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EXTREMAL PROPERTIES OF THE CHROMATIC POLYNOMIALS OF CONNECTED 3-CHROMATIC GRAPHS

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Abstract. In this paper the greatest $\lceil n/2 \rceil$ values of $P(G;3)$ in the class of connected 3-chromatic graphs G of order n are found, where $P(G;\lambda)$ denotes the chromatic polynomial of G.

1. Preliminary definitions and results

Let G be a graph of order n and let $P(G;\lambda)$ be its chromatic polynomial [1]. A k-color partition of G is a partition of the vertex set $V(G)$ into k classes where each class is an independent set of vertices. The number of k-color partitions of G and the chromatic number of G will be denoted by $\mathrm{Col}_k(G)$ and by $\chi(G)$, respectively. It is well known that $P(G;\lambda)$ can be expressed in terms of the number of k-color partitions as follows

$$P(G;\lambda) = \sum_{k=1}^{n} (\lambda)_k \, \mathrm{Col}_k(G),$$

where $(\lambda)_k = \lambda(\lambda-1)\cdots(\lambda-k+1)$.

It follows that if $\chi(G) = k$, then $\mathrm{Col}_k(G) = P(G;\lambda)/k!$. Let xy be an edge of G. By $G - xy$ we mean the graph obtained from G by deleting edge xy. Also G/xy denotes the graph obtained from G by identifying vertices x and y, i.e., (i) by deleting both x and y and all the edges incident to them, and (ii) by introducing a new vertex z and joining z to both all the neighbors of x different from y and all the neighbors of y different from x in G.

The following lemma describes some properties of $P(G;\lambda)$, which we will use later [2].

LEMMA 1.1. *The following properties hold:*

(i) Reduction Formula. Let a and b be two adjacent vertices of G. Then $P(G;\lambda) = P(G - ab;\lambda) - P(G/ab;\lambda)$.

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(ii) Let G and H be two graphs that overlap in a complete graph K_r on r vertices. Then the chromatic polynomial of this overlap graph is

$$P(G;\lambda)P(H;\lambda)/P(K_r;\lambda).$$

Let G be a graph and H an induced subgraph of G. The graph obtained from G by the contraction of H is the graph G_1 derived from G by the following operations: suppress all vertices of H and the edges incident with them, and replace them with a new vertex $w \notin V(G)$ and edges wx such that $wx \in E(G_1)$ if and only if there exists $y \in V(G)$ such that $xy \in E(G)$ and $x \in V(G) - V(H)$.

The cycle with n vertices will be denoted by C_n and C_n^1 will denote the graph consisting of C_n and one more vertex adjacent to only one vertex of C_n. The following theorem was proved in [4].

THEOREM 1.2. *The maximum number of 3-color partitions of a connected graph G having n vertices and chromatic number $\chi(G) = 3$ is $(2^{n-1} - 1)/3$ for odd n, and $(2^{n-1} - 2)/3$ for even n. Moreover, if n is odd, the unique connected graph that achieves the maximum number of 3-color partitions is C_n, while if n is even, the unique graph is C_{n-1}^1.*

By $H(n, 2r+1)$ we denote the class of connected graphs G of order n containing n edges and a unique cycle C_{2r+1}, where $3 \leqslant 2r+1 \leqslant n$. It is clear that the graph deduced from $G \in H(n, 2r+1)$ by contracting C_{2r+1} is a tree on $n - 2r$ vertices. By Rényi's formula [3], the number of labeled graphs in $H(n, 2r+1)$ is equal to $(n-1)_{2r} n^{n-2r-1}/2$.

Let D_n $(n \geqslant 5)$ be the graph consisting of a 4-cycle in which two nonadjacent vertices are connected by a newly added path of length $n-3$. Note that $\chi(D_n) = 3$ for even n and $\chi(D_n) = 2$ for odd n. If "nonadjacent" is replaced by "adjacent", the resulting graph is denoted by F_n. Hence, F_n consists of two cycles C_4 and C_{n-2} having a common edge. Also, $\chi(F_n) = 3$ for odd n and $\chi(F_n) = 2$ for even n.

The following two properties were deduced in [5].

LEMMA 1.3. *For every $n \geqslant 5$, the following equalities hold: $P(D_n;3) = 2^n - 2^{n-2} + (-1)^{n-1}6$ and $P(F_n;3) = 2^n - 2^{n-2} + (-1)^n 6$.*

THEOREM 1.4. *(a) If G is a 2-connected graph of order n, $n \geqslant 5$, such that $P(G;3)$ is maximum in the class $\mathcal{F}_n \setminus \{C_n, K_{2,n-2}, D_n\}$, where $\mathcal{F}_n$ denotes the class of all 2-connected graphs of order n, then $G \cong F_n$ for odd n.*

(b) If G is a 2-connected graph of order 6 such that $P(G;3)$ is maximum in the class $\mathcal{F}_6 \setminus \{C_6, K_{2,4}, F_6, K_{3,3} - e\}$, then $G \cong K_{3,3}$ or D_6.

(c) If G is a 2-connected graph of order n, $n \geqslant 8$, such that $P(G;3)$ is maximum in the class $\mathcal{F}_n \setminus \{C_n, K_{2,n-2}, F_n\}$, then $G \cong D_n$ for even n; for $n = 8$ there exists another extremal graph, $E_{8,3}$.

Note that the graph $E_{8,3}$, described in [5], has $\chi(E_{8,3}) = 2$; also $\chi(K_{2,n-2}) = \chi(K_{3,3} - e) = \chi(K_{3,3}) = 2$.

LEMMA 1.5. *Let G be a graph of order $n \geqslant 5$ consisting of two cycles C_{2r+1} and C_{n-2r} having exactly one vertex in common. Then $P(G;3) < 2^n - 2^{n-2} - 6$.*

Proof. By Lemma 1.1(ii) we get

$$P(G;\lambda) = ((\lambda-1)^{2r+1} - (\lambda-1))((\lambda-1)^{n-2r} + (-1)^{n-2r}(\lambda-2))/\lambda$$

since $P(C_n;\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1)$. It follows that

$$\begin{aligned} P(G;3) &= (2^{2r+1}-2)(2^{n-2r} + (-1)^{n-2r} + (-1)^{n-2r}2)/3 \\ &\leqslant (2^{2r+1}-2)(2^{n-2r}+2)/3 = 2(2^n + 2^{2r+1} - 2^{n-2r} - 2)/3. \end{aligned}$$

Since $n - 2r \geqslant 3$, we shall consider two subcases: Case I. $2r \leqslant n-4$, and Case II. $2r = n-3$.

Case I. If $2r \leqslant n-4$ we deduce $2(2^n + 2^{2r+1} - 2^{n-2r} - 2)/3 \leqslant 2(2^n + 2^{n-3} - 2^4 - 2)/3 = 2^n - 2^{n-2} - 12 < 2^n - 2^{n-2} - 6$.

Case II. In this case $n - 2r = 3$ and $P(G;3) = (2^{n-2} - 2)(2^3 - 2)/3 < 2^n - 2^{n-2} - 6$. ■

We define the skeleton $S(G)$ of a connected graph G as follows:

(α) If G has no vertex of degree one, then $S(G) = G$.

(β) Otherwise, let x be a vertex of degree one of G; then G is replaced by $G - x$. Repeat (α).

For example, $S(T)$ consists of a unique vertex if T is a tree, and $S(G) = C_{2r+1}$ for any graph $G \in H(n, 2r+1)$.

LEMMA 1.6. *Let G be a graph of order n such that its skeleton $S(G)$ has order r. Then $P(G;\lambda) = P(S(G);\lambda)(\lambda-1)^{n-r}$.*

Proof. One applies Lemma 1.1(ii) since $P(K_2;\lambda) = \lambda(\lambda-1)$. ■

COROLLARY 1.7. *For every $G \in H(n, 2r+1)$, where $3 \leqslant 2r+1 \leqslant n$, we have $P(G;\lambda) = (\lambda-1)^n - (\lambda-1)^{n-2r}$.*

LEMMA 1.8. *Let G be a connected graph of order n consisting of two vertex disjoint cycles C_r and C_s, joined by a path of length t $(r+s+t = n+1)$. Then*

$$P(G;\lambda) = P(H;\lambda)(\lambda-1)^t,$$

where H is the graph of order $r+s-1$ consisting of cycles C_r and C_s having a unique common vertex.

Proof. This equality is a consequence of Lemma 1.1(ii). ■

LEMMA 1.9. *Let G be a graph of order $2r+s+p$ consisting of two cycles—one cycle with $s \geqslant 3$ vertices and another odd cycle with $2r+1 \geqslant 3$ vertices, having in common a path of length $p \geqslant 1$. Then*

$$P(G;3) < P(H;3) = 2^{2r+s-p} - 2^{2r+s-p-2}, \tag{1}$$

where $H \in H(2r+s-p, 3)$.

Proof. Suppose that the common path with $p+1$ vertices of the two cycles of G has extremities a and b. It follows that $1 \leqslant p \leqslant 2r-1$ and $p \leqslant s-2$. If $p \geqslant 2$ then vertices a and b are not adjacent and by Lemma 1.1 we deduce

$$\begin{aligned}P(G;\lambda) &= P(G_1;\lambda) + P(G_2;\lambda) = \\ &= ((\lambda-1)^{s-p} + (-1)^{s-p}(\lambda-1))((\lambda-1)^p + (-1)^p(\lambda-1))\times \\ &\times((\lambda-1)^{2r-p+1} + (-1)^{2r-p+1}(\lambda-1))/\lambda^2 + \\ &+((\lambda-1)^{s-p+1} + (-1)^{s-p+1}(\lambda-1))((\lambda-1)^{p+1} + (-1)^{p+1}(\lambda-1))\times \\ &\times((\lambda-1)^{2r-p+2} + (-1)^{2r-p}(\lambda-1))/(\lambda^2(\lambda-1)^2),\end{aligned}$$

where G_1 consists of three cycles with p, $s-p$ and $2r-p+1$ vertices having a common vertex and G_2 of three cycles with $p+1$, $s-p+1$ and $2r-p+2$ vertices having a common edge. Hence (1) is equivalent to

$$2^{2r+s-p} > (-1)^s 2^{2r-p+4} - 2^{s-p+3} + (-1)^{s+1}2^{p+3} + (-1)^{s-p+1}8. \qquad (2)$$

For $s=3$ we deduce $p=1$ which contradicts our hypothesis. If $s \geqslant 4$ we can write $2^{2r+s-p} + (-1)^{s+1}2^{2r-p+4} \geqslant 2^{2r+s-p} - 2^{2r-p+4} = 2^{2r-p+4}(2^{s-4}-1) \geqslant 2^5(2^{s-4}-1) = 2^{s+1} - 2^5$ since $p \leqslant 2r-1$. Since $p \leqslant s-2$, $2^{s-p+3} + (-1)^s 2^{p+3} \geqslant 2^{s-p+3} - 2^{p+3} = 2^5 - 2^{s+1}$ for $p = s-2$ and $2^{s-p+3} - 2^{p+3} \geqslant 2^6 - 2^s$ for $p \leqslant s-3$, and (2) is verified.

If $p=1$ then cycles C_s and C_{2r+1} have an edge in common and $P(G;\lambda) = P(C_{2r+s-1};\lambda) - P(G_3;\lambda)$, where G_3 consists of two cycles with $s-1$ and $2r$ vertices having a common vertex. It follows that

$$P(G;3) = 2^{2r+s-1} + (-1)^{s-1}2 - (2^{s-1} + (-1)^{s-1}2)(2^{2r}+2)/3$$

and (1) is equivalent to $2^{2r+s-3} > (-1)^s 2^{2r+1} - 2^s + (-1)^{s-1}2$. But this inequality can be deduced from (2) for $p=1$ and it is also true for $s=3$. ■

2. Main result

We shall denote by $\mathcal{C}_{n,3}$ the class of connected 3-chromatic graphs of order n. The following theorem is an extension of Theorem 1.2.

THEOREM 2.1. *Let $n \geqslant 5$. Then:*

(a) For every $r = \lceil n/2 \rceil - 1$, $r = \lceil n/2 \rceil - 2$, ..., 1, if G is a connected 3-chromatic graph of order n, such that $P(G;3)$ is maximum in the class of graphs

$$\mathcal{C}_{n,3} \setminus \bigcup_{s \geqslant r+1} H(n, 2s+1),$$

then $G \in H(n, 2r+1)$ and $P(G;3) = 2^n - 2^{n-2r}$.

(b) If $P(G;3)$ is maximum in the class of graphs

$$\mathcal{C}_{n,3} \setminus \bigcup_{s \geqslant 1} H(n, 2s+1),$$

then $G \cong F_n$ for odd n, $G \cong D_n$ for even n and in this case $P(G;3) - 2^n - 2^{n-2} - 6$.

Proof. (a) Let $G \in \mathcal{C}_{n,3}$. It follows that G contains an odd cycle C_{2r+1}. If for every edge $e \in E(G) \setminus E(C_{2r+1})$ the graph $G - e$ is not connected then $G \in H(n, 2r+1)$. Otherwise, by Lemma 1.1(ii) we have

$$P(G-e;3) = P(G;3) + P(G/e;3). \tag{3}$$

But $\chi(G/e) = 3$ since G/e contains an odd cycle even if e is a chord of C_{2r+1}. It follows that $P(G/e;3) > 0$ and (3) implies that $P(G-e;3) > P(G;3)$. By applying several times this operation of deleting edges not belonging to C_{2r+1} without disconnecting the resulting graph, one obtains a graph $H \in H(n, 2r+1)$ such that $P(H;3) > P(G;3)$. By Corollary 1.7 if $3 \leqslant 2j+1 < 2i+1 \leqslant n$ then $G_1 \in H(n, 2i+1)$ and $G_2 \in H(n, 2j+1)$ imply

$$P(G_1;3) = 2^n - 2^{n-2i} > 2^n - 2^{n-2j} = P(G_2;3)$$

and (a) is proved for $r = \lceil n/2 \rceil - 1$ (this is the property expressed by Theorem 1.2).

Let $G \in \bigcup_{s \geqslant 2} H(n, 2s+1)$ and a, b be two nonadjacent vertices of G. We shall prove that if $e = ab$ then

$$P(G+e;3) < 2^n - 2^{n-2} = P(H;3), \tag{4}$$

where $H \in H(n,3)$.

It is clear that the skeleton $S(G+e)$ consists of: I. Two vertex disjoint cycles joined by a path of length $t \geqslant 1$; II. Two cycles having exactly one common vertex; III. Two cycles having in common a path of length $p \geqslant 1$. In all cases at least one cycle is odd. Suppose that $|S(G+e)| = m$.

Case I. In this case by Lemmas 1.6 and 1.8 one deduces

$$P(G+e;\lambda) = P(S(G+e);\lambda)(\lambda-1)^{n-m} = P(H;\lambda)(\lambda-1)^{n-m+t},$$

where H has order $m-t$ and consists of two cycles (one is odd) having one vertex in common. By Lemma 1.5 we get

$$P(G+e;3) = P(H;3)2^{n-m+t} < (2^{m-t} - 2^{m-t-2} - 6)2^{n-m+t} < 2^n - 2^{n-2}.$$

Cases II, III. We have $P(G+e;3) < (2^m - 2^{m-2})2^{n-m} = 2^n - 2^{n-2}$ by Lemmas 1.5, 1.6 and 1.9. Let now r be such that $1 \leqslant r \leqslant \lceil n/2 \rceil - 2$ and G be such that $P(G;3)$ is maximum in the class $\mathcal{C}_{n,3} \setminus \bigcup_{s \geqslant r+1} H(n, 2s+1)$. If $G \in \bigcup_{s=1}^{r} H(n, 2s+1)$ it follows that $G \in H(n, 2r+1)$ and the property is proved. Otherwise, there exists an edge $e \in E(G)$ such that $G - e \in \mathcal{C}_{n,3}$. Since $P(G;3)$ is maximum in the class $\mathcal{C}_{n,3} \setminus \bigcup_{s \geqslant r+1} H(n, 2s+1)$, it follows that $G - e \in \bigcup_{s \geqslant r+1} H(n, 2s+1)$, i.e., there exists a graph H in $\bigcup_{s \geqslant r+1} H(n, 2s+1)$ such that $G \cong H + e$. By (4) this leads to a contradiction.

(b) Let $G \in \mathcal{C}_{n,3} \setminus \bigcup_{s \geqslant 1} H(n, 2s+1)$ be such that $P(g;3)$ is maximum. We have seen that the greatest values of $P(G;3)$ in the class $\mathcal{C}_{n,3}$ are obtained for graphs in $\bigcup_{s \geqslant 1} H(n, 2s+1)$, and for graphs not belonging to this class the greatest values of $P(G;3)$ are obtained for graphs of the form $H+e$, where $H \in \bigcup_{s \geqslant 1} H(n, 2s+1)$ and

$e \notin E(H)$. It follows that $G \cong H + e$, where $H \in \bigcup_{s \geqslant 1} H(n, 2s+1)$ and $e \notin E(H)$. Suppose that $|S(H+e)| = m$. As for the case (a) we may distinguish cases I–III concerning the structure of $S(H+e)$. Using the same notation, in the case I one obtains $P(H+e;3) < (2^{m-t} - 2^{m-t-2} - 6)2^{n-m+t} < 2^n - 2^{n-2} - 6$ since $n - m + t \geqslant 1$. In the case II by Lemma 1.5, $P(H+e;3) < (2^m - 2^{m-2} - 6)2^{n-m} \leqslant 2^n - 2^{n-2} - 6$.

In the case III the skeleton $S(H+e)$ is 2-connected and by Lemmas 1.3, 1.6 and Theorem 1.4 one deduces

$$P(H+e;3) \leqslant (2^m - 2^{m-2} - 6)2^{n-m} \leqslant 2^n - 2^{n-2} - 6$$

and equality holds if and only if $m = n$ and $G \cong F_n$ for odd n and $G \cong D_n$ for even n. ■

Note that $\mathrm{Col}_3(F_n)$ for odd n, resp. $\mathrm{Col}_3(D_n)$ for even n is equal to $\mathrm{Col}_3(H) - 1 = 2^{n-3} - 1$ for any $H \in H(n, 3)$.

REFERENCES

[1] G.D. Birkhoff, *A determinantal formula for the number of ways of coloring a map*, Ann. Math. (2) **14** (1912), 42–46.

[2] R.C. Read, *An introduction to chromatic polynomials*, J. Combinatorial Theory **4** (1968), 52–71.

[3] A. Rényi, *On connected graphs*, Magyar Tud. Akad. Mat. Kutató Int. Közl **4** (1959), 385–388.

[4] I. Tomescu, *Le nombre maximal de 3-coloration d'un graphe connexe*, Discrete Math. **1** (1972), 351–356.

[5] I. Tomescu, *Maximum chromatic polynomial of 3-chromatic blocks*, Discrete Math. **172** (1997), 131–139.

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