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# SOME SUBSETS OF IDEAL TOPOLOGICAL SPACES

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**Abstract.** In ideal topological spaces, $\star$-dense in itself subsets are used to characterize ideals and mappings. In this note, properties of $\mathcal{A}_{\mathcal{I}}$-sets, $\mathcal{I}$-locally closed sets and almost strong $\mathcal{I}$-open sets are discussed. We characterize codense ideals by the collection of these sets. Also, we give a decomposition of continuous mappings and deduce some well-known results.

## 1. Introduction and preliminaries

Ideals in topological spaces have been considered since 1930. This topic has won its importance by the paper of Vaidyanathaswamy [19]. In this note, we discuss the properties of the $\star-$dense in itself sets, namely, $\mathcal{A}_{\mathcal{I}}$-sets, regular $\mathcal{I}$-closed sets and almost strong $\mathcal{I}$-open sets in ideal topological spaces.

An *ideal* $\mathcal{I}$ on a topological space $(X,\tau)$ is a nonempty collection of subsets of $X$ which satisfies: (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space $(X,\tau)$ with an ideal $\mathcal{I}$ on $X$ and if $\mathcal{P}(X)$ is the set of all subsets of $X$, a set operator $(\cdot)^\star \colon \mathcal{P}(X) \to \mathcal{P}(X)$, called a *local function* [13] of $A$ with respect to $\tau$ and $\mathcal{I}$ is defined as follows: for $A \subset X$, $A^\star(\mathcal{I},\tau) = \{ x \in X \mid U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x) \}$ where $\tau(x) = \{ U \in \tau \mid x \in U \}$. We will make use of the basic facts about the local functions [9, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^\star(\cdot)$ for a topology $\tau^\star(\mathcal{I},\tau)$, called the $\star$-*topology*, finer than $\tau$ is defined by $cl^\star(A) = A \cup A^\star(\mathcal{I},\tau)$ [19]. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{I},\tau)$ and $\tau^\star$ or $\tau^\star(\mathcal{I})$ for $\tau^\star(\mathcal{I},\tau)$. If $\mathcal{I}$ is an ideal on $X$, then $(X,\tau,\mathcal{I})$ is called an *ideal space*. $\mathcal{I}$ is said to be *codense* [4] if $\tau \cap \mathcal{I} = \{\emptyset\}$. $\mathcal{N}$ is the ideal of all nowhere dense subsets in $(X,\tau)$.

By a space, we always mean a topological space $(X,\tau)$ with no separation properties assumed. If $A \subset X$, $cl(A)$ and $int(A)$ will, respectively, denote the closure and interior of $A$ in $(X,\tau)$ and $cl^\star(A)$ and $int^\star(A)$ will, respectively, denote the closure and interior of $A$ in $(X,\tau^\star)$. An open subset $A$ of a space $(X,\tau)$ is

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said to be *regular open* if $A = int(cl(A))$. The complement of a regular open set is *regular closed*. The family of all regular open (resp. regular closed) set is denoted by $RO(X,\tau)$ (resp. $RC(X,\tau)$). A subset $A$ of a space (X, $\tau$) is an $\alpha$-*open* [16] (resp. *semiopen* [14], *preopen* [15], $\beta$-*open* or *semipreopen* [1]) set if $A \subset int(cl(int(A)))$ (resp. $A \subset cl(int(A))$, $A \subset int(cl(A))$, $A \subset cl(int(cl(A)))$). The complement of a semiopen (resp. preopen) set is *semiclosed* (resp. *preclosed*). The family of all $\alpha$-open (resp. semiopen, preopen) sets in $(X,\tau)$ is denoted by $\tau^\alpha$ (resp. $SO(X,\tau)$, $PO(X,\tau)$). The smallest preclosed set containing $A$ is called the *preclosure* of $A$ and is denoted by $pcl(A)$. Also, $pcl(A) = A \cup cl(int(A))$ [1, Theorem 1.5(e)]. A subset $A$ of a space $(X,\tau)$ is *locally closed* [2] (resp. $\mathcal{A}$-set [18]) if $A = U \cap V$ where $U$ is open and $V$ is closed (resp. regular closed). A subset $A$ of an ideal space $(X,\tau,\mathcal{I})$ is said to be $\mathcal{I}$-*open* [10] if $A \subset int(A^\star)$. The largest $\mathcal{I}$-open set contained in $A$ is called the $\mathcal{I}$-interior of $A$ and is denoted by $\mathcal{I}int(A)$. The family of all $\mathcal{I}$-open sets is denoted by $IO(X,\tau)$. A subset $A$ of an ideal space $(X,\tau,\mathcal{I})$ is $\tau^\star$-*closed* [8] (resp. $\star$-*dense in itself* [7], $\star$-*perfect* [7]) if $A^\star \subset A$ (resp. $A \subset A^\star$, $A = A^\star$). Clearly, $A$ is $\star$-perfect if and only if $A$ is $\tau^\star$-closed and $\star$-dense in itself. A subset $A$ of an ideal space $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-*locally closed* [3] if $A = G \cap V$, where $G$ is open and $V$ is $\star$-perfect. We will denote the collection of all $\mathcal{I}$-locally closed sets in $(X,\tau,\mathcal{I})$ by $\mathcal{I}LC(X,\tau)$. Clearly, every $\star$-perfect set is $\mathcal{I}$-locally closed. A subset $A$ of an ideal space $(X,\tau,\mathcal{I})$ is called an $f_{\mathcal{I}}$-*set* [12] (resp. *regular* $\mathcal{I}$-*closed* [11]) if $A \subset (int(A))^\star$ (resp. $A = (int(A))^\star$). The family of all $f_{\mathcal{I}}$-sets in a space $(X,\tau,\mathcal{I})$ will be denoted by $f_{\mathcal{I}}(X,\tau)$. A subset $A$ of an ideal space $(X,\tau,\mathcal{I})$ is called *pre*-$\mathcal{I}$-*open* [3] (resp. $\alpha - \mathcal{I}$-*open* [6], *semi*-$\mathcal{I}$-*open* [6])if $A \subset int(cl^\star(A))$ (resp. $A \subset int(cl^\star(int(A)))$, $A \subset cl^\star(int(A))$). The family of all pre-$\mathcal{I}$-open (resp. $\alpha$-$\mathcal{I}$-open, semi-$\mathcal{I}$-open) sets is denoted by $P_{\mathcal{I}}O(X,\tau)$ (resp. $\alpha_{\mathcal{I}}O(X,\tau)$, $S\mathcal{I}O(X,\tau)$). Given a space $(X,\tau)$ and ideals $\mathcal{I}$ and $\Im$ on $X$, the *extension* of $\mathcal{I}$ via $\Im$ [10], denoted by $\mathcal{I} \star \Im$, is the ideal given by $\mathcal{I} \star \Im = \{ A \subset X \mid A^\star(\mathcal{I}) \in \Im \}$. In particular, $\mathcal{I} \star \mathcal{N} = \{ A \subset X \mid int(A^\star(\mathcal{I})) = \emptyset \}$ is an ideal containing both $\mathcal{I}$ and $\mathcal{N}$ and $\mathcal{I} \star \mathcal{N}$ is usually denoted by $\widetilde{\mathcal{I}}$. The following lemmas will be useful in the sequel.

LEMMA 1.1. [9, Theorem 6.1] *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then the following are equivalent.*
(a) *$\mathcal{I}$ is codense.*
(b) $X = X^\star$.
(c) *$G \subset G^\star$ for every open set $G$.*
(d) *$G \subset G^\star$ for every semiopen set $G$* [17, Lemma 1(c)].

LEMMA 1.2. [17, Lemma 2] *Let $(X,\tau,\mathcal{I})$ be an ideal space. If $A$ is $\star$-dense in itself, then $A^\star = cl(A) = cl^\star(A)$.*

LEMMA 1.3. [17, Theorem 3.1(b)] *Let $(X,\tau,\mathcal{I})$ be an ideal space. A subset $A$ of $X$ is $\mathcal{I}$-locally closed if and only if $A = G \cap A^\star$ for some open set $G$.*

LEMMA 1.4. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $\Delta = \{A \subset X \mid A \subset A^\star\}$. Then $\Delta \cap \mathcal{I} = \{\emptyset\}$.*

*Proof.* Suppose $A \in \Delta \cap \mathcal{I}$. Then $A \in \mathcal{I}$ implies $A^\star = \emptyset$ and $A \in \Delta$ implies that $A \subset A^\star$. Therefore, $A = \emptyset$ which implies that $\Delta \cap \mathcal{I} = \{\emptyset\}$.

## 2. $\mathcal{A}_\mathcal{I}$-sets.

A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is called an $\mathcal{A}_\mathcal{I}$-*set* [11] if $A = U \cap V$ where $U \in \tau$ and $V$ is regular $\mathcal{I}$-closed. The family of all $\mathcal{A}_\mathcal{I}$-sets in a space $(X, \tau, \mathcal{I})$ will be denoted by $\mathcal{A}_\mathcal{I}(X, \tau)$. The following Theorem 2.1 gives some properties of $\mathcal{A}_\mathcal{I}$-sets.

THEOREM 2.1. (i) *If $A$ is an $\mathcal{A}_\mathcal{I}$-set of an ideal space $(X, \tau, \mathcal{I})$, then the following hold.*
(a) *$A$ and $int(A)$ are $\star$-dense in itself.*
(b) *$A^\star = cl(A) = cl^\star(A)$ and $(int(A))^\star = cl(int(A))$.*
(c) *$A$ is an $f_\mathcal{I}$-set.*
(d) *$A^\star = (int(A))^\star = ((int(A))^\star)^\star = (A^\star)^\star$.*
(e) *$A^\star$ and $(int(A))^\star$ are $\star$-perfect and $\mathcal{I}$-locally closed.*
(f) *$A^\star(\mathcal{I}) = cl(int(A)) = A^\star(\widetilde{\mathcal{I}})$ is regular closed.*
(g) *$A^\star = pcl(A)$.*
(h) *$A^\star$ is regular $\mathcal{I}$-closed.*
(ii) *In any ideal space $(X, \tau, \mathcal{I})$, $\mathcal{A}_\mathcal{I}(X, \tau) \cap \mathcal{I} = \{\emptyset\}$.*

*Proof.* (i) (a) If $A$ is an $\mathcal{A}_\mathcal{I}$-set, then $A = U \cap V$ where $U \in \tau$ and $V$ is regular $\mathcal{I}$-closed. Therefore, $A = U \cap V = U \cap (int(V))^\star \subset (U \cap int(V))^\star = (int(U \cap V))^\star = (int(A))^\star \subset A^\star$ which implies that $int(A) \subset A \subset (int(A))^\star \subset A^\star$. Therefore, $A$ and $int(A)$ are $\star$-dense in itself.

(b) By Lemma 1.2, we have $A^\star = cl(A) = cl^\star(A)$ and $(int(A))^\star = cl(int(A))$.

(c) From (a), $A \subset (int(A))^\star$ and so $A$ is an $f_\mathcal{I}$-set.

(d) From (a), we have $int(A) \subset A \subset (int(A))^\star \subset A^\star$ and so $(int(A))^\star \subset A^\star \subset ((int(A))^\star)^\star \subset (int(A))^\star \subset A^\star$ and so $A^\star = (int(A))^\star = ((int(A))^\star)^\star = (A^\star)^\star$.

(e) From (d), it follows that $A^\star$ and $(int(A))^\star$ are $\star$-perfect and hence are $\mathcal{I}$-locally closed.

(f) From (d), $A^\star = (int(A))^\star$ and so by (b), $A^\star = cl(int(A))$. Since $A$ is $\star$-dense in itself, $A^\star \subset cl(int(A^\star))$. Since $cl(int(A^\star)) = A^\star(\widetilde{\mathcal{I}}) \subset A^\star(\mathcal{I})$, we have $A^\star(\mathcal{I}) = cl(int(A)) = A^\star(\widetilde{\mathcal{I}})$ and each is regular closed, since $A^\star(\widetilde{\mathcal{I}})$ is regular closed [10, Theorem 3.2].

(g) Since $A^\star = cl^\star(A) = A \cup A^\star = A \cup cl(int(A))$ by (f), $A^\star = pcl(A)$.

(h) From (d), $A^\star = (int(A))^\star$. Let $B = (int(A))^\star$. Then $(int(B))^\star = (int(int(A))^\star)^\star = (int(A^\star))^\star \supset (int(A))^\star = B$, since $A$ is $\star$-dense in itself. Therefore, $B \subset (int(B))^\star$. Also, $int(B) \subset B$ implies that $(int(B))^\star \subset B^\star = ((int(A))^\star)^\star \subset (int(A))^\star = B$ and so $(int(B))^\star \subset B$. Therefore, $B = (int(B))^\star$ which implies that $B$ is regular $\mathcal{I}$-closed. Therefore, $A^\star$ is regular $\mathcal{I}$-closed.

(ii) The proof follows from Lemma 1.4. ∎

The following Theorems 2.2 and 2.4 give characterizations of codense ideals in terms of $\mathcal{A}_{\mathcal{I}}$-sets.

THEOREM 2.2. *Let* $(X, \tau, \mathcal{I})$ *be an ideal space. Then* $\mathcal{I}$ *is codense if and only if* $\tau \subset \mathcal{A}_{\mathcal{I}}(X, \tau)$.

*Proof.* If $\mathcal{I}$ is codense, by Proposition 4(a) of [11], $\tau \subset \mathcal{A}_{\mathcal{I}}(X, \tau)$. Conversely, suppose the condition holds. By Theorem 2.1(ii), $\mathcal{A}_{\mathcal{I}}(X, \tau) \cap \mathcal{I} = \{\emptyset\}$ and so $\tau \cap \mathcal{I} = \{\emptyset\}$. Therefore, $\mathcal{I}$ is codense. ■

COROLLARY 2.3. *Let* $(X, \tau, \mathcal{I})$ *be an ideal space. Then the following are equivalent.*

(a) $\mathcal{I}$ *is codense.*
(b) $\tau = P_{\mathcal{I}}O(X, \tau) \cap \mathcal{A}_{\mathcal{I}}(X, \tau)$.
(c) $\tau = \alpha_{\mathcal{I}}O(X, \tau) \cap \mathcal{A}_{\mathcal{I}}(X, \tau)$.
(d) $\tau \subset \mathcal{A}_{\mathcal{I}}(X, \tau)$.

*Proof.* (a) implies (b) and (a) implies (c) follow from Proposition 6 of [11]. It is clear that (b) implies (d) and (c) implies (d). (d) implies (a) by Theorem 2.2. ■

Every $\mathcal{A}_{\mathcal{I}}$-set is an $\mathcal{A}$-set [11, Proposition 5(b)] but not the converse [11, Example 5(3)]. Theorem 2.5 below shows that these two collection of sets are equal, if the ideal is codense and also it gives another characterization of codense ideals in terms of $\mathcal{A}_{\mathcal{I}}$-sets. Before that, we prove the following Theorem 2.4 which gives a characterization of codense ideals in terms of regular $\mathcal{I}$-closed sets.

THEOREM 2.4. *Let* $(X, \tau, \mathcal{I})$ *be an ideal space. Then* $\mathcal{I}$ *is codense if and only if* $\mathcal{R}_{\mathcal{I}}C(X, \tau) = RC(X, \tau)$ *where* $\mathcal{R}_{\mathcal{I}}C(X, \tau)$ *is the collection of all regular* $\mathcal{I}$*-closed sets in* $(X, \tau, \mathcal{I})$.

*Proof.* Suppose $\mathcal{I}$ is codense. Then $A \in \mathcal{R}_{\mathcal{I}}C(X, \tau)$ if and only if $A = (int(A))^\star$ if and only if $A = cl(int(A))$, by Lemma 1.1(c) and Lemma 1.2, if and only if $A \in RC(X, \tau)$. Conversely, suppose $\mathcal{R}_{\mathcal{I}}C(X, \tau) = RC(X, \tau)$. Since $X$ is regular closed, $X$ is regular $\mathcal{I}$-closed and so $X = (int(X))^\star = X^\star$ which implies that $\mathcal{I}$ is codense, by Lemma 1.1(b). ■

THEOREM 2.5. *Let* $(X, \tau, \mathcal{I})$ *be an ideal space. Then* $\mathcal{I}$ *is codense if and only if* $\mathcal{A}_{\mathcal{I}}(X, \tau) = \mathcal{A}(X, \tau)$ *where* $\mathcal{A}(X, \tau)$ *is the collection of all* $\mathcal{A}$*-sets in* $(X, \tau)$.

*Proof.* Suppose $\mathcal{I}$ is codense. $\mathcal{A}_{\mathcal{I}}(X, \tau) \subset \mathcal{A}(X, \tau)$ by [11, Proposition 5(b)]. On the other hand, $A \in \mathcal{A}(X, \tau)$ implies that $A = U \cap V$ where $U \in \tau$ and $V \in RC(X, \tau)$ and so $A = U \cap V$ where $U \in \tau$ and $V \in \mathcal{R}_{\mathcal{I}}C(X, \tau)$, by Theorem 2.4. So, $A \in \mathcal{A}_{\mathcal{I}}(X, \tau)$. Hence $\mathcal{A}_{\mathcal{I}}(X, \tau) = \mathcal{A}(X, \tau)$. Conversely, suppose $\mathcal{A}_{\mathcal{I}}(X, \tau) = \mathcal{A}(X, \tau)$. Since $X$ is an $\mathcal{A}$-set, $X$ is an $\mathcal{A}_{\mathcal{I}}$-set and so $X \subset X^\star$, by Theorem 2.1(a). Therefore $X = X^\star$ which implies that $\mathcal{I}$ is codense. ■

A function $f: (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be $\mathcal{A}_{\mathcal{I}} -$ *continuous* [11] (resp. $\mathcal{A}$-*continuous* [18]) if $f^{-1}(V)$ is an $\mathcal{A}_{\mathcal{I}}$-set(resp. $\mathcal{A}$-set) in $X$ for every open set $V$ in $Y$. Every $\mathcal{A}_{\mathcal{I}}$-continuous function is $\mathcal{A}$-continuous [11, Proposition 7(c)] but not the converse [11, Example 6(3)]. The following Theorem 2.6 shows that the two concepts are equivalent, if the ideal $\mathcal{I}$ is codense.

THEOREM 2.6. *Let $f\colon(X,\tau,\mathcal{I})\to(Y,\sigma)$ be a mapping and $\mathcal{I}$ be codense. Then $f$ is $\mathcal{A}_{\mathcal{I}}$-continuous if and only if $f$ is $\mathcal{A}$-continuous.*

*Proof.* The proof follows from Theorem 2.5. ■

Every $\mathcal{A}_{\mathcal{I}}$-set is $\mathcal{I}$-locally closed [11, Proposition 5(a)] but not the converse [11, Example 5(2)]. The following Theorem 2.7 shows that every $\mathcal{A}_{\mathcal{I}}$-set is an $f_{\mathcal{I}}$-set and characterizes $\mathcal{A}_{\mathcal{I}}$-set in terms of $f_{\mathcal{I}}$-set and $\mathcal{I}$-locally closed set. Example 2.8 below shows that $f_{\mathcal{I}}$-sets need not be $\mathcal{A}_{\mathcal{I}}$-sets.

THEOREM 2.7. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then $A$ is an $\mathcal{A}_{\mathcal{I}}$-set if and only if $A$ is both an $f_{\mathcal{I}}$-set and an $\mathcal{I}$-locally closed set.*

*Proof.* Suppose $A$ is an $\mathcal{A}_{\mathcal{I}}$-set. Then $A$ is $\mathcal{I}$-locally closed by [11, Proposition 5(a)]. Also, $A=U\cap V$ where $U\in\tau$ and $V\in\mathcal{R}_{\mathcal{I}}C(X,\tau)$ and so $int(A)=int(U\cap V)=U\cap int(V)$. Now $A=U\cap V$ implies that $A=U\cap(int(V))^\star\subset(U\cap int(V))^\star=(int(A))^\star$. Therefore, $A$ is an $f_{\mathcal{I}}$-set. Conversely, suppose $A$ is both an $f_{\mathcal{I}}$-set and an $\mathcal{I}$-locally closed set. Since $A$ is an $f_{\mathcal{I}}$-set, $A\subset(int(A))^\star$ implies that $A^\star\subset((int(A))^\star)^\star\subset(int(A))^\star\subset A^\star$ and so $A^\star=(int(A))^\star$. As in the proof of Theorem 2.1(h), we can prove that $A^\star$ is regular $\mathcal{I}$-closed. $A$ is $\mathcal{I}$-locally closed implies that $A=U\cap A^\star$ for some $U\in\tau$, by Lemma 1.3. Since $A^\star$ is regular $\mathcal{I}$-closed, it follows that $A$ is an $\mathcal{A}_{\mathcal{I}}$-set. ■

EXAMPLE 2.8. Let $X=\{a,b,c,d\}$, $\tau=\{\emptyset,\{a\},\{b\},\{a,b\},\{a,b,d\},X\}$ and $\mathcal{I}=\{\emptyset,\{c\}\}$. If $A=\{a,c\}$, then $(int(A))^\star=\{a,c,d\}$ and so $A$ is an $f_{\mathcal{I}}$-set. Since $X$ is the only open containing $A$ and $A$ is not regular $\mathcal{I}$-closed, $A$ is not an $\mathcal{A}_{\mathcal{I}}$-set.

A function $f\colon(X,\tau,\mathcal{I})\to(Y,\sigma)$ is said to be *$f_{\mathcal{I}}$-continuous* [12] (resp. *$\mathcal{I}LC$-continuous* [3], *semicontinuous* [14], *$LC$-continuous* [5]) if $f^{-1}(V)$ is an $f_{\mathcal{I}}$-set (resp. $\mathcal{I}$-locally closed set, semiopen set, locally closed set) in $X$ for every open set $V$ in $Y$. Every $\mathcal{A}_{\mathcal{I}}$-continuous function is $\mathcal{I}LC$-continuous [11, Proposition 7(b)] but not the converse [11, Example 6(2)]. The following Theorem 2.9 shows that the converse is true, if $f$ is $f_{\mathcal{I}}$-continuous and hence we have a decomposition of $\mathcal{A}_{\mathcal{I}}$-continuous functions.

THEOREM 2.9. *A function $f\colon(X,\tau,\mathcal{I})\to(Y,\sigma)$ is $\mathcal{A}_{\mathcal{I}}$-continuous if and only if $f$ is both $f_{\mathcal{I}}$-continuous and $\mathcal{I}LC$-continuous.*

*Proof.* The proof follows from Theorem 2.7. ■

THEOREM 2.10. *Let $f\colon(X,\tau,\mathcal{I})\to(Y,\sigma)$ be a mapping and $\mathcal{I}$ be codense. Then the following are equivalent.*

(a) *$f$ is $\mathcal{A}$-continuous.*
(b) *$f$ is $\mathcal{A}_{\mathcal{I}}$-continuous.*
(c) *$f$ is both $f_{\mathcal{I}}$-continuous and $\mathcal{I}LC$-continuous.*
(d) *$f$ is both semicontinuous and $LC$-continuous.*

*Proof.* (a) and (b) are equivalent, by Theorem 2.5. (b) and (c) are equivalent, by Theorem 2.9. (d) and (a) are equivalent by [5, Theorem 4(i)]. The proof will be over, if we prove (c) implies (d). From Lemma 1.3, it follows that every $\mathcal{I}$-locally closed set in $(X, \tau, \mathcal{I})$ is locally closed. Suppose $A$ is an $f_{\mathcal{I}}$-set. Then $A \subset (int(A))^\star = cl(int(A))$ and so $A \in SO(X, \tau)$. This completes the proof. ∎

The following Theorem 2.11 shows that the three collection of sets namely, $f_{\mathcal{I}}$-sets, $\mathcal{A}_{\mathcal{I}}$-sets and $\mathcal{I}$-locally closed sets coincide for the collection of open sets. The Example 2.12 below show that the condition *open* cannot be dropped.

THEOREM 2.11. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset X$ be open. Then the following hold.*

(a) *$A$ is an $f_{\mathcal{I}}$-set if and only if $A$ is an $\mathcal{A}_{\mathcal{I}}$-set.*

(b) *$A$ is an $\mathcal{I}$-locally closed set if and only if $A$ is an $\mathcal{A}_{\mathcal{I}}$-set.*

*Proof.* (a) Suppose $A$ is an open, $f_{\mathcal{I}}$-set. Then $A \subset (int(A))^\star \subset A^\star$ and so $A^\star = (int(A))^\star$ which implies that $A^\star$ is regular $\mathcal{I}$-closed. Since $A = A \cap A^\star$, it follows that $A$ is an $\mathcal{A}_{\mathcal{I}}$-set. Conversely, if $A$ is an $\mathcal{A}_{\mathcal{I}}$-set, by Theorem 2.7, $A$ is an $f_{\mathcal{I}}$-set.

(b) Suppose $A$ is an open, $\mathcal{I}$-locally closed set. Then $A = G \cap A^\star$ for some $G \in \tau$, by Lemma 1.3 and so $A \subset A^\star$. We prove that $A^\star$ is regular $\mathcal{I}$-closed. Since $int(A^\star) \subset A^\star$, we have $(int(A^\star))^\star \subset (A^\star)^\star \subset A^\star$. Therefore, $(int(A^\star))^\star \subset A^\star$. On the other hand, since $A$ is open and $\star$-dense in itself, $A^\star = (int(A))^\star \subset (int(A^\star))^\star$ and so $A^\star = (int(A^\star))^\star$ which implies that $A^\star$ is regular $\mathcal{I}$-closed. Therefore, $A$ is an $\mathcal{A}_{\mathcal{I}}$-set. Conversely, if $A$ is an $\mathcal{A}_{\mathcal{I}}$-set, by Theorem 2.7, $A$ is an $\mathcal{I}$-locally closed set. ∎

EXAMPLE 2.12. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. If $A = \{a, c\}$, then $int(A) = \{a\}$ and $(int(A))^\star = \{a, c, d\}$ and so $A$ is an $f_{\mathcal{I}}$-set which is not open. Since $X$ is the only open set containing $A$ and $\mathcal{R}_{\mathcal{I}}C(X, \tau) = \{\emptyset, \{a, c, d\}, \{b, c, d\}, X\}$, $A$ is not an $\mathcal{A}_{\mathcal{I}}$-set. If $B = \{d\}$, then $B^\star = \{c, d\}$ and $B = \{a, b, d\} \cap B^\star$ and so $B$ is $\mathcal{I}$-locally closed which is not open. Since $X$ and $\{a, b, d\}$ are the only open sets containing $B$, it follows that $B$ is not an $\mathcal{A}_{\mathcal{I}}$-set.

## 3. $\mathcal{I}$-locally closed sets

In this section, we characterize codense ideals in terms of $\mathcal{I}$-locally closed sets. Theorem 3.3 gives a decomposition of continuity. We deduce some results established in [5] as corollaries to Theorem 3.3.

THEOREM 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $\mathcal{I}LC(X, \tau) \cap \mathcal{I} = \{\emptyset\}$.*

*Proof.* Let $A \in \mathcal{I}LC(X, \tau)$. Then by Lemma 1.3, $A \subset A^\star$ and so by Lemma 1.4, $\mathcal{I}LC(X, \tau) \cap \mathcal{I} = \{\emptyset\}$. ∎

THEOREM 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following are equivalent.*

(a) $\mathcal{I}$ *is codense.*
(b) $\tau = P_{\mathcal{I}}O(X,\tau) \cap \mathcal{I}LC(X,\tau)$.
(c) $\tau = \alpha_{\mathcal{I}}O(X,\tau) \cap \mathcal{I}LC(X,\tau)$.
(d) $\tau \subset \mathcal{I}LC(X,\tau)$.

*Proof.* (a) implies (b) follows from [3, Proposition 4.1]. (b) implies (d) and (c) implies (d) are clear. (d) implies (a) follows from Theorem 3.1. Therefore, the proof will be over, if we prove (a) implies (c). Suppose $\mathcal{I}$ is codense. If $A$ is open, then $A$ is $\alpha - \mathcal{I}$-open and $A \subset A^\star$. By Lemma 1.3, it follows that $A$ is $\mathcal{I}$-locally closed. Conversely, suppose $A$ is both $\alpha - \mathcal{I}$-open and $\mathcal{I}$-locally closed. $A$ is $\mathcal{I}$-locally closed implies $A = U \cap A^\star$ for some open set $U$. $A$ is $\alpha - \mathcal{I}$-open implies $A \subset int(cl^\star(int(A))) \subset int(cl^\star(A)) = int(cl^\star(U \cap A^\star)) \subset int(cl^\star(A^\star)) = int(A^\star)$. Since $A \subset U$, $A \subset U \cap int(A^\star) = int(U \cap A^\star) = int(A)$ and so $A$ is open. This completes the proof of the theorem. ∎

A function $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is said to be *$\alpha - \mathcal{I}$-continuous* [6] (resp. *pre-$\mathcal{I}$-continuous* [3], *$\alpha$-continuous* [16], *pre-continuous* [15]) if $f^{-1}(V)$ is an $\alpha - \mathcal{I}$-open (resp. *pre*-$\mathcal{I}$-open, $\alpha$-open, preopen) set in $X$ for every open set $V$ in $Y$. The following Theorem 3.3, which is a decomposition of continuous function in ideal topological spaces, follows from Theorem 3.2.

THEOREM 3.3. *Let $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ be a mapping and $\mathcal{I}$ be codense. Then the following are equivalent.*
(a) *$f$ is continuous.*
(b) *$f$ is $\alpha$-$\mathcal{I}$-continuous and $\mathcal{I}LC$-continuous.*
(c) *$f$ is pre-$\mathcal{I}$-continuous and $\mathcal{I}LC$-continuous [3, Theorem 4.3].*

COROLLARY 3.4. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be a mapping. Then the following are equivalent.*
(a) *$f$ is continuous.*
(b) *$f$ is $\alpha$-continuous and $LC$-continuous [5, Theorem 4(ii)].*
(c) *$f$ is pre-continuous and $LC$-continuous [5, Theorem 4(iv)].*

*Proof.* Suppose $\mathcal{I} = \{\emptyset\}$ in Theorem 3.3. If $\mathcal{I} = \{\emptyset\}$, then $\alpha$-open sets coincide with $\alpha - \mathcal{I}$-open sets, preopen sets coincide with pre-$\mathcal{I}$-open sets and locally closed sets coincide with $\mathcal{I}$-locally closed sets. Hence the proof follows from Theorem 3.3. ∎

COROLLARY 3.5. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be a mapping. Then the following are equivalent.*
(a) *$f$ is continuous.*
(b) *$f$ is $\alpha$-continuous and $\mathcal{A}$-continuous.*
(c) *$f$ is pre-continuous and $\mathcal{A}$-continuous [5, Theorem 4(v)].*

*Proof.* Suppose $\mathcal{I} = \mathcal{N}$ in Theorem 3.3. If $\mathcal{I} = \mathcal{N}$, then $\alpha$-open sets coincide with $\alpha$-$\mathcal{I}$-open sets, preopen sets coincide with pre-$\mathcal{I}$-open sets and $\mathcal{A}$-sets coincide with $\mathcal{I}$-locally closed sets [3]. Hence the proof follows from Theorem 3.3. ∎

## 4. Almost strong $\mathcal{I}$-open sets

A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be *almost strong $\mathcal{I}$-open* [7] if $A \subset cl^\star(int(A^\star))$. Every $\mathcal{I}$-open set is an almost strong $\mathcal{I}$-open set but not the converse [7]. We will denote the family of all almost strong $\mathcal{I}$-open sets by $as\mathcal{I}O(X, \tau)$. The following Theorem 4.1 gives some properties of almost strong $\mathcal{I}$-open sets.

THEOREM 4.1. (i) *If $A$ is an almost strong $\mathcal{I}$-open set of an ideal space $(X, \tau, \mathcal{I})$, then the following hold:*

(a) *$A$ is $\star$-dense in itself.*

(b) $A^\star = cl(A) = cl^\star(A)$.

(c) $A^\star(\mathcal{I}) = (cl^\star(int(A^\star)))^\star = (cl(int(A^\star)))^\star = (A^\star)^\star = (A^\star(\widetilde{\mathcal{I}}))^\star(\mathcal{I})$.

(d) *$A^\star$ is $\star$-perfect, regular closed and $\mathcal{I}$-locally closed.*

(e) $A^\star = A^\star(\widetilde{\mathcal{I}})$.

(f) *$(cl^\star(int(A^\star)))^\star$ is $\star$-perfect and $\mathcal{I}$-locally closed.*

(ii) *In any ideal space $(X, \tau, \mathcal{I})$, $as\mathcal{I}O(X, \tau) \cap \mathcal{I} = \{\emptyset\}$.*

*Proof.* (i)(a) Since $A \subset cl^\star(int(A^\star)) \subset cl(int(A^\star)) \subset cl(A^\star) = A^\star$, $A$ is $\star$-dense in itself.

(b) Follows from Lemma 1.2 and (a).

(c) Follows from the inequality in (a) and the fact that $cl(int(A^\star)) = A^\star(\widetilde{\mathcal{I}})$ [10, Theorem 3.2].

(d) $A^\star$ is $\star$-perfect by (c) and hence it is $\mathcal{I}$-locally closed. Since every almost strong $\mathcal{I}$-open set is $\beta$-open [7] and the closure of a $\beta$-open set is regular closed, by (b), $A^\star$ is regular closed.

(e) Since $A^\star$ is regular closed, by (d), we have $A^\star = cl(int(A^\star)) = A^\star(\widetilde{\mathcal{I}})$.

(f) Since $A \subset cl^\star(int(A^\star)) \subset A^\star$, it follows that $cl^\star(int(A^\star))$ is $\star$-perfect and hence it is $\mathcal{I}$-locally closed.

(ii) Follows from Lemma 1.4. ■

COROLLARY 4.2. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $\mathcal{I}$ be codense. If $A \subset X$ is almost strong $\mathcal{I}$-open, then $A^\star$ is regular $\mathcal{I}$-closed and an $f_\mathcal{I}$-set. Moreover, if $A$ is $\star$-closed, then $A$ is regular $\mathcal{I}$-closed and an $f_\mathcal{I}$-set.*

*Proof.* By Theorem 4.1(d), $A^\star$ is regular closed. By Theorem 2.4, $A^\star$ is regular $\mathcal{I}$-closed and hence an $f_\mathcal{I}$-set. If $A$ is $\star$-closed, then $A$ is $\star$-perfect and so $A = A^\star$. Therefore, $A$ is regular $\mathcal{I}$-closed and hence $A$ is an $f_\mathcal{I}$-set. ■

The following Theorem 4.3 gives a characterization of codense ideals in terms of almost strong $\mathcal{I}$-open sets. Theorem 4.4 below gives another property of almost strong $\mathcal{I}$-open sets.

THEOREM 4.3. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $\mathcal{I}$ is codense if and only if $S\mathcal{I}O(X, \tau) \subset as\mathcal{I}O(X, \tau)$.*

*Proof.* Suppose $\mathcal{I}$ is codense. If $A \in S\mathcal{I}O(X, \tau)$, then $A \subset cl^\star(int(A)) \subset cl(int(A))$ and so $A$ is semiopen. By Lemma 1.1(d), $A \subset A^\star$. Therefore, $A \subset$

$cl^\star(int(A^\star))$ and so $A \in as\mathcal{I}O(X,\tau)$. Conversely, suppose the condition holds. Since $\tau \subset S\mathcal{I}O(X,\tau) \subset as\mathcal{I}O(X,\tau)$, by Theorem 4.1(ii), $\tau \cap \mathcal{I} = \{\emptyset\}$ and so $\mathcal{I}$ is codense. ■

Theorem 4.4. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A$ and $B$ be subsets of $X$ such that $A \subset B \subset A^\star$. If $A$ is almost strong $\mathcal{I}$-open, then $B$ is almost strong $\mathcal{I}$-open and so $cl^\star(int(A^\star))$ is almost strong $\mathcal{I}$-open.*

*Proof.* If $A \subset B \subset A^\star$, then $A^\star \subset B^\star \subset (A^\star)^\star \subset A^\star$ and so $A^\star = B^\star$ which implies that $B$ is $\star$-dense in itself and $B^\star$ is $\star$-perfect. If $A \in as\mathcal{I}O(X,\tau)$, then $A \subset cl^\star(int(A^\star)) = cl^\star(int(B^\star))$. Now $B \subset A^\star$ implies $B \subset (cl^\star(int(B^\star)))^\star \subset cl^\star(cl^\star(int(B^\star))) = cl^\star(int(B^\star))$ and so $B$ is an almost strong $\mathcal{I}$-open set. Since $A \subset cl^\star(int(A^\star)) \subset A^\star$, $cl^\star(int(A^\star))$ is an almost strong $\mathcal{I}$-open set. ■

We define the *almost strong $\mathcal{I}$-interior* of any subset $A$ of $X$ as the largest almost strong $\mathcal{I}$-open set contained in $A$ and denote it by $as\mathcal{I}int(A)$. The following Theorem 4.5 deals with the almost strong $\mathcal{I}$-interior of subsets of $X$. Moreover, Theorem 4.5(b) is a generalization of Theorem 4.1(4) of [10].

Theorem 4.5. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then the following hold.*
(a) *$as\mathcal{I}int(A) = A \cap cl^\star(int(A^\star))$.*
(b) *$as\mathcal{I}int(A) = \emptyset$ if and only if $A \in \widetilde{\mathcal{I}}$.*

*Proof.* (a) $A \cap cl^\star(int(A^\star)) \subset cl^\star(int(A^\star)) = cl^\star(int(int(A^\star))) = cl^\star(int(A^\star \cap int(A^\star))) \subset cl^\star(int(A \cap int(A^\star))^\star) \subset cl^\star(int(A \cap cl^\star(int(A^\star)))^\star)$. Therefore, $A \cap cl^\star(int(A^\star))$ is an almost strong $\mathcal{I}$-open set contained in $A$. Hence $A \cap cl^\star(int(A^\star)) \subset as\mathcal{I}int(A)$. Since $as\mathcal{I}int(A)$ is almost strong $\mathcal{I}$-open, $as\mathcal{I}int(A) \subset cl^\star(int(as\mathcal{I}int(A))^\star) \subset cl^\star(int(A^\star))$ and so $A \cap as\mathcal{I}int(A) \subset A \cap cl^\star(int(A^\star))$ which implies that $as\mathcal{I}int(A) \subset A \cap cl^\star(int(A^\star))$. Therefore, $as\mathcal{I}int(A) = A \cap cl^\star(int(A^\star))$.

(b) $as\mathcal{I}int(A) = \emptyset$ implies $A \cap cl^\star(int(A^\star)) = \emptyset$ implies $A \cap int(A^\star) = \emptyset$ implies $\mathcal{I}int(A) = \emptyset$ implies $A \in \widetilde{\mathcal{I}}$, by [10, Theorem 4.1(4)]. Conversely, $A \in \widetilde{\mathcal{I}}$ implies $int(A^\star) = \emptyset$ implies $cl^\star(int(A^\star)) = \emptyset$ implies $A \cap cl^\star(int(A^\star)) = \emptyset$ implies $as\mathcal{I}int(A) = \emptyset$. ■

In [7], it is established that the intersection of an almost strong $\mathcal{I}$-open set with an open set is always an almost strong $\mathcal{I}$-open set. The following Theorem 4.6 shows that, in the above result, open set can be replaced by $\alpha$-$\mathcal{I}$-open set.

Theorem 4.6. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If $A$ is almost strong $\mathcal{I}$-open and $B$ is $\alpha$-$\mathcal{I}$-open, then $A \cap B$ is almost strong $\mathcal{I}$-open.*

*Proof.* $A$ is almost strong $\mathcal{I}$-open implies $A \subset cl^\star(int(A^\star))$ and $B$ is $\alpha$-$\mathcal{I}$-open implies $B \subset int(cl^\star(int(B)))$. Now, $A \cap B \subset cl^\star(int(A^\star)) \cap int(cl^\star(int(B))) = (int(A^\star) \cup (int(A^\star))^\star) \cap int(cl^\star(int(B))) = (int(A^\star) \cap int(cl^\star(int(B)))) \cup ((int(A^\star))^\star \cap int(cl^\star(int(B)))) \subset int(int(A^\star) \cap cl^\star(int(B))) \cup (int(A^\star) \cap int(cl^\star(int(B))))^\star \subset int(cl^\star(int(A^\star) \cap int(B))) \cup (int(int(A^\star) \cap cl^\star(int(B))))^\star \subset int(cl^\star(int(A^\star \cap int(B)))) \cup (int(cl^\star(int(A^\star) \cap int(B))))^\star \subset int(cl^\star(int((A \cap int(B))^\star))) \cup (int(cl^\star(int(A^\star \cap

$int(B)))))^\star \subset int(cl^\star(int((A \cap int(B))^\star))) \cup (int(cl^\star(int((A \cap int(B))^\star))))^\star =$
$cl^\star(int(cl^\star(int((A \cap int(B))^\star)))) \subset cl^\star(int(cl^\star((A \cap int(B))^\star))) =$
$cl^\star(int((A \cap int(B))^\star)) \subset cl^\star(int((A \cap B)^\star))$. Therefore, $A \cap B$ is almost strong $\mathcal{I}$-open. ■

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