МАТЕМАТИЧКИ ВЕСНИК
61 (2009), 159–163

UDK 515.124
оригинални научни рад
research paper

ε-APPROXIMATION IN GENERALIZED 2-NORMED SPACES

Mehmet Açıkgöz

Abstract. The notion of generalized 2-normed spaces was introduced by Lewandowska in 1999 [5]. One can obtain a generalized 2-normed space from a normed space. We shall define the notions of 1-type ε-quasi Chebyshev subspaces and give some results in this field.

1. Introduction

The concept of linear 2-normed spaces has been investigated by Gahler in 1965 [3] and has been developed extensively in different subjects by others. Z. Lewandowska published a series of papers on 2-normed sets and generalized 2-normed spaces in 1999–2003 [5]–[9]. There are some works on characterization of 2-normed spaces, extension of 2-functionals and approximation in 2-normed spaces ([1], [2] and [4]). Also Sh. Rezapour has some works in ε-approximation theory [10]–[12].

Let X be a linear space of dimension greater than 1 over K, where K is the real or complex numbers field. Suppose $\|\cdot,\cdot\|$ be a non-negative real-valued function on $X \times X$ satisfying the following conditions:

(i) $\|x,y\| = 0$ if and only if x and y are linearly dependent vectors.

(ii) $\|x,y\| = \|y,x\|$ for all $x, y \in X$.

(iii) $\|\lambda x, y\| = |\lambda|\, \|x,y\|$ for all $\lambda \in K$ and all $x, y \in X$.

(iv) $\|x+y, z\| \le \|x,z\| + \|y,z\|$ for all $x, y, z \in X$.

Then $\|\cdot,\cdot\|$ is called a 2-norm on X and $(X, \|\cdot,\cdot\|)$ is called a linear 2-normed space. Every 2-normed space is a locally convex topological vector space. In fact for a fixed $b \in X$, $p_b(x) = \|x,b\|$ for all $x \in X$, is a seminorm and the family $P = \{p_b : b \in X\}$ generates a locally convex topology on X. There are no remarkable relations between normed spaces and 2-normed spaces and we can not construct a 2-norm by using a norm.

AMS Subject Classification: 46A15, 41A65.

Keywords and phrases: Generalized 2-normed spaces; y-subadditive map; 1-type ε-quasi Chebyshev subspace.

Definition 1. ([5] and [7]) Let X and Y be linear spaces, D be a non-empty subset of $X \times Y$ such that for every $x \in X$, $y \in Y$ the sets

$$D_x = \{y \in Y : (x,y) \in D\}, \quad D^y = \{x \in X : (x,y) \in D\}$$

are linear subspaces of the spaces Y and X, respectively. A function $\|\cdot,\cdot\| : D \longrightarrow [0,\infty)$ is called a generalized 2-norm on D if it satisfies the following conditions:

(N_1) $\|x, \alpha y\| = |\alpha| \, \|x,y\| = \|\alpha x, y\|$, for all $(x,y) \in D$ and every scalar α.

(N_2) $\|x, y+z\| \leq \|x,y\| + \|x,z\|$, for all $(x,y), (x,z) \in D$.

(N_3) $\|x+y, z\| \leq \|x,z\| + \|y,z\|$, for all $(x,z), (y,z) \in D$.

Then $(D, \|\cdot,\cdot\|)$ is called a 2-normed set. In particular, if $D = X \times Y$, $(X \times Y, \|\cdot,\cdot\|)$ is called a generalized 2-normed space. Moreover, if $X = Y$, then the generalized 2-normed space is denoted by $(X, \|\cdot,\cdot\|)$.

For example, let A be a Banach algebra and $\|a,b\| = \|ab\|$ for all $a, b \in A$. Then, $(A, \|\cdot,\cdot\|)$ is a generalized 2-normed space.

Let us consider linear spaces X and Y and $D \subseteq X \times Y$ a 2-normed set. A map $f \colon D \longrightarrow R$ is called 2-linear if it satisfies the following conditions [5]–[9]:

(i) $f(x_1 + x_2, y_1 + y_2) = f(x_1, y_1) + f(x_1, y_2) + f(x_2, y_1) + f(x_2, y_2)$, for all $(x_1, y_1), (x_2, y_2) \in D$.

(ii) $f(\delta x, \lambda y) = \delta \lambda f(x,y)$ for all scalars δ, λ and $(x,y) \in D$.

A 2-linear map f is said to be bounded if there exists a non-negative real number M such that $\|f(x,y)\| \leq M \|x,y\|$ for all $(x,y) \in D$. Also, the norm of a 2-linear map f is defined by

$$\|f\| = \inf\{M \geq 0 : \|f(x,y)\| \leq M \|x,y\| \text{ for all } (x,y) \in D\}.$$

2. ε-approximation in generalized 2-normed spaces

Definition 2. Let $(X \times Y, \|\cdot,\cdot\|)$ be a generalized 2-normed space, W be a subpace of X, $0 \neq y \in Y$ and $\varepsilon > 0$ be given.

(i) $w_0 \in W$ is called ε-best approximation of $x \in X$ in W respect to y, if

$$\|x - w_0, y\| \leq \inf\{\|x - w, y\| : w \in W\} + \varepsilon.$$

The set of all ε-best approximations of x in W respect to y is denoted by $P^y_{W,\varepsilon}(x)$.

Note that every subspace W of X is ε-proximinal, that is $P^y_{W,\varepsilon}(x)$ is nonempty for all $x \in X$ and all $y \in Y$.

(ii) W is called 1-type ε-pseudo Chebyshev if $P^y_{W,\varepsilon}(x)$ is finite dimensional for all $x \in X$ and all $0 \neq y \in Y$. Also, W is called 1-type ε-quasi Chebyshev if $P^y_{W,\varepsilon}(x)$ is compact in (X, p_y) for all $x \in X$ and all $0 \neq y \in Y$.

(iii) Let y be a non-zero element of Y and $\langle y \rangle$ be the subspace of Y generated by y. A mapping $f \colon W \times \langle y \rangle \longrightarrow R$ is called y-subadditive if

$$f(w_1 + w_2, y) \leq f(w_1, y) + f(w_2, y) \quad \text{and} \quad f(w_1, \lambda y) = \lambda f(w_1, y)$$

for all $w_1, w_2 \in W$ and for every scalar λ.

A y-subadditive map f is said to be bounded if there exists a non-negative real number M such that $|f(w,t)| \leq M \|w,t\|$ for all $w \in W$ and all $t \in \langle y \rangle$. Also, the norm of a y-subadditive map f is defined by

$$\|f\| = \inf \{M \geq 0 : |f(w,t)| \leq M \|w,t\| \text{ for all } (w,t) \in W \times \langle y \rangle\}.$$

We will denote by $S(W,y)$ the set of all bounded y-subadditive maps on $W\text{x}\langle y \rangle$.

Theorem 1. *Let $(X \times Y, \|\cdot,\cdot\|)$ be a generalized 2-normed space, $x \in X$, W be a subspace of X, $w_0 \in W$, $0 \neq y \in Y$ and $\varepsilon > 0$ be given. Then, $w_0 \in P^y_{W,\varepsilon}(x)$ if and only if there exists $f \in S(X,y)$ such that $f|_{W \times \langle y \rangle} = 0$, $\|f\| = 1$ and $f(x - w_0, y) \geq \|x - w_0, y\| - \varepsilon$.*

Proof. First suppose that there exists $f \in S(X,y)$ such that $f|_{W \times \langle y \rangle} = 0$, $\|f\| = 1$ and $f(x - w_0, y) \geq \|x - w_0, y\| - \varepsilon$. Then $\|x - w_0, y\| \leq f(x_0 - w_0, y) + \varepsilon = f(x - w, y) + \varepsilon \leq \|x - w, y\| \cdot \|f\| + \varepsilon = \|x - w, y\| + \varepsilon$ for all $w \in W$. Hence, $w_0 \in P^y_{W,\varepsilon}(x)$. Conversely, define $f(x,t) = \inf\{\|x - w, t\| : w \in W\}$. Then, $f \in S(X,y)$, $f|_{W \times \langle y \rangle} = 0$, $\|f\| = 1$ and $f(x - w_0, y) + \varepsilon \geq \|x - w_0, y\|$. ∎

Theorem 2. *Let $(X \times Y, \|\cdot,\cdot\|)$ be a generalized 2-normed space, $x \in X$, W be a subspace of X, $w_0 \in W$, $0 \neq y \in Y$ and $\varepsilon > 0$ be given. Then, $M \subseteq P^y_{W.\varepsilon}(x)$ if and only if there exists $f \in S(X,y)$ such that $f|_{W \times \langle y \rangle} = 0$, $\|f\| = 1$ and $f(x - w, y) \geq \|x - w, y\| - \varepsilon$ for all $m \in M$.*

Proof. Let $M \subseteq P^y_{W,\varepsilon}(x)$ and choose $w_0 \in P^y_{W,\varepsilon}(x)$ with $\|x - w_0, y\| = \lambda + \varepsilon$, $\lambda = \inf\{\|x - w, y\| : w \in W\}$. By Theorem 1, there exists $f \in S(X,y)$ such that $f|_{W \times \langle y \rangle} = 0$, $\|f\| = 1$ and $f(x - w_0, y) \geq \|x - w_0, y\| - \varepsilon$. Then, $f(x - m, y) = f(x - w_0, y) \geq \|x - w_0, y\| - \varepsilon = \lambda \geq \|x - m, y\| - \varepsilon$, for all $m \in M$. ∎

Definition 3. Let $(X \times Y, \|\cdot,\cdot\|)$ be a generalized 2-normed space, $0 \neq y \in Y$, $\varepsilon > 0$ be given and $f \in S(X,y)$. Define

$$M^y_{f,\varepsilon} = \{x \in X : f(x,y) \geq \|x,y\| - \varepsilon, \ \|x,y\| \leq 1 + \varepsilon\}.$$

Theorem 3. *Let $(X \times Y, \|\cdot,\cdot\|)$ be a generalized 2-normed space, W be a subspace of X, and $\varepsilon > 0$ be given.*

(i) W is 1-type ε-pseudo Chebyshev if and only if there do not exist $0 \neq y \in Y$, $f \in S(X,y)$, $x \in X$ with $\|x,y\| \leq 1$ and infinitely many linearly independent elements $w_1, w_2, \ldots$ in W such that $f|_{W \times \langle y \rangle} = 0$, $\|f\| = 1$ and $f(x - w_n, y) \geq \|x - w, y\| - \varepsilon$ for all $n \geq 1$

(ii) W is 1-type ε-quasi Chebyshev if and only if there do not exist $0 \neq y \in Y$, $f \in S(X,y)$, $x \in X$ with $\|x,y\| \leq 1$ and a sequence $\{w_n\}_{n \geq 1}$ in W without a convergent subsequence such that $f|_{W \times \langle y \rangle} = 0$, $\|f\| = 1$ and $f(x - w_n, y) \geq \|x - w_n, y\| - \varepsilon$ for all $n \geq 1$.

Proof. (i) First assume that there exist $0 \neq y \in Y$, $f \in S(X,y)$, $x \in X$ with $\|x,y\| \leq 1$ and infinitely many linearly independent elements $w_1, w_2, \ldots$ in

W such that $f\mid_{W\times\langle y\rangle}=0$, $\|f\|=1$ and $f(x-w_n,y)\geq\|x-w_n,y\|-\varepsilon$ for all $n\geq 1$. It follows that $\dim P^y_{W,\varepsilon}=\infty$ and hence W is not 1-type ε-pseudo Chebyshev subspace of X. Now, suppose that W is not 1-type ε-pseudo Chebyshev subspace of X. Since $P^y_{W,\varepsilon}(\lambda x)=\lambda P^y_{W,\frac{\varepsilon}{\lambda}}(x)$ and $P^y_{W,\varepsilon_1}(x)\subseteq P^y_{W,\varepsilon_2}(x)$ for all $0<\varepsilon_1\leq\varepsilon_2$, $x\in X$ and $\lambda>0$, there exists $x\in X$ with $\|x,y\|\leq 1$ such that $\dim P^y_{W,\varepsilon}=\infty$. Hence, $P^y_{W,\varepsilon}$ contains infinitely many linearly independent elements $g_1,g_2,\ldots$. By Theorem 2, there exists $f\in S(X,y)$ such that $f\mid_{W\times\langle y\rangle}=0$, $\|f\|=1$ and $f(x-g_n,y)\geq\|x-g_n,y\|-\varepsilon$ for all $n\geq 1$. The proof of part (ii) is similar that of (i). ■

Theorem 4. *Let $(X\times Y,\|\cdot,\cdot\|)$ be a generalized 2-normed space, W be a subspace of X and $\varepsilon>0$ be given. If $M^y_{f,\varepsilon}$ is finite dimensional for all $0\neq y\in Y$, and all $f\in\Lambda_y=\left\{h\in S(X,y):\|h\|=1 \text{ and } h\mid_{W\times\langle y\rangle}=0\right\}$, then W is 1-type ε-pseudo Chebyshev subspace of X.*

Proof. Assume that W is not 1-type ε-pseudo Chebyshev subspace of X. Then by Theorem 3, there exist $0\neq y\in Y$, $f\in S(X,y)$, $x_0\in X$ with $\|x_0,y\|\leq 1$ and infinitely many linearly independent elements $w_1,w_2,\ldots$ in W such that $\|f\|=1$, $f\mid_{W\times\langle y\rangle}=0$, and $f(x_0-w_n,y)\geq\|x_0-w_n,y\|-\varepsilon$ for all $n\geq 1$. Since $\|x_0-w_n,y\|\leq f(x_0-w_n,y)+\varepsilon=f(x_0,y)+\varepsilon\leq 1+\varepsilon$, $x_0-w_n\in M^y_{f,\varepsilon}$ for all $n\geq 1$. This is a contradiction. ■

Definition 4. Let $(X\times Y,\|\cdot,\cdot\|)$ be a generalized 2-normed space, $0\neq y\in Y$, $\varepsilon>0$ be given and let M be a subspace of $S(X,y)$. For each $x\in X$, put

$$D^{M,y}_{x,\varepsilon}=\{t\in X: f(t,y)=f(x,y)\ \text{for all}\ f\in M\ \text{and}\ \|t,y\|\leq\|x,y\|_M+\varepsilon\},$$

where $\|x,y\|_M=\sup\{|f(x,y)|:\|f\|\leq 1,\ f\in M\}$.

It is clear that $D^{M,y}_{x,\varepsilon}$ is a non-empty, closed and convex subset of (X,p_y), for all $x\in X$.

We say that M has the property $(y,\varepsilon)-F^*$ if $D^{M,y}_{x,\varepsilon}$ is finite dimensional for all $x\in X$. Also,we say that M has the property $(y,\varepsilon)-C^*$ if $D^{M,y}_{x,\varepsilon}$ is compact for all $x\in X$.

Theorem 5. *Let $(X\times Y,\|\cdot,\cdot\|)$ be a generalized 2-normed space, $0\neq y\in Y$, W be a closed subspace of (X,p_b), $\varepsilon>0$ be given and let $M_0=\{f\in S(X,y): f\mid_{W\times\langle y\rangle}=0\}$. Then, $\dim P^y_{W,\varepsilon}(x)<\infty$ if and only if M_0 has the property $(y,\varepsilon)-F^*$.*

Proof. If $D^{M_0,y}_{x,\varepsilon}=\infty$ for some $x\in X$, then there exist infinitely many linearly independent elements $t_1,t_2,\ldots$ in $D^{M_0,y}_{x,\varepsilon}$. Hence, $t_1-t_2\in W$ for all $n\geq 1$ and

$$\|t_1-(t_1-t_n),y\|=\|t_n,y\|=\|x,y\|_{M_0}+\varepsilon=\|t_1-(t_1-t_n),y\|_{M_0}+\varepsilon$$

for all $n\geq 1$. Therefore, $t_1-t_n\in P^y_{W,\varepsilon}(t_1)$ for all $n\geq 1$. Now, suppose that $\dim P^y_{W,\varepsilon}(x_0)=\infty$ for some $x_0\in X$. Then, there exist infinitely many linearly independent elements $g_1, g_2, \dots$ in $P^y_{W,\varepsilon}(x_0)$. It is easy to see that, $\|x_0 - g_n, y\| \leq \|x_0 - g_n, y\|_{M_0} + \varepsilon = \|x_0, y\|_{M_0} + \varepsilon$ for all $n \geq 1$. It follows that $x_0 - g_n \in D^{M_0,y}_{x_0,\varepsilon}$ for all $n \geq 1$, which is a contradiction. ■

THEOREM 6. *Let $(X \times Y, \|\cdot,\cdot\|)$ be a generalized 2-normed space, $0 \neq y \in Y$, W be a closed subspace of (X, p_b), $\varepsilon > 0$ be given and let $M_0 = \{f \in S(X, y) : f\,|_{W\times\langle y\rangle} = 0\}$. Then, $\dim P^y_{W,\varepsilon}(x)$ is compact if and only if M_0 has the property $(y, \varepsilon) - C^*$.*

REFERENCES

[**1**] S. Cobzas and C. Mustata, *Extension of bilinear functionals and best approximation in 2-normed spaces*, Studia Univ. Babes-Bolyai Math. **43**, 2 (1998), 1-1-3.

[**2**] S. Elumalai, M. Souruparani, *A characterization of best approximation and the operators in linear 2-normed spaces*, Bull. Cal. Math. Soc., **92**, 4 (2000), 235–248.

[**3**] S. Gahler, *Lineare 2-normierte Räume*, Math. Nachr, **28** (1965), 1–45.

[**4**] S. N. Lal, S. Bhattacharya, C. Sreedhar, *Complex 2-normed linear spaces and extension of linear 2-functionals*, Z. Anal. Anwendungen, **20**, 1 (2001), 35–53.

[**5**] Z. Lewandowska, *Linear operators on generalized 2-normed spaces*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **42(90)**, 4 (1999), 353–368.

[**6**] Z. Lewandowska, *Generalized 2-normed spaces*, Slupskie Prace Matematyczno-Fizyczne **1** (2001), 33–40.

[**7**] Z. Lewandowska, *On 2-normed sets*, Glasnik Mat. Ser. III **38** (58), 1 (2003), 99–110.

[**8**] Z. Lewandowska, *Banach-Steinhaus theorems for bounded linear operators with values in a generalized 2-normed space*, Glasnik Mat. Ser. III **38** (58), 2 (2003), 329–340.

[**9**] Z. Lewandowska, *Bounded 2-linear operators on 2-normed sets*, Glasnik Mat. Ser. III **39** (59), 2 (2004), 301–312.

[**10**] Sh. Rezapour, H. Mohebi, *ε-weakly Chebyshev subspaces and quotient spaces*, Bull. Iranian Math. Soc. **29**, 1 (2003), 13–20.

[**11**] Sh. Rezapour, *Weak compactness of the set of ε-extensions*, Bull. Iranian Math. Soc. **30**, 1 (2004), 13–20.

[**12**] Sh. Rezapour, *ε-pseudo Chebyshev and ε-quasi Chebyshev subspaces of Banach spaces*, Anal. Theory Appl. **20**, 4 (2004), 1–8.

[**13**] W. Raymond Freese and Yeol Je Cho, *Geometry of Linear 2-Normed Spaces*, Nova Science Publishers, Inc., New York, 2001.

[**14**] Yeol Je Cho, Paul C. S. Lin, Seong Sik Kim and Aleksander Misiak, *Theory of 2-Inner Product Spaces*, Nova Science Publishers, Inc. New York, 2001.

(received 03.04.2008, in revised form 19.11.2008)

University of Gaziantep, Faculty of Science and Arts, Department of Mathematics, 27310 Gaziantep, TURKEY

E-mail: acikgoz@gantep.edu.tr