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Q-PRIMITIVES AND EXPLICIT SOLUTIONS OF NON-HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS IN $L^p(\mathbb{T})$

Abstract. Let $\mathbb{T} = [-\pi, \pi]$, $1 \leq p \leq \infty$ and $Q(x)$ be a polynomial. In this paper, we introduce the notion called Q-primitives of a function in $\mathcal{S}'(\mathbb{R})$ and apply it to examine the existence and uniqueness of solutions in $L^p(\mathbb{T})$ of the non-homogeneous equation $Q(D)f = \psi \in L^p(\mathbb{T})$. The explicit solutions of the equation are given. In particular, we show that the condition $Q(x) \neq 0\ \forall\, x \in \operatorname{supp} \widehat{\psi}$ is the criterion for the existence of a Q-primitive in $L^p(\mathbb{T})$ of f. Note that every Q-primitive in $L^p(\mathbb{T})$ of f is a solution of the equation $Q(D)f = \psi$. Moreover, an inequality for higher order Q-primitives is also given.

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Key words and phrases. L^p-spaces, explicit solutions, periodic functions, Fourier transform.

რეზიუმე. დავუშვათ, $\mathbb{T} = [-\pi, \pi]$, $1 \leq p \leq \infty$ და $Q(x)$ არის მრავალწევრი. ნაშრომში შემოყვანილია ფუნქციის Q-პრიმიტივის ცნება $\mathcal{S}'(\mathbb{R})$-ში და ის გამოყენებულია $L^p(\mathbb{T})$-ში არაერთგვაროვანი $Q(D)f = \psi \in L^p(\mathbb{T})$ განტოლების ამონახსნის არსებობისა და ერთადერთობის გამოსაკვლევად. მოცემულია ამ განტოლების ცხადი ამონახსნები. კერძოდ, ნაჩვენებია, რომ პირობა $Q(x) \neq 0\ \forall x \in \operatorname{supp}\widehat{\psi}$ წარმოადგენს f-ის Q-პრიმიტივის არსებობის კრიტერიუმს $L^p(\mathbb{T})$-ში. შევნიშნოთ, რომ f-ის ყოველი Q-პრიმიტივი $L^p(\mathbb{T})$-ში არის $Q(D)f = \psi$ განტოლების ამონახსნი. უფრო მეტიც, აგრეთვე მოცემულია უტოლობა უმაღლესი რიგის Q-პრიმიტივებისთვის.

1 Introduction

Denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing functions and by $\mathcal{S}'(\mathbb{R})$ the dual space of $\mathcal{S}(\mathbb{R})$, the space of tempered distributions on $\mathbb{R}$. Let $\mathbb{T}$ be the torus, $1 \le p \le \infty$, $Q(x) = \sum\limits_{k=0}^{n} a_k x^k$ be a polynomial and $f \in \mathcal{S}'(\mathbb{R})$. The differential operator $Q(D)$ is obtained from $Q(x)$ by substituting $x \to -iD = -id/dx$,

$$Q(D)f = \sum_{k=0}^{n} a_k (-i)^k D^k f.$$

Denote by $L^p(\mathbb{T})$ the set of all 2π-periodic functions f on $\mathbb{R}$ such that the norm

$$\|f\|_{p,\mathbb{T}} := \begin{cases} \left(\int\limits_{-\pi}^{\pi} |f(x)|^p \, dx \right)^{1/p} & \text{if } \ p < \infty, \\ \operatorname*{ess\,sup}\limits_{x \in \mathbb{T}} |f(x)| & \text{if } \ p = \infty \end{cases}$$

is finite. Clearly, $L^p(\mathbb{T}) \subset \mathcal{S}'(\mathbb{R})$. Let $\psi \in L^p(\mathbb{T})$. We consider the following non-homogeneous equation with constant coefficients in $L^p(\mathbb{T})$:

$$Q(D)f = \psi. \tag{1.1}$$

In the theory of ordinary differential equations, there are several methods for solving the non-homogeneous equation with constant coefficients:

$$y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots + a_{n-1} y'(x) + a_n y(x) = h(x).$$

The best method depends on the nature of the function h that makes the equation non-homogeneous. If h is a linear combination of exponential and sinusoidal functions, then the exponential response formula may be used. If, more generally, h is a linear combination of functions of the form $x^k e^{ax}, x^k \cos(ax)$, and $x^k \sin(ax)$, where k is a nonnegative integer, and a is a constant (which need not be the same in each term), then the *Method of undetermined coefficients* may be used. Still more general, the *Annihilator method* is applied when h satisfies a homogeneous linear differential equation. The most general method is the *Variation of constants*. Note that, in general, f and ψ in (1.1) are generalized functions, and so the traditional methods have no effect in our case. Hörmander proved the following result [13, Theorem 7.3.2]: *Let $\psi \in \mathcal{E}'(\mathbb{R})$. Then the equation $Q(D)f = \psi$ has a solution in $\mathcal{E}'(\mathbb{R})$ if and only if $\widehat{\psi}(\xi)/Q(\xi)$ is an entire function, where $\mathcal{E}'(\mathbb{R})$ is the space of distributions with compact supports.*

In this paper, by using another method, we solve the non-homogeneous equation $Q(D)f = \psi$ in the space $L^p(\mathbb{T})$. Since $L^p(\mathbb{T})$ is not contained in $\mathcal{E}'(\mathbb{R})$, $\widehat{\psi}(\xi)$ is not an entire function in general, and so we cannot apply the just mentioned result to solve (1.1).

The paper consists of three sections. In Section 2, we introduce the notion called Q-primitives of a function in $\mathcal{S}'(\mathbb{R})$ and use it to examine the existence and uniqueness of solutions of equation (1.1). The explicit solutions of the equation are given. In Section 3, we obtain an inequality for higher order Q-primitives.

Notations

Let $f \in L^1(\mathbb{R})$ and $\widehat{f} = \mathcal{F}f$ be its Fourier transform

$$\widehat{f}(\zeta) = (2\pi)^{-1/2} \int\limits_{\mathbb{R}} e^{-ix\zeta} f(x) \, dx.$$

The Fourier transform of a tempered distribution f can be defined via the formula

$$\langle \mathcal{F}f, \varphi \rangle = \langle f, \mathcal{F}\varphi \rangle, \ \ \varphi \in \mathcal{S}(\mathbb{R}).$$

Let K be a compact in $\mathbb{R}$ and $\epsilon > 0$.
Denote by

$$K_\epsilon := \big\{\zeta \in \mathbb{R} : \ \exists\, x \in K : \ |x - \zeta| \le \epsilon\big\}.$$

Recall that $\operatorname{supp} \widehat{f} \subset \mathbb{Z}$ for any $f \in L^p(\mathbb{T})$.

We have the following Young inequality for 2π-periodic functions [14]: *Let $1 \le p \le \infty$, $f \in L^p(\mathbb{T})$ and $g \in L^1(\mathbb{R})$. Then $f * g \in L^p(\mathbb{T})$ and*

$$\|f * g\|_{p,\mathbb{T}} \le \|f\|_{p,\mathbb{T}} \|g\|_{L^1(\mathbb{R})},$$

where the convolution $f * g$ is defined as

$$(f * g)(x) = \int\limits_{\mathbb{R}} f(x-y) g(y)\, dy.$$

Definition. Let $f \in \mathcal{S}'(\mathbb{R})$ and Q be a polynomial. The tempered distribution $\mathcal{Q}f$ is termed a Q-primitive of f if $Q(D)(\mathcal{Q}f) = f$.

When $Q(x) = x$, then Q-primitives become primitives of tempered distributions defined in [7–9], and the notion of primitives of distributions in $\mathcal{D}'(a,b)$, $a, b \in \mathbb{R}$, can be found in [18].

2 The necessary and sufficient conditions for the unique existence of Q-primitives of a function in $L^p(\mathbb{T})$

We have the following result.

Theorem 2.1. *Let $1 \le p \le \infty, Q(x)$ be a polynomial and $f \in L^p(\mathbb{T})$. Then there exists a unique Q-primitive in $L^p(\mathbb{T})$ of f denoted by $\mathcal{Q}f$ satisfying $Q(x) \ne 0\ \forall\, x \in \operatorname{supp} \widehat{\mathcal{Q}f}$ if and only if $Q(x) \ne 0$ $\forall\, x \in \operatorname{supp} \widehat{f}$.*

To prove Theorem 2.1, we need the following result in [18]:

Lemma 2.2. *Let $1 \le p \le \infty$ and $f \in L^p(\mathbb{T})$. Then the Fourier series of f converges to f in $\mathcal{S}'(\mathbb{R})$:*

$$f(x) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \bigg(e^{ikx} \int\limits_{-\pi}^{\pi} f(t) e^{-ikt}\, dt \bigg).$$

(The functional series $\sum\limits_{k\in\mathbb{Z}} f_k(x)$ is called convergent to f in $\mathcal{S}'(\mathbb{R})$ if the functional sequence $S_n(x) = \sum\limits_{k=-n}^{n} f_k(x)$ converges to f in $\mathcal{S}'(\mathbb{R})$.)

Proof of Theorem 2.1. Necessity. It follows from Lemma 2.2 that the Fourier series of $\mathcal{Q}f$ converges to $\mathcal{Q}f$ in $\mathcal{S}'(\mathbb{R})$,

$$\mathcal{Q}f(x) = \sum_{k \in \mathbb{Z}} (b_k e^{ikx}), \quad b_k = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \mathcal{Q}f(t) e^{-ikt}\, dt.$$

Therefore, the functional series $\sum\limits_{k\in\mathbb{Z}} b_k (ik)^m e^{ikx}$ converges to $D^m(\mathcal{Q}f)$ in $\mathcal{S}'(\mathbb{R})$ for all $m \in \mathbb{Z}_+$. That gives

$$Q(D)\mathcal{Q}f(x) = \sum_{k\in\mathbb{Z}} b_k Q(k) e^{ikx}.$$

Then, since $f = Q(D)\mathcal{Q}f$,

$$f(x) = \sum_{k\in\mathbb{Z}} b_k Q(k) e^{ikx}.$$

Put $A := \{z \in \mathbb{Z} : \ Q(z) = 0\}$ and let ν be a number in A. We have for $\varphi \in \mathcal{S}'(\mathbb{R})$, $\operatorname{supp} \mathcal{F}^{-1}\varphi \subset [\nu - \frac{1}{2}, \nu + \frac{1}{2}]$:

$$\langle \widehat{f}, \mathcal{F}^{-1}\varphi\rangle = \langle f, \varphi\rangle = \sum_{k\in\mathbb{Z}} b_k Q(k)\langle e^{ikx}, \varphi\rangle = \sum_{k\in\mathbb{Z}} \sqrt{2\pi} b_k Q(k)\mathcal{F}^{-1}\varphi(k). \tag{2.1}$$

From $\nu \in A$ and $\operatorname{supp} \mathcal{F}^{-1}\varphi \subset [\nu - \frac{1}{2}, \nu + \frac{1}{2}]$, we obtain $Q(k) = 0$ for $k = \nu$, and $\mathcal{F}^{-1}\varphi(k) = 0$ for $k \in \mathbb{Z} \setminus \{\nu\}$. So, $Q(k)\mathcal{F}^{-1}\varphi(k) = 0 \ \forall k \in \mathbb{Z}$. Combining these and (2.1), we deduce that

$$\langle \widehat{f}, \mathcal{F}^{-1}\varphi\rangle = 0.$$

Hence, $\nu \notin \operatorname{supp} \widehat{f}$ for all $\nu \in A$. So, $Q(x) \neq 0 \ \forall x \in \operatorname{supp} \widehat{f}$.

Sufficiency. Now, we assume that $Q(x) \neq 0 \quad \forall x \in \operatorname{supp} \widehat{f}$. We first prove the existence of Q-primitives in $L^p(\mathbb{T})$ of f. To do this, we put

$$\gamma_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\, dt, \quad k = 0, 1, 2, \dots$$

and consider three cases as follows.

Case 1 **($Q(x) = x - d$, $d \neq 0$).** Then $d \notin \operatorname{supp} \widehat{f}$. We define the following function $I_1 f$:

$$I_1 f(x) = \int_{-\pi}^{x} (f(t) - \gamma_0(f))\, dt - \tau, \quad x \in \mathbb{R},$$

where the constant τ is chosen such that $\int_{-\pi}^{\pi} I_1 f(x)\, dx = 0$. Then $I_1 f$ is well defined and bounded on $\mathbb{R}$. Moreover,

$$I_1 f(x + 2\pi) - I_1 f(x) = \int_{x}^{x+2\pi} (f(t) - \gamma_0(f))\, dt = \int_{\mathbb{T}} (f(t) - \gamma_0(f))\, dt = 0$$

and

$$\begin{aligned}
\langle I_1 f, \varphi'\rangle &= \int_{\mathbb{R}} \Big(\int_{-\pi}^{x} (f(t) - \gamma_0(f))\, dt - \tau \Big) \varphi'(x)\, dx \\
&= \int_{\mathbb{R}} \Big(\int_{-\pi}^{x} (f(t) - \gamma_0(f))\, dt \Big) \varphi'(x)\, dx = \int_{-k\pi}^{k\pi} \Big(\int_{-k\pi}^{x} (f(t) - \gamma_0(f))\, dt \Big) \varphi'(x)\, dx \\
&= \int_{-k\pi}^{k\pi} \Big(\int_{t}^{k\pi} \varphi'(x)\, dx \Big) (f(t) - \gamma_0(f))\, dt = \int_{-k\pi}^{k\pi} -\varphi'(t)(f(t) - \gamma_0(f))\, dt = -\langle f - \gamma_0(f), \varphi\rangle
\end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R})$, where k is an odd natural number such that $\operatorname{supp} \varphi \subset (-k\pi, k\pi)$. Therefore, $I_1 f \in L^p(\mathbb{T})$ and $D(I_1 f) = f - \gamma_0(f)$.

Further,

$$\begin{aligned}
\int_{-\pi}^{\pi} I_1 f(x) e^{-ikx}\, dx &= \int_{-\pi}^{\pi} \Big(\int_{-\pi}^{x} (f(t) - \gamma_0(f))\, dt \Big) e^{-ikx}\, dx - \int_{-\pi}^{\pi} \tau e^{-ikx}\, dx \\
&= \int_{-\pi}^{\pi} \Big(\int_{t}^{\pi} e^{-ikx}\, dx \Big) (f(t) - \gamma_0(f))\, dt = \int_{-\pi}^{\pi} \Big(\frac{e^{-ikt} - e^{-ik\pi}}{ik} \Big) (f(t) - \gamma_0(f))\, dt = \frac{2\pi\gamma_k(f)}{ik}
\end{aligned}$$

for all $k \in \mathbb{Z}_*$, where $\mathbb{Z}_* = \{k \in \mathbb{Z} : \ k \neq 0\}$. Then it follows from Lemma 2.2 and $\int\limits_{-\pi}^{\pi} I_1 f(x)\, dx = 0$ that the Fourier series of 2π-periodic function $I_1 f$ converges to $I_1 f$ in $\mathcal{S}'(\mathbb{R})$:

$$I_1 f(x) = \sum_{k\in\mathbb{Z}_*} \frac{\gamma_k(f)}{ik}\, e^{ikx}. \tag{2.2}$$

Put

$$I_2 f(x) = \frac{\gamma_0(f)}{-d} + \sum_{k\in \operatorname{supp} \widehat{f}\cap\mathbb{Z}_*} \frac{d\gamma_k(f)}{k(k-d)}\, e^{ikx}, \quad \mathcal{Q}f = iI_1 f + I_2 f.$$

Observe that

$$|\gamma_k(f)| = \frac{1}{2\pi} \left| \int\limits_{-\pi}^{\pi} f(t) e^{-ikt}\, dt \right| \leq (2\pi)^{-1/p} \|f\|_{p,\mathbb{T}},$$

and then

$$\left| \frac{\gamma_k(f)}{k(k-d)}\, e^{ikx} \right| = \left| \frac{\gamma_k(f)}{k(k-d)} \right| \leq \frac{(2\pi)^{-1/p} \|f\|_{p,\mathbb{T}}}{|k(k-d)|}.$$

Therefore, it follows from

$$\sum_{k\in \operatorname{supp} \widehat{f}\cap\mathbb{Z}_*} \frac{1}{|k(k-d)|} < \infty$$

that the functional series

$$\sum_{k\in \operatorname{supp} \widehat{f}\cap\mathbb{Z}_*} \frac{\gamma_k(f)}{k(k-d)}\, e^{ikx}$$

uniformly converges on $\mathbb{R}$.

Hence $I_2 f$ is well defined and bounded on $\mathbb{R}$. Clearly, $I_2 f \in L^p(\mathbb{T})$, which together with $I_1 f \in L^p(\mathbb{T})$ imply $\mathcal{Q}f \in L^p(\mathbb{T})$. From (2.2) and the definition of $I_2 f$, we deduce that the functional series

$$\sum_{k\in \operatorname{supp} \widehat{f}\cap\mathbb{Z}_*} \frac{\gamma_k(f)(k-d)}{ik}\, e^{ikx}$$

converges to $Q(D)I_1 f$ in $\mathcal{S}'(\mathbb{R})$ and

$$\gamma_0(f) + \sum_{k\in \operatorname{supp} \widehat{f}\cap\mathbb{Z}_*} \frac{d\gamma_k(f)}{k}\, e^{ikx}$$

converges to $Q(D)I_2 f$ in $\mathcal{S}'(\mathbb{R})$. Therefore,

$$\gamma_0(f) + \sum_{k\in \operatorname{supp} \widehat{f}\cap\mathbb{Z}_*} \gamma_k(f)\Big(\frac{k-d}{k} + \frac{d}{k}\Big) e^{ikx} = \sum_{k\in \operatorname{supp} \widehat{f}} \gamma_k(f) e^{ikx}$$

converges to $Q(D)\mathcal{Q}f$ in $\mathcal{S}'(\mathbb{R})$. Hence $Q(D)\mathcal{Q}f = f$.

Case 2 ($Q(x) = x$)**.** We put $\mathcal{Q}f = iI_1 f$. Then $\mathcal{Q}f \in L^p(\mathbb{T})$ and $Q(D)\mathcal{Q}f = f$.

Case 3 ($\deg Q(x) \geq 2$)**.** Clearly,

$$\left| \frac{\gamma_k(f)}{Q(k)}\, e^{ikx} \right| = \frac{|\gamma_k(f)|}{|Q(k)|} \leq \frac{(2\pi)^{-1/p} \|f\|_{p,\mathbb{T}}}{|Q(k)|}$$

and the number series $\sum\limits_{k\in \operatorname{supp} \widehat{f}} \frac{1}{|Q(k)|}$ converges. Hence the functional series

$$\sum_{k\in \operatorname{supp} \widehat{f}} \frac{\gamma_k(f)}{Q(k)}\, e^{ikx}, \quad \gamma_k(f) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(t) e^{-ikt}\, dt,$$

uniformly converges on $\mathbb{R}$. So, $\mathcal{Q}f$ is well defined in $\mathbb{R}$, $\mathcal{Q}f \in L^p(\mathbb{T})$, where

$$\mathcal{Q}f = \sum_{k\in \operatorname{supp}\widehat{f}} \frac{\gamma_k(f)}{Q(k)}\, e^{ikx}.$$

Then the functional series

$$\sum_{k\in \operatorname{supp}\widehat{f}} \frac{\gamma_k(f)}{Q(k)}\, e^{ikx}$$

converges to $\mathcal{Q}f$ in $\mathcal{S}'(\mathbb{R})$. Consequently, the series $\sum\limits_{k\in \operatorname{supp}\widehat{f}} \gamma_k(f)e^{ikx}$ converges to $Q(D)\mathcal{Q}f$. Therefore, $Q(D)\mathcal{Q}f = f$.

Combining Cases 1–3, we get the existence of Q-primitives in $L^p(\mathbb{T})$ of f.

Next, we prove the uniqueness of Q-primitives in $L^p(\mathbb{T})$ of f, which is denoted by $\mathcal{Q}f$, satisfying $Q(x) \neq 0\ \forall\, x \in \operatorname{supp}\widehat{\mathcal{Q}f}$. Indeed, let $\mathcal{Q}_1 f$ and $\mathcal{Q}_2 f$ be Q-primitives in $L^p(\mathbb{T})$ of f such that $\mathcal{Q}_j f \in L^p(\mathbb{T})$, $Q(D)\mathcal{Q}_j f = f$ and $Q(x) \neq 0\ \forall\, x \in \operatorname{supp}\widehat{\mathcal{Q}_j f}$, $j = 1, 2$. Put $\phi = \mathcal{Q}_1 f - \mathcal{Q}_2 f$. Then the series

$$\sum_{k\in\mathbb{Z}} \gamma_k(\phi)e^{ikx}, \quad \gamma_k(\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(t)e^{-ikt}\, dt,$$

converges to ϕ in $\mathcal{S}'(\mathbb{R})$, and then $\sum\limits_{k\in\mathbb{Z}} \gamma_k(\phi)Q(k)e^{ikx}$ converges to $Q(D)\phi$ in $\mathcal{S}'(\mathbb{R})$. Hence it follows from $Q(D)\phi = 0$ that $\gamma_k(\phi)Q(k) = 0$ for all $k \in \mathbb{Z}$. Using $Q(x) \neq 0\ \forall\, x \in \operatorname{supp}\widehat{\phi}$, we have $\gamma_k(\phi) = 0$ for all $k \in \mathbb{Z}$. That gives $\phi \equiv 0$ and then $\mathcal{Q}_1 f = \mathcal{Q}_2 f$.

To end the proof, we have to show that $Q(x) \neq 0\ \forall\, x \in \operatorname{supp}\widehat{\mathcal{Q}f}$. This is obtained from the fact that the functional series $\sum\limits_{k\in \operatorname{supp}\widehat{f}} \frac{\gamma_k(f)}{Q(k)}\, e^{ikx}$ converges to $\mathcal{Q}f$ in $\mathcal{S}'(\mathbb{R})$. The proof is complete. $\square$

From the proof of Theorem 2.1 we get the following necessary and sufficient conditions for the existence of Q-primitives of a function in $L^p(\mathbb{T})$.

Theorem 2.3. *Let $1 \leq p \leq \infty$, $Q(x)$ be a polynomial and $f \in L^p(\mathbb{T})$. Then there exists a Q-primitive $\mathcal{Q}f \in L^p(\mathbb{T})$ of f if and only if $Q(x) \neq 0\ \forall\, x \in \operatorname{supp}\widehat{f}$.*

Remark 2.1. It should be noticed that the assumption $Q(x) \neq 0\ \forall\, x \in \operatorname{supp}\widehat{f}$ does not guarantee the uniqueness of Q-primitives of f in $L^p(\mathbb{T})$, but there is exactly one Q-primitive in $L^p(\mathbb{T})$ of f denoted by $\mathcal{Q}f$ satisfying $Q(x) \neq 0\ \forall\, x \in \operatorname{supp}\widehat{\mathcal{Q}f}$. For example, $f(x) = \cos x$, $Q(x) = x$, $p = \infty$ then $\sin x + c$ is a Q-primitive in $L^\infty(\mathbb{T})$ of $\cos x$ for all $c \in \mathbb{C}$.

Applying Theorem 2.1 and Theorem 2.3, we have the following corollary.

Corollary. *Let $1 \leq p \leq \infty, Q(x)$ be a polynomial and $\psi \in L^p(\mathbb{T})$. Then equation* (1.1) *has a solution in $L^p(\mathbb{T})$ if and only if $Q(x) \neq 0\ \forall\, x \in \operatorname{supp}\widehat{\psi}$. Moreover, if equation* (1.1) *has solutions, then there is exactly one solution f satisfying $Q(x) \neq 0\ \forall\, x \in \operatorname{supp}\widehat{f}$, which has the following explicit form:*

$$f(x) = \sum_{k\in \operatorname{supp}\widehat{\psi}} \frac{\gamma_k(\psi)}{Q(k)}\, e^{ikx}, \quad \gamma_k(\psi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(t)e^{-ikt}\, dt.$$

By induction, we obtain the following result.

Theorem 2.4. *Let $1 \leq p \leq \infty$, $Q(x)$ be a polynomial and $f \in L^p(\mathbb{T})$. Then there exists only one sequence of functions $(\mathcal{Q}^m f)_{m=0}^\infty \subset L^p(\mathbb{T})$ such that $\mathcal{Q}^0 f = f$, $Q(D)(\mathcal{Q}^{m+1} f) = \mathcal{Q}^m f$ and $Q(x) \neq 0$ for all $x \in \operatorname{supp}\widehat{\mathcal{Q}^m f}$, $m = 0, 1, \ldots,$ if and only if $Q(x) \neq 0\ \forall\, x \in \operatorname{supp}\widehat{f}$.*

3 An inequality for higher order Q-primitives

We give now an inequality for higher order Q-primitives.

Theorem 3.1. *Let $1 \leq p \leq \infty$, K be a subset of $\mathbb{Z}$ with a finite number of elements, $Q(x)$ be a polynomial and $f \in L^p(\mathbb{T})$. Assume that $\operatorname{supp} \widehat{f} \subset K$ and $Q(x) \neq 0 \ \forall x \in K$. Then there exists a constant $C = C(K, Q)$, independent of f, such that*

$$\|\mathcal{Q}^m f\|_{p,\mathbb{T}} \leq Cm \sup_{x \in K} \Big| \frac{1}{Q^m(x)} \Big| \, \|f\|_{p,\mathbb{T}} \tag{3.1}$$

for all $m \in \mathbb{N}$, $p \in [1, \infty]$.

Proof. Since $Q(x) \neq 0 \ \forall x \in K$, there is a positive number ℓ such that $Q(x) \neq 0 \ \forall x \in K_\ell$. We define the function χ as follows:

$$\chi(z) = \begin{cases} ce^{1/(z^2-1)} & \text{if } \ |z| < 1, \\ 0 & \text{if } \ |z| \geq 1, \end{cases}$$

where the positive constant c satisfies $\int_{\mathbb{R}} \chi(z)\, dz = 1$. For $m \in \mathbb{N}$, we define the function $\Psi_m \in C_0^\infty(\mathbb{R})$ via the formula

$$\Psi_m = 1_{K_{\ell/(2m)}} * \chi_{\ell/(4m)},$$

where

$$\chi_{\ell/(4m)}(z) = \frac{4m}{\ell}\, \chi\Big(z\, \frac{4m}{\ell}\Big).$$

Clearly, $\Psi_m(z) = 1$ if $z \in K_{\ell/(4m)}$ and $\Psi_m(z) = 0$ if $z \notin K_{\ell/m}$. Using

$$\widehat{f} = \widehat{\mathcal{Q}^m f} Q^m(z) \quad \forall\, m \in \mathbb{Z}_+$$

and $\operatorname{supp} \widehat{\mathcal{Q}^m f} \subset K$, $\Psi_m(z) = 1 \ \forall\, z \in K_{\ell/(4m)}$, we have

$$\widehat{\mathcal{Q}^m f} = \widehat{f}\, \frac{\Psi_m(z)}{Q^m(z)}\,.$$

Hence

$$\mathcal{Q}^m f = \frac{1}{\sqrt{2\pi}}\, f * \mathcal{F}^{-1}\Big(\frac{\Psi_m(z)}{Q^m(z)}\Big).$$

So, it follows from the Young inequality for 2π-periodic functions that

$$\|\mathcal{Q}^m f\|_{p,\mathbb{T}} \leq \frac{1}{\sqrt{2\pi}}\, \|f\|_{p,\mathbb{T}} \Big\| \mathcal{F}^{-1}\Big(\frac{\Psi_m(z)}{Q^m(z)}\Big)\Big\|_{L^1(\mathbb{R})}. \tag{3.2}$$

Now, we define

$$\mathcal{J}_m(x) := \Big(\mathcal{F}^{-1}\Big(\frac{\Psi_m(z)}{Q^m(z)}\Big)\Big)(x).$$

Clearly,

$$\mathcal{J}_m(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixz}\, \frac{\Psi_m(z)}{Q^m(z)}\, dz = \frac{1}{\sqrt{2\pi}} \int_{z \in K_{\ell/m}} e^{ixz}\, \frac{\Psi_m(z)}{Q^m(z)}\, dz.$$

Hence

$$\sup_{x \in \mathbb{R}} |\mathcal{J}_m(x)| \leq \frac{1}{\sqrt{2\pi}} \int_{z \in K_{\ell/m}} \Big| \frac{\Psi_m(z)}{Q^m(z)} \Big|\, dz \leq \frac{1}{\sqrt{2\pi}} \sup_{x \in K_{\ell/m}} \Big| \frac{1}{Q^m(x)} \Big|\, \|\Psi_m\|_{L^1(\mathbb{R})}. \tag{3.3}$$

Moreover,

$$\|\Psi_m\|_{L^1(\mathbb{R})} \leq \|1_{K_{\ell/(2m)}}\|_{L^1(\mathbb{R})} \|\chi_{\ell/(4m)}\|_{L^1(\mathbb{R})} \leq \operatorname{meas}(K_\ell) \|\chi\|_{L^1(\mathbb{R})} = \operatorname{meas}(K_\ell),$$

where $\mathrm{meas}(K_\ell)$ is the Lebesgue measure of K_ℓ. Combining this with (3.3), we obtain

$$\sup_{x\in\mathbb{R}} |\mathcal{J}_m(x)| \le \frac{\mathrm{meas}(K_\ell)}{\sqrt{2\pi}} \sup_{x\in K_{\ell/m}} \Big|\frac{1}{Q^m(x)}\Big| . \tag{3.4}$$

Since $K_{\ell/m}$ is compact, there is a number $\sigma_m \in K_{\ell/m}$ such that

$$\sup_{x\in K_{\ell/m}} |Q(x)| = |Q(\sigma_m)|.$$

By $\sigma_m \in K_{\ell/m}$, there exists $\gamma_m \in K$ such that $|\sigma_m - \gamma_m| \le \ell/m$. Then from

$$|Q(\sigma_m) - Q(\gamma_m)| = |\sigma_m - \gamma_m| \cdot |Q'(z_m)|$$

for some $z_m \in K_{\ell/m}$, we derive

$$|Q(\sigma_m) - Q(\gamma_m)| \le \frac{\ell |Q'(z_m)|}{m} \le \frac{\ell \sup_{x\in K_{\ell/m}} |Q'(x)|}{m} \le \frac{C_1}{m} \,,$$

where $C_1 := \ell \sup\limits_{x\in K_\ell} |Q'(x)|$. Therefore,

$$|Q(\sigma_m)| \ge |Q(\gamma_m)| - \frac{C_1}{m} \le \sup_{x\in K} |Q(x)| + \frac{C_1}{m} \,.$$

Consequently,

$$\inf_{x\in K_{\ell/m}} |Q(x)| \ge \inf_{x\in K} |Q(x)| - \frac{C_1}{m} \,.$$

Hence

$$\inf_{x\in K_{\ell/m}} |Q^m(x)| \ge \Big(\inf_{x\in K} |Q(x)| - \frac{C_1}{m}\Big)^m = \inf_{x\in K} |Q^m(x)| \Big(1 - \frac{C_1}{m \inf\limits_{x\in K} |Q(x)|}\Big)^m$$

and then

$$\sup_{x\in K_{\ell/m}} \Big|\frac{1}{Q^m(x)}\Big| \le \Big(\inf_{x\in K} |Q(x)| - \frac{C_1}{m}\Big)^{-m} = \Big(\sup_{x\in K} \Big|\frac{1}{Q^m(x)}\Big|\Big) \Big(1 - \frac{C_1}{m \inf\limits_{x\in K} |Q(x)|}\Big)^{-m} .$$

Combining this and $(1+t)^{1/t} \le e \;\forall\, t > -1/2$, we obtain

$$\sup_{x\in K_{\ell/m}} \Big|\frac{1}{Q^m(x)}\Big| \ge C_2 \sup_{x\in K} \Big|\frac{1}{Q^m(x)}\Big| \quad \forall\, m \ge m_0, \tag{3.5}$$

where

$$C_2 := e^{C_1/\inf\limits_{x\in K} |Q(x)|}, \quad m_0 = \frac{2C_1}{\inf\limits_{x\in K} |Q(x)|} \,.$$

Hence, by (3.4), we deduce that

$$\sup_{x\in\mathbb{R}} |\mathcal{J}_m(x)| \le C_3 \sup_{x\in K} \Big|\frac{1}{Q^m(x)}\Big| \quad \forall\, m \ge m_0, \tag{3.6}$$

where

$$C_3 := \frac{C_2\, \mathrm{meas}(K_\ell)}{\sqrt{2\pi}} \,.$$

Moreover,

$$\sup_{x\in\mathbb{R}} |x^2 \mathcal{J}_m(x)| \le \frac{1}{\sqrt{2\pi}} \sup_{x\in\mathbb{R}} \Big| \int\limits_{\mathbb{R}} e^{ixz} D^2\Big(\frac{\Psi_m(z)}{Q^m(z)}\Big)\, dz \Big| \le \frac{1}{\sqrt{2\pi}} \int\limits_{z\in K_{\ell/m}} \Big| D^2\Big(\frac{\Psi_m(z)}{Q^m(z)}\Big)\Big|\, dz,$$

which gives

$$\sup_{x\in\mathbb{R}}|x^2\mathcal{J}_m(x)| \le \frac{1}{\sqrt{2\pi}} \int\limits_{z\in K_{\ell/m}} \Big|\frac{(D^2\Psi_m(z))}{Q^m(z)}\Big| + 2m\Big|\frac{(D\Psi_m(z))(DQ(z))}{Q^{m+1}(z)}\Big|$$

$$+ m\Big|\Psi_m(z)\,\frac{(D^2Q(z))}{Q^{m+1}(z)}\Big| + m(m+1)\Big|\Psi_m(z)\,\frac{(DQ(z))^2}{Q^{m+2}(z)}\Big|\,dz.$$

Therefore,

$$\sup_{x\in\mathbb{R}}|x^2\mathcal{J}_m(x)| \le C_{4,m} \sup_{x\in K_{\ell/m}}\Big|\frac{1}{Q^{m+2}(x)}\Big|, \tag{3.7}$$

where

$$C_{4,m} = \frac{1}{\sqrt{2\pi}} \int\limits_{z\in K_\ell} \Big(\big|(D^2\Psi_m(z))Q^2(z)\big| + 2m\big|D\Psi_m(z)(DQ(z))Q(z)\big|$$

$$+ m\big|\Psi_m(z)(D^2Q(z))Q(z)\big| + m(m+1)\big|\Psi_m(z)(DQ(z))^2\big|\Big)\,dz.$$

To evaluate $C_{4,m}$, we observe that

$$\|D^2\Psi_m\|_{L^\infty(\mathbb{R})} = \big\|1_{K_{\ell/(2m)}} * (D^2\chi_{\ell/(4m)})\big\|_{L^\infty(\mathbb{R})} \le \|D^2(\chi_{\ell/(4m)})\|_{L^1(\mathbb{R})} = \Big(\frac{4m}{\ell}\Big)^2\|D^2\chi\|_{L^1(\mathbb{R})},$$

$$\|D\Psi_m\|_{L^\infty(\mathbb{R})} = \big\|1_{K_{\ell/(2m)}} * (D\chi_{\ell/(4m)})\big\|_{L^\infty(\mathbb{R})} \le \|D(\chi_{\ell/(4m)})\|_{L^1(\mathbb{R})} = \Big(\frac{4m}{\ell}\Big)\|D\chi\|_{L^1(\mathbb{R})}$$

and

$$\|\Psi_m\|_{L^\infty(\mathbb{R})} = \big\|1_{K_{\ell/(2m)}} * (\chi_{\ell/(4m)})\big\|_{L^\infty(\mathbb{R})} \le \|\chi_{\ell/(4m)}\|_{L^1(\mathbb{R})} = \|\chi\|_{L^1(\mathbb{R})} = 1.$$

So,

$$C_{4,m} \le C_5 m^2,$$

where

$$C_5 = \frac{16}{\sqrt{2\pi}} \int\limits_{z\in K_\ell} \Big(\frac{\|D^2\chi\|_{L^1(\mathbb{R})}|Q^2(z)|}{\ell^2} + \frac{\|D\chi\|_{L^1(\mathbb{R})}|D(Q(z))Q(z)|}{\ell}$$

$$+ \big|(D^2Q(z))Q(z)\big| + 2\big|(DQ(z))^2\big|\Big)\,dz.$$

Combining this with (3.5) and (3.7), we obtain

$$\sup_{x\in\mathbb{R}}|x^2\mathcal{J}_m(x)| \le C_6 m^2 \sup_{x\in K}\Big|\frac{1}{Q^m(x)}\Big| \quad \forall\, m \ge m_0, \tag{3.8}$$

where

$$C_6 = C_2C_5 \sup_{x\in K}\Big|\frac{1}{Q^2(x)}\Big|.$$

Further, we have

$$\|\mathcal{J}_m\|_{L^1(\mathbb{R})} = \int\limits_{|x|\le m} |\mathcal{J}_m(x)|\,dx + \int\limits_{|x|\ge m} |\mathcal{J}_m(x)|\,dx$$

$$\le \int\limits_{|x|\le m} |\mathcal{J}_m(x)|\,dx + \Big(\sup_{x\in\mathbb{R}}|x^2\mathcal{J}_m(x)|\Big)\Big(\int\limits_{|x|\ge m} \frac{1}{x^2}\,dx\Big),$$

that gives

$$\|\mathcal{J}_m\|_{L^1(\mathbb{R})} \le 2m \sup_{x\in\mathbb{R}}|\mathcal{J}_m(x)| + \frac{2}{m}\sup_{x\in\mathbb{R}}|x^2\mathcal{J}_m(x)|. \tag{3.9}$$

Combining (3.6), (3.8) and (3.9), we obtain

$$\|\mathcal{J}_m\|_{L^1(\mathbb{R})} \leq (2C_3 + 2C_6)m \sup_{x \in K} |Q^m(x)| \quad \forall\, m \geq m_0. \tag{3.10}$$

Clearly, the constants C_1, C_2, C_3, C_5, C_6 are not dependent on f, m, p. Note that, for $m \leq m_0$, $\mathcal{J}_m$ are not dependent on f, p and $\|\mathcal{J}_m\|_{L^1(\mathbb{R})} < \infty$. From (3.2) and (3.10), we conclude that there exists a constant C, independent of f, m, p, such that

$$\|\mathcal{Q}^m f\|_{p,\mathbb{T}} \leq Cm\|f\|_{p,\mathbb{T}} \sup_{x \in K} \left|\frac{1}{Q^m(x)}\right| \quad \forall\, m \in \mathbb{N}.$$

The proof is complete. □

Remark 3.1. Note that (3.1) cannot be obtained by using itself (for $m = 1$) m times.

It was shown in Section 2 that the condition $Q(x) \neq 0\ \forall\, x \in \operatorname{supp} \widehat{f}$ is a criterion for the existence of Q-primitives in $L^p(\mathbb{T})$ of f. Moreover, this condition guarantees the unique existence of one sequence $(\mathcal{Q}^m f)_{m=0}^\infty \subset L^p(\mathbb{T})$ satisfying $\mathcal{Q}^0 f = f$, $Q(D)(\mathcal{Q}^{m+1} f) = \mathcal{Q}^m f$ and $Q(x) \neq 0$ for all $x \in \operatorname{supp} \widehat{\mathcal{Q}^m f}$, $m = 0, 1, \ldots$. By the same method as in [10] and using Theorem 3.1, we have the following behavior of the sequence $(\|\mathcal{Q}^m f\|_{p,\mathbb{T}})_{m=0}^\infty$ based on the spectrum of f. Note that the behavior of the sequences of norm of derivatives and primitives of functions based on their spectra was studied in [1–17].

Theorem 3.2. *Let $1 \leq p \leq \infty$, $Q(x)$ be a polynomial and $(\mathcal{Q}^m f)_{m=0}^\infty \subset L^p(\mathbb{T})$. Assume that $Q(x) \neq 0$ $\forall\, x \in \operatorname{supp} \widehat{f}$. Then $\operatorname{supp} \widehat{\mathcal{Q}^m f} = \operatorname{supp} \widehat{f}$ for all $m \in \mathbb{Z}_+$ and*

$$\liminf_{m \to \infty} \|\mathcal{Q}^m f\|_{p,\mathbb{T}}^{1/m} \geq \sup_{k \in \operatorname{supp} \widehat{f}} \left|\frac{1}{Q(k)}\right|.$$

Moreover, if $\operatorname{supp} \widehat{f}$ is compact, then we have the following limit:

$$\lim_{m \to \infty} \|\mathcal{Q}^m f\|_{p,\mathbb{T}}^{1/m} = \sup_{k \in \operatorname{supp} \widehat{f}} \left|\frac{1}{Q(k)}\right|.$$

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