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ON THE CONVERGENCE OF FINITE DIFFERENCE SCHEME FOR ELLIPTIC EQUATION WITH COEFFICIENTS CONTAINING DIRAC DISTRIBUTION

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Abstract. First boundary value problem for elliptic equation with youngest coefficient containing Dirac distribution concentrated on a smooth curve is considered. For this problem a finite difference scheme on a special quasiregular grid is constructed. The finite difference scheme converges in discrete W_2^1 norm with the rate $O(h^{3/2})$. Convergence rate is compatible with the smoothness of input data.

1. Introduction

Problems with concentrated factors arise in different physical applications. Such problems can be modelled by partial differential equations containing singular coefficients, e. g. Dirac delta distribution (see [3], [7]). In the present paper we consider two-dimensional Dirichlet boundary value problem (BVP) with Dirac distribution involved in the youngest coefficient. The support of Dirac distribution is a smooth curve S, which splits region Ω into two parts. The solution of BVP has discontinous derivatives on the interface S and can not be well approximated by standard difference schemes.

For the numerical solution of the considered problem a five-point finite difference scheme (FDS) with averaged right hand side and youngest coefficient on special quasilinear mesh is proposed. The convergence of FDS on generalized solutions of BVP is proved in discrete Sobolev $W_{2,h}^1$ norm.

Analogous one-dimensional problems were considered in [6] and [11]. A survey of different numerical methods for the solution of such problems is given in [5]. In particular, in the present paper we improve convergence rate estimate obtained in [5].

AMS Subject Classification: 65 N 15

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2. Statement of the problem

In the rectangle $\Omega = (0, l_1) \times (0, l_2)$ we consider the Dirichlet boundary value problem

$$-\Delta\, u + c(x,y)\, \delta_S(x,y)\, u = f(x,y), \quad (x,y) \in \Omega; \qquad u = 0, \quad (x,y) \in \Gamma = \partial\Omega, \tag{1}$$

where S is a smooth curve in Ω and $\delta_S(x,y)$ is Dirac distribution concentrated on S. We assume that $f(x,y) \in W_2^{-1}(\Omega)$, $c(x,y) \in L_\infty(S)$ and $0 < C_0 \le c(x,y) \le C_1$ almost everywhere in S. We also assume that curve S is defined by equation $y = g(x)$, where function $g(x)$ satisfies the following conditions: $g'(x) > 0$, $g''(x) > 0$, $g(0) = a$, $g(l_1) = b$ and $0 < a < b < l_2$. In such a manner S splits domain Ω into two parts, denoted by Ω^- and Ω^+ (fig. 1).

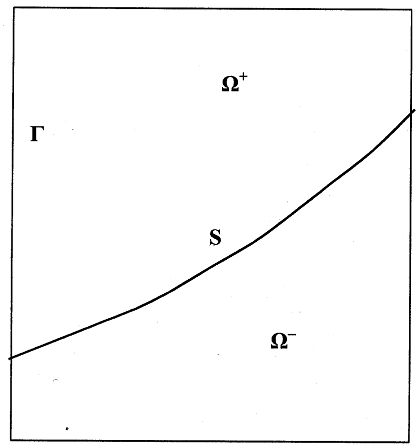

Fig. 1

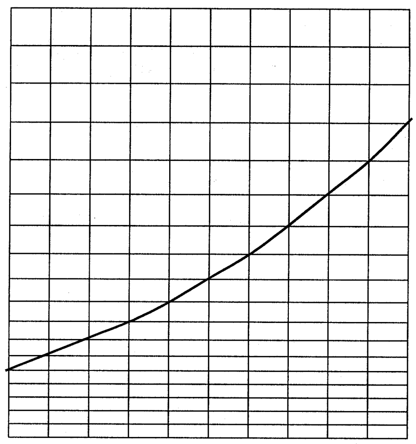

Fig. 2

It can be easily checked that the solution of the BVP (1) satisfies the following conditions:

$$-\Delta\, u = f(x,y) \;\text{ in }\; \Omega^- \text{ and } \Omega^+; \quad u = 0 \;\text{ on }\; \Gamma; \quad [u]_S = 0; \quad \left[\frac{\partial u}{\partial \nu}\right]_S = c\, u \;\text{ on }\; S, \tag{2}$$

where $[u]_S$ is the jump of u over S and $\partial u/\partial \nu$ is normal derivative on S.

The convergence of the FDS approximating BVP (1) will be proved under stronger assumptions on the smoothness of input data, i.e. $c(x,y) \in W_2^1(S)$, $f(x,y) \in W_2^s(\Omega)$, $s > 0$.

3. Finite difference scheme

In $\bar\Omega$ we introduce quasiuniform mesh $\bar\Omega_h$ in the following manner. Let $\omega_1 = \{x_i = ih \,:\, i = 0, 1, \ldots, n\}$ be a uniform mesh with the step size $h = l_1/n$ on $[0, l_1]$

(on variable x). On $[a,b]$ we define nonuniform mesh $\bar{\omega}_2^0 = \{y_{i+m^-} = g(x_i) \,:\, i = 0,1,\ldots,n\}$, where $m^- = \left[\frac{a}{g(h)-g(0)}\right]^+$ and $[\alpha]^+$ is the smallest integer, greater or equal α. On $[0,a]$ we define uniform mesh $\bar{\omega}_2^- = \{y_j = jk^- \,:\, j = 0,1,\ldots,m^-\}$ with the step size $k^- = a/m^-$. In analogous manner we set $m^+ = \left[\frac{l_2-b}{g(l_1)-g(l_1-h)}\right]$, where $[\alpha]$ is the integer part of α. On $[b,l_2]$ we define uniform mesh $\bar{\omega}_2^+ = \{y_{j+n+m^-} = b + jk^+ \,:\, j = 0,1,\ldots,m^+\}$ with the step size $k^+ = (l_2 - b)/m^+$. We define $\bar{\omega}_2 = \bar{\omega}_2^- \cup \bar{\omega}_2^0 \cup \bar{\omega}_2^+$, $\bar{\Omega}_h = \bar{\omega}_1 \times \bar{\omega}_2$, $\Omega_h = \bar{\Omega}_h \cap \Omega$, $\Gamma_h = \bar{\Omega}_h \cap \Gamma$, $\Omega_{1h} = \{(x,y) \in \bar{\Omega}_h \,:\, 0 < x \le l_1,\ 0 < y < l_2\}$, $\Omega_{2h} = \{(x,y) \in \bar{\Omega}_h \,:\, 0 < x < l_1,\ 0 < y \le l_2\}$ and $S_h = S \cap \Omega_h$ (fig. 2).

We also denote $k_j = y_j - y_{j-1}$ and $\hbar_j = (k_j + k_{j+1})/2$. It can be easily checked that $k_j \le k_{j+1}$, $k_j = O(h)$ and $k_{j+1} - k_j = O(h^2)$. For a mesh function $v = v(x,y)$ defined on $\bar{\Omega}_h$ we introduce divided differences v_x, $v_{\bar{x}}$, $v_{\hat{y}}$ and $v_{\bar{y}}$ in a usual way (see [9]). In the sequel, we shall use the standard denotation of the theory of difference schemes [9]: $v = v_{ij} = v(x_i, y_j)$, $k = k_j$, $k_+ = k_{j+1}$, $v_x = v_{x,ij} = (v_{i+1,j} - v_{ij})/h$, $v_{\bar{x}} = v_{\bar{x},ij} = (v_{ij} - v_{i-1,j})/h$, $v_{\hat{y}} = v_{\hat{y},ij} = (v_{i,j+1} - v_{ij})/k_{j+1}$, $v_{\bar{y}} = v_{\bar{y},ij} = (v_{ij} - v_{i,j-1})/k_j$ etc.

Let H_h be the set of mesh function defined on $\bar{\Omega}_h$ which vanish on Γ_h. We introduce the inner products

$$(v,\, w)_h = \sum_{(x,y)\in\Omega_h} vwh\hbar \quad \text{and} \quad (v,\, w)_{ih} = \sum_{(x,y)\in\Omega_{ih}} vwhk$$

and the corresponding norms $\|v\|_h$ and $\|v\|_{ih}$. We also define discrete Sobolev norm W_2^1:

$$\|v\|^2_{W^1_{2,h}} = |v|^2_{W^1_{2,h}} + \|v\|^2_h\,, \qquad |v|^2_{W^1_{2,h}} = \|v_{\bar{x}}\|^2_{1h} + \|v_{\bar{y}}\|^2_{2h}\,.$$

We define Steklov averaging operators [10, p. 56]:

$$T_1 f(x,y) = \frac{1}{h}\int_{x-h/2}^{x+h/2} f(\tilde{x},y)\,d\tilde{x}, \qquad T_2 f(x,y) = \frac{1}{\hbar}\int_{y-k/2}^{y+k_+/2} f(x,\tilde{y})\,d\tilde{y}.$$

On the mesh $\bar{\Omega}_h$ we approximate BVP (1) with standard homogeneous FDS with averaged right hand side and youngest coefficient:

$$-\Delta_h v + \alpha\, v = \varphi \quad \text{in} \quad \Omega_h\,; \qquad v = 0 \quad \text{on} \quad \Gamma_h\,, \tag{3}$$

where $\Delta_h v = v_{\bar{x}x} + v_{\bar{y}\hat{y}}$, $\varphi = T_1 T_2 f$ and $\alpha = T_1 T_2 (c\,\delta_S)$.

For an internal node $x \in \Omega_h$ we define elementary cell $e = e(x,y) = (x - h/2, x + h/2) \times (y - k/2, y + k_+/2)$. The coefficient α in (3) can be represented in the following manner (fig. 3):

$$\alpha(x,y) = \begin{cases} \dfrac{1}{h\hbar}\displaystyle\int_{S\cap e} c\,dS\,, & (x,y) \in S_h\,, \\ 0\,, & (x,y) \in \Omega_h \setminus S_h\,. \end{cases}$$

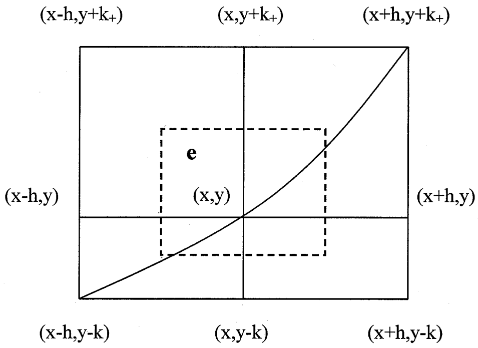

Fig. 3

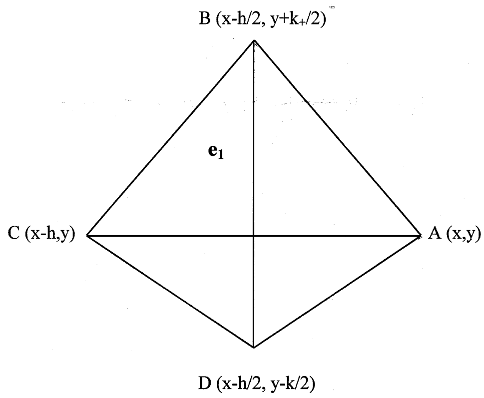

Fig. 4

4. Convergence of finite difference scheme

The error $z = u - v$ satisfies the following conditions

$$-\Delta_h z + \alpha\, z = \eta_{1,x} + \eta_{2,\hat{y}} + \zeta \quad \text{in} \quad \Omega_h\, ; \qquad z = 0 \quad \text{on} \quad \Gamma_h \tag{4}$$

where

$$\eta_1 = T_2 \frac{\partial u}{\partial x}\bigg|_{(x-h/2,y)} - u_{\bar{x}}\, , \qquad \eta_2 = T_1 \frac{\partial u}{\partial y}\bigg|_{(x,y-k/2)} - u_{\bar{y}}\, ,$$

$$\zeta = [T_1 T_2(c\delta_S)]u - T_1 T_2(c\delta_S u)$$
$$= \begin{cases} \dfrac{1}{h\hbar k}\left(u \displaystyle\int_{S\cap e} c\, dS - \int_{S\cap e} cu\, dS\right), & (x,y) \in S_h\, , \\ 0\, , & (x,y) \in \Omega_h \setminus S_h\, . \end{cases}$$

Taking inner product of (4) with z and using summation by parts, we obtain

$$\|z_{\bar{x}}\|_{1h}^2 + \|z_{\bar{y}}\|_{2h}^2 + \sum_{(x,y)\in S_h} \alpha\, z^2\, h\hbar k = -(\eta_1,\, z_{\bar{x}})_{1h} - (\eta_2,\, z_{\bar{y}})_{2h} + \sum_{(x,y)\in S_h} \zeta\, z\, h\hbar k\, .$$

From here, using the discrete analogue of Friedrichs inequality [10, p. 55], we obtain the a priori estimate

$$\|z\|_{W_{2,h}^1} \le C\left\{ \|\eta_1\|_{1h} + \|\eta_2\|_{2h} + \left(\sum_{(x,y)\in S_h} \frac{\zeta^2}{\alpha}\, h\hbar k \right)^{1/2} \right\}. \tag{5}$$

In such a way, to obtain the convergence rate estimates for FDS (3) it is sufficient to estimate the right hand side terms in (5).

Let us first estimate the norm of η_1. For each node $(x,y) \in \Omega_{1h}$ we define the elementary cell $e_1 = e_1(x,y)$ with vertices $A = (x,y)$, $B = (x - h/2, y + k_+/2)$,

$C = (x-h, y)$ and $D = (x-h/2, y-k/2)$ (fig. 4). In the case when $e_1 \cap S = \emptyset$ we set

$$
\begin{aligned}
\eta_1 &= \eta_{11} + \eta_{12},\\
\eta_{11} &= T_2 \frac{\partial u}{\partial x}\bigg|_{(x-h/2,y)} - u_{\bar{x}} - \frac{k_+ - k}{4}\, T_2 \frac{\partial^2 u}{\partial x \partial y}\bigg|_{(x-h/2,y)},\\
\eta_{12} &= \frac{k_+ - k}{4}\, T_2 \frac{\partial^2 u}{\partial x \partial y}\bigg|_{(x-h/2,y)}\\
&= \frac{k_+ - k}{4\bar{k}} \left[\frac{\partial u}{\partial x}(x-h/2, y+k_+/2) - \frac{\partial u}{\partial x}(x-h/2, y-k/2)\right].
\end{aligned}
$$

The value η_{11} in the node $(x,y) \in \Omega_{1h}$ is a bounded linear functional of $u \in W_2^s(e_1)$, $s > 2$, which vanishes on polynomials of the second degree. Using Bramble-Hilbert lemma [1], [2] and the methodology proposed in [10] and [4] we easily obtain

$$
|\eta_{11}(x,y)| \le C\, h^{s-2}\, |u|_{W_2^s(e_1)}\,, \qquad 2 < s \le 3\,. \tag{6}
$$

The value $\eta_{12}(x,y)$ is a bounded linear functional of $\frac{\partial u}{\partial x}(x-h/2,\cdot) \in W_2^\sigma(y-k/2, y+k_+/2)$, $\sigma > 1/2$, which vanishes on constants. Using Bramble-Hilbert lemma we obtain

$$
|\eta_{12}(x,y)| \le C\, h^{\sigma+1/2} \left|\frac{\partial u}{\partial x}(x-h/2,\cdot)\right|_{W_2^\sigma(y-k/2,y+k_+/2)}, \qquad 1/2 < \sigma \le 1\,. \tag{7}
$$

Let now $e_1 \cap S \ne \emptyset$. Then the vertex $C = (x-h, y)$ of e_1 belongs to S_h. By $B' = (x-h/2, y')$ we denote intersection of BD with S, and by $B'' = (x-h/2, y'')$ we denote intersection of BD with tangent on S in the point C (fig. 5). By $\bar{k}''$ we denote the distance between B'' and D. Finally, we set

$$
\begin{aligned}
\eta_1 &= \eta_{11}^* + \eta_{12}^* + \eta_{13}^* + \eta_{14}^* + \eta_{15}^*,\\
\eta_{11}^* &= \frac{1}{\bar{k}''} \int_{y-k/2}^{y''} \frac{\partial u}{\partial x}(x-h/2, \tilde{y})\, d\tilde{y} - u_{\bar{x}}\\
&\quad - \frac{y'' - y - k/2}{2\bar{k}''} \left[\frac{\partial u}{\partial x}(x-h/2, y'') - \frac{\partial u}{\partial x}(x-h/2, y-k/2)\right],\\
\eta_{12}^* &= \frac{y'' - y - k/2}{2\bar{k}''} \left[\frac{\partial u}{\partial x}(x-h/2, y'') - \frac{\partial u}{\partial x}(x-h/2, y-k/2)\right],\\
\eta_{13}^* &= \left(\frac{1}{\bar{k}} - \frac{1}{\bar{k}''}\right) \int_{y-k/2}^{y''} \frac{\partial u}{\partial x}(x-h/2, \tilde{y})\, d\tilde{y},\\
\eta_{14}^* &= \frac{1}{\bar{k}} \int_{y''}^{y'} \frac{\partial u}{\partial x}(x-h/2, \tilde{y})\, d\tilde{y},\\
\eta_{15}^* &= \frac{1}{\bar{k}} \int_{y'}^{y+k_+/2} \frac{\partial u}{\partial x}(x-h/2, \tilde{y})\, d\tilde{y}.
\end{aligned}
$$

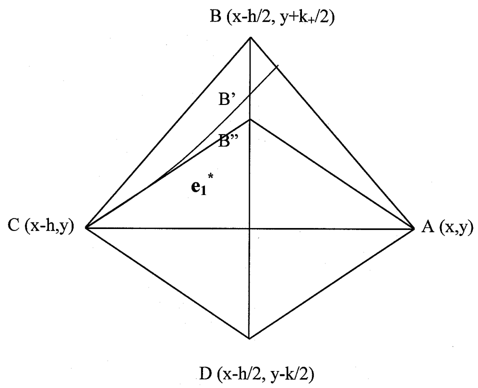

Fig. 5

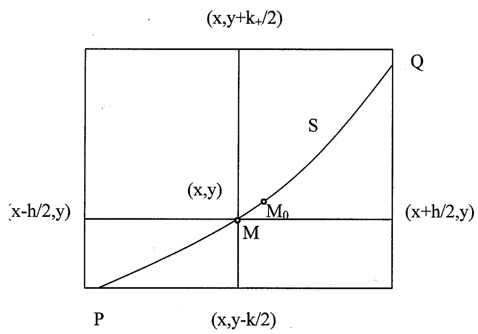

Fig. 6

Terms η_{11}^* and η_{12}^* can be estimated in the same manner as η_{11} and η_{12}:

$$|\eta_{11}^*(x,y)| \le C\,h^{s-2}\,|u|_{W_2^s(e_1^*)}\,, \qquad 2 < s \le 3\,. \tag{8}$$

$$|\eta_{12}^*(x,y)| \le C\,h^{\sigma+1/2}\left|\frac{\partial u}{\partial x}(x-h/2,\cdot)\right|_{W_2^\sigma(y-k/2,y'')}, \qquad 1/2 < \sigma \le 1\,, \tag{9}$$

where e_1^* is quadrangle $AB''CD$.

We have $\bar{k} - \bar{k}'' = O(h^2)$, wherefrom follows

$$|\eta_{13}^*| \le C\,h^{1/2}\left\|\frac{\partial u}{\partial x}(x-h/2,\cdot)\right\|_{L_2(y-k/2,y'')}.$$

Using inequality [8]

$$\|\varphi\|_{L_2(0,\varepsilon)} \le C\,\varepsilon^{1/2}\,\|\varphi\|_{W_2^\tau(0,1)}\,, \qquad 0 < \varepsilon < 1, \quad \tau > 1/2 \tag{10}$$

and applying imbedding theorem, from here follows

$$|\eta_{13}^*| \le C\,h\left\|\frac{\partial u}{\partial x}(x-h/2,\cdot)\right\|_{W_2^\tau(0,y'')} \le C\,h\,\|u\|_{W_2^{\tau+3/2}(\Omega-)}\,, \qquad \tau > 1/2. \tag{11}$$

Analogously

$$|\eta_{14}^*| \le C\left\|\frac{\partial u}{\partial x}(x-h/2,\cdot)\right\|_{L_2(y'',y')} \le C\,h\left\|\frac{\partial u}{\partial x}(x-h/2,\cdot)\right\|_{W_2^\tau(0,y')} \le C\,h\,\|u\|_{W_2^{\tau+3/2}(\Omega-)} \tag{12}$$

and

$$|\eta_{15}^*| \le C\left\|\frac{\partial u}{\partial x}(x-h/2,\cdot)\right\|_{L_2(y',y+k+/2)} \le C\,h\left\|\frac{\partial u}{\partial x}(x-h/2,\cdot)\right\|_{W_2^\tau(y',l_2)} \le C\,h\,\|u\|_{W_2^{\tau+3/2}(\Omega+)}\,, \tag{13}$$

where $\tau > 1/2$.

From (6)–(9) and (11)–(13) after summation over the mesh Ω_{1h} and application of imbedding theorem, we immediately obtain the estimate

$$\|\eta_1\|_{1h} \le C\, h^{s-1}\Big(\|u\|_{W_2^s(\Omega^-)} + \|u\|_{W_2^s(\Omega^+)}\Big) + C\, h^{\sigma+1}\Big(\|u\|_{W_2^{\sigma+3/2}(\Omega^-)} + \\ + \|u\|_{W_2^{\sigma+3/2}(\Omega^+)}\Big) + C\, h^{3/2}\Big(\|u\|_{W_2^{\tau+3/2}(\Omega^-)} + \|u\|_{W_2^{\tau+3/2}(\Omega^+)}\Big),$$

where $2 < s \le 3$, $1/2 < \sigma \le 1$ and $\tau > 1/2$. Setting

$$\sigma + 3/2 = \tau + 3/2 = s \in (2,\, 5/2]$$

and using obvious inequalities

$$h^{\sigma+1} = h^{s-1/2} \le h^{s-1}\,, \qquad h^{3/2} \le h^{s-1}$$

(for $h < 1$ and $s \le 5/2$), one finally obtains

$$\|\eta_1\|_{1h} \le C\, h^{s-1}\Big(\|u\|_{W_2^s(\Omega^-)} + \|u\|_{W_2^s(\Omega^+)}\Big)\,, \qquad 2 < s \le 5/2. \tag{14}$$

In an analogous manner:

$$\|\eta_2\|_{2h} \le C\, h^{s-1}\Big(\|u\|_{W_2^s(\Omega^-)} + \|u\|_{W_2^s(\Omega^+)}\Big)\,, \qquad 2 < s \le 5/2. \tag{15}$$

Now, let us estimate ζ. Let us consider the node $(x,y) = (x_i, g(x_i)) \in S_h$ and the corresponding arc $\overset{\frown}{PQ} = S \cap e(x,y)$ (fig. 6). The length of $\overset{\frown}{PQ}$ is equal to

$$l = \int_{x_0}^{x+h/2} \sqrt{1 + [g'(\tilde{x})]^2}\, d\tilde{x}, \qquad \text{where} \quad g(x_0) = y - k/2.$$

Obviously, $l = O(h)$. On $\overset{\frown}{PQ}$ we introduce the local system of coordinates, taking P as the zero point and denoting by

$$\sigma = \int_{x_0}^{x'} \sqrt{1 + [g'(\tilde{x})]^2}\, d\tilde{x}, \qquad x' \in (x_0, x + h/2)$$

the distance from P to point $(x', g(x')) \in S \cap e$. Let us denote $u(\sigma) = u(x', g(x'))$. Also, let $u(\sigma_*) = u(x, g(x))$ be the value of the function u in the point $M = (x, g(x))$, and M_0 – the middle point of the arc $\overset{\frown}{PQ}$ (corresponding to the value $\sigma = l/2$). It can be easily checked that $l/2 - \sigma_* = O(h^2)$ and $\alpha(x,y) = O(h^{-1})$.

The value $\zeta = \zeta(x,y)$ can be represented in the following way

$$\zeta = \frac{1}{h\bar{k}} \int_0^l c(\sigma)\,[u(\sigma_*) - u(\sigma)]\, d\sigma = \zeta_1 + \zeta_2 + \zeta_3\,,$$

where

$$\zeta_1 = \frac{1}{h\bar{k}} \int_0^l [c(\sigma) - c(l/2)]\,[u(\sigma_*) - u(\sigma)]\, d\sigma,$$

$$\zeta_2 = \frac{c(l/2)}{h\bar{k}} \int_0^l [u(\sigma_*) - u(l/2)]\, d\sigma = \frac{l}{h\bar{k}}\, c(l/2)\,[u(\sigma_*) - u(l/2)]\,,$$

$$\zeta_3 = \frac{c(l/2)}{h\bar{k}} \int_0^l [u(l/2) - u(\sigma)]\, d\sigma.$$

We have

$$\zeta_1 = \frac{1}{h\hbar}\int_0^l \left(\int_{l/2}^{\sigma} \frac{dc}{d\sigma}(\sigma')\,d\sigma'\right)[u(\sigma_*) - u(\sigma)]\,d\sigma$$
$$= \frac{1}{h\hbar}\int_0^l \left(\int_{l/2}^{\sigma} \frac{dc}{d\sigma}(\sigma')\,d\sigma'\right)\left(\int_{\sigma}^{\sigma_*} \frac{du}{d\sigma}(\sigma'')\,d\sigma''\right)d\sigma,$$

wherefrom follows

$$|\zeta_1| \le C\,h^{-1/2}\left\|\frac{\partial c}{\partial S}\right\|_{L_2(S\cap e)} \|u\|_{C(S)} \le C\,h^{-1/2}\left\|\frac{\partial c}{\partial S}\right\|_{L_2(S\cap e)} \|u\|_{W_2^1(S)}\,, \tag{16}$$

where as usual $\|u\|_{C(S)} = \max_{(x,y)\in S}|u(x,y)|$, and

$$|\zeta_1| \le C\,h^{1/2}\left\|\frac{\partial c}{\partial S}\right\|_{L_2(S\cap e)}\left\|\frac{\partial u}{\partial S}\right\|_{C(S)} \le C\,h^{1/2}\left\|\frac{\partial c}{\partial S}\right\|_{L_2(S\cap e)} \|u\|_{W_2^2(S)}\,. \tag{17}$$

Analogously we obtain

$$|\zeta_2| = \left|\frac{l}{h\hbar}\,c(l/2)\int_{l/2}^{\sigma_*} \frac{du}{d\sigma}(\sigma)\,d\sigma\right| \le C\,\|c\|_{C(S)}\left\|\frac{\partial u}{\partial S}\right\|_{L_2(\widehat{MM_0})}$$
$$\le C\,\|c\|_{W_2^1(S)}\,\|u\|_{W_2^1(S)}\,. \tag{18}$$

From (18), using (10) one obtains

$$|\zeta_2| \le C\,h\,\|c\|_{W_2^1(S)}\,\|u\|_{W_2^2(S)}\,. \tag{19}$$

Finally, from integral representations

$$\zeta_3 = \frac{c(l/2)}{h\hbar}\int_0^l\int_{\sigma}^{l/2}\frac{du}{d\sigma}(\sigma')\,d\sigma'\,d\sigma = \frac{c(l/2)}{h\hbar}\int_0^l\int_{\sigma}^{l/2}\int_{l/2}^{\sigma'}\frac{d^2u}{d\sigma^2}(\sigma'')\,d\sigma''\,d\sigma'\,d\sigma$$

we immediately obtain

$$|\zeta_3| \le C\,h^{-1/2}\,\|c\|_{C(S)}\left\|\frac{\partial u}{\partial S}\right\|_{L_2(S\cap e)} \le C\,h^{-1/2}\,\|c\|_{W_2^1(S)}\,\|u\|_{W_2^1(S\cap e)} \tag{20}$$

and

$$|\zeta_3| \le C\,h^{1/2}\,\|c\|_{C(S)}\left\|\frac{\partial^2 u}{\partial S^2}\right\|_{L_2(S\cap e)} \le C\,h^{1/2}\,\|c\|_{W_2^1(S)}\,\|u\|_{W_2^2(S\cap e)}\,. \tag{21}$$

From (16), (18) and (20) by summing over S_h we obtain

$$\left(\sum_{(x,y)\in S_h}\frac{\zeta^2}{\alpha}\,h\hbar\right)^{1/2} \le C\,h\,\|c\|_{W_2^1(S)}\,\|u\|_{W_2^1(S)}\,. \tag{22}$$

Analogously, from (17), (19) and (21) follows

$$\left(\sum_{(x,y)\in S_h}\frac{\zeta^2}{\alpha}\,h\hbar\right)^{1/2} \le C\,h^2\,\|c\|_{W_2^1(S)}\,\|u\|_{W_2^2(S)}\,. \tag{23}$$

From (22) and (23) by interpolation one obtains

$$\Big(\sum_{(x,y)\in S_h}\frac{\zeta^2}{\alpha}\,h\bar{k}\Big)^{1/2}\le C\,h^{s-1}\,\|c\|_{W_2^1(S)}\,\|u\|_{W_2^{s-1}(S)}\,,\qquad 2\le s\le 3. \tag{24}$$

Finally, from (5), (14), (15) and (24) we obtain the following assertion:

THEOREM 1. *Under previous assumptions FDS (3) converges and the following convergence rate estimate holds*

$$\|z\|_{W_{2,h}^1}\le C\,h^{s-1}\Big(\|u\|_{W_2^s(\Omega^-)}+\|u\|_{W_2^s(\Omega^+)}+\|c\|_{W_2^1(S)}\,\|u\|_{W_2^{s-1}(S)}\Big)\,,\quad 2<s\le 5/2. \tag{25}$$

Estimate (25) is compatible with the smoothness of generalized solution of BVP (1).

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