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NEW RESULTS ON THE WEIGHTED PSEUDO ALMOST AUTOMORPHIC SOLUTIONS OF THE MACKEY–GLASS MODEL WITH MIXED DELAYS

Abstract. Using the weighted pseudo almost automorphic concept and the Banach fixed point theorem, in this paper we study the existence and uniqueness of a new class of hematopoiesis model. The exponential and asymptotic stabilities are also established. In addition, we provide two numerical examples and computer simulations in order to demonstrate the feasibility of theoretical results. It should be mentioned that the methods and definitions introduced in the paper can be applied to study other types of delayed dynamic models.

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რეზიუმე. შეწონილი ფსევდო-თითქმის ავტომორფული კონცეფციისა და ბანახის უძრავი წერტილის თეორემის გამოყენებით, ნაშრომში შესწავლილია ახალი კლასის სისხლწარმოქმნის მოდელის არსებობა და ერთადერთობა. აგრეთვე დადგენილია ექსპონენციალური და ასიმპტოტური მდგრადობა. გარდა ამისა, მოყვანილია ორი რიცხვითი მაგალითი და კომპიუტერული სიმულაცია თეორიული შედეგების პრაქტიკაში განხორციელებადობის დემონსტრირებისთვის. უნდა აღინიშნოს, რომ ნაშრომში წარმოდგენილი მეთოდები და განმარტებები შეიძლება გამოყენებულ იქნას სხვა ტიპის დაგვიანებული დინამიკური მოდელების შესასწავლად.

1 Introduction

The process that results in producing and regulating blood cells is named hematopoiesis. The mathematical modeling of hematopoiesis dynamics has been extensively studied for the past 40 years, in order to understand and explain the reason leading to a number of repetitive blood cell diseases. Hence, in 1978 (see [20]), Mackey described the production process of all kinds of blood cells by a notable self-regulated system given by

$$\Gamma'(t) = -\Theta\Gamma(t) + \frac{Q}{1+\Gamma^n(t-e)}\,, \quad t \geq 0,$$

where at time t in blood circulation, $\Gamma(t)$ represents the mature cell density, and e denotes the time delay between the immature cell production within the marrow of the bone and the maturation of these cells to be released within the circulating bloodstream. The cells can be in fact lost from the circulation at an Θ rate. Added to that, the flux of the cells from the stem cell compartment into the circulation, particularly depends on the mature cell density at previous time $t-e$. Since any change in the environment plays a significant role, in [19], the authors considered the next hematopoiesis model:

$$\Gamma'(t) = -\Theta(t)\Gamma(t) + \sum_{i=1}^{D} \frac{Q_i(t)\Gamma^m(t-e_i(t))}{1+\Gamma^n(t-e_i(t))}\,, \quad t \geq 0,$$

where $0 \leq m \leq n$, $m \leq 1$, and they studied the global exponential stability, uniqueness and existence of a positive almost periodic solution.

Besides, the concept of pseudo almost automorphic functions had been first introduced by Xiao et al. [23]. Actually, it naturally generalizes the almost periodicity, almost automorphy, and pseudo almost periodicity. Lately, the existence and stability of pseudo almost automorphic/pseudo almost periodic solutions of differential equations have been a great deal studied (see [1–4, 9, 10, 14, 17, 18, 21, 22]). In [8], Blot et al. introduced the concept of weighted pseudo almost automorphic functions using the measure theory, which is a natural generalization of the classical pseudo almost periodicity (see [24]) and pseudo almost automorphic periodicity, which is the central tool in this paper.

However, the model within biological, physical, and social sciences is in certain cases obligatory to consider the different time delays. Indeed, such an explicit inclusion of delays within equations is in many instances the simplification or idealization introduced on account of the detailed description of underlying processes that are very complex to be modeled mathematically. In this case, it may be necessary to choose a model with a distributed and/or continuous delay.

Moreover, it is well known that the optimal management of renewable resources has a direct link with the population development to sustain. Then we can assume that population models are subject to harvesting. Thus, the term of harvesting plays a great role in protecting the balanced development.

Until now and to the best of our knowledge, the hematopoiesis model studies have been devoted to the case where $0 \leq m, n \leq 1$ (i.e., the existence, uniqueness and stability). In addition to that, most papers have been dedicated to the study of the hematopoiesis model within almost or pseudo almost periodic functions (see [4, 12, 16, 19]). However, in [13], Haifa et al. established the existence and exponential stability of the pseudo almost periodic solution of the hematopoiesis model in the case where $m, n > 1$. Nevertheless, if $m, n > 1$, there are many questions that can be asked: How do things work? What is happening? Is there a solution or not? Is the solution unique? Is it stable or not? Furthermore, if we consider the hematopoiesis model in this case with mixed delays and with the harvesting term, this work becomes more and more complicated. In this context, our objective in this paper is to study the model of hematopoiesis with harvesting term and mixed delays in the case where $m, n > 1$ in the space of weighted pseudo almost automorphic functions. The model is given as follows:

$$\begin{aligned}\Gamma'(t) = &-\Theta(t)\Gamma(t) + \sum_{i=1}^{D} Q_i(t)\frac{\Gamma^m(t-e_i(t))}{1+\Gamma^n(t-e_i(t))}\\ &+ \sum_{i=1}^{D} p_i(t)\int_0^{\delta_i} k_i(u)\frac{\Gamma^m(t-u)}{1+\Gamma^n(t-u)}\,du - \Xi\big(t, \Gamma(t-j(t))\big), \quad t \geq 0, \quad 1 \leq m \leq n, \end{aligned} \tag{1.1}$$

where $\Theta(\cdot)$, $e_i(\cdot)$ and $j(\cdot)$ are almost automorphic functions. In addition for all $1 \le i \le D$, $Q_i(\cdot)$, $p_i(\cdot)$, they are weighted pseudo almost automorphic functions. Moreover, $k_i(\cdot)$ is the kernel, and $\Xi(\cdot,\cdot)$ is the harvesting term.

Let

$$K = \sup_t \left\{ e_i(t),\ \delta_i,\ j(t),\ i = 1, 2, \dots, n \right\}.$$

We assume that equation (1.1) is supplemented with the initial conditions as follows:

$$\begin{aligned} &\Gamma(t) = \psi(t), \quad \psi \in \mathrm{BC}([-K, 0], \mathbb{R}_+), \\ &\Gamma(0) = \psi(0) > 0. \end{aligned} \tag{1.2}$$

We assume in the remainder of the paper that the following conditions hold:

(H1) $M[\Theta] > 0$.

(H2) For all $1 \le i \le D$, the delay kernels satisfy

$$\int_0^{\delta_i} k_i(u)\,du = 1.$$

(H3) $\Xi : t \mapsto \Xi(t,\cdot)$ is almost automorphic in t for each $z \in \mathcal{V}$, where $\mathcal{V}$ represents a compact subset of $\mathbb{R}$, and there exists a number $L > 0$ such that

$$|\Xi(t,a) - \Xi(t,b)| \le L|a - b|$$

for all $t \in \mathbb{R}$ and $a, b \in \mathbb{R}_+$.

The principal results of this paper are:

1. We show the positivity and boundedness of the solution when $m, n > 1$ which corresponds to the biological reality, since the blood cells have a life length.
2. A serious and important issue within this kind of nonlinear differential equation is: What are the sufficient conditions to obtain a weighted pseudo almost automorphic solution if the coefficients are weighted pseudo almost automorphic? Moreover, if $m, n > 1$, does the solution exist or not? What method proves its existence? If the solution exists, is it unique? Hence, in this paper we obtain the uniqueness and existence of the weighted pseudo almost automorphic solution of equations (1.1), (1.2) by using the Banach fixed point theorem and some sufficient conditions.
3. What is the impact of continuous and distributed delays on the stability of the solutions? The answer to this question is given by two new theorems.

Our paper is organized as follows. First of all, Section 2 establishes the required definitions and fundamental properties of the space $\mathrm{PAA}(\mathbb{R}, \mathbb{R}^N, \eta)$, which will be used to obtain our main results. Next, Section 3 establishes the necessary criteria for the existence and uniqueness of the weighted pseudo almost automorphic solution for equation (1.1). Lastly, in Sections 4 and 5, we obtain the globally exponential and asymptotic stabilities of the weighted pseudo almost automorphic solution. After that, two numerical examples are given in Section 6 in order to depict the feasibility of theoretical results that we have got.

2 Definitions and lemmas

Let Σ be a Lebesgue ξ-field of $\mathbb{R}$, Υ denote the set of all positive measures η on Σ satisfying $\eta(\mathbb{R}) = \infty$ and $\eta([c, d]) < \infty$ for all $c, d \in \mathbb{R}$ $(c \le d)$. Let us consider the class of measures $\eta \in \Upsilon$ which satisfy

(A1) (i) For all $\varsigma \in \mathbb{R}$, there exists $\epsilon > 0$ and a bounded interval I such that

$$\eta\big(\{d + \varsigma : \ d \in H\}\big) \leq \epsilon \eta(H)$$

when $H \in \Sigma$ satisfies $H \cap I = \varnothing$.

(ii) There exists a continuous function $\lambda : \mathbb{R} \to \mathbb{R}_+$ as follows:

$$d\eta_\theta(s) = \lambda(s)\, d\eta(s), \quad \forall\, s \in \mathbb{R},$$

with $\eta_\theta(Q) = \eta((\Upsilon - \theta)^{-1}(Q))$ for all $Q \in \Sigma$.

Lemma 2.1 ([7])**.** *Under* (A1)*, for all* $\alpha > 0$*,*

$$\limsup_{c\to\infty} \frac{\eta([-c-\alpha, c+\alpha])}{\eta([-c,c])} < \infty.$$

Definition 2.1 ([8])**.** A continuous function $P : \mathbb{R} \to \mathbb{R}^N$ is called almost automorphic in the case that for each sequence $(O_n)_{n\in\mathbb{N}}$ there exists a subsequence $(\varsigma_n)_{n\in\mathbb{N}} \subset (O_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} P(t + \varsigma_n) = D(t)$$

exists for all $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} D(t - \varsigma_n) = P(t) \ \text{ for all } \ t \in \mathbb{R},$$

are well defined for each $t \in \mathbb{R}$.

Let $\mathrm{AA}(\mathbb{R}, \mathbb{R}^N)$ denote the collection from $\mathbb{R}$ to $\mathbb{R}^N$ of all almost automorphic functions.

Definition 2.2 ([17])**.** Let $\eta \in \Upsilon$. We can say that a bounded continuous function $J : \mathbb{R} \to \mathbb{R}^N$ will be η-ergodic if

$$\lim_{c\to\infty} \frac{1}{\eta([-c,c])} \int\limits_{[-c,c]} \|J(t)\|\, d\eta(t) = 0.$$

We denote the collection of all such functions by $\mathrm{PAP}_0(\mathbb{R}, \mathbb{R}^N, \eta)$.

Definition 2.3 ([17])**.** Let $\eta \in \Upsilon$. A function $J \in \mathrm{BC}(\mathbb{R}, \mathbb{R}^N)$ is called weighted pseudo almost automorphic if it can be expressed as $J = D + S$, where $D \in \mathrm{AA}(\mathbb{R}, \mathbb{R}^N)$ and $D \in \mathrm{PAP}_0(\mathbb{R}, \mathbb{R}^N, \eta)$. We denote the collection of such functions by $\mathrm{PAA}(\mathbb{R}, \mathbb{R}^N, \eta)$.

Theorem 2.1 ([17])**.** *Let* $\eta \in \Upsilon$*.* $(\mathrm{PAA}(\mathbb{R}, \mathbb{R}^N, \eta), \|\cdot\|_\infty)$ *is a Banach space.*

Corollary 2.1 ([7])**.** *Let* $\eta \in \Upsilon$ *and let* $G : \mathbb{R} \to \mathbb{R}$ *be a continuous function. Assume that for each bounded subset* $\breve{B}$ *of* $\mathbb{R}$*,* ψ *is bounded on* $\breve{B}$*. If* $\Gamma \in \mathrm{PAA}(\mathbb{R}, \mathbb{R}, \eta)$*, then* $t \mapsto G(\Gamma(t))$ *belongs to* $\mathrm{PAA}(\mathbb{R}, \mathbb{R}, \eta)$*.*

3 Existence and uniqueness of the weighted pseudo almost automorphic solution

Our first result is given as follows:

Lemma 3.1. *Let* (H1) *hold. Assume that there exist two positive constants* R_1 *and* R_2 *such that* $R_2 > R_1$ *and*

$$R_2 > \frac{\sum\limits_{i=1}^{D} (\overline{p}_i + \overline{Q}_i)}{\underline{\Theta}}, \quad \overline{\Theta} R_2 < \sum_{i=1}^{D} \underline{Q}_i \frac{R_1^m}{1 + R_2^n} - \Xi.$$

Then there exists $T_s > 0$ *in such a way that for* $t \geq T_s$*, we get*

$$R_1 \leq \Gamma(t) \leq R_2(t).$$

Proof. Firstly, let us demonstrate that for all $t \geq T_s$ the solution of (1.1)–(1.2) satisfies $\Gamma(t) \leq R_2$. Let us assume that there exists a number $t_1 \in [T_s, \infty)$ such that

$$\Gamma(t_1) = R_2,$$
$$\Gamma(t) < R_2, T_s - K \leq t < t_1.$$

Hence

$$0 \leq \Gamma'(t_1) \leq -\Theta(t_1)\Gamma(t_1) + \sum_{i=1}^{D}\left(Q_i(t_1)\,\frac{\Gamma^m(t_1 - e_i(t_1))}{1+\Gamma^n(t_1 - e_i(t_1))} + p_i(t_1)\int_0^{\delta_i} k_i(u)\,\frac{\Gamma^m(t_1-u)}{1+\Gamma^n(t_1-u)}\right) du$$
$$\leq -\Theta(t_1)R_2 + \sum_{i=1}^{D}\left(\overline{Q}_i\,\frac{\Gamma^m(t_1 - e_i(t_1))}{1+\Gamma^n(t_1 - e_i(t_1))} + \overline{p}_i\int_0^{\delta_i} k_i(u)\,\frac{\Gamma^m(t_1-u)}{1+\Gamma^n(t_1-u)}\, du\right).$$

Since $\sup_a \frac{a^m}{1+a^n} \leq 1$ for $m \leq n$, we have

$$0 \leq \Gamma'(t_1) \leq -\Theta(t_1)R_2 + \sum_{i=1}^{D}(\overline{Q}_i + \overline{p}_i) < 0,$$

which is a contradiction. So $\Gamma(t) \leq R_2$ for all $t \geq T_s$.

On the next step in the proof, it should be demonstrated that there exists $T_s > 0$ in a certain way that for $t \geq T_s$, $R_2 \leq \Gamma(t)$. In fact, suppose that there exists $t_2 \geq T_s$ as follows:

$$\Gamma(t_2) = R_1,$$
$$\Gamma(t) > R_1,\ \ T_s - K \leq t < t_2,$$
$$0 \geq \Gamma'(t_2) \geq -\Theta(t_2)\Gamma(t_2) + \sum_{i=1}^{D} Q_i(t_2)\,\frac{\Gamma^m(t_2 - e_i(t_2))}{1+\Gamma^n(t_2 - e_i(t_2))} - \Xi\big(t_2, \Gamma(t - j(t_2))\big)$$
$$\geq -\Theta(t_2)R_2 + \sum_{i=1}^{D} \underline{Q}_i\,\frac{R_1^m}{1+R_2^n} - \overline{\Xi} \geq -\overline{\Theta}R_2 + R_1^m \sum_{i=1}^{D} \frac{\underline{Q}_i}{1+R_2^n} - \overline{\Xi} > 0,$$

which is a contradiction. As a consequence, there exists $T_s > 0$ such that for $t \geq T_s$, we have $R_1 \leq \Gamma(t) \leq R_2$. □

Lemma 3.2. *Every solution of* (1.1)–(1.2) *is strictly positive and bounded for all* $t \in [-K, \zeta(\psi))$*, and* $\zeta(\psi) = \infty$.

Proof. From (1.1), for any $\psi \in \mathrm{BC}([-K, 0], \mathbb{R}_+)$, $\psi(0) > 0$ and for $t \geq 0$, we get

$$\Gamma(t) = \Gamma(0)\, e^{-\int_0^t \Theta(u)\,du}$$
$$+ \int_0^t e^{-\int_s^t \Theta(u)\,du}\left[\sum_{i=1}^{D}\left(Q_i(s)\,\frac{\Gamma^m(s-e_i(s))}{1+\Gamma^n(s-e_i(s))} + p_i(s)\int_0^{\delta_i} k_i(u)\,\frac{\Gamma^m(s-u)}{1+\Gamma^n(s-u)}\,du\right) - \Xi\big(s, \Gamma(s-j(s))\big)\right] ds$$
$$\geq \Gamma(0)\, e^{-\int_0^t \Theta(u)\,du} + \int_0^t e^{-\int_s^t \Theta(u)\,du}\left[\sum_{i=1}^{D} Q_i(s)\,\frac{\Gamma^m(s-e_i(s))}{1+\Gamma^n(s-e_i(s))} - \Xi\big(s, \Gamma(s-j(s))\big)\right] ds$$
$$\geq \Gamma(0)\, e^{-\int_0^t \Theta(u)\,du} + \int_0^t e^{-\int_s^t \Theta(u)\,du}\left[\sum_{i=1}^{D} \underline{Q}_i\,\frac{R_1^m}{1+R_2^n} - \overline{\Xi}\right] ds$$
$$\geq \Gamma(0)\, e^{-\int_0^t \Theta(u)\,du} + \int_0^t e^{-\int_s^t \Theta(u)\,du}\,\overline{\Theta}\, ds\, R_1 > 0.$$

Then $\Gamma(t) > 0$ for all $t \in [-K, \zeta(\psi))$.

Now, we have to prove that every solution of equation (1.1) is bounded. In all other respects, there will exist an unbounded solution $\Gamma(t)$ of (1.1) satisfying

$$\Gamma'(t) \leq -\underline{\Theta}\,\Gamma(t) + \sum_{i=1}^{D}(\overline{Q}_i + \overline{p}_i) \leq 0. \tag{3.1}$$

In addition, there exits $0 < t_a < t_b$ such that

$$\Gamma(t_a) < \Gamma(t_b) \ \text{ and } \ -\underline{\Theta}\,\Gamma(t_b) + \sum_{i=1}^{D}(\overline{Q}_i + \overline{p}_i) \leq 0.$$

By (3.1), we get that $\Gamma'(t_b) < 0$. Let $\Gamma(t_c) = \max\limits_{t_a \leq t \leq t_b} \Gamma(t)$. It can be easily seen that $t_c \neq t_a$ and $t_c \neq t_b$. Therefore, $\Gamma'(t_c) = 0$ and from (3.1) we get

$$\Gamma'(t_c) \leq -\underline{\Theta}\,\Gamma(t_c) + \sum_{i=1}^{D}(\overline{Q}_i + \overline{p}_i) \leq -\underline{\Theta}\,\Gamma(t_b) + \sum_{i=1}^{D}(\overline{Q}_i + \overline{p}_i) < 0.$$

Consequently, $\Gamma(t)$ is bounded. □

Lemma 3.3. *Let $\eta \in \Upsilon$ and $\Gamma \in \mathrm{PAA}(\mathbb{R}, \mathbb{R}_+, \eta)$. Since $\Gamma(t)$ is bounded, we have $\frac{1}{\Gamma} \in \mathrm{PAA}(\mathbb{R}, \mathbb{R}_+, \eta)$.*

Proof. Since Γ is bounded, there are $R_1, R_2 > 0$ such that $0 < R_1 \leq |\Gamma(t)| \leq R_2$. The function $F(z) = \frac{1}{z}$ is bounded in $R_1 \leq |z(t)| \leq R_2$, and according to Corollary 2.1, $\frac{1}{\Gamma} \in \mathrm{PAA}(\mathbb{R}, \mathbb{R}, \eta)$. □

Lemma 3.4. *Let $\eta \in \Upsilon$ and $\Gamma \in \mathrm{PAA}(\mathbb{R}, \mathbb{R}, \eta)$. Under* (H2)*, for all $1 \leq i \leq D$, the function*

$$\breve{A}_i : t \longmapsto \int_0^{\delta_i} k_i(s)\, \frac{\Gamma^m(t-s)}{1+\Gamma^n(t-s)}\, ds$$

belongs to $\mathrm{PAA}(\mathbb{R}, \mathbb{R}, \eta)$.

Proof. Since $\Gamma(\,\cdot\,)$ is a weighted pseudo almost automorphic function and $\Gamma(\,\cdot\,)$ is bounded, the function $t \mapsto \Gamma^m(t)$ is bounded, too. By Corollary 2.1, the function $t \mapsto \Gamma^m(t)$ is weighted pseudo almost automorphic. In addition, by Lemma 3.3, the function $J(t-s) = \frac{\Gamma^m(t-s)}{1+\Gamma^n(t-s)} \in \mathrm{PAA}(\mathbb{R}, \mathbb{R}, \eta)$, then J is expressed as follows:

$$J = J_1 + J_2,$$

with $J_1(\,\cdot\,) \in \mathrm{AA}(\mathbb{R}, \mathbb{R})$ and $J_2(\,\cdot\,) \in \mathrm{PAP}_0(\mathbb{R}, \mathbb{R}, \eta)$. The function $\breve{A}_i$ is written as

$$\breve{A}_i(t) = \int_0^{\delta_i} k_i(s) J_1(t-s)\, ds + \int_0^{\delta_i} k_i(s) J_2(t-s)\, ds = \breve{A}_i^1(t) + \breve{A}_i^2(t)$$

with

$$\breve{A}_i^1(t) = \int_0^{\delta_i} k_i(s) J_1(t-s)\, ds, \quad \breve{A}_i^2(t) = \int_0^{\delta_i} k_i(s) J_2(t-s)\, ds.$$

Now, it should be proven that $\breve{A}_i^1(\,\cdot\,) \in \mathrm{AA}(\mathbb{R}, \mathbb{R})$. Since $J_1(\,\cdot\,)$ is almost automorphic, we can extract a subsequence of real numbers $(\varsigma_n) \subset (O'_n)$ as follows:

$$\lim_{n\to\infty} J_1(t + \varsigma_n - s) = v_1(t-s), \quad \lim_{n\to\infty} v_1(t - \varsigma_n - s) = J_1(t-s)$$

for all $s \in [0, \delta_i]$ and $t \in \mathbb{R}$. Note that

$$G_i(t) = \int_0^{\delta_i} k_i(s) v^1(t-s)\, ds,$$

$$\begin{aligned}\left|\breve{A}_i^1(t+\varsigma_n) - G_i(t)\right| &= \left| \int_0^{\delta_i} k_i(s) J_1(t+\varsigma_n - s)\, ds - \int_0^{\delta_i} k_i(s) v_1(t-s)\, ds \right| \\ &= \left| \int_0^{\delta_i} k_i(s) \big[J_1(t+\varsigma_n - s) - v_1(t-s)\big]\, ds \right| \\ &\leq \int_0^{\delta_i} k_i(s) \big|J_1(t+\varsigma_n - s) - v_1(t-s)\big|\, ds.\end{aligned}$$

Using Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty} \breve{A}_i^1(t+\varsigma_n) = G_i(t).$$

For all $t \in \mathbb{R}$, the above-mentioned approach shows that

$$\lim_{n\to\infty} \tilde{G}_i(t-\varsigma_n) = \breve{A}_i^1(t),$$

thus demonstrating that $\breve{A}_i^1(\,\cdot\,) \in \mathrm{AA}(\mathbb{R},\mathbb{R})$.

On the following step of this proof, it should be proven that $\breve{A}_i^2(\,\cdot\,) \in \mathrm{PAP}_0(\mathbb{R},\mathbb{R},\eta)$.

$$\begin{aligned}\lim_{C\to\infty} \frac{1}{\eta([-C,C])} \int_{-C}^{C} |\breve{A}_i^2(t)|\, d\eta(t) &\leq \lim_{C\to\infty} \frac{1}{\eta([-C,C])} \int_{-C}^{C} \int_0^{\delta_i} k_i(s) J_2(t-s)\, ds\, d\eta(t) \\ &\leq \lim_{C\to\infty} \frac{1}{\eta([-C,C])} \int_0^{\delta_i} k_i(s) \left(\int_{-C}^{C} J_2(t-s)\, d\eta(t) \right) ds \\ &\leq \int_0^{\delta_i} k_i(s) \left(\lim_{C\to\infty} \frac{1}{\eta([-C,C])} \int_{-C}^{C} J_2(t-s)\, d\eta(t) \right) ds.\end{aligned}$$

Since $J_2(\,\cdot\,) \in \mathrm{PAP}_0(\mathbb{R},\mathbb{R},\eta)$ and $\mathrm{PAP}_0(\mathbb{R},\mathbb{R},\eta)$ is invariant by translation, we have

$$\lim_{C\to\infty} \frac{1}{\eta([-C,C])} \int_{-C}^{C} J_2(t-s)\, d\eta(t) = 0.$$

Consequently,

$$\lim_{C\to\infty} \frac{1}{\eta([-C,C])} \int_{-C}^{C} |\breve{A}_i^2(t)|\, d\eta(t) = 0. \qquad \square$$

Lemma 3.5. *Let $\eta \in \Upsilon$. By Assumption* (A1)-(ii)*, if $\Gamma(\,\cdot\,) \in \mathrm{PAA}(\mathbb{R},\mathbb{R}_+,\eta)$ and $e(\,\cdot\,) \in \mathrm{AA}(\mathbb{R},\mathbb{R}_+)$, then $\Gamma(\,\cdot\, - e(\,\cdot\,)) \in \mathrm{PAA}(\mathbb{R},\mathbb{R}_+,\eta)$.*

Proof. $\Gamma(\,\cdot\,) \in \mathrm{PAA}(\mathbb{R},\mathbb{R}_+,\eta)$, through the use of the composition theorem, $\Gamma(\,\cdot\,)$ is expressed by $\Gamma = \Gamma_1 + \Gamma_2$, with $\Gamma_1(\,\cdot\,) \in \mathrm{AA}(\mathbb{R},\mathbb{R}_+)$ and $\Gamma_2(\,\cdot\,) \in \mathrm{PAP}_0(\mathbb{R},\mathbb{R}_+,\eta)$:

$$\Gamma(t-e(t)) = \Gamma_1(t-e(t)) + \Gamma(t-e(t)) - \Gamma_1(t-e(t)).$$

After that, it shall be proven that $\Gamma_1(\cdot - e(\cdot)) \in \mathrm{AA}(\mathbb{R}, \mathbb{R}_+)$. $\Gamma_1 \in \mathrm{BC}(\mathbb{R}, \mathbb{R}_+)$, so Γ_1 is uniformly continuous in each compact $\mathfrak{S} \subset \mathrm{BC}(\mathbb{R}, \mathbb{R}_+)$, that is, for all $\varepsilon > 0$, there exists $\iota > 0$ such that for all $s_1, s_2 \in \mathfrak{S}$, one has

$$\|s_1 - s_2\| < \iota \Longrightarrow \|\Gamma(s_1) - \Gamma(s_2)\| < \varepsilon.$$

$\Gamma_1(\cdot - a), e(\cdot) \in \mathrm{AA}(\mathbb{R}, \mathbb{R}_+)$, then we have a subsequence $(\varsigma_n) \subset (\zeta_n)$ such that

$$\lim_{n\to\infty} \Gamma_1(t + \varsigma_n - a) = R(t - a), \quad \lim_{n\to\infty} e(t + \varsigma_n) = \omega(t)$$

and

$$\lim_{n\to\infty} R(t - \varsigma_n - a) = \Gamma_1(t - a), \lim_{n\to\infty} \omega(t - \varsigma_n) = e(t).$$

On the other hand,

$$\begin{aligned}&\big|\Gamma(t + \varsigma_n - e(t + \varsigma_n)) - R(t - \omega(t))\big| \\ &\qquad \le \big|\Gamma(t + \varsigma_n - e(t + \varsigma_n)) - \Gamma(t + \varsigma_n - \omega(t))\big| + \big|\Gamma(t + \varsigma_n - \omega(t)) - R(t - \omega(t))\big|.\end{aligned}$$

Consequently, $\Gamma_1(\cdot - e(\cdot))$ is almost automorphic.

It is shown that $\Gamma(\cdot - e(\cdot)) - \Gamma_1(\cdot - e(\cdot)) \in \mathrm{PAP}_0(\mathbb{R}, \mathbb{R}_+, \eta)$,

$$\begin{aligned}&\lim_{C\to\infty} \frac{1}{\eta([-C, C])} \int_{-C}^{C} \big|\Gamma(t - e(t)) - \Gamma_1(t - e(t))\big|\, d\eta(t) \\ &= \lim_{C\to\infty} \frac{1}{\eta([-C, C])} \int_{-C}^{C} |\Gamma_2(t - e(t))|\, d\eta(t) \le \lim_{C\to\infty} \frac{1}{\eta([-C, C])} \int_{-(C+\overline{e})}^{C+\overline{e}} |\Gamma_2(t)|\, d\eta(t + e(t)) \\ &\le \lim_{C\to\infty} \frac{\eta([-C - \overline{e}, C + \overline{e}])}{\eta([-C, C])} \frac{1}{\eta([-C - \overline{e}, C + \overline{e}])} \int_{-(C+\overline{e})}^{C+\overline{e}} |\Gamma_2(t)| \lambda(t)\, d\eta(t) \\ &\le \lim_{C\to\infty} \frac{\eta([-C - \overline{e}, C + \overline{e}])}{\eta([-C, C])} \sup_{\xi \in [-(C+\overline{e}), C+\overline{e}]} \lambda(\xi) \frac{1}{\eta([-C - \overline{e}, C + \overline{e}])} \int_{-(C+\overline{e})}^{C+\overline{e}} |\Gamma_2(t)|\, d\eta(t).\end{aligned}$$

Thus $\Gamma(\cdot - e(\cdot)) - \Gamma_1(\cdot - e(\cdot)) \in \mathrm{PAP}_0(\mathbb{R}, \mathbb{R}_+, \eta)$. So, $\Gamma(\cdot - e(\cdot)) \in \mathrm{PAA}(\mathbb{R}, \mathbb{R}_+, \eta)$. □

Lemma 3.6. *Let $\eta \in \Upsilon$. By assumption* (H3)*, the function $t \mapsto \Xi(t, \Gamma(t - j(t)))$ is weighted pseudo almost automorphic.*

Proof. $\Gamma(\cdot) \in \mathrm{PAA}(\mathbb{R}, \mathbb{R}_+, \eta)$, using the composition theorem, $\Gamma(\cdot)$ is written as $\Gamma = \Gamma_1 + \Gamma_2$ with $\Gamma_1(\cdot) \in \mathrm{AA}(\mathbb{R}, \mathbb{R}_+)$ and $\Gamma_2(\cdot) \in \mathrm{PAP}_0(\mathbb{R}, \mathbb{R}_+, \eta)$. Therefore, the function $\Xi(\cdot, \cdot)$ is expressed by

$$\Xi\big(t, \Gamma(t - j(t))\big) = \Xi\big(t, \Gamma_1(t - j(t))\big) + \Xi\big(t, \Gamma(t - j(t))\big) - \Xi\big(t, \Gamma_1(t - j(t))\big).$$

It should be proven now that the function $t \mapsto \Xi(t, \Gamma_1(t - j(t)))$ is almost automorphic. $\Gamma_1(\cdot) \in \mathrm{AA}(\mathbb{R}, \mathbb{R}_+)$ and $\Xi(\cdot, \Gamma) \in \mathrm{AA}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}_+)$, so, for each sequence (O_n), we have a subsequence $(\varsigma_n) \subset (O_n)$ such that

$$\begin{aligned}&\lim_{n\to\infty} \Gamma_1(\varsigma_n + t) = F(t), \quad \lim_{n\to\infty} \Xi(t + \varsigma_n, u) = N(t, u), \\ &\lim_{n\to\infty} F(t - \varsigma_n) = \Gamma_1(t), \quad \lim_{n\to\infty} N(t - \varsigma_n, u) = \Xi(t, u).\end{aligned}$$

Therefore,

$$\begin{aligned}&\big|\Xi(t + \varsigma_n, \Gamma_1(t + \varsigma_n)) - N(t, F(t))\big| \\ &\qquad \le \big|\Xi(t + \varsigma_n, \Gamma_1(t + \varsigma_n)) - \Xi(t + \varsigma_n, F(t))\big| + \big|\Xi(t + \varsigma_n, F(t)) - N(t, F(t))\big| \\ &\qquad \le L|\Gamma_1(t + \varsigma_n) - F(t)| + \big|\Xi(t + \varsigma_n, F(t)) - N(t, F(t))\big|.\end{aligned}$$

Since

$$\lim_{n\to\infty} \Xi(t+\varsigma_n, F(t)) = N(t, F(t)),$$

we obtain

$$\lim_{n\to\infty} \Xi(t+\varsigma_n, \Gamma_1(t+\varsigma_n)) = N(t, F(t)).$$

Similarly, we give

$$\lim_{n\to\infty} N(t-\varsigma_n, F(t-\varsigma_n)) = \Xi(t, \Gamma_1(t)).$$

Accordingly, the function $t \mapsto \Xi(t, \Gamma_1(t-j(t)))$ is almost automorphic. Next, we shall prove that $\Xi(\cdot, \Gamma(\cdot - j(\cdot))) - \Xi(\cdot, \Gamma_1(\cdot - j(\cdot)))$ is η-ergodic,

$$\begin{aligned}
&\lim_{C\to\infty} \frac{1}{\eta([-C,C])} \int_{-C}^{C} \left|\Xi\big(t, \Gamma(t-j(t))\big) - \Xi\big(t, \Gamma_1(t-j(t))\big)\right| d\eta(t) \\
&\quad \le \lim_{C\to\infty} \frac{L}{\eta([-C,C])} \int_{-C}^{C} \left|\Gamma(t-j(t)) - \Gamma_1(t-j(t))\right| d\eta(t) \\
&\quad \le \lim_{C\to\infty} \frac{L}{\eta([-C,C])} \int_{-(C+\bar{j})}^{C+\bar{j}} |\Gamma_2(t)|\, d\eta(t+j(t)) \\
&\quad \le \lim_{C\to\infty} \frac{\eta([-C-\bar{j}, C+\bar{j}])}{\eta([-C,C])} \frac{L}{\eta([-C,C])} \int_{-(C+\bar{j})}^{C+\bar{j}} |\Gamma_2(t)|\lambda(t)\, d\eta(t) \\
&\quad \le \lim_{C\to\infty} \sup_{\xi\in[-(C+\bar{j}), C+\bar{j}]} \lambda(\xi) \frac{\eta([-C-\bar{j}, C+\bar{j}])}{\eta([-C,C])} \frac{L}{\eta([-C-\bar{j}, C+\bar{j}])} \int_{-(C+\bar{j})}^{C+\bar{j}} |\Gamma_2(t)|\, d\eta(t).
\end{aligned}$$

Thus $t \mapsto \Xi(t, \Gamma(t-j(t))) - \Xi(t, \Gamma_1(t-j(t)))$ is η-ergodic. Consequently, $\Xi(t, \Gamma(\cdot - j(\cdot))) \in \mathrm{PAA}(\mathbb{R}, \mathbb{R}_+, \eta)$. ☐

Theorem 3.1. *Consider $\eta \in \Upsilon$. Under assumption* (H1)–(H3)*, the operator Π defined as*

$$\begin{aligned}
(\Pi\Gamma)(t) = &\int_{\infty}^{t} e^{-\int_s^t \Theta(u)\,du} \\
&\times \left[\sum_{i=1}^{D} \left(Q_i(s) \frac{\Gamma^m(s-e_i(s))}{1+\Gamma^n(s-e_i(s))} + p_i(s) \int_0^{\delta_i} k_i(u) \frac{\Gamma^m(u+s)}{1+\Gamma^n(u+s)}\, du\right) - \Xi\big(s, \Gamma(s-j(s))\big)\right] ds
\end{aligned}$$

is a mapping in $\mathrm{PAA}(\mathbb{R}, \mathbb{R}, \eta)$ *into itself.*

Proof. As regards Lemmas 3.4, 3.5, besides the use of Lemma 3.6, the function

$$\Lambda : s \longmapsto \sum_{i=1}^{D} \left(Q_i(s) \frac{\Gamma^m(s-e_i(s))}{1+\Gamma^n(s-e_i(s))} + p_i(s) \int_0^{\delta_i} k_i(u) \frac{\Gamma^m(s-u)}{1+\Gamma^n(s-u)}\, du\right) - \Xi\big(s, \Gamma(s-j(s))\big)$$

is weighted pseudo almost automorphic. Consequently, Λ is written as

$$\Lambda(s) = \Lambda_1(s) + \Lambda_2(s),$$

where $\Lambda_1(\cdot) \in \mathrm{AA}(\mathbb{R},\mathbb{R})$ and $\Lambda_2(\cdot) \in \mathrm{PAP}_0(\mathbb{R},\mathbb{R},\eta)$. Then

$$(\Pi\Gamma)(t) = \int_{\infty}^{t} e^{-\int_s^t \Theta(u)\,du} \Lambda_1(s)\,ds + \int_{\infty}^{t} e^{-\int_s^t \Theta(u)\,du} \Lambda_2(s)\,ds = (\Pi\Lambda_1)(t) + (\Pi\Lambda_2)(t).$$

Now, it should be proven that $(\Pi\Lambda_1)(\cdot) \in \mathrm{AA}(\mathbb{R},\mathbb{R})$. Since $\Theta(\cdot)$, $\Lambda_1(\cdot)$ are almost automorphic, so for every sequence of real numbers (O_n), we can extract a subsequence (ς_n) such that

$$\lim_{n\to\infty} \Lambda_1(t+\varsigma_n) = \upsilon_1(t), \quad \lim_{n\to\infty} \upsilon_1(t-\varsigma_n) = \Lambda_1(t),$$
$$\lim_{n\to\infty} \Theta(t+\varsigma_n) = \iota(t), \quad \lim_{n\to\infty} \iota(t-\varsigma_n) = \Theta(t)$$

for all $t \in \mathbb{R}$. Let us consider

$$(\breve{B}\upsilon_1)(t) = \int_{\infty}^{t} e^{-\int_s^t \iota(u)\,du} \upsilon_1(s)\,ds,$$

$$\begin{aligned}
\big|(\Pi\Lambda_1)(t+\varsigma_n) - (\breve{B}\upsilon_1)(t)\big| &= \Bigg| \int_{\infty}^{t+\varsigma_n} e^{-\int_s^{t+s_n} \Theta(u)\,du} \Lambda_1(s)\,ds - \int_{\infty}^{t} e^{-\int_s^t \iota(u)\,du} \upsilon_1(s)\,ds \Bigg| \\
&= \Bigg| \int_{\infty}^{t} e^{-\int_{s+\varsigma_n}^{t+\varsigma_n} \Theta(u)\,du} \Lambda_1(s+\varsigma_n)\,ds - \int_{\infty}^{t} e^{-\int_s^t \iota(u)\,du} \upsilon_1(s)\,ds \Bigg| \\
&\leq \int_{\infty}^{t} e^{-(t-s)\underline{\Theta}} |\Lambda_1(s+\varsigma_n) - \upsilon_1(s)|\,ds \\
&\qquad + \|\upsilon_1\|_\infty \int_{\infty}^{t} \Big| e^{-\int_s^t \Theta(u+\varsigma_n)} - e^{-\int_s^t \iota(u)\,du} \Big|\,ds.
\end{aligned}$$

Consequently, one can find $\theta \in]0,1[$ satisfying

$$\begin{aligned}
&\big|(\Pi\Lambda_1)(t+s_n) - (\breve{B}\upsilon_1)(t)\big| \\
&\qquad \leq \int_{\infty}^{t} e^{-(t-s)\underline{\Theta}} |\Lambda_1(s+\varsigma_n) - \upsilon_1(s)|\,ds \\
&\qquad\quad + \|\upsilon_1\|_\infty \int_{\infty}^{t} e^{-\int_s^t \Theta(u+\varsigma_n)\,du + \theta\int_s^t [\Theta(u+\varsigma_n) - \iota(u)]\,du} \Big| \int_s^t \Theta(u+\varsigma_n) - \iota(u)\,du \Big|\,ds \\
&\qquad \leq \int_{\infty}^{t} e^{-(t-s)\underline{\Theta}} |\Lambda_1(s+\varsigma_n) - \upsilon_1(s)|\,ds + \|\varsigma_1\|_\infty \int_{\infty}^{t} \Big| \int_s^t \Theta(u+\varsigma_n) - \iota(u)\,du \Big| e^{-(t-s)\underline{\Theta}}\,ds.
\end{aligned}$$

By Lebesgue's dominated convergence theorem, we immediately obtain that for all $t \in \mathbb{R}$,

$$\lim_{n\to\infty} (\Pi\Lambda_1)(t+s_n) = (\breve{B}\upsilon_1)(t).$$

Similarly, for all $t \in \mathbb{R}$,

$$\lim_{n\to\infty} (\breve{B}\upsilon_1)(t-s_n) = (\Pi\Lambda_1)(t).$$

which implies that $(\Pi\Lambda_1)(\cdot) \in \mathrm{AA}(\mathbb{R},\mathbb{R}_+)$. Now, it is necessary to demonstrate that $(\Pi\Lambda_2)(\cdot) \in$

$\mathrm{PAP}_0(\mathbb{R},\mathbb{R}_+,\eta)$. We have

$$\begin{aligned}\lim_{C\to\infty}\frac{1}{\eta([-C,C])}\int_{-C}^{C}\Big|\int_{\infty}^{t} e^{(-\int_s^t \Theta(u)\,du)}\Lambda_2(s)\,ds\Big|\,d\eta(t)\\ &\le \lim_{C\to\infty}\frac{1}{\eta([-C,C])}\int_{-C}^{C}\int_{\infty}^{t}|\Lambda_2(s)|e^{(-\int_s^t \Theta(u)\,du)}\,ds\,d\eta(t)\\ &\le \lim_{C\to\infty}\frac{1}{\eta([-C,C])}\int_{-C}^{C}\int_{\infty}^{t}e^{-(t-s)\underline{\Theta}}|\Lambda_2(s)|\,ds\,d\eta(t)\\ &\le \lim_{C\to\infty}\frac{1}{\eta([-C,C])}\int_{-C}^{C}\int_{0}^{\infty}e^{-s\underline{\Theta}}|\Lambda_2(t-s)|\,ds\,d\eta(t)\\ &\le \lim_{C\to\infty}\int_{0}^{\infty}e^{-s\underline{\Theta}}\Big(\frac{1}{\eta([-C,C])}\int_{-C}^{C}|\Lambda_2(t-s)|\,d\eta(t)\Big)\,ds.\end{aligned}$$

Utilizing Lebesgue's dominated convergence theorem as well as hypothesis (A1), we get

$$\begin{aligned}\lim_{C\to\infty}\int_{0}^{\infty}e^{-s\underline{\Theta}}\Big(\frac{1}{\eta([-C,C])}\int_{-C}^{C}|\Lambda_2(t-s)|\,d\eta(t)\Big)\,ds\\ &=\int_{0}^{\infty}e^{-s\underline{\Theta}}\lim_{C\to\infty}\Big(\frac{1}{\eta([-C,C])}\int_{-C}^{C}|\Lambda_2(t-s)|\,d\eta(t)\Big)\,ds=0.\end{aligned}$$

Consequently, $(\Pi\Lambda_2)(\,\cdot\,)$ belongs to $\mathrm{PAP}_0(\mathbb{R},\mathbb{R}_+,\eta)$, which implies the result. $\square$

Theorem 3.2. *Let* $\eta\in\Upsilon$. *Under* (H1)–(H3) *and*

(H4) $$\frac{\sum_{i=1}^{D}(\overline{Q}_i+\overline{p}_i)\big[\big(\frac{(n-m)}{4}+\frac{m}{(1+R_1^n)^2}\big)R_2^{m-1}\big]+L}{\underline{\Theta}}<1,$$

equations (1.1)–(1.2) *have one weighted pseudo almost automorphic solution within a convex subset* $\mathfrak{B}=\{f\in\mathrm{PAA}(\mathbb{R},\mathbb{R},\eta),\ R_1\le\Gamma(t)\le R_2\}$.

Proof. Firstly, by Theorem 3.1, the operator Π noted as

$$\begin{aligned}(\Pi\Gamma)(t)=\int_{\infty}^{t}e^{-\int_s^t\Theta(u)\,du}\\ \times\Big[\sum_{i=1}^{D}\Big(Q_i(s)\frac{\Gamma^m(s-e_i(s))}{1+\Gamma^n(s-e_i(s))}+p_i(s)\int_0^{\delta_i}k_i(u)\frac{\Gamma^m(s-u)}{1+\Gamma^n(s-u)}\,du\Big)-\Xi(s,\Gamma(s-j(s)))\Big]\,ds\end{aligned}$$

is a mapping in $\mathrm{PAA}(\mathbb{R},\mathbb{R},\eta)$ into itself. Next, we have to prove that for $\Gamma\in\mathfrak{B}$, $R_1\le(\Pi\Gamma)(t)\le R_2$.

For $\Gamma \in \mathfrak{B}$,

$$
\begin{aligned}
(\Pi\Gamma)(t) = \int_{\infty}^{t} e^{-\int_s^t \Theta(u)\,du} \\
\times \Big[\sum_{i=1}^{D} \Big(Q_i(s)\, \frac{\Gamma^m(s-e_i(s))}{1+\Gamma^n(s-e_i(s))} + p_i(s) \int_0^{\delta_i} k_i(u)\, \frac{\Gamma^m(s-u)}{1+\Gamma^n(s-u)}\,du \Big) - \Xi(s,\Gamma(s-j(s))) \Big]\,ds \\
\geq \int_{\infty}^{t} e^{-(t-s)\overline{\Theta}} \Big[\sum_{i=1}^{D} \underline{Q}_i\, \frac{R_1^m}{1+R_2^n} - \overline{\Xi} \Big]\,ds \geq \int_{\infty}^{t} e^{-(t-s)\overline{\Theta}} R_1 \overline{\Theta} = R_1.
\end{aligned}
$$

However,

$$
\begin{aligned}
(\Pi\Gamma)(t) = \int_{\infty}^{t} e^{-\int_s^t \Theta(u)\,du} \\
\times \Big[\sum_{i=1}^{D} \Big(Q_i(s)\, \frac{\Gamma^m(s-e_i(s))}{1+\Gamma^n(s-e_i(s))} + p_i(s) \int_0^{\delta_i} k_i(u)\, \frac{\Gamma^m(s-u)}{1+\Gamma^n(s-u)}\,du \Big) - \Xi(s,\Gamma(s-j(s))) \Big]\,ds \\
\leq \int_{\infty}^{t} e^{-(t-s)\underline{\Theta}} \Big[\sum_{i=1}^{D} (\overline{Q}_i + \overline{p}_i) \Big]\,ds \leq R_2.
\end{aligned}
$$

Consequently, Π is a mapping in $\mathfrak{B}$ into itself. Now, we shall prove that Π is a contraction mapping. For $A, B \in \mathfrak{B}$,

$$
\begin{aligned}
&\big|(\Pi A)(t) - (\Pi B)(t)\big| \\
&= \Big| \int_{\infty}^{t} e^{-\int_s^t \Theta(u)\,du} \Big[\sum_{i=1}^{D} Q_i(s) \Big[\frac{A^m(s-e_i(s))}{1+A^n(s-e_i(s))} - \frac{B^m(s-e_i(s))}{1+B^n(s-e_i(s))} \Big] \\
&\quad + \sum_{i=1}^{D} p_i(s) \int_0^{\delta_i} k_i(u) \Big[\frac{A^m(s-u)}{1+A^n(s-u)} - \frac{B^m(s-u)}{1+B^n(s-u)} \Big]\,du \\
&\quad + \Xi(s,B(s-j(s))) - \Xi(s,A(s-j(s))) \Big]\,ds \Big| \\
&\leq \int_{\infty}^{t} e^{-(s-t)\underline{\Theta}} \Big[\sum_{i=1}^{D} \overline{Q}_i \Big| \frac{A^m(s-e_i(s))}{1+A^n(s-e_i(s))} - \frac{B^m(s-e_i(s))}{1+B^n(s-e_i(s))} \Big| \\
&\quad + \sum_{i=1}^{D} \overline{p}_i \int_0^{\delta_i} k_i(u) \Big| \frac{A^m(s-u)}{1+A^n(s-u)} - \frac{B^m(s-u)}{1+B^n(s-u)} \Big|\,du + L\big|B(s-j(s)) - A(s-j(s))\big| \Big]\,ds.
\end{aligned}
$$

From the elementary mean value theorem of differential calculus, we have

$$
\frac{a^m}{1+a^n} - \frac{b^m}{1+b^n} = \frac{(m-n)\varrho^{m+n-1} + m\varrho^{m-1}}{(1+\varrho^n)^2}\,(a-b),
$$

where ϱ lies between a and b. Then

$$
\Big| \frac{A^m}{1+A^n} - \frac{B^m}{1+B^n} \Big| \leq \Big[\frac{(n-m)\varrho^{m-1}}{4} + \frac{m\varrho^{m-1}}{(1+\varrho^n)^2} \Big]\,|A-B|.
$$

Besides, we have

$$\begin{aligned}&\big|(\Pi A)(t)-(\Pi B)(t)\big|\\&\quad\le \|A-B\|_\infty\int_{\infty}^{t}e^{-(s-t)\underline{\Theta}}\Big[\sum_{i=1}^{D}(\overline{Q}_i+\overline{p}_i)\Big[\frac{(n-m)R_2^{m-1}}{4}+\frac{mR_2^{m-1}}{(1+R_1^n)^2}\Big]+L\Big]\,ds\\&\quad\le\frac{\sum_{i=1}^{D}(\overline{Q}_i+\overline{p}_i)\big[\big(\frac{(n-m)}{4}+\frac{m}{(1+R_1^n)^2}\big)R_2^{m-1}+L\big]}{\underline{\Theta}}\|A-B\|_\infty\le\|A-B\|_\infty.\end{aligned}$$

Consequently, the mapping Π has a unique positive weighted pseudo almost automorphic solution $\Gamma^*\in\mathfrak{B}$ via the contraction principle. □

4 Exponential stability of the weighted pseudo almost automorphic solution

The exponential stability of equations (1.1), (1.2) is established in this section.

Theorem 4.1. *Let $\eta\in\Upsilon$. Beneath* (H1)–(H4). *The weighted pseudo almost automorphic solution Γ^* of equations* (1.1)–(1.2) *is globally exponentially stable in the region $\mathfrak{B}$.*

Proof. Let Σ be as follows:

$$\Sigma(t)=t-\underline{\Theta}+Le^{tK}+\Big[\Big(\frac{(n-m)}{4}+\frac{m}{(1+R_1^n)^2}\Big)R_2^{m-1}\Big]\sum_{i=1}^{D}\Big(\overline{Q}_ie^{tr}+\overline{p}_i\int_0^{\delta_i}k_i(s)\,e^{ts}\,ds\Big).$$

Clearly, the function $t\mapsto\Sigma(t)$ is continuous on $\mathbb{R}_+$, and

$$\Sigma(0)=-\underline{\Theta}+L+\Big[\Big(\frac{(n-m)}{4}+\frac{m}{(1+R_1^n)^2}\Big)R_2^{m-1}\Big]\sum_{i=1}^{N}\Big(\overline{Q}_i+\overline{p}_i\int_0^{\delta_i}k_i(s)\,ds\Big).$$

By hypothesis (H4), we get $\Sigma(0)<0$. Therefore, we have one sufficiently small constant $\varpi>0$ as $\Sigma(\varpi)<0$.

Let $\Gamma(t)$ be another solution of (1.1)–(1.2) and $\Lambda(t)=|\Gamma^*(t)-\Gamma(t)|e^{\varpi t}$. For $-K\le t\le 0$, we have

$$\Lambda(t)\le\sup_{-K\le t\le 0}|\Gamma^*(t)-\Gamma(t)|e^{\varpi t}\le\sup_{-K\le t\le 0}|\Gamma^*(t)-\Gamma(t)|=M\le M+\zeta.$$

In the remaining of this proof, it should be proven that $\Lambda(t)\le M$, for $t\ge 0$. In all other respects, for $\zeta>0$, $\{t>0:\ \Lambda(t)>M+\zeta\}\ne\varnothing$. Let $t_s=\inf\{t>0:\ \Lambda(t)>M+\zeta\}$. Then $\Lambda(t_S)=M+\zeta$, $D^+\Lambda(t)\ge 0$ and $\Lambda(t)\le M+\zeta$, for $-K\le t\le t_s$. Calculating the upper derivative of $\Lambda(t)$, we obtain

$$\begin{aligned}0&\le D^+\Lambda(t_s)=D^+\big[|\Gamma^*(t_s)-\Gamma(t_s)|e^{\varpi t_s}\big]\\&=e^{\varpi t_s}\Big[\varpi|\Gamma^*(t_s)-\Gamma(t_s)|-\Theta(t_s)|\Gamma^*(t_s)-\Gamma(t_s)|\\&\quad+\sum_{i=1}^{D}Q_i(t_s)\Big[\Big(\frac{(n-m)}{4}+\frac{m}{(1+R_1^n)^2}\Big)R_2^{m-1}\Big]\big|\Gamma^*(t_s-e_i(t_s))-\Gamma(t_s-e_i(t_s))\big|\\&\quad+\sum_{i=1}^{D}p_i(t)\Big[\Big(\frac{(n-m)}{4}+\frac{m}{(1+R_1^n)^2}\Big)R_2^{m-1}\Big]\int_0^{\delta_i}k_i(s)|\Gamma^*(t_0-s)-\Gamma(t_0-s)|\,ds\\&\quad+\Big|\Xi\big(t_s,\Gamma^*(t_s-j(t_s))\big)-\Xi\big(t_s,\Gamma(t_s-j(t_s))\big)\Big|\Big]\end{aligned}$$

$$\begin{aligned}
&\leq (\varpi - \underline{\Theta})\Gamma(t_s) + \sum_{i=1}^{D} \overline{Q}_i \Big[\Big(\frac{(n-m)}{4} + \frac{m}{(1+R_1^n)^2}\Big) R_2^{m-1}\Big] \Gamma(t_s - e_i(t_s))\, e^{\varpi r} \\
&\quad + \sum_{i=1}^{D} \overline{p}_i \Big[\Big(\frac{(n-m)}{4} + \frac{m}{(1+R_1^n)^2}\Big) R_2^{m-1}\Big] \int_0^{\delta_i} k_i(s)\, e^{\varpi s} \Gamma(t_s - s)\, ds + L\Gamma(t_s - j(t_s))\, e^{\varpi r} \\
&\leq \Bigg[\varpi - \underline{\Theta} + \sum_{i=1}^{D} \overline{Q}_i \Big[\Big(\frac{(n-m)}{4} + \frac{m}{(1+R_1^n)^2}\Big) R_2^{m-1}\Big] \\
&\quad + \sum_{i=1}^{D} \overline{p}_i \Big[\Big(\frac{(n-m)}{4} + \frac{m}{(1+R_1^n)^2}\Big) R_2^{m-1}\Big] \int_0^{\delta_i} k_i(s)\, e^{\varpi s}\, ds + L e^{\varpi r}\Bigg] (M + \varepsilon) \leq 0,
\end{aligned}$$

which is a contradiction. Thus, for every $\varepsilon > 0$, we have $\Lambda(t) \leq M + \varepsilon$ for all $t \geq 0$, and it can be concluded that $\Lambda(t) \leq M + \varepsilon$, for all $t \geq -K$. Consequently,

$$|\Gamma(t) - \Gamma^*(t)| \leq (M + \varepsilon) e^{-\varpi t}, \quad t \geq -K,$$

for $\varepsilon \to 0$, and we get

$$|\Gamma(t) - \Gamma^*(t)| \leq M e^{-\varpi t}, \quad t \geq -K,$$

□

5 Asymptotic stability of the weighted pseudo almost atutomorphic solutions

Theorem 5.1. *Let $\eta \in \Upsilon$. Suppose that* (H1)–(H4) *are satisfied, and let Γ^* be the unique positive weighted pseudo almost automorphic solution of equations* (1.1)–(1.2)*. Then Γ^* will be globally asymptotically stable in the region $\mathfrak{B}$.*

Proof. Let Γ be an arbitrary solution of equations (1.1)–(1.2) in $\mathfrak{B}$. Consider the Lyapunov function

$$W(t) = |\Gamma^*(t) - \Gamma(t)|.$$

The calculation of an upper Dini derivative $D^+W(t)$ of W leads to

$$\begin{aligned}
&0 \leq D^+W(t) = D^+|\Gamma^*(t) - \Gamma(t)| \\
&\leq -\Theta(t)|\Gamma^*(t) - \Gamma(t)| + \sum_{i=1}^{D} b_i(t) \Big| \frac{(\Gamma^*)^m(s - e_i(s))}{1 + (\Gamma^*)^n(s - e_i(s))} - \frac{\Gamma^m(s - e_i(s))}{1 + \Gamma^n(s - e_i(s))} \Big| \\
&\quad + \sum_{i=1}^{D} \overline{p}_i \int_0^{\delta_i} k_i(u) \Big| \frac{(\Gamma^*)^m(u+s)}{1 + (\Gamma^*)^n(u+s)} - \frac{\Gamma^m(u+s)}{1 + \Gamma^n(u+s)} \Big| + \big|\Xi\big(t, \Gamma^*(t - j(t))\big) - \Xi\big(t, \Gamma(t - j(t))\big)\big| \\
&\leq -\Theta(t)|\Gamma^*(t) - \Gamma(t)| + \Bigg(\sum_{i=1}^{D} (\overline{Q}_i + \overline{p}_i) \Big[\Big(\frac{(n-m)}{4} + \frac{m}{(1+R_1^n)^2}\Big) R_2^{m-1}\Big] + L\Bigg) |\Gamma^*(t) - \Gamma(t)| \\
&\leq -\Bigg(\underline{\Theta} - \sum_{i=1}^{D} (\overline{Q}_i + \overline{p}_i) \Big[\Big(\frac{(n-m)}{4} + \frac{m}{(1+R_1^n)^2}\Big) R_2^{m-1}\Big] - L\Bigg) |\Gamma^*(t) - \Gamma(t)|.
\end{aligned}$$

By hypothesis (H4),

$$\xi = \underline{\Theta} - \sum_{i=1}^{D} (\overline{Q}_i + \overline{p}_i) \Big[\Big(\frac{(n-m)}{4} + \frac{m}{(1+R_1^n)^2}\Big) R_2^{m-1}\Big] - L > 0.$$

Then

$$D^+W(t) \leq -\xi |\Gamma^*(t) - \Gamma(t)| < 0. \tag{5.1}$$

Hence, equation (1.1)–(1.2) is stable in the Laypunov sense. Integrating (5.1) over $[T,t]$, gives

$$\xi \int_T^t |\Gamma^*(s) - \Gamma(s)|\, ds \le W(T) - W(t),$$

which implies

$$\int_T^t |\Gamma^*(s) - \Gamma(s)|\, ds \le \frac{W(T)}{\xi}\,.$$

It follows that

$$\limsup_{t\to\infty} \int_T^t |\Gamma^*(s) - \Gamma(s)|\, ds < \infty.$$

Therefore,

$$\lim_{t\to\infty} |\Gamma^*(t) - \Gamma(t)| = 0. \qquad \square$$

Remark. In this paper, we provide new criteria concerning the hematopoiesis model with $1 < m \le n$. The existence of a unique point is proved by the Banach fixed point theorem. The kinds of stability exponential and asymptotic criteria are derived by using suitable Lyapunov functionals and other techniques. Note that our methods can be applicable when $\Xi(\,\cdot\,,\,\cdot\,) = 0$ or $p(\,\cdot\,) = 0$, or $\Xi(\,\cdot\,,\,\cdot\,) = p(\,\cdot\,) = 0$. Furthermore, if $\eta = 1$, we obtain $\mathrm{PAA}(\mathbb{R},\mathbb{R},1) = \mathrm{PAA}(\mathbb{R},\mathbb{R})$, then our method and results are more general, since $\mathrm{PAP}(\mathbb{R},\mathbb{R}) \subseteq \mathrm{PAA}(\mathbb{R},\mathbb{R}) \subseteq \mathrm{PAA}(\mathbb{R},\mathbb{R},\eta)$ and $\mathrm{PAP}(\mathbb{R},\mathbb{R},\eta) \subseteq \mathrm{PAA}(\mathbb{R},\mathbb{R},\eta)$.

In fact, the stability plays an important role in many areas. That is why we prove in this paper the global exponential stability and the global asymptotic stability of equations (1.1)–(1.2) when $n \ge m > 1$. Hence our finding on the existence and some kinds of stability of equations (1.1)–(1.2) is new.

In this paper, the continuous time delays and the distributed delays when $n \ge m > 1$ are considered to provide a generally more realistic description of our model, since it is a model of mathematical biology.

6 Example 1

In this section, we give two examples with the goal of verifying the validity of the results from previous sections.

Consider the following hematopoiesis model:

$$\begin{aligned}
\Gamma'(t) = -\Big(8 + \cos\Big(\frac{1}{2+\sin(t)+\cos(t)}\Big)\Big)\Gamma(t) \\
+ \Big(2 + \frac{1}{2}\cos\Big(\frac{1}{2+\sin(t)+\cos(t)}\Big) + \exp(-t)\Big)\,\frac{\Gamma^2(t-0.9)}{1+\Gamma^3(t-0.9)} \\
+ \frac{1}{8}\Big(1 + \frac{1}{2}\sin\Big(\frac{1}{2+\sin(t)+\cos(t)}\Big) + \frac{1}{100(1+t^2)}\Big)\int_0^{\frac{\pi}{2}} \cos(u)\,\frac{\Gamma^2(t-u)}{1+\Gamma^3(t-u)}\,du \\
- \frac{1}{10}\cos\Big(\frac{1}{2+\sin(t)+\cos(t)}\Big)^2\,\frac{|\Gamma(t-0.8)|}{1+\Gamma^2(t-0.8)}\,,
\end{aligned} \tag{6.1}$$

where $n = 3$, $m = 2$, $D = 1$, $e(t) = 0.9$, $j(t) = 0.8$, and $\delta = \frac{\pi}{2}$. Let us take into account measure η with the result that the Radon–Nikodym derivative is $\rho = \exp(t)$, and $\rho(t)$ satisfies (A1), i.e.,

$$\limsup_{|t|\to\infty} \frac{\rho(t+r)}{\rho(t)} < \infty.$$

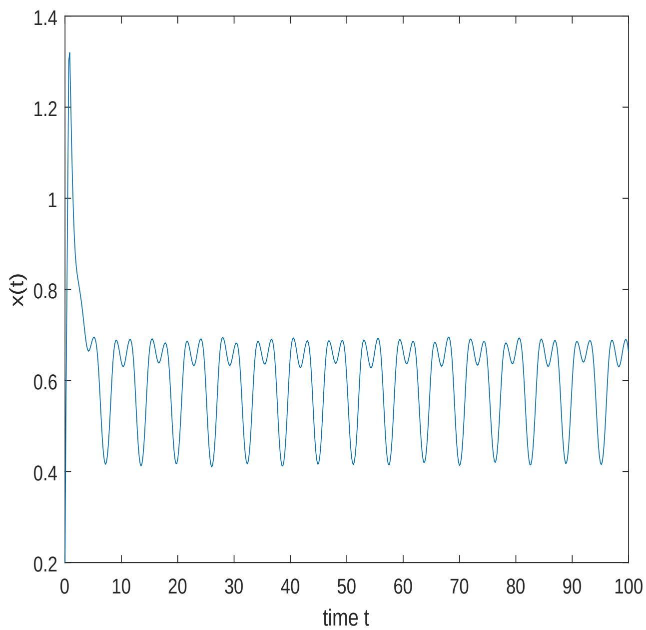

Figure 6.1. Curve of the weighted pseudo almost automorphic solution $\Gamma(t)$ of equation (6.1) for initial value 0.2, 0.7 .

Then η satisfies hypothesis (A1).

Moreover, for

$$\Theta(t) = 8 + \cos\Big(\frac{1}{2+\sin(t)+\cos(t)}\Big),$$
$$Q(t) = 2 + \frac{1}{2}\cos\Big(\frac{1}{2+\sin(t)+\cos(t)}\Big) + \exp(-t),$$
$$p(t) = \frac{1}{8}\Big(1 + \frac{1}{2}\sin\Big(\frac{1}{2+\sin(t)+\cos(t)}\Big) + \frac{1}{100}\exp(-t)\Big),$$
$$\Xi\big(t,\Gamma(t-j(t))\big) = \frac{1}{10}\cos\Big(\frac{1}{2+\sin(t)+\cos(t)}\Big)^2 \frac{|\Gamma(t-0.8)|}{1+\Gamma^2(t-0.8)},$$

and $k(t) = \cos\left(\frac{\pi}{2}t\right)$, all hypotheses (H1)–(H3) are satisfied. Furthermore,

$$\frac{(\overline{Q}+\overline{p})[(\frac{(n-m)}{4} + \frac{m}{(1+R_1^n)^2})R_2^{m-1}] + L}{\underline{\Theta}} \sim 0.016 < 1.$$

Then hypothesis (H4) is verified. Consequently, all conditions of Theorems 3.2, 4.1, and 5.1 hold, and equation (1.1) has in the region $\mathfrak{B}$ a unique weighted pseudo almost automorphic solution.

7 Example 2

Consider the following model of hematopoiesis

$$\Gamma'(t) = -\Theta(t)\Gamma(t) + \sum_{i=1}^{D} b_i(t)\frac{\Gamma^m(t-e_i(t))}{1+\Gamma^n(t-e_i(t))}$$
$$+ \sum_{i=1}^{D} p_i(t)\int_0^{\delta_i} k_i(u)\frac{\Gamma^m(t-u)}{1+\Gamma^n(t-u)}\,du - \Xi\big(t,\Gamma(t-j(t))\big), \tag{7.1}$$

where $n = 2$, $m = 2$, $D = 1$, $e(t) = 0.2$, $j(t) = 0.4$, and $\delta = 1$. Let us take into consideration measure η with the result that the Radon–Nikodym derivative is $\rho = \exp(t)$, and $\rho(t)$ satisfies (A1), i.e.,

$$\limsup_{|t|\to\infty} \frac{\rho(t+r)}{\rho(t)} < \infty.$$

Then η satisfies hypothesis (A1).

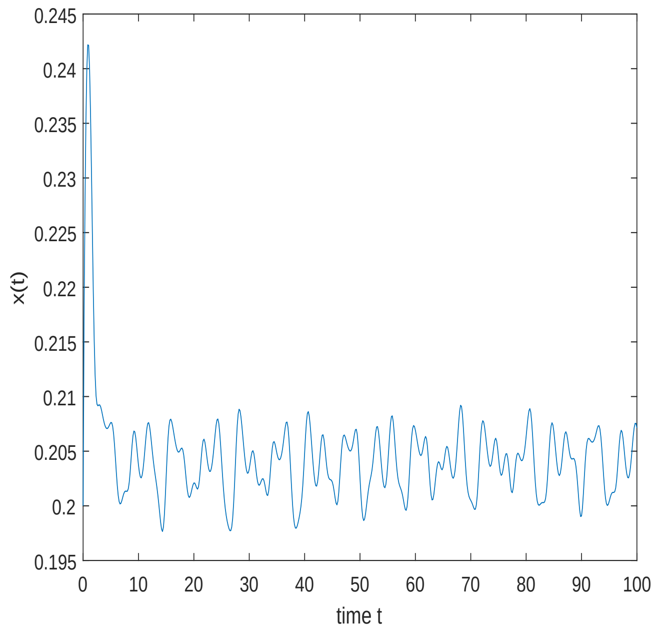

Figure 7.1. Curve of the weighted pseudo almost automorphic solution $\Gamma(t)$ of the equation (7.1) for initial value 0.2, 0.7 .

Besides, for

$$\begin{aligned}
\Theta(t) &= 2 + \cos\Big(\frac{1}{2+\sin(t)+\cos(t)}\Big)^2,\\
Q(t) &= \frac{1}{10}\Big(2 + \frac{1}{2}\sin\Big(\frac{1}{2+\sin(t)+\cos(t)}\Big) + \frac{1}{10}\exp(-t)\Big),\\
p(t) &= \frac{1}{8}\Big(1 + \frac{1}{2}\sin\Big(\frac{1}{2+\sin(t)+\cos(t)}\Big) + \frac{1}{100}\exp(-t)\Big),\\
\Xi(t,\Gamma(t-j(t))) &= \frac{1}{100}\cos\Big(\frac{1}{2+\sin(t)+\cos(t)}\Big)^2 \frac{|\Gamma(t-0.8)|}{1+\Gamma^2(t-0.8)},
\end{aligned}$$

and $k(t) = \cos(\frac{\pi}{2}t)$, all hypotheses (H1)–(H3) are satisfied. Furthermore,

$$\frac{(\overline{Q}+\overline{p})[(\frac{(n-m)}{4} + \frac{m}{(1+R_1^n)^2})R_2^{m-1}] + L}{\underline{\Theta}} \le 0.16 < 1.$$

Therefore, all conditions of Theorems 3.2, 4.1 and 5.1 are satisfied. Then the hematopoiesis model with a harvesting term and mixed delays (7.1) has one unique weighted pseudo almost automorphic solution (as illustrated in Figure 7.1) in the region $\mathfrak{B} = \{\Gamma \in \mathrm{PAA}(\mathbb{R},\mathbb{R},\eta), R_1 \le \Gamma(t) \le R_2\}$.

Conclusion

In this paper, one novel Mackey–Glass model which has been studied. By using the theory of the exponential dichotomy along with the Banach fixed point theorem, some conditions have been provided, enough to obtain the existence and uniqueness of the weighted pseudo almost automorphic solution. In addition to that, we have established the global exponential and asymptotic stabilities of the unique solution. An example is given with the purpose of showing the efficiency of the obtained results. Up to our knowledge, there has not been any paper published considering the model of Mackey–Glass with a harvesting term as well as mixed delays when $1 < m \leq n$. Moreover, no research has been published for this model in the space of weighted pseudo almost automorphic functions.

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