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ABOUT ONE OF SCHAUDER'S THEOREMS

Dedicated to the blessed memory of Professor N. V. Azbelev

Abstract. The parameter continuation method plays an important role in the theory of linear operators and linear boundary value problems. This method is based on Schauder's theorem on the invertibility of the sum of two linear operators, the first of which is invertible and the sum satisfies a certain condition. In the present paper, the mentioned assertion is extended for arbitrary steady properties of linear operators. The corresponding information related to steady properties and examples of steady properties are given. An example of applying the obtained assertions is presented.

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რეზიუმე. პარამეტრის გაგრძელების მეთოდი მნიშვნელოვან როლს ასრულებს წრფივ ოპერატორთა თეორიასა და წრფივ სასაზღვრო ამოცანებში. ეს მეთოდი ეფუძნება შაუდერის თეორემას ორი წრფივი ოპერატორის ჯამის შექცევადობის შესახებ, რომელთაგან პირველი შექცევადია, და ჯამი აკმაყოფილებს გარკვეულ პირობას. წინამდებარე ნაშრომში აღნიშნული დებულება გავრცელებულია წრფივი ოპერატორების ნებისმიერი მდგრადი თვისებისთვის. მოყვანილია შესაბამისი ინფორმაცია მდგრად თვისებებთან დაკავშირებით და მდგრადი თვისებების მაგალითები. წარმოდგენილია მიღებული დებულებების გამოყენების მაგალითი.

Perhaps Schauder's most famous assertion is the fixed point theorem, called "Schauder's principle". In this paper, we discuss another Schauder's assertion, which played a significant role in the theory of linear operators and boundary value problems. This statement is the basis of the "parameter continuation method". The main goal of the work is to extend the mentioned Schauder's theorem to arbitrary steady properties.

Let X, Y be Banach spaces, $\mathbf{B}(X,Y)$ be the space of linear bounded operators $A : X \to Y$. We present the theorem from [7] in the following edition. Let $A, B \in \mathbf{B}(X,Y)$. Put $A(\lambda) = (1-\lambda)A + \lambda B$.

Theorem 1 (Schauder)**.** *Let the operator A be invertible and let there exist a positive constant γ such that the inequality*

$$\gamma\|x\| \leqslant \|A(\lambda)x\|$$

is true for any $\lambda \in [0,1]$, $x \in X$. Then the operator $B : X \to Y$ is invertible.

This statement has been extended to the case of an arbitrary continuous operator-function $A(\lambda)$, but the presented version of Schauder's theorem remains the most popular in the parameter continuation method. Therefore, the main emphasis in this paper is on this version of Schauder's theorem.

For the sake of simplicity, the further content of the work is divided into sections.

1. In this section, following [1], we define the concept of a steady property of a linear operator. Let $R = (-\infty, +\infty)$ and $R^+ = [0, +\infty)$.

Consider a non-negative functional $\mu : \mathbf{B}(X,Y) \to R^+$ with the following properties:

(1) $\mu(\lambda A) = |\lambda|\mu(A)$, $\lambda \in R$, $A \in \mathbf{B}(X,Y)$,

(2) $\mu(A+B) \le \mu(A) + \|B\|$, $A, B \in \mathbf{B}(X,Y)$.

Put $R(\mu) = \{A \in \mathbf{B}(X,Y) : \ \mu(A) > 0\}$. The elements $R(\mu)$ and only they are called having a steady property with respect to the functional (μ). It is natural to call this functional a functional that generates a steady property. As a rule, the known steady properties have their names without connection with the generating functional. Some well-known steady properties of linear operators are listed below.

As is shown in [1], the set $R(\mu)$ in $\mathbf{B}(X,Y)$ is characterized by the following features:

(1) zero operator does not belong to $R(\mu)$,

(2) if $A \in R(\mu)$ and $\lambda \neq 0$, then $\lambda A \in R(\mu)$,

(3) $R(\mu)$ is open.

In the sequel, we will consider only non-trivial steady properties, i.e., $R(\mu) \neq \mathbf{B}(X,Y)$.

Consider the functional $m : \mathbf{B}(X,Y) \to R^+$ defined by the equality

$$m(A) = \inf\big\{\|Ax\|/\|x\| = 1\big\}.$$

This functional generates a steady property known as the correct solvability of a linear operator equation with an operator A [4]. In terms of functional $m(\cdot)$, the inequality in Schauder's theorem admits the following equivalent formulation: there exists a positive constant γ such that the inequality $\gamma \le m(A(\lambda))$ holds for all $\lambda \in [0,1]$. As is known, in the case $m(A) > 0$, the operator A has the bounded left inverse operator.

The surjectivity property of the operator $A : X \to Y$ is characterized by the functional

$$q(A) = \inf\Big\{\|A^*\omega\|/\|\omega\| = 1\Big\},$$

where $A : Y^* \to X^*$ is the adjoint to $A : X \to Y$.

Herewith, $q(A) > 0$ if and only if the operator $A : X \to Y$ is surjective. The fact that the functionals $m(A)$ and $q(A)$ meet the required conditions can be checked directly (see also [6]).

Put

$$\eta(A) = \min\big\{m(A), q(A)\big\}.$$

It is clear that $\eta(A) > 0$ if and only if the operator $A : X \to Y$ has the linear bounded inverse operator $A^{-1} : Y \to X$ (i.e., invertible) and $R(\eta) \subset \mathbf{B}(X, Y)$ forms the set of all invertible operators. Herewith,

$$m(A) = q(A) = \|A^{-1}\|^{-1}, \quad A \in R(\eta).$$

In the development of functional analysis, the invertibility as a steady property (meaning the existence of a linear bounded inverse operator) played an exceptional role. It suffices to remind Banach's theorems on the invertibility.

2. A key element in the proof of Schauder's theorem is the use of the equivalence of two functionals that generate steady properties. Let us define the corresponding concept. Let us consider two steady properties generated by functionals $\mu_1(\,\cdot\,)$ and $\mu_2(\,\cdot\,)$ accordingly and let $\mathbf{B}_0 \subset \mathbf{B}(X, Y)$ be some family of operators. We say that the steady properties (μ_1) and (μ_2) are equivalent on $\mathbf{B}_0$ if there exist the constants $0 < c_1 \le c_2$ such that for any $A \in \mathbf{B}_0$, the following inequalities hold:

$$c_1\mu_2(A) \le \mu_1(A) \le c_2\mu_2(A).$$

Consider in the space $\mathbf{B}(X, Y)$ two non-trivial steady properties generated by functionals $\mu_1(\,\cdot\,)$ and $\mu_2(\,\cdot\,)$, respectively, and put

$$R(\mu_i) = \big\{A \in \mathbf{B}(X, Y) \mid \ \mu_i(A) > 0\big\}.$$

The functional $\mu_0(A) = \min\{\mu_1(A), \mu_2(A)\}$ satisfies also the conditions of the generating functional and at the same time

$$R(\mu_0) = R(\mu_1) \cap R(\mu_2).$$

For $A, B \in \mathbf{B}(X, Y)$, we put $A(\lambda) = (1 - \lambda)A + \lambda B$, $\lambda \in [0, 1]$.

Theorem 2. *Let the following conditions be fulfilled:*

(1) $\mu_1(A) > 0$*;*

(2) μ_1 *and* μ_2 *are equivalent on* $R(\mu_0)$*;*

(3) $\mu_2(A(\lambda)) \ge \gamma > 0$ *for all* $\lambda \in [0; 1]$.

Then $B = A(1) \in R(\mu_0)$.

Proof. First, we note that condition (2) allows us to use the fact that in the case $A(\lambda_0) \in R(\mu_0)$ for some $\lambda_0 \in [0, 1]$, we have

$$c_1\mu_2(A(\lambda_0)) \leqslant \mu_1(A(\lambda_0)).$$

Assume that $\delta = \frac{c_1\gamma}{2\|B-A\|}$ and let $\lambda \in [0, \delta]$. Then

$$\begin{aligned}\mu_1(A(\lambda)) = \mu_1((1 - \lambda)A + \lambda B) = \mu_1(A + \lambda(B - A)) \ge \mu_1(A) - \|\lambda(B - A)\| \\ \ge c_1\mu_2(A) - \delta\|B - A\| \ge c_1\gamma - \frac{c_1\gamma}{2\|B - A\|}\,\|B - A\| = \frac{c_1\gamma}{2} > 0.\end{aligned}$$

Therefore, $A(\delta) \in R(\mu_0)$.

If $\delta \ge 1$, then the statement is proved. If $\delta < 1$, then consider a segment $[\delta; 2\delta]$ for the parameter λ. For arbitrary $\lambda \in [\delta; 2\delta]$, we have

$$\begin{aligned}\mu_1(A(\lambda)) = \mu_1\big((1 - \lambda)A + \lambda B\big) = \mu_1\big(A(\delta) + (\lambda - \delta)(B - A)\big) \ge \mu_1(A(\delta)) - |\lambda - \delta|\|B - A\| \\ \ge c_1\mu_2(A(\delta)) - \delta\|B - A\| \ge c_1\gamma - \frac{c_1\gamma}{2\|B - A\|}\,\|B - A\| = \frac{c_1\gamma}{2} > 0.\end{aligned}$$

Thus $A(\lambda) \in R(\mu_0)$ for all $\lambda \in [\delta, 2\delta]$.

If $2\delta \ge 1$, then the statement is proved. In case $2\delta < 1$, we repeat the described procedure n times, while the product $n\delta$ will not become more than one. □

Note that a similar result for the elements of Banach algebras was announced in [1].

In our opinion, the above-proved statement will be useful for researchers engaged in linear operator equations and boundary value problems.

If in the proved statement we assume $\mu_1(A) = m(A)$ and $\mu_2(A) = q(A)$, then we find ourselves in the conditions of Schauder's theorem. Herewith, on $R(\mu_0) = R(m) \cap R(q)$, we have

$$\mu_0(A) = \eta(A) = m(A) = q(A) = \|A^{-1}\|^{-1}.$$

Therefore, Theorem 1 follows from this theorem.

The statement of the theorem remains valid if $A(\lambda)$ is a continuous operator-function.

3. As an example of application of Theorem 2, we consider a question of the existence and uniqueness of the solution to the integral equation

$$x(t) + \frac{1}{t}\int_0^t x(s)\,ds = y(t), \quad t \in [0,1], \tag{0.1}$$

in the Hilbert space $L_2 = L_2[0;1]$ of functions $x : [0;1] \to R$, Lebesgue measurable and square summable, with the inner product defined by the equality

$$(x_1, x_2) = \int_0^1 x_1(t)x_2(t)\,dt$$

and generating a norm

$$\|x\| = \sqrt{(x,x)} = \sqrt{\int_0^1 x^2(t)\,dt}\,.$$

Let us write the integral equation (0.1) in the form of the operator equation

$$(I + C)x = y, \tag{0.2}$$

where $I : L_2 \to L_2$ is the identical operator, $C : L_2 \to L_2$, $(Cx)(t) = \frac{1}{t}\int_0^t x(s)\,ds$ is an integral operator known in the literature as the Cesàro operator [2,3,5].

Let us prove the invertibility of the operator $I + C$. For this purpose, on the space of linear bounded operators acting from the space L_2 in L_2 let us consider the following generating functionals $\mu_1(A) = \eta(A)$ and $\mu_2(A) = q(A)$. Note that with this choice, $\mu_0(A) = \mu_1(A)$.

Assuming that

$$A(\lambda) = (1 - \lambda)I + \lambda(I + C),$$

we verify the fulfillment of the conditions of Theorem 2.

As $A(0) = I$, $A(0)$ is an invertible operator, i.e., $\mu_1(A(0)) > 0$.

It has previously been noted that for any invertible operator, the equality $m(A) = q(A) = \|A^{-1}\|^{-1}$ holds. This means the equivalence of functionals $\eta(A)$ and $q(A)$ on the set of invertible operators with constants $c_1 = c_2 = 1$.

Let us show the validity of the third condition of the theorem. For the adjoint operator $C^* : L_2 \to L_2$ defined by the equality $(C^*\omega) = \int_t^1 \frac{\omega(s)}{s}\,ds$, we have

$$(C^*\omega, \omega) = \int_0^1 \omega(t)\int_t^1 \frac{\omega(s)}{s}\,ds\,dt = -\frac{1}{2}s\left(\int_t^1 \frac{\omega(s)}{s}\,ds\right)^2\Bigg|_0^1 + \frac{1}{2}\int_0^1\left(\int_t^1 \frac{\omega(s)}{s}\,ds\right)^2 dt = \frac{1}{2}\|C^*\omega\|^2.$$

Taking into account the obtained ratio for an arbitrary $\lambda \in [0;1]$, we can write

$$\begin{aligned}
&\big\|(1-\lambda)I+\lambda(I+C))\omega\big\|^2 = \big\|((2-\lambda)I+\lambda C)^*\omega\big\|^2 \\
&= \Big(((2-\lambda)I+\lambda C)^*\omega, ((2-\lambda)I+\lambda C)^*\omega\Big) = (2-\lambda)^2\|\omega\|^2 + 2\lambda(2-\lambda)(C^*\omega,\omega) + \lambda^2\|C^*\omega\|^2 \\
&= (2-\lambda)^2\|\omega\|^2 + 2\lambda(2-\lambda)\,\frac{1}{2}\|C^*\omega\|^2 + \lambda^2\|C^*\omega\|^2 \\
&= (2-\lambda)^2\|\omega\|^2 + 2\lambda\|C^*\omega\|^2 \geq (2-\lambda)^2\|\omega\|^2 \geq \|\omega\|^2.
\end{aligned}$$

Moving to the infimum with respect to all elements of space L_2 with the unit norm, we get $\mu_2(A(\lambda)) \geq 1 > 0$.

Thus, all the conditions of Theorem 2 are met and its application allows us to conclude that the operator $I + C$ is invertible. In other words, the integral equation (0.1) is uniquely solvable for any right-hand side $y \in L_2$.

Note that since the norm of the Cesàro operator $C : L_2 \to L_2$ equals two, the application of the Banach theorem on the invertibility of the sum does not allow us to obtain this result.

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