МАТЕМАТИЧКИ ВЕСНИК
60 (2008), 155–163

UDK 512.558
оригинални научни рад
research paper

# GENERALIZED COHERENT RINGS BY GORENSTEIN PROJECTIVE DIMENSION

**Linlin Li and Jiaqun Wei**

**Abstract.** In this paper, we introduce a new generalization of coherent rings using the Gorenstein projective dimension. Let $n$ be a positive integer or $n = \infty$. A ring $R$ is called a left $G_n$-coherent ring in case every finitely generated submodule of finitely generated free left $R$-modules whose Gorenstein projective dimension $\leq n-1$ is finitely presented. We characterize $G_n$-coherent rings in various ways, using $G_n$-flat, $G_n$-injective modules and cotorsion theory.

## 1. Introduction

A ring $R$ is called left coherent if every finitely generated left ideal of $R$ is finitely presented. Recently, various generalizations of coherent rings were studied in [1, 9, 19] etc. Among them, Lee [19] introduced the class of left $n$-coherent rings: a ring is said to be left $n$-coherent, provided every finitely generated submodule of a finitely generated free left module whose projective dimension (denoted by pd) $\leq n-1$ is finitely presented. It is easy to see that all rings are left 1-coherent and left $\infty$-coherent rings are just left coherent rings. Many characterizations of left coherent rings were successfully extended to left $n$-coherent rings in [19].

On the other hand, as a natural refinement of the projective dimension, the Gorenstein projective dimension, originally from the G-dimension in Auslander and Bridger[15], was now studied extensively, especially over commutative algebras and $n$-Gorenstein rings(see [6, 8] and their references). Following [8] a left module $M$ is called Gorenstein projective, if $M$ is an image of a homomorphism in a complete projective resolution, i.e., there exists an infinite exact sequence $\cdots \to P_{-1} \xrightarrow{f_{-1}} P_0 \xrightarrow{f_0} P_1 \xrightarrow{f_1} \cdots$ $(*)$, with each $P_i$ projective, such that $\mathrm{Hom}_R(*, P)$ is exact for all projective modules $P$ and $M \cong \mathrm{Im}(f_i)$ for some $i$. A left module $M$ is called Gorenstein projective dimension $\leq n$ (denoted by $\mathrm{Gpd}\,(M) \leq n$), if there is an exact sequence $0 \to G_n \to \cdots \to G_0 \to M \to 0$ with each $G_i$ Gorenstein projective. If no such exact sequence exists, then denote $\mathrm{Gpd}\,(M) = \infty$. It is easy to see that all projective modules are Gorenstein projective. Since the projective resolution

*AMS Subject Classification*: 16E99
*Keywords and phrases*: Gorenstein projective dimension, coherent, cotorsion theory.

exists for all modules, so does the Gorenstein projective resolution. Also it is easy to see that $\mathrm{pd}\, M \leq n$ implies $\mathrm{Gpd}\,(M) \leq n$. The converse does not hold in general. To see this, consider modules over a QF ring which is not semisimple. Then all modules are Gorenstein projective, while all non-projective modules have projective dimension $\infty$.

In view of the above observations, it is then a question if we can study the generalizations of coherency by the Gorenstein projective dimension, using the idea of defining left $n$-coherent ring in [19]. We introduce here left $G_n$-coherent rings: a ring $R$ is called left $G_n$-coherent if every finitely generated submodule of a finitely generated free module whose Gorenstein projective dimension is $\leq n-1$, is finitely presented. Note left coherent rings are just left $G_\infty$-coherent and left $G_n$-coherent rings are left $n$-coherent rings in [19]. However, as contrast to left $n$-coherent rings in [19], we do not know if all rings are left $G_1$-coherent, since it is an open question whether a finitely generated Gorenstein projective module is finitely presented see [17], while a finitely generated projective module is always finitely presented. Because of the differences between Gorenstein projective dimension and projective dimension, we cannot yet expect all left $n$-coherent rings are left $G_n$-coherent rings. Thus the left $G_n$-coherent rings lie between left coherent rings and left $n$-coherent rings.

The aim of the paper is to extend characterizations of left coherent rings to left $G_n$-coherent rings. To this end, $G_n$-injective and $G_n$-flat modules are introduced and their properties are studied in section 3. In particular, we show that $({}^{\perp}\mathcal{I}_n, \mathcal{I}_n)$ is a complete cotorsion theory and $(\mathcal{F}_n, \mathcal{F}_n^{\perp})$ is a perfect cotorsion theory, where $\mathcal{I}_n$ denotes the class of all $G_n$-injective left $R$-modules and $\mathcal{F}_n$ denotes the class of all $G_n$-flat right $R$-modules. Then, in section 4, we extend successfully characterizations of left coherent rings to left $G_n$-coherent rings. Especially, we show that a ring is left $G_n$-coherent if and only if any direct product of $R$ has a right $R$-module is $G_n$-flat, if and only if every right $R$-module has an $\mathcal{F}_n$-preenvelope, if and only if $({}^{\perp}\mathcal{I}_n, \mathcal{I}_n)$ is a hereditary cotorsion theory, if and only if $\mathcal{I}_n$ is a coresolving subcategory.

## 2. Notations

Throughout this paper, all rings are associative with identity, all left modules are unitary and all modules without explicit mentions will always mean left modules. In this section we recall some known notions and definitions needed in the sequel. Let $R$ be a ring. A left $R$-module $M$ is called $n$-presented if it has a finite $n$-presentation, i.e., there is an exact sequence $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ where every $F_i$ is finitely generated free left $R$-module.

Let $\mathcal{C}$ be a class of $R$-modules and $M$ an $R$-module. Following [2], we say that a homomorphism $\phi\colon M \to C$ is a $\mathcal{C}$-preenvelope if $C \in \mathcal{C}$ and the abelian group homomorphism $\mathrm{Hom}_R(\phi, C')\colon \mathrm{Hom}_R(C, C') \to \mathrm{Hom}_R(M, C')$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-preenvelope $\phi\colon M \to C$ is said to be a $\mathcal{C}$-envelope if every endomorphism $g\colon C \to C'$ such that $g\phi = \phi$ is an isomorphism.

Given a class $\mathcal{L}$ of $R$-modules, we will denote by $\mathcal{L}^{\perp} = \{C : \mathrm{Ext}_R^1(L, C) = 0$ for all $L \in \mathcal{L}\}$ the right orthogonal class of $\mathcal{L}$, and by ${}^{\perp}\mathcal{L} = \{C : \mathrm{Ext}_R^1(C, L) = 0$ for all $L \in \mathcal{L}\}$ the left orthogonal class of $\mathcal{L}$. Following [4, Definition 7.1.6], a monomorphism $\alpha\colon M \to C$ with $C \in \mathcal{C}$ is said to be a special $\mathcal{C}$-preenvelope of $M$ if $\mathrm{coker}(\alpha) \in {}^{\perp}\mathcal{C}$. Dually, we have the definitions of a (special) $\mathcal{C}$-precover and a $\mathcal{C}$-cover. Special $\mathcal{C}$-preenvelope (resp. special $\mathcal{C}$-precovers) are obviously $\mathcal{C}$-preenvelopes (resp. $\mathcal{C}$-precovers). $\mathcal{C}$-envelopes ($\mathcal{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

A pair $(\mathcal{F}, \mathcal{C})$ of classes of $R$-modules is called a cotorsion theory [4] if $\mathcal{F}^{\perp} = \mathcal{C}$ and ${}^{\perp}\mathcal{C} = \mathcal{F}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is said to be complete (resp. pefect) if every $R$-module has a special $\mathcal{C}$-preenvelope and a special $\mathcal{F}$-precover (resp. $\mathcal{C}$-envelope and $\mathcal{F}$-cover) (see [5, 11]). A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called hereditary [5] if whenever $0 \to L' \to L \to L'' \to 0$ is exact with $L, L'' \in \mathcal{F}$, then $L'$ is also in $\mathcal{F}$. Note $(\mathcal{F}, \mathcal{C})$ is hereditary if and only if whenever $0 \to C' \to C \to C'' \to 0$ is exact with $C, C' \in \mathcal{C}$, then $C''$ is also in $\mathcal{C}$.

## 3. $G_n$-injective and $G_n$-flat modules

Definition 3.1. Let $R$ be a ring, $n$ a non-negative integer. A left $R$-module $A$ is said to be $G_n$-injective if $\mathrm{Ext}_R^1(N, A) = 0$ for any finitely presented left $R$-module $N$ with $\mathrm{Gpd}\,(N) \leq n$. A right $R$-module $M$ is said to be $G_n$-flat if $\mathrm{Tor}_1^R(M, N) = 0$ for any finitely presented left $R$-module $N$ with $\mathrm{Gpd}\,(N) \leq n$.

Let $\mathcal{I}_n$ denote the class of all $G_n$-injective left $R$-modules and $\mathcal{F}_n$ denote the class of all $G_n$-flat right $R$-modules.

Remark 3.2. (1) Recall that a left $R$-module $A$ is called FP-injective if for all finitely presented left $R$-modules $N$, $\mathrm{Ext}_R^1(N, A) = 0$. Obviously, we have that injective modules $\Rightarrow$ FP-injective modules $\Rightarrow$ $G_n$-injective modules.

(2) flat modules $\Rightarrow$ $G_n$-flat modules.

(3) It is easy to see that $\mathcal{I}_n$ and $\mathcal{F}_n$ are closed under direct summands.

The character module of the left (right) $R$-module $M$ is the right(left) $R$-module $M^+ = \mathrm{Hom}_Z(M, Q/Z)$. A well known theorem of Lambek [13] states that $M$ is a flat right $R$-module exactly if $M^+$ is an injective left $R$-module.

Lemma 3.3. *A right $R$-module $B$ is $G_n$-flat if and only if its character module $B^+$ is a $G_n$-injective left $R$-module.*

*Proof.* By the natural isomorphism

$$\mathrm{Ext}_R^1(A, \mathrm{Hom}_Z(B, Q/Z)) \cong \mathrm{Hom}_Z(\mathrm{Tor}_1^R(B, A), Q/Z),$$

where $A$ is a left $R$-module and $B$ is a right $R$-module.

Proposition 3.4. *Let $R$ be a ring. Then*

(1) *$\mathcal{I}_n$ and $\mathcal{F}_n$ are closed under pure submodules.*

(2) *$\mathcal{I}_n$ is closed under direct products and $\mathcal{F}_n$ is closed under direct sums.*

(3) *$\mathcal{I}_n$ is closed under direct sums.*

(4) *$\mathcal{I}_n$ is closed under extension.*

*Proof.* (1) Let $M$ be a $G_n$-injective module and $M_1$ a pure submodule of $M$. Then we obtain a pure exact sequence $0 \to M_1 \to M \xrightarrow{\pi} M/M_1 \to 0$. Let $N$ be any finitely presented left $R$-module with $\mathrm{Gpd}\,(N) \leq n$. By applying the functor $\mathrm{Hom}_R(N, -)$ to the above sequence, we can get an induced exact sequence $0 \to \mathrm{Hom}_R(N, M_1) \to \mathrm{Hom}_R(N, M) \xrightarrow{\mathrm{Hom}_R(N,\pi)} \mathrm{Hom}_R(N, M/M_1) \to \mathrm{Ext}^1_R(N, M_1) \to \mathrm{Ext}^1_R(N, M)$. By [20, Theorem 4.89], $\mathrm{Hom}_R(N, \pi)$ is epic. Note also $\mathrm{Ext}^1_R(N, M) = 0$ for $M \in \mathcal{I}_n$, so $\mathrm{Ext}^1_R(N, M_1) = 0$. It follows that $M_1 \in \mathcal{I}_n$, by the definition.

Let $M$ be a $G_n$-flat module and $N$ a pure submodule of $M$. Then applying the functor $\mathrm{Hom}_Z(-, Q/Z)$ to the pure exact sequence $0 \to N \to M \to M/N \to 0$, we obtain a split exact sequence $0 \to (M/N)^+ \to M^+ \to N^+ \to 0$ by [20]. Thus $N^+$ is $G_n$-injective since direct summands of $G_n$-injective modules are also $G_n$-injective and $M^+$ is $G_n$-injective by lemma 3.3. So $N$ is $G_n$-flat by Lemma 3.3 again.

(2) follows from the two natural isomorphisms $\mathrm{Ext}^1_R(N, \prod M_i) \cong \prod \mathrm{Ext}^1_R(N, M_i)$ and $\mathrm{Tor}^R_1(\bigoplus M_i, N) \cong \bigoplus \mathrm{Tor}^R_1(M_i, N)$.

(3) Let $M_i$ $(i \in I)$ be a family of $G_n$-injective left $R$-module. For any finitely presented left $R$-module $N$ with $\mathrm{Gpd}\,(N) \leq n$, there is an exact sequence $0 \to H \to F \to N \to 0$, where $F$ is finitely generated free and $H$ is finitely generated. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(F, \bigoplus M_i) & \to & \mathrm{Hom}_R(H, \bigoplus M_i) & \to & \mathrm{Ext}^1_R(N, \bigoplus M_i) & \to & \mathrm{Ext}^1_R(F, \bigoplus M_i) = 0 \\
\alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \\
\bigoplus \mathrm{Hom}_R(F, M_i) & \to & \bigoplus \mathrm{Hom}_R(H, M_i) & \to & \bigoplus \mathrm{Ext}^1_R(N, M_i) & \to & 0
\end{array}$$

Since $\alpha$ and $\beta$ are isomorphisms by [7, Exercise 16.3, p.189] (for $F$ and $H$ are finitely generated), $\gamma$ is an isomorphism by Five Lemma. Thus $\mathrm{Ext}^1_R(N, \bigoplus M_i) \cong \bigoplus \mathrm{Ext}^1_R(N, M_i) = 0$. So $\bigoplus M_i \in \mathcal{I}_n$ .

(4) For any exact sequence $0 \to A \to B \to C \to 0$, where $A$ and $C$ are $G_n$-injective modules, we have an induced exact sequence $\cdots \to \mathrm{Ext}^1_R(N, A) \to \mathrm{Ext}^1_R(N, B) \to \mathrm{Ext}^1_R(N, C) \to \cdots$, where $N$ is finitely presented with $\mathrm{Gpd}\,(N) \leq n$. Because $\mathrm{Ext}^1_R(N, A) = 0 = \mathrm{Ext}^1_R(N, C)$, $\mathrm{Ext}^1_R(N, B) = 0$. Hence $B$ is $G_n$-injective. ■

THEOREM 3.5. *Let $R$ be a ring. Then*

(1) *$({}^{\perp}\mathcal{I}_n, \mathcal{I}_n)$ is a complete cotorsion theory.*

(2) *$(\mathcal{F}_n, \mathcal{F}_n^{\perp})$ is a perfect cotorsion theory.*

*Proof.* (1) Let $\mathcal{X}$ denotes the class of all finitely presented left $R$-module $N$ with $\mathrm{Gpd}\,(N) \leq n$. Note that $\mathcal{I}_n = \mathcal{X}^{\perp}$, so the result follows from [18, Theorem 10] and [4, Definition 7.1.5].

(2) Denote by $\mathcal{B}$ the class of all left $R$-modules $B$ with $\mathrm{Tor}_1^R(N,B)=0$ for all $N\in\mathcal{F}_n$. Then we can known that $N\in\mathcal{F}_n$ if and only if $\mathrm{Tor}_1^R(N,B)=0$ for all $B\in\mathcal{B}$. So (2) follows from [11, Lemma 1.11 and Theorem 2.8].

## 4. $G_n$-coherent rings

Definition 4.1. Let $n$ be a non-negative integer or $n=\infty$. A ring $R$ is called left $G_n$-coherent in case every finitely generated submodule of a finitely generated free left $R$-modules whose Gorenstein projective dimension $\leq n-1$ is finitely presented.

Here we recall the definition of $k$-presentence. Let $R$ be a ring. A left $R$-module $M$ is called $k$-presented if it has a finite $k$-presentation, i.e., there is an exact sequence $F_k\to F_{k-1}\to\cdots\to F_1\to F_0\to M\to 0$ where every $F_i$ is finitely generated free left $R$-module.

Remark 4.2. (1) Left $G_n$-coherent rings are left $n$-coherent rings in [19] and left $G_\infty$-coherent rings are left coherent.

(2) Let $N$ be finitely presented left $R$-module with $\mathrm{Gpd}\,(N)\leq n$. If $R$ is $G_n$-coherent, then $N$ is finitely $k$-presented for any nonnegative $k$.

Theorem 4.3. *For a ring $R$, the following conditions are equivalent.*

(1) *$R$ is left $G_n$-coherent.*

(2) *Direct products of $G_n$-flat right $R$-module are $G_n$-flat.*

(3) *Direct products of copies of $R_R$ are $G_n$-flat.*

(4) $\mathrm{Tor}_m^R(\prod N_\alpha,A)\cong\prod\mathrm{Tor}_m^R(N_\alpha,A)$ *($m$ is any non-negative integer) for any family $N_\alpha$ of right $R$-modules and any finitely presented left $R$-module $A$ with* $\mathrm{Gpd}\,(A)\leq n$.

(5) $\lim_\to\mathrm{Ext}_R^m(A,M_i)\to\mathrm{Ext}_R^m(A,\lim_\to M_i)$ *($m$ is any non-negative integer) is an isomorphism for any finitely presented left $R$-module $A$ with* $\mathrm{Gpd}\,(A)\leq n$.

(6) *Any direct limit of $G_n$-injective left $R$-modules is $G_n$-injective.*

*Proof.* (4) $\Rightarrow$ (2) $\Rightarrow$ (3) and (5) $\Rightarrow$ (6) are obvious.

(3) $\Rightarrow$ (1). Let $H$ be a finitely generated submodule of a finitely generated free module $F$ with $\mathrm{Gpd}\,(H)\leq n-1$. We have an exact sequence $0\to H\to F\to N\to 0$. Then $N$ is finitely presented with $\mathrm{Gpd}\,(N)\leq n$ by [8]. Consider now the following exact commutative diagram:

$$\begin{array}{ccccccccc}
\mathrm{Tor}_1^R(R^I,N) & \longrightarrow & R^I\bigotimes_R H & \longrightarrow & R^I\bigotimes_R F & \longrightarrow & R^I\bigotimes_R N & \longrightarrow & 0\\
 & & f\big\downarrow & & g\big\downarrow & & h\big\downarrow & & \\
0 & \longrightarrow & H^I & \longrightarrow & F^I & \longrightarrow & N^I & \longrightarrow & 0
\end{array}$$

By hypothesis, $R^I$ is an $G_n$-flat right $R$-module, thus $\mathrm{Tor}_1^R(N,R^I)=0$. Since $g$ and $h$ are isomorphisms by [4, Theorem 3.2.22] (for $F$ and $N$ are finitely presented),

$f$ is an isomorphism. Hence, $H$ is finitely presented by [16, Proposition 2.5] and $R$ is $G_n$-coherent by the definition.

(1) $\Rightarrow$ (4). Since $R$ is $G_n$-coherent, every finitely presented left $R$-module $A$ with $\mathrm{Gpd}\,(A) \le n$ is finitely $k$-presented for any $k$. By [9, Lemma 2.10 (2)], we get (4).

(1) $\Rightarrow$ (5). Since $R$ is $G_n$-coherent, every finitely presented left $R$-module $A$ with $\mathrm{Gpd}\,(A) \le n$ is finitely $k$-presented for any $k$. Then we get (5) by [9, Lemma 2.9 (2)].

(6) $\Rightarrow$ (1). Let $H$ be a finitely generated submodule of a finitely generated free module $F$ with $\mathrm{Gpd}\,(H) \le n-1$. We can get an exact sequence $0 \to H \to F \to N \to 0$. Then $N$ is finitely presented with $\mathrm{Gpd}\,(N) \le n$ by [8]. Let $(M_i)_{i\in I}$ be a family of injective left $R$-modules, where $I$ is a directed set. Then $\lim_\to M_i$ is $G_n$-injective by (6). Note that $\mathrm{Ext}^1_R(N, \lim_\to M_i) = 0$, so we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(N, \lim_\to M_i) & \to & \mathrm{Hom}_R(F, \lim_\to M_i) & \to & \mathrm{Hom}_R(H, \lim_\to M_i) & \to & 0 \\
\alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\
\lim_\to \mathrm{Hom}_R(N, M_i) & \to & \lim_\to \mathrm{Hom}_R(F, M_i) & \to & \lim_\to \mathrm{Hom}_R(H, M_i) & \to & 0
\end{array}$$

Since $\alpha$ and $\beta$ are isomorphisms by [16, Proposition 2.5] (for $N$ and $F$ are finitely presented), $\gamma$ is an isomorphism. So $H$ is finitely presented by [16, Proposition 2.5] again. Therefore $R$ is $G_n$-coherent. ∎

Let $\mathfrak{C}$ be a subcategory of category of all $R$-modules. $\mathfrak{C}$ is called coresolving if it satisfies the following conditions: (1) all injective modules are in $\mathfrak{C}$; (2) $\mathfrak{C}$ is closed under cokernal of monomorphism; (3) $\mathfrak{C}$ is closed under extension; (4) $\mathfrak{C}$ is closed under direct summands. Coresolving submodules are of important in many areas(see for instance [7, 14]).

Theorem 4.4 *The following are equivalent for a ring $R$:*

(1) *$R$ is a left $G_n$-coherent ring.*

(2) *Every right $R$-module has an $\mathcal{F}_n$-preenvelope.*

(3) $\mathrm{Ext}^k_R(N, A) = 0$ *($k$ is a positive integer) for any finitely presented left $R$-module $N$ with* $\mathrm{Gpd}\,(N) \le n$ *and any $A \in \mathcal{I}_n$.*

(4) *$\mathcal{I}_n$ is a coresolving subcategory.*

(5) *$({}^\perp\mathcal{I}_n, \mathcal{I}_n)$ is a hereditary cotorsion theory.*

(6) *Quotients of $G_n$-injective left $R$-modules modulo pure submodules are $G_n$-injective.*

(7) *Quotients of injective left $R$-modules modulo pure submodules are $G_n$-injective.*

(8) *A left $R$-module $A$ is $G_n$-injective if and only if its character module $A^+$ is a $G_n$-flat right $R$-module.*

(9) *A left $R$-module $A$ is $G_n$-injective if and only if its double character module $A^{++}$ is a $G_n$-injective left $R$-module.*

(10) *For every injective left $R$-module $E$, the character module $E^+$ is a $G_n$-flat right $R$-module.*

(11) *A right $R$-module $M$ is $G_n$-flat if and only if its double character module $M^{++}$ is a $G_n$-flat right $R$-module.*

*Proof.* (1) $\Rightarrow$ (2). Let $N$ be any right $R$-module. By [4, Lemma 5.3.12], there is an infinite cardinal number $\mathfrak{N}_\alpha$ such that for any $R$-homorphisim $f\colon N \to L$ with $L$ $G_n$-flat, there is a pure submodule $Q$ of $L$ such that $Card(Q) \leq \mathfrak{N}_\alpha$ and $f(N) \subseteq Q$. Note that $Q$ is $G_n$-flat by Proposition 3.4 and $\mathcal{F}_n$ is closed under products by Theorem 4.3, so $N$ has an $\mathcal{F}_n$-preenvelope by [4. Proposition 6.2.1].

(2) $\Rightarrow$ (1). Since $\mathcal{F}_n$ is a preenveloping class and $\mathcal{F}_n$ is closed under direct summands, $\mathcal{F}_n$ is closed under products by [10, Lemma 1], so (1) follows from Theorem 4.3.

(1) $\Rightarrow$ (3). Suppose $R$ is $G_n$-coherent, and $A$ is a $G_n$-injective left $R$-module. For any finitely presented left $R$-module $N$ with $\mathrm{Gpd}\,(N) \leq n$, we have the exact sequence $0 \to H \to F \to N \to 0$, where $F$ is a finitely generated free module and $H$ is finitely generated. By [8], we have $\mathrm{Gpd}\,(H) \leq n-1$, so $H$ is finitely presented by hypothesis. Hence $\mathrm{Ext}^1_R(H,A) = 0$. From the induced exact sequence $\mathrm{Ext}^1_R(H,A) \to \mathrm{Ext}^2_R(N,A) \to \mathrm{Ext}^2_R(F,A) = 0$, we have $\mathrm{Ext}^2_R(N,A) = 0$. Note $H$ is also finitely presented with $\mathrm{Gpd}\,(H) \leq n$, so $\mathrm{Ext}^2_R(H,A) = 0$. From the exact sequence $\mathrm{Ext}^2_R(H,A) \to \mathrm{Ext}^3_R(N,A) \to \mathrm{Ext}^3_R(F,A) = 0$, we have $\mathrm{Ext}^3_R(N,A) = 0$, and so on, by induction, we get the conclusion.

(3) $\Rightarrow$ (4). For any exact sequence $0 \to N \to M \to L \to 0$ with $N \in \mathcal{I}_n$ and $M \in \mathcal{I}_n$, we get the induced exact sequence $0 = \mathrm{Ext}^1_R(A,M) \to \mathrm{Ext}^1_R(A,L) \to \mathrm{Ext}^2_R(A,N$, where $A$ is finitely presented with $\mathrm{Gpd}\,(A) \leq n$. Note $\mathrm{Ext}^2_R(A,N) = 0$ by (3), so $\mathrm{Ext}^1_R(A,L) = 0$. It follows $L \in \mathcal{I}_n$. By the definition of coresolving subcategory, Remark 3.2 (1) and (3) and Proposition 3.4 (4), we can get (4).

(4) $\Rightarrow$ (5) is obvious by definitions.

(5) $\Rightarrow$ (6). Suppose $B$ is a pure submodule of a $G_n$-injective left $R$-module $A$. We have the pure exact sequence $0 \to B \to A \to A/B \to 0$, by (5) and the definition of hereditary cotorsion theory, we get $A/B$ is $G_n$-injective.

(6) $\Rightarrow$ (7) is obvious.

(7) $\Rightarrow$ (3). Take any finitely presented left $R$-module $N$ with $\mathrm{Gpd}(N) \leq n$, and $A \in \mathcal{I}_n$. Evidently, $\mathrm{Ext}^1_R(N,A) = 0$. Consider the exact sequence $0 \to A \to E \to E/A \to 0$ with $E$ is injective, we have the induced exact sequence $\mathrm{Ext}^1_R(N,E/A) \to \mathrm{Ext}^2_R(N,A) \to \mathrm{Ext}^2_R(N,E) = 0$. By hypothesis, $E/A \in \mathcal{I}_n$. So, $\mathrm{Ext}^1_R(N,E/A) = 0$. It follows that $\mathrm{Ext}^2_R(N,A) = 0$. By induction, we can get $\mathrm{Ext}^k_R(N,A) = 0$($k$ is any positive integer).

(3) $\Rightarrow$ (1). Assume every $G_n$-injective left $R$-module $A$ satisfy $\mathrm{Ext}^k_R(N,A) = 0$($k$ is any positive integer), for all finitely presented left $R$-module $N$ with $\mathrm{Gpd}\,(N) \leq n$. Suppose $I$ is a finitely generated submodule of a finitely generated free left $R$-module $F$ with $\mathrm{Gpd}\,(I) \leq n-1$. By the exact sequence $0 \to I \to F \to F/I \to 0$, we have that $F/I$ is finitely presented with $\mathrm{Gpd}\,(F/I) \leq n$ by [8]. Consider the induced exact sequence $0 = \mathrm{Ext}^1_R(F,A) \to \mathrm{Ext}^1_R(I,A) \to \mathrm{Ext}^2_R(F/I,A)$.

By hypothesis $\mathrm{Ext}_R^2(F/I, A) = 0$, hence $\mathrm{Ext}_R^1(I, A) = 0$ for all $G_n$-injective left $R$-module $A$. In particular, $\mathrm{Ext}_R^1(N, A) = 0$ for all FP-injective left $R$-module $A$ by Remark 3.2 (1). As $I$ is also finitely generated, we obtain $I$ is finitely presented by Enochs [3]. It follows $R$ is left $G_n$-coherent by the definition.

(1) $\Rightarrow$ (8). Let $N$ be a finitely presented left $R$-module with $\mathrm{Gpd}\,(N) \leq n$. Since $R$ is a left $G_n$-coherent ring, then $N$ is finitely $k$-presented for any $k$ by Remark 4.2 (2). We can get an isomorphism: $\mathrm{Tor}_n^R(\mathrm{Hom}_Z(B, Q/Z), N) \cong \mathrm{Hom}_Z(\mathrm{Ext}_R^n(N, B), Q/Z)$ by [9]. Now (8) is easily obtained by the above isomorphism.

(8) $\Leftrightarrow$ (9). By Lemma 3.3.

(8) $\Rightarrow$ (11). By assumption and Lemma 3.3.

(11) $\Rightarrow$ (10). Let $E$ be an injective left $R$-module. By Lambek [12], every right (left) $R$-module $M$ embeds as a submodule in the character module of a free left (right) $R$-module, so $E$ is a direct summand of the character module $F^+$ of a free right $R$-module $F$. Then $E^+$ is a direct summand of $F^{++}$. Since $F^{++}$ is $G_n$-flat by hypothesis, $E^+$ is a $G_n$-flat right $R$-module by Remark 3.2 (3).

(10) $\Rightarrow$ (7). Let $E$ be an injective left $R$-module containing a pure submodule $A$. Then the pure exact sequence $0 \to A \to E \to E/A \to 0$ induces the split exact sequence $0 \to (E/A)^+ \to E^+ \to A^+ \to 0$. By hypothesis, $E^+$ is $G_n$-flat, so $(E/A)^+$ is $G_n$-flat by Remark 3.2 (3). Hence $(E/A)^{++}$ is $G_n$-injective by Lemma 3.3. Since $E/A$ is a pure submodule of $(E/A)^{++}$, it follows that $E/A$ is $G_n$-injective by Proposition 3.4. ■

REFERENCES


[**1**] D. X. Zhou, *On n-coherent rings and (n, d)-Rings*, Comm. Algebra **32** (2004), 2425–2441.

[**2**] E. E. Enochs, *Injective and flat covers, envelopes and resolvents*, Israel J. Math. **39** (1981), 189–209.

[**3**] E. E. Enochs, *A note on absolutely pure modules*, Canand. Math. Bull. **19** (1976), 361–362.

[**4**] E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, Walter de Gruyter, Belin-New York, 2000.

[**5**] E. E. Enochs, O. M. G. Jenda and J. A. Lopez-Ramos, *The exisence of Gorenstein flat covers*, Math. Scand. **94** (2004), 46–62.

[**6**] E. E. Enochs and O. M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220** (4) (1995), 611–633.

[**7**] F. W. Anderson and K. R. Fuller, *Rings Categories of Modules*, Springer-Verlag, New York, 1974.

[**8**] H. Holm, *Gorenstein homological dimensions*, J. of Pure and Applied Algebra **189** (2004), 167–193.

[**9**] J. L. Chen and N. Q. Ding, *On n-coherent rings*, Comm. Algebra **24** (1996), 3211–3216.

[**10**] J. L. Chen and N. Q. Ding, *A note on existence of envelopes and covers*, Bull. Austral. Math. Soc. **54** (1996), 383–390.

[**11**] J. Trlifaj, *Covers, envelopes, and cotorsion theories*, Lectures Notes for the Workshop "Homological Methods in Module Theory", Cortona, September 10–16, 2000.

[**12**] J. Lambek, *Lectures on Rings and Modules*, Blaisdell Publ. Co., Waltham, 1966.

[**13**] J. Lambek, *A Module is flat if and only if its character modules are injective*, Canad. Math. Bull. **7** (1964), 237–243.

[**14**] M. Auslander and I. Reiten, *Applications of contravariantly finite subcategories*, Adv. Math. **86** (1991), 221–245.

[**15**] M. Auslander and M. Bridge, *Stable Module Theory*, American Mathematical Society, Providence, RI, 1969, Memories of the American Mathematical Society, No. 94.

[**16**] M. F. Jones, *Coherence relative to a hereditary torsion theory*, Comm. Algebra **10** (1982), 719–739.

[**17**] N. Mahdou and D. Bennis, *Strongly Gorenstein projective, injective, and flat Modules*, J. of Pure and Applied Algebra, in press, avaliable online 29 November 2006.

[**18**] P. C. Eklof and J. Trlifaj, *How to make Ext vanish*, Bull. London Math. Soc. **33** (1) (2001), 41–51.

[**19**] S. B. Lee, *n-coherent rings*, Comm. Algebra **30** (2002), 1119–1126.

[**20**] T. Y. Lam, *Lectures on Modules and Rings*, Press, New York, 1999.

(received 30.03.2007, in revised form 22.07.2008)

Institute of Mathematics, School of Mathematics and Computer Sciences, Nanjing Normal University, Nanjing 210097, China

*E-mail*: `Li13754717926@126.com, weijiaqun@njun.edu.cn`