t \geq t_0$ and $\forall\, x \in \mathcal{B}_r$,*

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|), \tag{2.2}$$
$$\dot{V}(t,x) \leq -\alpha_3(\|x\|), \tag{2.3}$$

then the origin $x = 0$ is uniformly asymptotically stable (UAS). If $\mathcal{B}_r = \mathbb{R}^n$ and α_1 and α_2 are two functions of the class $\mathcal{K}_\infty$, then the origin $x = 0$ is globally uniformly asymptotically stable (GUAS).

To solve the stabilization problem of the bilinear system (1.3) the RDE (1.2) is useful.

Theorem 2.2 ([6]). *The following statements are equivalent:*

(i) *the pair $(A(t), B(t))$ is GNC;*

(ii) *for $Q \in \mathbf{M}([0,\infty), \mathbb{R}^n_+)$, the RDE (1.2) has a solution $P \in \mathbf{M}([0,\infty), \mathbb{R}^n_+)$.*

3 The main results

Let us consider the BTV control system (1.3)

$$\dot{x}(t) = A(t)x(t) + u(t)B(t)x(t), \quad t \in \mathbb{R}^+,$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $A(t)$, $B(t)$ are matrix functions, continuous and bounded on $[0,\infty)$. Suppose that the pair $(A(t), B(t))$ is GNC. Then for $Q \in \mathbf{M}([0,\infty), \mathbb{R}^n_+)$, the RDE (1.2) has a solution $P \in \mathbf{M}([0,\infty), \mathbb{R}^n_+)$. Denote

$$b = \sup_{t\geq 0} \|B(t)\|, \quad p = \sup_{t\geq 0} \|P(t)\|.$$

In what follows, we need the following assumptions:

(H_1) The BTV control system (1.3) is GNC.

(H_2) $\eta = \inf_{t\geq 0} \|Q(t)\|$ satisfies $\eta > p^2b^2$.

Proposition 3.1. *Let $B(t)$ and $P(t)$ be bounded matrix functions. Then for $r > 0$, the function*

$$g(t,x) = -r\Big(\frac{\|B(t)\|\,\|P(t)\|\,\|x\|}{1+\|B(t)\|\,\|P(t)\|\,\|x\|}\Big)B(t)x$$

is globally Lipschitz with respect to $x \in \mathbb{R}^n$.

Proof. Let $x_1, x_2 \in \mathbb{R}^n$, $t \geq 0$. We have

$$\begin{aligned}\|g(t,x_1)-g(t,x_2)\| &= r\Big\|\frac{\|B(t)\|\,\|P(t)\|\,\|x_2\|}{1+\|B(t)\|\,\|P(t)\|\,\|x_2\|}B(t)x_2 - \frac{\|B(t)\|\,\|P(t)\|\,\|x_1\|}{1+\|B(t)\|\,\|P(t)\|\,\|x_1\|}B(t)x_1\Big\| \\ &\leq r\|B(t)\|^2\|P(t)\|\,\Big\|\frac{\|x_2\|x_2}{1+\|B(t)\|\,\|P(t)\|\,\|x_2\|} - \frac{\|x_1\|x_1}{1+\|B(t)\|\,\|P(t)\,\|\|x_1\|}\Big\| \\ &\leq r\|B(t)\|^2\|P(t)\|\,\Big\|\frac{\|x_2\|x_2 - \|x_1\|x_1 + \|B(t)\|\,\|P(t)\|\,\|x_1\|\,\|x_2\|(x_2-x_1)}{(1+\|B(t)\|\,\|P(t)\|\,\|x_1\|)(1+\|B(t)\|\,\|P(t)\|\,\|x_2\|)}\Big\|.\end{aligned}$$

Since

$$\begin{aligned}\Big\|\|x_2\|x_2 - \|x_1\|x_1\Big\| &= \Big\|\|x_2\|x_2 - \|x_1\|x_2 + \|x_1\|x_2 - \|x_1\|x_1\Big\| \\ &\leq \|x_2\|\,\|x_2-x_1\| + \|x_1\|\,\|x_2-x_1\| \\ &\leq \|x_2-x_1\|\,\big(\|x_2\|+\|x_1\|\big),\end{aligned}$$

we get

$$\begin{aligned}&\|g(t,x_1)-g(t,x_2)\| \\ &\quad\leq r\|B(t)\|^2\|P(t)\|\,\|x_1-x_2\|\,\Big[\frac{\|x_2\|+\|x_1\|+\|B(t)\|\,\|P(t)\|\,\|x_1\|\,\|x_2\|}{(1+\|B(t)\|\,\|P(t)\|\,\|x_1\|)(1+\|B(t)\|\,\|P(t)\|\,\|x_2\|)}\Big] \\ &\quad\leq r\|B(t)\|^2\|P(t)\|\,\|x_1-x_2\|\,\Big[\frac{\|x_1\|+\|x_2\|(1+\|B(t)\|\,\|P(t)\|\,\|x_1\|)}{(1+\|B(t)\|\,\|P(t)\|\,\|x_1\|)(1+\|B(t)\|\,\|P(t)\|\,\|x_2\|)}\Big] \\ &\quad\leq r\|B(t)\|\,\|x_1-x_2\|\,\Big[\frac{\|B(t)\|\,\|P(t)\|\,\|x_1\|}{(1+\|B(t)\|\,\|P(t)\|\,\|x_1\|)(1+\|B(t)\|\,\|P(t)\|\,\|x_2\|)} \\ &\qquad\qquad + \frac{\|B(t)\|\,\|P(t)\|\,\|x_2\|}{1+\|B(t)\|\,\|P(t)\|\,\|x_2\|}\Big] \\ &\quad\leq 2rb\|x_1-x_2\|.\end{aligned}$$

Therefore the function $g(t,x)$ is a globally Lipschitz function with respect to x. □

Theorem 3.1. *Suppose that the conditions (H_1) and (H_2) are fulfilled. Then if we choose $0 < r < \frac{\eta - p^2b^2}{2pb}$, the feedback function*

$$u(t,x) = -r\Big(\frac{\|B(t)\|\,\|P(t)\|\,\|x\|}{1+\|B(t)\|\,\|P(t)\|\,\|x\|}\Big), \;\; t\in\mathbb{R}^+, \;\; x\in\mathbb{R}^n, \tag{3.1}$$

is bounded and makes system (1.3) *GUAS.*

Proof. Let us consider the Lyapunov function

$$V(t,x) = \langle P(t)x, x\rangle, \;\; t\in\mathbb{R}^+, \;\; x\in\mathbb{R}^n.$$

Since P is a positive definite symmetric matrix, we can reduce condition (2.2) of Theorem 2.1 by choosing $\alpha_1(\|x\|) = c\|x\|^2$ and $\alpha_2(\|x\|) = p\|x\|^2$. Furthermore, the derivative of $V(t,x)$ along the solutions of the closed-loop system (1.1) by the feedback (3.1) is

$$\begin{aligned}\dot V(t,x) &= \langle \dot{P(t)}x, x\rangle + 2\langle P(t)\dot x, x\rangle \\ &\leq -\eta\|x\|^2 + \big\langle P(t)B(t)B(t)^TP(t)x, x\big\rangle - 2r\,\frac{\|B(t)\|\,\|P(t)\|\,\|x\|}{1+\|B(t)\|\,\|P(t)\|\,\|x\|}\,\langle P(t)B(t)x, x\rangle.\end{aligned}$$

Since

$$\big|\langle P(t)B(t)x, x\rangle\big| \le \|P(t)\|\,\|B(t)\|\,\|x\|^2,$$

we get

$$\begin{aligned}\dot V(t,x) &\le -\eta\|x\|^2 + \|P(t)\|^2\|B(t)\|^2\|x\|^2 + 2r\big|\langle P(t)B(t)x, x\rangle\big|\\ &\le -\eta\|x\|^2 + \|P(t)\|^2\|B(t)\|^2\|x\|^2 + 2r\|P(t)\|\,\|B(t)\|\,\|x\|^2\\ &\le (p^2b^2 - \eta + 2rpb)\|x\|^2.\end{aligned}$$

By choosing $\alpha_3(\|x\|) = (\eta - p^2b^2 - 2rpb)\|x\|^2$, condition (2.3) of Theorem 2.1 is well checked. So, the closed loop system (1.3) is globally uniformly asymptotically stable. Moreover, $|u(t,x)| \le r$, $\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. □

Now, let us consider the dynamical control system

$$\dot x(t) = A(t)x(t) + u(t)B(t)x(t) + F(t,x),\ \ t \in \mathbb{R}^+, \tag{3.2}$$

where $x(t) \in \mathbb{R}^n$, $F : [0, +\infty[\, \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear continuous function which is locally Lipschitz with respect to x.

Theorem 3.2. *If $F(t,x)$ satisfies the condition*

$$\|F(t,x)\| \le \gamma\|x\,\|,\ \ \forall\, t \ge 0,\ \ \forall\, x \in \mathbb{R}^n,$$

where γ is a positive number satisfying

$$0 < \gamma < \frac{\eta - p^2b^2 - 2rpb}{2p}\,, \tag{3.3}$$

then the closed loop system (3.2) *by the feedback function* (3.1) *is GUAS.*

Proof. Let us consider the Lyapunov function

$$V(t,x) = \langle P(t)x, x\rangle,\ \ t \in \mathbb{R}_+,\ \ x \in \mathbb{R}^n,$$

and let the feedback control be of form (3.1). The derivative of V along the solutions of the system (3.2) by using the chosen feedback control (3.1) and the RDE (1.2), results in

$$\begin{aligned}\dot V(t,x) &= \big\langle \dot P(t)x, x\rangle + 2\langle P(t)\dot x, x\big\rangle\\ &\le (p^2b^2 - \eta + 2rpb)\|x\|^2 + 2\big\langle P(t)F(t,x), x(t)\big\rangle\\ &\le (p^2b^2 - \eta + 2rpb)\|x\|^2 + 2\|P(t)\|\,\|F(t,x)\|\,\|x(t)\|\\ &\le (p^2b^2 - \eta + 2rpb)\|x\|^2 + 2\gamma\|P(t)\|\,\|x(t)\|\|x(t)\|\\ &\le (p^2b^2 - \eta + 2rpb + 2p\gamma)\|x\|^2.\end{aligned}$$

The proof of the theorem is completed by using condition (3.3) and Theorem 2.1. □

Theorem 3.3. *If $F(t,x)$ satisfies*

$$\|F(t,x)\| \le \gamma\|x\|^d,\ \ \forall\, t \ge 0,\ \ \forall\, x \in \mathbb{R}^n,$$

where $d > 1$, then the closed loop system (3.2) *by the feedback function* (3.1) *is locally uniformly asymptotically stable.*

Proof. Let us consider the Lyapunov function

$$V(t,x) = \langle P(t)x, x\rangle,\ \ t \in \mathbb{R}^+,\ \ x \in \mathbb{R}^n,$$

and let the feedback control be of form (3.1).

The derivative of V along the solutions of system (3.2) by using the chosen feedback control (3.1) and the RDE (1.2) gives

$$\begin{aligned}\dot{V}(t,x) &= \langle \dot{P}(t)x, x\rangle + 2\langle P(t)\dot{x}, x\rangle \\ &\leq (p^2b^2 - \eta + 2rpb)\|x\|^2 + 2\big\langle P(t)F(t,x), x(t)\big\rangle \\ &\leq (p^2b^2 - \eta + 2rpb)\|x\|^2 + 2\|P(t)\|\,\|F(t,x)\|\,\|x(t)\| \\ &\leq (p^2b^2 - \eta + 2rpb)\|x\|^2 + 2\gamma\|x\|^d\|P(t)\|\,\|x(t)\| \\ &\leq \big(p^2b^2 - \eta + 2rpb + 2p\gamma\|x\|^{d-1}\big)\|x\|^2.\end{aligned}$$

So, for x in a small neighborhood of the origin, $p^2b^2 - \eta + 2rpb + 2p\gamma\|x\|^{d-1} < -\rho < 0$. Then $\dot{V}(t,x) \leq -\rho\|x\|^2$, which implies that the origin is locally uniformly asymptotically stable. □

4 Example

Let us consider the bilinear time-varying control system

$$\dot{x}(t) = A(t)x(t) + u(t)B(t)x(t), \tag{4.1}$$

where $x(t) \in \mathbb{R}^2$,

$$A(t) = \begin{pmatrix} -e^{-t} & 1 \\ -1 & e^{-t}\end{pmatrix} \quad \text{and} \quad B(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t}\end{pmatrix}.$$

To verify the global null-controllablity of system (4.1), we apply Proposition (2.1)(ii). Denote

$$M(t) = \begin{bmatrix} e^{-t} & 0 & e^{-2t} - e^{-t} & -e^{-t} \\ 0 & e^{-t} & e^{-t} & -e^{-2t} - e^{-t}\end{bmatrix}.$$

It is easy to verify that $rank(M(t)) = 2$ for all $t \geq 0$. By taking $Q = 100I_2 \in \mathbf{M}([0,\infty), \mathbb{R}^2_+)$, the RDE (1.2) has a solution $P(t) \in \mathbf{M}([0,\infty), \mathbb{R}^2_+)$.

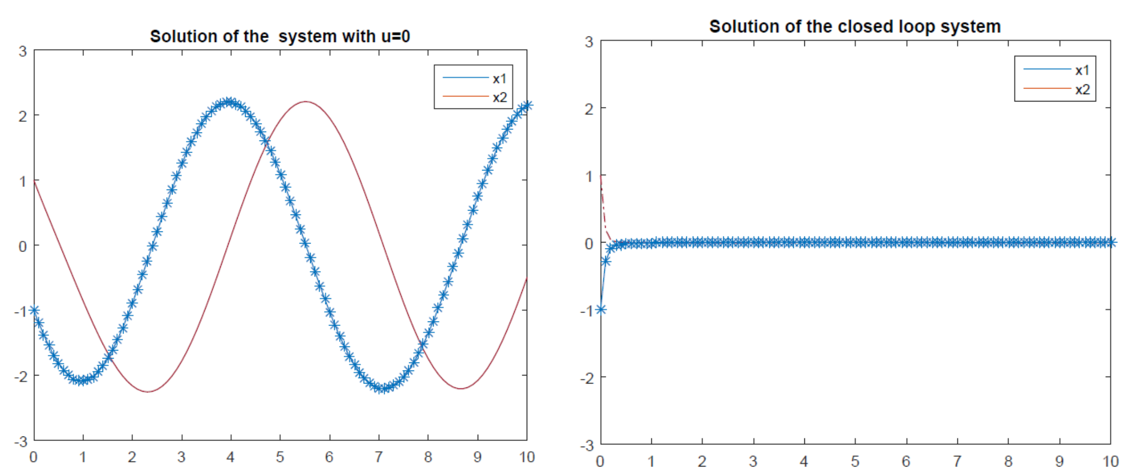

Figure 1. Dynamics of the closed BTV system $\dot{x}(t) = A(t)x(t) + u(t,x)B(t)x(t)$.

Using the Lyapunov function

$$V(t,x) = \langle P(t)x, x\rangle$$

and the feedback function

$$u(t,x) = -20\,\frac{\|B(t)\|\,\|P(t)\|\,\|x\|}{1 + \|B(t)\|\,\|P(t)\|\,\|x\|},$$

we verify that there exists $\alpha > 0$ such that

$$\dot{V}(t,x) \leq -\alpha\|x\|^2, \quad \forall t \in \mathbb{R}_+, \quad \forall x \in \mathbb{R}^n.$$

So, according to Theorem 3.1, system (4.1) is GUAS (see Figure 1).

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