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SINGULAR STRONGLY INCREASING SOLUTIONS OF HIGHER-ORDER QUASILINEAR ORDINARY DIFFERENTIAL EQUATIONS

Abstract. In the present paper, we consider quasilinear ordinary differential equations of the form

$$D(\alpha_n, \alpha_{n-1}, \dots, \alpha_1)x = p(t)|x|^{\beta} \operatorname{sgn} x, \quad t \geq a, \tag{1.1}$$

where $D(\alpha_n, \alpha_{n-1}, \dots, \alpha_1)$ is the nth-order iterated differential operator such that

$$D(\alpha_n, \alpha_{n-1}, \dots, \alpha_1)x \equiv D(\alpha_n)D(\alpha_{n-1}) \cdots D(\alpha_1)x$$

and, in general, $D(\alpha)$ is the first-order differential operator defined by $D(\alpha)x = (d/dt)(|x|^{\alpha} \operatorname{sgn} x)$ for $\alpha > 0$. For the case where $\alpha_1\alpha_2 \cdots \alpha_n < \beta$, we present a new sufficient condition for all strongly increasing solutions of (1.1) to be singular. If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$, then one of the main results, Corollary 3.2, gives an extension of the well-known theorem of Kiguradze and Chanturia [2].

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რეზიუმე. სტატიაში განხილულია შემდეგი სახის კვაზიწრფივი ჩვეულებრივი დიფერენციალური განტოლებები:

$$D(\alpha_n, \alpha_{n-1}, \dots, \alpha_1)x = p(t)|x|^{\beta} \operatorname{sgn} x, \quad t \geq a, \tag{1.1}$$

სადაც $D(\alpha_n, \alpha_{n-1}, \dots, \alpha_1)$ არის n რიგის იტერაციული დიფერენციალური ოპერატორი ისეთი, რომ

$$D(\alpha_n, \alpha_{n-1}, \dots, \alpha_1)x \equiv D(\alpha_n)D(\alpha_{n-1}) \cdots D(\alpha_1)x$$

და, ზოგადად, $D(\alpha)$ პირველი რიგის დიფერენციალური ოპერატორია $\alpha > 0$-თვის, განსაზღვრული $D(\alpha)x = (d/dt)(|x|^{\alpha} \operatorname{sgn} x)$ ფორმულით. იმ შემთხვევაში, როცა $\alpha_1\alpha_2 \cdots \alpha_n < \beta$, წარმოდგენილია ახალი საკმარისი პირობა იმისათვის, რომ (1.1)-ის ყველა მკაცრად ზრდადი ამონახსნი იყოს სინგულარული. თუ $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$, მაშინ ერთ-ერთი მთავარი შედეგი, კერძოდ, შედეგი 3.2, გვაძლევს კიღურაძისა და ჭანტურიას [2] ცნობილი თეორემის განზოგადებას.

1 Introduction

For a positive constant α, let $D(\alpha)$ be the first-order differential operator defined by

$$D(\alpha)x = \frac{d}{dt}\left(|x|^{\alpha}\operatorname{sgn} x\right),$$

and for n positive constants $\alpha_1, \alpha_2, \dots, \alpha_n$ let $D(\alpha_n, \alpha_{n-1}, \dots, \alpha_1)$ be the nth-order iterated differential operator defined by

$$D(\alpha_n, \alpha_{n-1}, \dots, \alpha_1)x = D(\alpha_n)D(\alpha_{n-1})\cdots D(\alpha_1)x.$$

Further, for an interval $I \subset \boldsymbol{R}$, we denote by $C(\alpha_i, \alpha_{i-1}, \dots, \alpha_1)(I)$ the set of all real-valued continuous functions $x(t)$ which are defined on I such that

$$D(\alpha_1)x(t), D(\alpha_2, \alpha_1)x(t), \dots, D(\alpha_i, \alpha_{i-1}, \dots, \alpha_1)x(t)$$

exist and are continuous on I $(i = 1, 2, \dots, n)$.

In this paper, we consider nth-order quasilinear ordinary differential equations of the form

$$D(\alpha_n, \alpha_{n-1}, \dots, \alpha_1)x = p(t)|x|^{\beta}\operatorname{sgn} x, \quad t \ge a, \tag{1.1}$$

where it is assumed that

(a) $n \ge 2$ is an integer;

(b) $\alpha_1, \alpha_2, \dots, \alpha_n$ and β are positive constants;

(c) $p(t)$ is a continuous function on an interval $[a, \infty)$, and $p(t) > 0$ on $[a, \infty)$.

By a solution $x(t)$ of (1.1) on a subinterval $I \subset [a, \infty)$, we mean that $x(t)$ belongs to the set $C(\alpha_n, \alpha_{n-1}, \dots, \alpha_1)(I)$ and satisfies (1.1) at every point $t \in I$. A solution $x(t)$ of (1.1) on an interval I $(\subset [a, \infty))$ is said to be *strongly increasing* on I if $x(t) \not\equiv 0$ on I and

$$D(\alpha_i, \dots, \alpha_1)x(t) \ge 0 \ \ (t \in I) \ \text{ for all } \ i = 0, 1, 2, \dots, n-1. \tag{1.2}$$

Here, if $i = 0$, then $D(\alpha_i, \dots, \alpha_1)x(t)$ is interpreted as $x(t)$. To make our idea definite, we will restrict our attention to solutions $x(t)$ of (1.1) which exist on some intervals of the form $I = [a, b)$, $a < b \le \infty$. Here, b may depend on the particular solution $x(t)$.

By the definition, a strongly increasing solution $x(t)$ of (1.1) on $[a, b)$, $a < b \le \infty$, satisfies

$$D(\alpha_i, \dots, \alpha_1)x(a) \ge 0 \ \text{ for all } \ i = 0, 1, 2, \dots, n-1.$$

Conversely, if $x(t)$ is a solution of (1.1) defined on a right neighborhood of a such that

$$\begin{cases} D(\alpha_i, \dots, \alpha_1)x(a) \ge 0 \ \text{ for all } \ i \in \{0, 1, 2, \dots, n-1\}, \\ D(\alpha_{i_0}, \dots, \alpha_1)x(a) > 0 \ \text{ for some } \ i_0 \in \{0, 1, 2, \dots, n-1\}, \end{cases}$$

then $x(t)$ is strongly increasing on the maximal interval $[a, b)$ of existence. The right end point b may be finite or infinite.

Suppose that $x(t)$ is a strongly increasing solution of (1.1) on $[a, b)$, and let $[a, b)$ be the maximal interval of existence of $x(t)$. If b is finite, then $x(t)$ is called *singular*. A singular strongly increasing solution is often said to be a *second kind singular* solution of (1.1).

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be positive constants and I $(\subset \boldsymbol{R})$ be an interval. To shorten the notation, we denote the set $C(\alpha_i, \dots, \alpha_1)(I)$ briefly by $C_i(I)$ $(i = 0, 1, 2, \dots, n)$. The set $C_0(I)$ is interpreted as $C(I)$: the set of all real-valued continuous functions on I. Furthermore, we set

$$D(\alpha_i, \dots, \alpha_1)x(t) = D_i x(t) \ \text{ for } \ i = 0, 1, 2, \dots, n.$$

Then equation (1.1) may be expressed as

$$D_n x = p(t)|x|^{\beta}\operatorname{sgn} x, \quad t \ge a, \tag{1.3}$$

and, for the case $I = [a, b)$ $(a < b \le \infty)$, condition (1.2) becomes

$$D_i x(t) \ge 0 \ \ (a \le t < b) \ \text{ for } \ i = 0, 1, 2, \ldots, n-1. \tag{1.4}$$

It can be proved [6, Theorem 4.1] that, for the case $\alpha_1\alpha_2\cdots\alpha_n \ge \beta$, all solutions of (1.3) can be continued to ∞. Therefore, in this case, none of strongly increasing solutions of (1.3) are singular.

On the other hand, the case $\alpha_1\alpha_2\cdots\alpha_n < \beta$ has a different aspect. A strongly increasing solution $x(t)$ of (1.3) may be singular. It is known [6, Theorem 4.3] that if $\alpha_1\alpha_2\cdots\alpha_n < \beta$, then (1.3) always has a singular solution. Now, consider the initial value problem of the form

$$\begin{cases} D_n x = p(t)|x|^\beta \operatorname{sgn} x, \\ x(a) = \lambda > 0 \ \text{ and } \ D_i x(a) = 0 \ \ (i = 1, 2, \ldots, n-1), \end{cases} \tag{1.5}$$

where the value of $x(a) = \lambda$ is regarded as a positive parameter. We know [6, Section 3] that the initial value problem (1.5) has a unique solution $x = x_\lambda(t)$ defined on a right neighborhood of a, and that $x_\lambda(t)$ is strongly increasing on the maximal interval of existence. Furthermore, we have the following result (see [6, Theorem 6.1]). Suppose that $\alpha_1\alpha_2\cdots\alpha_n < \beta$ and

$$\int_a^\infty (t-a)^{r_{n-1}\beta} p(t)\, dt < \infty, \tag{1.6}$$

where

$$r_{n-1} = \frac{1}{\alpha_1} + \frac{1}{\alpha_1\alpha_2} + \cdots + \frac{1}{\alpha_1\alpha_2\cdots\alpha_{n-2}} + \frac{1}{\alpha_1\alpha_2\cdots\alpha_{n-2}\alpha_{n-1}}\,. \tag{1.7}$$

Then there exists $\lambda^* > 0$ such that

(i) if $\lambda \in (0, \lambda^*]$, then $x_\lambda(t)$ exists on $[a, \infty)$, i.e., $x_\lambda(t)$ is not singular

and

(ii) if $\lambda \in (\lambda^*, \infty)$, then $x_\lambda(t)$ is singular.

Roughly speaking, condition (1.6) means that $p(t)$ is small enough in a neighborhood of ∞. Conversely, if $p(t)$ is large enough in a neighborhood of ∞, then all of strongly increasing solutions of (1.1) are singular. Actually, Naito and Usami [6, Theorem 6.2] have proved the following theorem.

Theorem A. *Let $\alpha_1\alpha_2\cdots\alpha_n < \beta$. If*

$$\liminf_{t\to\infty} t^{r_{n-1}\beta+1} p(t) > 0, \tag{1.8}$$

then all of strongly increasing solutions of (1.1) *are singular.*

The main purpose of this paper is to show that Theorem A can be generalized in the following way.

For the positive constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ appearing in equation (1.1), we put

$$\mu_n = \alpha_2 + \big(\alpha_2\alpha_3 + \alpha_3\big) + \cdots + \big(\alpha_2\alpha_3\cdots\alpha_n + \alpha_3\alpha_4\cdots\alpha_n + \cdots + \alpha_{n-1}\alpha_n + \alpha_n\big), \tag{1.9}$$

$$\nu_n = \alpha_2\alpha_3\cdots\alpha_n + \alpha_3\alpha_4\cdots\alpha_n + \cdots + \alpha_{n-1}\alpha_n + \alpha_n, \tag{1.10}$$

$$\xi_n = \alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_3 + \cdots + \alpha_1\alpha_2\cdots\alpha_{n-1} + \alpha_1\alpha_2\cdots\alpha_n, \tag{1.11}$$

and

$$\begin{aligned} \zeta_{n-1} = &\Big(1 + \frac{1}{\alpha_2} + \frac{1}{\alpha_2\alpha_3} + \cdots + \frac{1}{\alpha_2\alpha_3\cdots\alpha_{n-1}}\Big) \\ &+ \Big(1 + \frac{1}{\alpha_3} + \frac{1}{\alpha_3\alpha_4} + \cdots + \frac{1}{\alpha_3\alpha_4\cdots\alpha_{n-1}}\Big) + \cdots \\ &+ \Big(1 + \frac{1}{\alpha_{n-2}} + \frac{1}{\alpha_{n-2}\alpha_{n-1}}\Big) + \Big(1 + \frac{1}{\alpha_{n-1}}\Big) + 1. \end{aligned} \tag{1.12}$$

Note that if $n = 2$, then $\zeta_1 = 1$. As an important relation between these numbers, we have

$$\zeta_{n-1} = -\mu_n + \xi_n r_{n-1}.$$

Further, we have $\nu_n = \alpha_1\alpha_2\cdots\alpha_n r_{n-1}$. Here, r_{n-1} is defined by (1.7).

Then we can show the following theorem.

Theorem 1.1. *Let $\alpha_1\alpha_2\cdots\alpha_n < \beta$. Let μ_n, ν_n, ξ_n, ζ_{n-1} and r_{n-1} be the numbers defined by* (1.9), (1.10), (1.11), (1.12) *and* (1.7), *respectively. Suppose that there exist $\sigma > 0$ and $\tau > 0$ such that*

$$(\nu_n + 1)\sigma + \mu_n\tau - 1 \le 0, \tag{1.13}$$

$$\Big(\frac{\beta}{\alpha_1\alpha_2\cdots\alpha_n}\nu_n + 1\Big)\sigma + \Big(\mu_n - \frac{\nu_n\xi_n}{\alpha_1\alpha_2\cdots\alpha_n}\Big)\tau - 1 \ge 0, \tag{1.14}$$

and either

$$\int\limits_{a^+}^{\infty} s^{-\zeta_{n-1}\tau-1+(r_{n-1}\beta+1)\sigma}p(s)^\sigma\,ds = \infty \ \ (a^+ > \max\{a,0\}) \tag{1.15}$$

or else

$$\limsup_{t\to\infty} t^{\zeta_{n-1}\tau}\int\limits_{t}^{\infty} s^{-\zeta_{n-1}\tau-1+(r_{n-1}\beta+1)\sigma}p(s)^\sigma\,ds > 0. \tag{1.16}$$

Then all of strongly increasing solutions of (1.1) *are singular.*

If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$, then

$$D_i x(t) = x^{(i)}(t) \ \ (i = 0, 1, 2, \dots, n),$$

and so equation (1.1) is reduced to

$$x^{(n)} = p(t)|x|^\beta \operatorname{sgn} x, \quad t \ge a. \tag{1.17}$$

If $n = 2$ and $\alpha_1 = 1$, $\alpha_2 = \alpha$ (> 0), then (1.1) is the second-order quasilinear differential equation

$$\big(|x'|^\alpha \operatorname{sgn} x'\big)' = p(t)|x|^\beta \operatorname{sgn} x, \quad t \ge a. \tag{1.18}$$

Results on the problem of existence and asymptotic behavior of strongly increasing solutions of (1.17) are summarized and proved in the book of Kiguradze and Chanturia [2]. This problem has also been studied by Mizukami, Naito and Usami [3] for equation (1.18), and by Naito and Usami [6] for the general equation (1.1).

The proof of Theorem 1.1 is given in Section 2. In Section 3, Theorem 1.1 is restated in different forms, and some important corollaries are mentioned.

A function $x(t)$ is said to be a *Kneser solution* of the equation

$$D_n x = (-1)^n p(t)|x|^\beta \operatorname{sgn} x, \quad t \ge a, \tag{1.19}$$

on $[a,\infty)$ if $x(t)$ is a nontrivial solution of (1.19) on $[a,\infty)$ and satisfies

$$(-1)^i D_i x(t) \ge 0 \ \ (t \ge a) \ \text{ for all } \ i = 0, 1, 2, \dots, n-1.$$

Moreover, a Kneser solution $x(t)$ of (1.19) is said to be *singular* if there is $b > a$ such that

$$x(t) > 0 \ \ (a \le t < b) \ \text{ and } \ x(t) \equiv 0 \ \ (t \ge b).$$

A singular Kneser solution of (1.19) is also said to be a *first kind singular* solution of (1.19). There is a remarkable duality between Kneser solutions of (1.19) and strongly increasing solutions of (1.3) (see [6, 7]). In [5], the first author has established a new sufficient condition for all Kneser solutions of (1.19) to be singular. The present paper corresponds to [5].

2 Proof of Theorem 1.1

For brevity, we use the notation

$$\xi^{\alpha*} = |\xi|^{\alpha} \operatorname{sgn} \xi, \quad \xi \in \boldsymbol{R}, \quad \alpha > 0.$$

It is clear that for each $\alpha > 0$, $f(\xi) = \xi^{\alpha*}$ is a continuous and strictly increasing function of $\xi \in \boldsymbol{R}$. Expediently, if $\xi = \infty$ [resp. $\xi = -\infty$], then $\xi^{\alpha*} = \infty$ [resp. $\xi^{\alpha*} = -\infty$].

Let $I \subset \boldsymbol{R}$ be an interval and let $x \in C(\alpha)(I)$, $\alpha > 0$. Then it follows from the equality $D(\alpha)x(t) = (d/dt)x(t)^{\alpha*}$ that

$$x(t) = \left(x(\tau)^{\alpha*} + \int_{\tau}^{t} D(\alpha)x(s)\, ds \right)^{(1/\alpha)*}, \quad t, \tau \in I. \tag{2.1}$$

Therefore, by (2.1), we see that if $D(\alpha)x(t) \geq 0$ [resp. > 0, ≤ 0, < 0] on I, then $x(t)$ is increasing [resp. strictly increasing, decreasing, strictly decreasing] on I.

Moreover, we have the following

Lemma 2.1. *Suppose that x, $y \in C(\alpha)[a, \infty)$, $\alpha > 0$. If*

$$\lim_{t\to\infty} y(t) = \infty \quad \textit{and} \quad \lim_{t\to\infty} \frac{D(\alpha)x(t)}{D(\alpha)y(t)} = \ell \in \boldsymbol{R} \cup \{\infty\} \cup \{-\infty\},$$

then

$$\lim_{t\to\infty} \frac{x(t)}{y(t)} = \ell^{(1/\alpha)*}.$$

Proof. Since $\lim_{t\to\infty} y(t)^{\alpha*} = \infty$, it follows from L'Hospital's rule that

$$\lim_{t\to\infty} \frac{x(t)}{y(t)} = \left[\lim_{t\to\infty} \frac{x(t)^{\alpha*}}{y(t)^{\alpha*}} \right]^{(1/\alpha)*} = \left[\lim_{t\to\infty} \frac{D(\alpha)x(t)}{D(\alpha)y(t)} \right]^{(1/\alpha)*} = \ell^{(1/\alpha)*}.$$

The proof of Lemma 2.1 is complete. □

In Lemma 2.1, take $y(t) = t^{1/\alpha}$, $\alpha > 0$. Then we find that

$$\lim_{t\to\infty} D(\alpha)x(t) = \ell \in \boldsymbol{R} \cup \{\infty\} \cup \{-\infty\}$$

implies

$$\lim_{t\to\infty} \frac{x(t)}{t^{1/\alpha}} = \ell^{(1/\alpha)*}.$$

Now, let $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ be fixed positive constants. Then we define the numbers $r(i)$ $(i = 0, 1, 2, \ldots, n-1)$ by

$$\begin{cases} r(0) = 0 \\ r(i) = \dfrac{1}{\alpha_{n-i}} + \dfrac{1}{\alpha_{n-i}\alpha_{n-i+1}} + \cdots + \dfrac{1}{\alpha_{n-i}\alpha_{n-i+1}\cdots\alpha_{n-1}}, \quad i = 1, \ldots, n-1. \end{cases} \tag{2.2}$$

Observe that the number $r(n-1)$ is identical with the number r_{n-1} which is given by (1.7). In addition, we define the positive numbers $k(i)$ $(i = 0, 1, 2, \ldots, n-1)$ by

$$\begin{cases} k(0) = 1, \quad k(1) = 1 \\ k(i) = [1 + r(1)]^{-1/(\alpha_{n-i}\alpha_{n-i+1}\cdots\alpha_{n-2})}[1 + r(2)]^{-1/(\alpha_{n-i}\alpha_{n-i+1}\cdots\alpha_{n-3})} \cdots \\ \qquad \times [1 + r(i-2)]^{-1/(\alpha_{n-i}\alpha_{n-i+1})}[1 + r(i-1)]^{-1/\alpha_{n-i}}, \quad i = 2, \ldots, n-1. \end{cases} \tag{2.3}$$

Applying Lemma 2.1 repeatedly, we have the following

Lemma 2.2. *Let $\alpha_i > 0$ $(i = 1, 2, \ldots, n-1)$ and suppose that $x \in C_{n-1}[a, \infty)$. If*

$$\lim_{t\to\infty} D_{n-1}x(t) = \ell \in \boldsymbol{R} \cup \{\infty\} \cup \{-\infty\},$$

then

$$\lim_{t\to\infty} \frac{D_{n-i-1}x(t)}{t^{r(i)}} = k(i)\ell^{(1/[\alpha_{n-i}\alpha_{n-i+1}\cdots\alpha_{n-1}])*}, \quad i = 1, 2, \ldots, n-1,$$

where $r(i)$ and $k(i)$ are defined by (2.2) *and* (2.3)*, respectively.*

Lemma 2.2 will be effectively used in the proof of Theorem 1.1.

Suppose that $\alpha_1\alpha_2\cdots\alpha_n < \beta$, and let $x(t) \not\equiv 0$ be a strongly increasing solution of equation (1.3) on $[a, b)$, $a < b \le \infty$. We have (1.4). In addition, it follows from (1.3) that $D_n x(t) \ge 0$, $\not\equiv 0$ on $[a, b)$. Therefore, $D_i x(t)$ $(i = 0, 1, 2, \ldots, n-1)$ are increasing on $[a, b)$. It can be found that

$$D_i x(t) > 0, \;\; a < t < b, \;\; i = 0, 1, 2, \ldots, n-1.$$

To see this, suppose that there is $i_0 = 0, 1, 2, \ldots, n-1$ such that $D_{i_0}x(c) = 0$ for some c, $a < c < b$. Since $D_{i_0}x(t)$ is nonnegative and increasing on $[a, c]$, this implies that $D_{i_0}x(t) = 0$ on $[a, c]$, and so $D_n x(t) = 0$ on $[a, c]$. Then, by (1.3), we have $x(t) = 0$ on $[a, c]$, and, in particular, $D_i x(a) = 0$ for all $i = 0, 1, 2, \ldots, n-1$. Therefore, we have $x(t) \equiv 0$ on $[a, b)$ (see [7, Theorem 5.1]). This is a contradiction.

Proof of Theorem 1.1. The proof is done by contradiction. Suppose that equation (1.1) has a strongly increasing solution $x(t) \not\equiv 0$ on the entire interval $[a, \infty)$. By the above remark, we see that

$$D_i x(t) > 0, \;\; t > a, \;\; i = 0, 1, 2, \ldots, n-1. \tag{2.4}$$

Since $D_{n-1}x(t)$ is positive and (strictly) increasing on (a, ∞), we have either

$$\lim_{t\to\infty} D_{n-1}x(t) = \infty \tag{2.5}$$

or

$$\lim_{t\to\infty} D_{n-1}x(t) \;\text{ exists and is a positive finite value.} \tag{2.6}$$

As the first case, assume that (2.5) holds. Then, by Lemma 2.2,

$$\lim_{t\to\infty} \frac{D_{n-i-1}x(t)}{t^{r(i)}} = \infty, \;\; i = 1, 2, \ldots, n-1. \tag{2.7}$$

Therefore, by (2.5) and (2.7), we see that there exists a number $a_1 > \max\{a, 0\}$ such that

$$\frac{D_{n-i-1}x(t)}{t^{r(i)}} \ge 1 \;\text{ for }\; t \ge a_1 \;\; (i = 0, 1, 2, \ldots, n-1).$$

In particular,

$$D_{n-1}x(t) \ge 1 \;\text{ and }\; x(t) \ge t^{r_{n-1}} \;\text{ for }\; t \ge a_1. \tag{2.8}$$

For $i = 0, 1, 2, \ldots, n-1$, define the functions $\omega_{n-i-1}(t)$ by

$$\omega_{n-i-1}(t) = \frac{D_{n-i-1}x(t)}{t^{r(i)}}\,, \;\; t \ge a_1. \tag{2.9}$$

Then we have

$$\lim_{t\to\infty} \omega_{n-i-1}(t) = \infty \;\; (i = 0, 1, 2, \ldots, n-1). \tag{2.10}$$

Now, define the numbers $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n$ by

$$\begin{aligned}
\lambda_1 &= \frac{1}{\nu_n}\,\alpha_2\cdots\alpha_n(1-\sigma-\mu_n\tau) + \big(\alpha_2+\alpha_2\alpha_3+\cdots+\alpha_2\cdots\alpha_n\big)\tau,\\
\lambda_2 &= \frac{1}{\nu_n}\,\alpha_3\cdots\alpha_n(1-\sigma-\mu_n\tau) + \big(\alpha_3+\alpha_3\alpha_4+\cdots+\alpha_3\cdots\alpha_n\big)\tau,\\
&\ \ \vdots\\
\lambda_{n-1} &= \frac{1}{\nu_n}\,\alpha_n(1-\sigma-\mu_n\tau) + \alpha_n\tau,\\
\lambda_n &= \sigma,
\end{aligned}$$

where σ and τ are positive constants satisfying (1.13) and (1.14), and μ_n and ν_n are given by (1.9) and (1.10), respectively. It is easy to see that

$$\lambda_1+\lambda_2+\cdots+\lambda_n = 1, \tag{2.11}$$

$$\lambda_i - \alpha_{i+1}\lambda_{i+1} = \alpha_{i+1}\tau \quad (i=1,2,\dots,n-2). \tag{2.12}$$

Further, we have

$$\lambda_i > 0 \quad (i=1,2,\dots,n). \tag{2.13}$$

In fact, it follows from (1.13) that

$$\lambda_{n-1} - \alpha_n(\sigma+\tau) = -\frac{\alpha_n}{\nu_n}\big[(\nu_n+1)\sigma+\mu_n\tau-1\big] \ge 0, \tag{2.14}$$

which yields

$$\lambda_{n-1} \ge \alpha_n(\sigma+\tau) > 0.$$

Therefore, by induction, (2.12) gives

$$\lambda_i = \alpha_{i+1}\lambda_{i+1} + \alpha_{i+1}\tau > 0 \ \text{ for } \ i = n-2, n-3, \dots, 1.$$

Clearly, $\lambda_n = \sigma > 0$. Thus we have (2.13).

Next, define the function $y(t)$ by

$$y(t) = x(t)^{\alpha_1}[D_1x(t)]^{\alpha_2}[D_2x(t)]^{\alpha_3}\cdots[D_{n-1}x(t)]^{\alpha_n}, \quad t \ge a_1. \tag{2.15}$$

By (2.4), we have $y(t) > 0$ ($t \ge a_1$). It is easy to find that the derivative $y'(t)$ of $y(t)$ is given by

$$y'(t) = \left[\frac{D_1x(t)}{x(t)^{\alpha_1}} + \frac{D_2x(t)}{[D_1x(t)]^{\alpha_2}} + \cdots + \frac{D_nx(t)}{[D_{n-1}x(t)]^{\alpha_n}}\right]y(t), \quad t \ge a_1. \tag{2.16}$$

As a general inequality, we have

$$u_1^{\lambda_1}u_2^{\lambda_2}\cdots u_n^{\lambda_n} \le \lambda_1u_1+\lambda_2u_2+\cdots+\lambda_nu_n$$

for $u_i \ge 0$, $\lambda_i > 0$, $\sum_{i=1}^{n}\lambda_i = 1$ (see, for example, [1, pp. 13–14]). This inequality may be written equivalently as

$$Lv_1^{\lambda_1}v_2^{\lambda_2}\cdots v_n^{\lambda_n} \le v_1+v_2+\cdots+v_n \ \text{ with } \ L = \lambda_1^{-\lambda_1}\lambda_2^{-\lambda_2}\cdots\lambda_n^{-\lambda_n} \tag{2.17}$$

for $v_i \ge 0$, $\lambda_i > 0$, $\sum_{i=1}^{n}\lambda_i = 1$. Therefore, by (2.16) and (2.17) of the case $v_i = D_ix(t)/[D_{i-1}x(t)]^{\alpha_i}$, we obtain

$$\begin{aligned}
y'(t) &\ge L\left[\frac{D_1x(t)}{x(t)^{\alpha_1}}\right]^{\lambda_1}\left[\frac{D_2x(t)}{[D_1x(t)]^{\alpha_2}}\right]^{\lambda_2}\cdots\left[\frac{D_nx(t)}{[D_{n-1}x(t)]^{\alpha_n}}\right]^{\lambda_n}y(t)\\
&= Lx(t)^{-\alpha_1\lambda_1}\big[D_1x(t)\big]^{\lambda_1-\alpha_2\lambda_2}\cdots\\
&\qquad\times\big[D_{n-2}x(t)\big]^{\lambda_{n-2}-\alpha_{n-1}\lambda_{n-1}}\big[D_{n-1}x(t)\big]^{\lambda_{n-1}-\alpha_n\lambda_n}[D_nx(t)]^{\lambda_n}y(t)
\end{aligned}$$

for $t \geq a_1$. Then, on account of (2.12), (1.3) and (2.15), we see that

$$\begin{aligned} y'(t) &\geq Lx(t)^{-\alpha_1\lambda_1-\alpha_1\tau+\beta\lambda_n} x(t)^{\alpha_1\tau}\big[D_1x(t)\big]^{\alpha_2\tau}\cdots \\ &\quad\times \big[D_{n-2}x(t)\big]^{\alpha_{n-1}\tau}\big[D_{n-1}x(t)\big]^{\alpha_n\tau}\big[D_{n-1}x(t)\big]^{-\alpha_n\tau+\lambda_{n-1}-\alpha_n\lambda_n} p(t)^{\lambda_n}y(t) \\ &= Lx(t)^{-\alpha_1\lambda_1-\alpha_1\tau+\beta\sigma}\big[D_{n-1}x(t)\big]^{\lambda_{n-1}-\alpha_n(\tau+\sigma)} p(t)^{\sigma}y(t)^{1+\tau} \end{aligned}$$

for $t \geq a_1$. Since

$$-\alpha_1\lambda_1-\alpha_1\tau+\beta\sigma = \frac{\alpha_1\alpha_2\cdots\alpha_n}{\nu_n}\bigg[\Big(\frac{\beta}{\alpha_1\alpha_2\cdots\alpha_n}\,\nu_n+1\Big)\sigma+\Big(\mu_n-\frac{\nu_n\xi_n}{\alpha_1\alpha_2\cdots\alpha_n}\Big)\tau-1\bigg] \geq 0$$

(see (1.14)) and $\lambda_{n-1}-\alpha_n(\tau+\sigma) \geq 0$ (see (2.14)), it follows from (2.8) that

$$y'(t) \geq Lt^{(-\alpha_1\lambda_1-\alpha_1\tau+\beta\sigma)r_{n-1}} p(t)^{\sigma}y(t)^{1+\tau} = Lt^{-\zeta_{n-1}\tau-1+(r_{n-1}\beta+1)\sigma}p(t)^{\sigma}y(t)^{1+\tau},\ \ t \geq a_1.$$

From this inequality it is seen that

$$-y(T)^{-\tau}+y(t)^{-\tau} \geq \tau L\int_t^T s^{-\zeta_{n-1}\tau-1+(r_{n-1}\beta+1)\sigma}p(s)^{\sigma}\,ds,\quad a_1 \leq t \leq T.$$

Then, letting $T \to \infty$, we find that

$$\int_{a_1}^{\infty} s^{-\zeta_{n-1}\tau-1+(r_{n-1}\beta+1)\sigma}p(s)^{\sigma}\,ds < \infty \tag{2.18}$$

and

$$y(t)^{-\tau} \geq \tau L\int_t^{\infty} s^{-\zeta_{n-1}\tau-1+(r_{n-1}\beta+1)\sigma}p(s)^{\sigma}\,ds,\ \ t \geq a_1. \tag{2.19}$$

By (2.9) and (2.15), we have

$$\begin{aligned} y(t) &= [t^{r(n-1)}\omega_0(t)]^{\alpha_1}[t^{r(n-2)}\omega_1(t)]^{\alpha_2}\cdots[t^{r(0)}\omega_{n-1}(t)]^{\alpha_n} \\ &= t^{r(n-1)\alpha_1+r(n-2)\alpha_2+\cdots+r(1)\alpha_{n-1}}\omega_0(t)^{\alpha_1}\omega_1(t)^{\alpha_2}\cdots\omega_{n-1}(t)^{\alpha_n} \\ &= t^{\zeta_{n-1}}\omega_0(t)^{\alpha_1}\omega_1(t)^{\alpha_2}\cdots\omega_{n-1}(t)^{\alpha_n}. \end{aligned}$$

Therefore, (2.19) gives

$$\big\{\omega_0(t)^{\alpha_1}\omega_1(t)^{\alpha_2}\cdots\omega_{n-1}(t)^{\alpha_n}\big\}^{-\tau} \geq \tau Lt^{\zeta_{n-1}\tau}\int_t^{\infty} s^{-\zeta_{n-1}\tau-1+(r_{n-1}\beta+1)\sigma}p(s)^{\sigma}\,ds,\ \ t \geq a_1.$$

Then it follows from (2.10) that

$$\lim_{t\to\infty} t^{\zeta_{n-1}\tau}\int_t^{\infty} s^{-\zeta_{n-1}\tau-1+(r_{n-1}\beta+1)\sigma}p(s)^{\sigma}\,ds = 0. \tag{2.20}$$

As the second case, assume that (2.6) holds. Let $\lim\limits_{t\to\infty} D_{n-1}x(t) = \ell \in (0,\infty)$. By Lemma 2.2, we have

$$\lim_{t\to\infty}\frac{x(t)}{t^{r_{n-1}}} = k(n-1)\ell^{1/[\alpha_1\alpha_2\cdots\alpha_{n-1}]} \in (0,\infty).$$

Then it can be concluded [4, Corollary 1.4] that

$$\int_{a_1}^{\infty} s^{r_{n-1}\beta}p(s)\,ds < \infty, \tag{2.21}$$

where a_1 is a number satisfying $a_1 > \max\{a, 0\}$. We will show that (2.21) implies (2.18) and (2.20). On account of (1.13), we obtain $\sigma < 1$. Then Hölder's inequality gives

$$\int_t^T s^{-\zeta_{n-1}\tau-1+(r_{n-1}\beta+1)\sigma}p(s)^\sigma\,ds \le \Big(\int_t^T s^{r_{n-1}\beta}p(s)\,ds\Big)^\sigma \Big(\int_t^T s^{-[\zeta_{n-1}\tau/(1-\sigma)]-1}\,ds\Big)^{1-\sigma}$$
$$= \Big(\int_t^T s^{r_{n-1}\beta}p(s)\,ds\Big)^\sigma \Big(-\frac{1-\sigma}{\zeta_{n-1}\tau}\big[T^{-\zeta_{n-1}\tau/(1-\sigma)} - t^{-\zeta_{n-1}\tau/(1-\sigma)}\big]\Big)^{1-\sigma}$$

for $a_1 \le t < T$. Letting $T \to \infty$ in the above inequality, we get (2.18) and

$$t^{\zeta_{n-1}\tau}\int_t^\infty s^{-\zeta_{n-1}\tau-1+(r_{n-1}\beta+1)\sigma}p(s)^\sigma\,ds \le \Big(\frac{1-\sigma}{\zeta_{n-1}\tau}\Big)^{1-\sigma}\Big(\int_t^\infty s^{r_{n-1}\beta}p(s)\,ds\Big)^\sigma, \quad t \ge a_1. \tag{2.22}$$

Then it is clear that (2.21) and (2.22) imply (2.20).

We have proved that if equation (1.1) has a strongly increasing solution on $[a, \infty)$, then (2.18) and (2.20) hold. Therefore, if either (1.15) or (1.16) holds, then a strongly increasing solution of (1.1) cannot exist on $[a, \infty)$. The proof of Theorem 1.1 is complete. □

For the case $n = 2$, $\alpha_1 = 1$ and $\alpha_2 = \alpha > 0$, equation (1.1) becomes (1.18). In this case, we have

$$r_1 = 1, \quad \mu_2 = \alpha, \quad \nu_2 = \alpha, \quad \xi_2 = 1+\alpha, \quad \zeta_1 = 1.$$

Therefore, Theorem 1.1 gives an extension of Theorem 2.7 of [3]. The lim inf in condition (2.4) of Theorem 2.7 of [3] can be replaced to lim sup.

3 Different forms of Theorem 1.1

It should be remarked that

$$\nu_n\xi_n - \alpha_1\alpha_2\cdots\alpha_n\mu_n > 0,$$

where μ_n, ν_n and ξ_n are defined by (1.9), (1.10) and (1.11), respectively. Therefore, the term $\mu_n - [(\nu_n\xi_n)/(\alpha_1\alpha_2\cdots\alpha_n)]$ appearing in (1.14) is a negative number. For simplicity of notation, we put

$$\eta_n = \frac{\nu_n\xi_n}{\alpha_1\alpha_2\cdots\alpha_n\mu_n} - 1.$$

By the above remark, η_n is a positive number.

Now, let $\alpha_1\alpha_2\cdots\alpha_n < \beta$. We easily find that $\sigma > 0$ and $\tau > 0$ satisfy (1.13) and (1.14) if and only if

$$\frac{1}{[\beta/(\alpha_1\alpha_2\cdots\alpha_n)]\nu_n + 1} < \sigma < \frac{1}{\nu_n+1} \tag{3.1}$$

and

$$0 < \tau \le \frac{1}{\mu_n}\min\Big\{1-(\nu_n+1)\sigma,\ \frac{1}{\eta_n}\Big[\Big(\frac{\beta}{\alpha_1\alpha_2\cdots\alpha_n}\,\nu_n+1\Big)\sigma - 1\Big]\Big\}. \tag{3.2}$$

Suppose first that $\sigma > 0$ satisfies (3.1). Next, choose $\tau > 0$ so that the equality holds in the latter half of (3.2), and put $\tau = \tau(\sigma)$. More precisely,

$$\tau(\sigma) = \frac{1}{\mu_n}\min\Big\{1-(\nu_n+1)\sigma,\ \frac{1}{\eta_n}\Big[\Big(\frac{\beta}{\alpha_1\alpha_2\cdots\alpha_n}\,\nu_n+1\Big)\sigma - 1\Big]\Big\}. \tag{3.3}$$

Then conditions (1.15) and (1.16) become

$$\int_{a^+}^\infty s^{-\zeta_{n-1}\tau(\sigma)-1+(r_{n-1}\beta+1)\sigma}p(s)^\sigma\,ds = \infty \quad (a^+ > \max\{a, 0\}) \tag{3.4}$$

and

$$\limsup_{t\to\infty} t^{\zeta_{n-1}\tau(\sigma)} \int_t^\infty s^{-\zeta_{n-1}\tau(\sigma)-1+(r_{n-1}\beta+1)\sigma} p(s)^\sigma \, ds > 0, \tag{3.5}$$

respectively. Therefore, Theorem 1.1 produces the following result.

Theorem 3.1. *Let $\alpha_1\alpha_2\cdots\alpha_n < \beta$. Suppose that σ satisfies* (3.1)*. Define $\tau(\sigma)$ by* (3.3)*. If either* (3.4) *or* (3.5) *holds, then all of strongly increasing solutions of* (1.1) *are singular.*

Example. Consider the fourth-order equation

$$\big(|x''|^\alpha \operatorname{sgn} x''\big)'' = \kappa t^{-1-[(2\alpha+1)\beta/\alpha]}\Big(1+\frac{t}{t+1}\sin t\Big)|x|^\beta \operatorname{sgn} x, \quad t \ge 1, \tag{3.6}$$

where $0<\alpha<\beta$, and κ is a positive constant. In this equation, $n=4$ and $\alpha_1=1$, $\alpha_2=1$, $\alpha_3=\alpha$, $\alpha_4=1$, and $p(t)=\kappa t^{-1-[(2\alpha+1)\beta/\alpha]}(1+\varphi(t)\sin t)$, where $\varphi(t)=t/(t+1)$. We have

$$r_3=\frac{2\alpha+1}{\alpha}\,, \quad \mu_4=2(2\alpha+1), \quad \nu_4=2\alpha+1, \quad \xi_4=2(\alpha+1), \quad \zeta_3=\frac{2(2\alpha+1)}{\alpha}\,, \quad \eta_4=\frac{1}{\alpha}\,.$$

If $\varepsilon_0>0$ is taken sufficiently small, then

$$\frac{1}{[\beta/\alpha](2\alpha+1)+1} < \frac{1-\varepsilon_0}{2(\alpha+1)} < \frac{1}{2(\alpha+1)}$$

and

$$\varepsilon_0 < \alpha\bigg[\Big(\frac{\beta}{\alpha}\,(2\alpha+1)+1\Big)\,\frac{1-\varepsilon_0}{2(\alpha+1)}-1\bigg].$$

For such an $\varepsilon_0>0$, put

$$\sigma=\frac{1-\varepsilon_0}{2(\alpha+1)}\,.$$

Then σ satisfies (3.1). Further, the number $\tau(\sigma)$ is given by

$$\tau(\sigma)=\frac{1}{2(2\alpha+1)}\,\min\Big\{1-2(\alpha+1)\sigma,\ \alpha\Big[\Big(\frac{\beta}{\alpha}\,(2\alpha+1)+1\Big)\sigma-1\Big]\Big\}=\frac{\varepsilon_0}{2(2\alpha+1)}\,.$$

Then we have

$$t^{\zeta_3\tau(\sigma)}\int_t^\infty s^{-\zeta_3\tau(\sigma)-1+(r_3\beta+1)\sigma}p(s)^\sigma\,ds$$
$$=\kappa^{(1-\varepsilon_0)/[2(\alpha+1)]}t^{\varepsilon_0/\alpha}\int_t^\infty s^{-1-(\varepsilon_0/\alpha)}(1+\varphi(s)\sin s)^{(1-\varepsilon_0)/[2(\alpha+1)]}\,ds.$$

It is seen that for $m=1,2,\dots,$

$$\int_{2m\pi}^\infty s^{-1-(\varepsilon_0/\alpha)}(1+\varphi(s)\sin s)^{(1-\varepsilon_0)/[2(\alpha+1)]}\,ds \ge \sum_{i=0}^\infty \int_{2(m+i)\pi}^{(2(m+i)+1)\pi} s^{-1-(\varepsilon_0/\alpha)}\,ds$$
$$\ge \sum_{i=0}^\infty \big[(2(m+i)+1)\pi\big]^{-1-(\varepsilon_0/\alpha)}\pi \ge \pi^{-\varepsilon_0/\alpha}\int_m^\infty (2s+1)^{-1-(\varepsilon_0/\alpha)}\,ds = \pi^{-\varepsilon_0/\alpha}\,\frac{\alpha}{2\varepsilon_0}\,(2m+1)^{-\varepsilon_0/\alpha},$$

and so,

$$\liminf_{m\to\infty}\,(2m\pi)^{\varepsilon_0/\alpha}\int_{2m\pi}^\infty s^{-1-(\varepsilon_0/\alpha)}(1+\varphi(s)\sin s)^{(1-\varepsilon_0)/[2(\alpha+1)]}\,ds \ge \frac{\alpha}{2\varepsilon_0} > 0.$$

Consequently, we find that

$$\limsup_{t\to\infty} t^{\zeta_3\tau(\sigma)} \int_t^\infty s^{-\zeta_3\tau(\sigma)-1+(r_3\beta+1)\sigma} p(s)^\sigma\, ds > 0.$$

By Theorem 3.1, it is concluded that all of strongly increasing solutions of (3.6) are singular. Note that Theorem A cannot be applied to equation (3.6), since $\liminf_{t\to\infty} t^{r_3\beta+1}p(t) = 0$.

Now, let $\alpha_1\alpha_2\cdots\alpha_n < \beta$, and put

$$\sigma_n = \frac{\eta_n + 1}{[\beta/(\alpha_1\alpha_2\cdots\alpha_n)]\nu_n + 1 + \eta_n(\nu_n+1)}\,. \tag{3.7}$$

We have

$$\frac{1}{[\beta/(\alpha_1\alpha_2\cdots\alpha_n)]\nu_n + 1} < \sigma_n < \frac{1}{\nu_n+1}\,.$$

It is easily seen that if σ satisfies

$$\sigma_n \le \sigma < \frac{1}{\nu_n+1}\,, \tag{3.8}$$

then the number $\tau(\sigma)$ which is defined by (3.3) is

$$\tau(\sigma) = \frac{1}{\mu_n}\big[1-(\nu_n+1)\sigma\big]. \tag{3.9}$$

Therefore, Theorem 3.1 produces the following result.

Theorem 3.2. *Let $\alpha_1\alpha_2\cdots\alpha_n < \beta$. Let σ be a number satisfying* (3.8)*, where σ_n is given by* (3.7)*, and define $\tau(\sigma)$ by* (3.9)*. If either* (3.4) *or* (3.5) *holds, then all of strongly increasing solutions of* (1.1) *are singular.*

We have derived Theorem 3.1 from Theorem 1.1, and Theorem 3.2 from Theorem 3.1. We remark here that Theorem 1.1 can be derived from Theorem 3.2. In this sense, these three theorems are essentially identical. The following is a brief proof of the fact that Theorem 1.1 is derived from Theorem 3.2. Let $\sigma > 0$ and $\tau > 0$ be numbers which satisfy (1.13) and (1.14). As mentioned before, this is equivalent to the statement that σ and τ satisfy (3.1) and (3.2). Take a number $\sigma^* > 0$ such that $\sigma = \sigma^*$ satisfies (3.8) and $\tau(\sigma^*)/\sigma^* < \tau/\sigma$ and $\sigma < \sigma^*$. Here, $\tau(\sigma^*)$ is defined by (3.9) with $\sigma = \sigma^*$. If σ^* is taken sufficiently close to $1/(\nu_n+1)$, then it is possible to take such a number σ^*. By an application of Hölder's inequality, it is seen that

$$\int_{a^+}^t s^{-\zeta_{n-1}\tau-1+(r_{n-1}\beta+1)\sigma} p(s)^\sigma\, ds \le C_1\left(\int_{a^+}^t s^{-\zeta_{n-1}\tau(\sigma^*)-1+(r_{n-1}\beta+1)\sigma^*} p(s)^{\sigma^*}\, ds\right)^{\sigma/\sigma^*},\ \ t \ge a^+,$$

and

$$t^{\zeta_{n-1}\tau}\int_t^\infty s^{-\zeta_{n-1}\tau-1+(r_{n-1}\beta+1)\sigma} p(s)^\sigma\, ds$$
$$\le C_2\left(t^{\zeta_{n-1}\tau(\sigma^*)}\int_t^\infty s^{-\zeta_{n-1}\tau(\sigma^*)-1+(r_{n-1}\beta+1)\sigma^*} p(s)^{\sigma^*}\, ds\right)^{\sigma/\sigma^*},\ \ t \ge a^+.$$

Here, C_1 and C_2 are certain positive constants. Therefore, (1.15) implies (3.4) with $\sigma = \sigma^*$, and (1.16) implies (3.5) with $\sigma = \sigma^*$. This shows that Theorem 1.1 is derived from Theorem 3.2.

Theorem A stated in Section 1 can be derived from Theorem 3.2. In fact, suppose that (1.8) holds. Let $\sigma > 0$ and $\tau(\sigma) > 0$ be the numbers in the statement of Theorem 3.2. By (1.8), there is a constant $c > 0$ such that $p(t) \ge ct^{-r_{n-1}\beta-1}$, and so,

$$t^{-\zeta_{n-1}\tau(\sigma)-1+(r_{n-1}\beta+1)\sigma} p(t)^\sigma \ge c^\sigma t^{-\zeta_{n-1}\tau(\sigma)-1}$$

for all large t. If (3.4) does not hold, then the above inequality implies

$$\int_t^\infty s^{-\zeta_{n-1}\tau(\sigma)-1+(r_{n-1}\beta+1)\sigma} p(s)^\sigma\,ds \geq \frac{c^\sigma}{\zeta_{n-1}\tau(\sigma)}\, t^{-\zeta_{n-1}\tau(\sigma)}$$

for all large t, and, in consequence, condition (3.5) is satisfied.

It is also clear that if σ satisfies

$$\frac{1}{[\beta/(\alpha_1\alpha_2\cdots\alpha_n)]\nu_n+1} < \sigma \leq \sigma_n, \tag{3.10}$$

then the number $\tau(\sigma)$ defined by (3.3) is

$$\tau(\sigma) = \frac{1}{\mu_n\eta_n}\Big[\Big(\frac{\beta}{\alpha_1\alpha_2\cdots\alpha_n}\,\nu_n+1\Big)\sigma-1\Big]. \tag{3.11}$$

Therefore, by Theorem 3.1, we have the following result.

Corollary 3.1. *Let $\alpha_1\alpha_2\cdots\alpha_n<\beta$. Let σ be a number satisfying* (3.10)*, where σ_n is given by* (3.7)*, and define $\tau(\sigma)$ by* (3.11)*. If either* (3.4) *or* (3.5) *holds, then all of strongly increasing solutions of* (1.1) *are singular.*

As mentioned before, if $\alpha_1=\alpha_2=\cdots=\alpha_n=1$, then $D_ix(t)=x^{(i)}(t)$ $(i=0,1,2,\dots,n)$, and equation (1.1) is reduced to (1.17). Note that condition (1.4) is rewritten in the form

$$x^{(i)}(t)\geq 0 \ \ (a\leq t<b) \ \text{ for } \ i=0,1,2,\dots,n-1.$$

Moreover, in the case $\alpha_1=\alpha_2=\cdots=\alpha_n=1$, we have

$$r_{n-1}=\nu_n=n-1, \quad \mu_n=\zeta_{n-1}=\frac{n(n-1)}{2}\,, \quad \xi_n=n, \quad \eta_n=1.$$

Therefore, Theorem 3.2 yields the following result.

Corollary 3.2. *Consider equation* (1.17) *for the superlinear case $\beta>1$. Let σ be a number satisfying $2/[n+1+(n-1)\beta]\leq\sigma<1/n$. If either*

$$\int_{a^+}^\infty s^{-2+[n+1+(n-1)\beta]\sigma}p(s)^\sigma\,ds=\infty \ \ (a^+>\max\{a,0\})$$

or

$$\limsup_{t\to\infty} t^{1-n\sigma}\int_t^\infty s^{-2+[n+1+(n-1)\beta]\sigma}p(s)^\sigma\,ds>0,$$

then all of strongly increasing solutions of (1.17) *are singular.*

Corollary 3.2 gives an extension of Theorem 11.4 ($m=0$, $k=1$) in the book of Kiguradze and Chanturia [2].

By Corollary 3.1, we have the following result.

Corollary 3.3. *Consider equation* (1.17) *for the superlinear case $\beta>1$. Let σ be a number satisfying $1/[1+(n-1)\beta]<\sigma\leq 2/[n+1+(n-1)\beta]$. If either*

$$\int_a^\infty p(s)^\sigma\,ds=\infty$$

or

$$\limsup_{t\to\infty} t^{-1+[1+(n-1)\beta]\sigma}\int_t^\infty p(s)^\sigma\,ds>0,$$

then all of strongly increasing solutions of (1.17) *are singular.*

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