МАТЕМАТИЧКИ ВЕСНИК
61, 3 (2009), 195–202
September 2009

UDK 515.122
оригинални научни рад
research paper

SELECTION PRINCIPLES AND BAIRE SPACES

Marion Scheepers

Abstract. We prove that if X is a separable metric space with the Hurewicz covering property, then the Banach-Mazur game played on X is determined. The implication is not true when "Hurewicz covering property" is replaced with "Menger covering property".

1. Introduction

The selection principle $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ states that there is for each sequence $(A_n : n \in \mathbb{N})$ with each $A_n \in \mathcal{A}$, a sequence $(B_n : n \in \mathbb{N})$ such that each $B_n \subset A_n$ is finite and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$. Letting $\mathcal{O}$ denote for the space X the set of all open covers of X, the statement $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ denotes the Menger property for X. Hurewicz [5] introduced the Menger property in 1925 and showed that a conjecture of Menger is equivalent to the statement that a metrizable space has the Menger property if, and only if, it is σ-compact. In 1927 Hurewicz [6] defined the following stronger version of the Menger property: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that each $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ is in all but finitely many of the sets $\bigcup \mathcal{V}_n$. This property is said to be the Hurewicz property. In [9] it was shown that the Hurewicz property can also be formulated in the form $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$, but we will not need that result here.

It is clear that σ-compactness implies the Hurewicz property in all finite powers, and that the Hurewicz property implies the Menger property. Early proofs that none of the converses hold used the Continuum Hypothesis. More recent proofs do not rely on additional set theoretic hypotheses: Fremlin and Miller [11] disproved Menger's Conjecture, thus showing that Menger's property is weaker than σ-compactness. Numerous examples in the literature show that Menger's property is not necessarily preserved by finite powers. Chaber and Pol [2] showed that the Menger property (even in all finite powers) does not imply the Hurewicz property, and in [7] it was shown that the Hurewicz property does not imply σ-compactness. See [16] for more details.

AMS Subject Classification: 03E99, 54D20, 54E52.

Keywords and phrases: Baire space; First category; Banach-Mazur game; Menger property; Hurewicz property.

This raises the possibility that theorems proven using the hypothesis that some space X is σ-compact, may be strengthened by proving it using the weaker hypothesis that for all n, X^n has Hurewicz's or Menger's property, or that X has Hurewicz's or Menger's property. Several examples of such work can be found in recent literature, for example: [1], [13] and [14]. We give such results in this paper in connection with Baire category.

A topological space is said to be Baire if the intersection of any sequence of dense open subsets is a dense set. It is said to be first category if it is a union of countably many nowhere dense sets. If it is not first category, it is said to be second category. In Exercise 25B of [17] the reader is asked to prove the following statement:

> If X is a σ-compact space then it is a second category (respectively Baire) space if, and only if X has an element (respectively, dense set of elements) with a compact neighborhood.

We examine weakening the hypothesis "X is a σ-compact space".

2. The Banach-Mazur game and selection principles

The Banach-Mazur game on X, $\mathsf{BM}(X)$, is played as follows: Players ONE and TWO play an inning per positive integer. In the n-th inning ONE chooses a nonempty open set O_n; TWO responds with a nonempty open set $T_n \subseteq O_n$. ONE must also obey the rule that for each n, $O_{n+1} \subseteq T_n$. A play

$$O_1, T_1, \cdots, O_n, T_n, \cdots$$

is won by TWO if $\bigcap_{n\in\mathbb{N}} T_n \neq \emptyset$; otherwise, ONE wins.

A strategy of a player is a function with domain the set of finite sequences of moves by the opponent, and with values legal moves for the strategy owner. A strategy σ for player TWO is said to be a *tactic* if it is of the form $T_n = \sigma(O_n)$ for all n. The notion of a tactic for ONE is defined analogously. In [5] tactics are also called stationary strategies. The following facts are well-known [15]:

1. X is a Baire space if, and only if, ONE has no winning strategy in $\mathsf{BM}(X)$.
2. If X is a separable metrizable space such that TWO has a winning strategy in $\mathsf{BM}(X)$, then X contains a homeomorphic copy of the Cantor set.
3. There are examples of X where neither player has a winning strategy in $\mathsf{BM}(X)$.
4. If TWO has a winning strategy in $\mathsf{BM}(X)$, then for each Baire space Y, X×Y is a Baire space.
5. If TWO has a winning strategy in $\mathsf{BM}(X)$, then all box powers of X are Baire spaces.

Regarding the above mentioned Exercise 25B of [17] one can indeed prove for σ-compact spaces X that the following statements are equivalent:

1. X is a Baire space.

2. X has a dense set of points with compact neighborhoods.
3. TWO has a winning strategy in BM(X).
4. TWO has a winning tactic in BM(X).

It follows that in σ-compact spaces BM(X) is determined. We show that this particular consequence of σ-compactness is not a consequence of the Menger property, but is a consequence of the Hurewicz property.

There is a natural game, $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$, that corresponds to the selection principle $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$: The game has an inning per positive integer n. In the n-th inning ONE first chooses an $O_n \in \mathcal{A}$, and TWO then responds with a finite set $T_n \subseteq O_n$. A play $(O_1, T_1, \cdots, O_n, T_n, \cdots)$ is won by TWO if $\bigcup_{n\in\mathbb{N}} T_n \in \mathcal{B}$. Otherwise, ONE wins.

The following equivalence, proved in Theorem 10 of [5], is very useful for applications involving the Menger property:

Theorem 1. [Hurewicz] *For topological space X the following are equivalent:*

(1) *The space has property $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$.*

(2) *ONE has no winning strategy in $\mathsf{G}_{fin}(\mathcal{O},\mathcal{O})$.*

Below we shall use this equivalence without specifically referencing Theorem 1.

Theorem 2. *For X a T_3-space with $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ the following are equivalent:*

(1) *TWO has a winning strategy in BM(X).*

(2) *$D = \{x \in X : x$ has a neighborhood with compact closure$\}$ is dense in X.*

(3) *TWO has a winning tactic in BM(X).*

Proof. The proof that (2) $\Rightarrow$ (3) does not require that X has property $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$. Here is a tactic for TWO: When ONE chooses a nonempty open set O, TWO first chooses an element $x \in O \cap D$. Then choose a neighborhood U of x with $\overline{U}$ compact. Then, as X is T_3, choose an open set $\sigma(O)$ with $x \in \sigma(O)$ and $\overline{\sigma(O)} \subset O \cap U$. To see that σ is a winning tactic for TWO, note that $\overline{\sigma(O)}$ is compact, and $\sigma(O) \subset \overline{\sigma(O)} \subset O$.

It is clear that (3) $\Rightarrow$ (1). We prove (1) $\Rightarrow$ (2) by proving the contrapositive: If D is not dense, then TWO does not have a winning strategy in BM(X). Thus: Assume D is not dense, and let F be a strategy for TWO in the game BM(X).

Define a strategy σ for ONE of the game $\mathsf{G}_{fin}(\mathcal{O},\mathcal{O})$ as follows: First, player ONE of BM(X) moves: B_1 is a nonempty open set disjoint from D. TWO's response is $W_1 = F(B_1)$. Each neighborhood of each x in W_1 has a non-compact closure. Choose $x_1 \in W_1$. Choose a neighborhood V_1 of x_1 with $\overline{V_1} \subset W_1$, and an open (in X) cover $\mathcal{A}_1$ of $\overline{V_1}$ such that no finite subset $\mathcal{F}$ of $\mathcal{A}_1$ satisfies $V_1 \subset \overline{\bigcup\mathcal{F}}$ as follows: First, since $\overline{V_1}$ is not compact, take an infinite cover $\mathcal{U}$ of $\overline{V_1}$ consisting of sets open in X, and which has no finite subset covering $\overline{V_1}$. Then using the fact that X is T_3, choose for each $x \in \overline{V_1}$ an open neighborhood U_x of x such that for some $U \in \mathcal{U}$ we have $\overline{U}_x \subseteq U$. Put $\mathcal{A}_1 := \{U_x : x \in \overline{V_1}\}$. Then we define ONE's move for the game $\mathsf{G}_{fin}(\mathcal{O},\mathcal{O})$ by

$$\sigma(\emptyset) = \mathcal{A}_1 \bigcup \{X \setminus \overline{V_1}\}.$$

When TWO responds with a finite set $T_1 \subset \sigma(\emptyset)$, ONE plays the move $\sigma(T_1)$ as follows: Player ONE of BM(X) responds with

$$B_2 = W_1 \setminus \overline{\bigcup T_1},$$

a nonempty open set. Then TWO of BM(X) plays $W_2 = F(B_1, B_2)$. Choose an $x_2 \in W_2$ and a neighborhood V_2 of x_2 with $\overline{V_2} \subset W_2$. Then choose an open (in X) cover $\mathcal{A}_2$ of $\overline{V_2}$ such that no finite subset $\mathcal{F} \subset \mathcal{A}_2$ has $V_2 \subset \overline{\bigcup \mathcal{F}}$. Then put

$$\sigma(T_1) = \mathcal{A}_2 \bigcup \{X \setminus \overline{V_2}\}.$$

When TWO now responds with a finite $T_2 \subset \sigma(T_1)$, then ONE plays the move $\sigma(T_1, T_2)$ as follows: Player ONE of BM(X) responds with

$$B_3 = W_2 \setminus \overline{\bigcup T_2},$$

a nonempty open set. TWO of BM(X) applies the strategy F to obtain $W_3 = F(B_1, B_2, B_3)$. Choose an $x_3 \in W_3$, and a neighborhood V_3 of x_3 with $\overline{V_3} \subset W_3$, and then an open (in X) cover $\mathcal{A}_3$ of $\overline{V_3}$ such that for no finite set $\mathcal{F} \subset \mathcal{A}_3$ do we have $\overline{\bigcup \mathcal{F}} \supset V_3$. Then ONE plays

$$\sigma(T_1, T_2) = \mathcal{A}_3 \bigcup \{X \setminus \overline{V_3}\},$$

and so on.

Since X has property $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$, σ is not a winning strategy for ONE of $\mathsf{G}_{fin}(\mathcal{O},\mathcal{O})$. Thus, consider a σ-play

$$\sigma(\emptyset),\ T_1,\ \sigma(T_1),\ \cdots,\ T_n,\ \sigma(T_1,\cdots,T_n),\ \cdots$$

lost by ONE. It corresponds to an F-play

$$B_1,\ F(B_1),\ B_2,\ F(B_1,B_2),\ \cdots,\ B_n,\ F(B_1,\cdots,B_n),\ \cdots$$

of BM(X) where for all n we have $B_{n+1} = F(B_1,\cdots,B_n) \setminus \overline{\bigcup T_n}$. Since ONE lost the σ-play, the set $\bigcup_{n\in\mathbb{N}} T_n$ is an open cover of X. For the corresponding play of BM(X) we have $\bigcap_{n\in\mathbb{N}} B_n \subseteq W_1 \setminus \bigcup_{n\in\mathbb{N}}(\bigcup T_n) = \emptyset$. Thus, F is not a winning strategy for TWO in BM(X). ■

Note. The referee pointed out that by essentially the same argument a third equivalent statement can be added, namely (in the notation of Theorem 2:

(4) $X \setminus D$ is nowhere dense.

In general, if TWO has a winning strategy in BM(X), then TWO need not have a winning tactic [3]. A number of conditions on X that ensures that TWO has a winning strategy if, and only if, TWO has a winning tactic, are known. These include various completeness properties. Theorem 2 gives another such condition. It follows that the $\mathsf{T}_{3\frac{1}{2}}$-space X of [3] in which TWO has a winning strategy, but not a winning tactic, in BM(X), does not have the Menger property.

THEOREM 3. (CH) *There is a subspace X of the real line such that:*

(1) *X has the property $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ in all finite powers, but*

(2) *Neither player has a winning strategy in BM(X).*

Proof. Consider a Lusin set $X \subset \mathbb{R}$ which has the property $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ in all finite powers. Such is constructed for example in [7] or [10]. We may assume that $X = X + \mathbb{Q}$. Then for each dense open $E_n \subset X$ there is a dense open $D_n \subset \mathbb{R}$ with $E_n = X \bigcap D_n$. Since $\mathbb{R} \setminus D_n$ is nowhere dense, it follows that $X \setminus E_n$ is countable. But then $\bigcap_{n\in\mathbb{N}} E_n$ is dense in X, showing that X is a Baire space. By the Banach-Oxtoby theorem, ONE has no winning strategy in BM(X). Since X contains no subset homeomorphic to the Cantor set, also TWO has no winning strategy in BM(X). ■

We now show that in separable metrizable spaces the Hurewicz property suffices as a replacement for σ-compactness in the following sense:

THEOREM 4. *For X a separable metric space with the Hurewicz property the following are equivalent:*

(1) *X is a Baire space.*

(2) *$D = \{x \in X : x$ has a neighborhood with compact closure$\}$ is dense in X.*

(3) *TWO has a winning strategy in BM(X).*

Proof. We already have (2) ⇒ (3) from Theorem 2, and (3) ⇒ (1) is folklore. We must prove that (1) ⇒ (2). We do this by proving the contrapositive: Assume D is not dense. We will show that ONE has a winning strategy in BM(X). The Banach-Oxtoby theorem implies that X is not Baire.

Here is how a winning strategy for ONE is defined. ONE's first move, $\sigma(X)$, is a nonempty open set $O_1 \subset X \setminus D$. Since O_1 is an F_σ subset of X, it has the Hurewicz property also. Fix a metric d on X and choose a countable base $(B_n : n \in \mathbb{N})$ for O_1 such that for each n, $\overline{B_n} \subset O_1$, B_n has d-diameter less than 1, and $\lim_{n\to\infty} diam(B_n) = 0$ (the latter is implied directly by the Menger property of O_1). Now no $\overline{B_n}$ is compact, so we may choose for each n an open (in O_1) cover $\mathcal{U}_n^1$ of $\overline{B_n}$ which does not contain any finite set $\mathcal{T}$ with $B_n \subset \overline{\bigcup \mathcal{T}}$. Then for each n the set $\mathcal{U}_n = \{O_1 \setminus \overline{B_n}\} \bigcup \mathcal{U}_n^1$ is an open cover of O_1. Choose, by the Hurewicz property, for each n a finite set $\mathcal{V}_n \subset \mathcal{U}_n$ such that for each $x \in O_1$, for all but finitely many n, $x \in \bigcup \mathcal{V}_n$. We are now ready to define ONE's strategy σ further. For each nonempty open set $U \subset O_1$ choose an $n = n(U)$ such that $\overline{B_n} \subset U$, and if U has finite diameter, then $diam_d(B_n) < \frac{1}{2} \cdot diam_d(U)$. When TWO plays an open set U, ONE responds with

$$\sigma(U) = B_{n(U)} \setminus \overline{\bigcup \mathcal{V}_{n(U)}}.$$

It is clear that when U is nonempty and open, so is $\sigma(U)$. We must see that σ is a winning strategy for ONE. Consider a σ-play of BM(X):

$$O_1 = \sigma(X),\, W_1,\, \sigma(W_1),\, W_2,\, \sigma(W_2),\, W_3,\, \cdots$$

For each W_k, put $m_k = n(W_k)$. Then by the definition of ONE's strategy $\sigma(W_1) = B_{m_1} \setminus \overline{\bigcup \mathcal{V}_{m_1}} \supseteq W_2$ and for each $k > 1$ $\sigma(W_k) = B_{m_k} \setminus \overline{\bigcup \mathcal{V}_{m_k}} \supseteq W_{k+1}$ and $diam_d(W_{k+1}) < \frac{1}{2} \cdot diam(B_{m_{k-1}})$. This implies that $\{m_k : k \in \mathbb{N}\}$ is infinite, so that $\bigcup_{k\in\mathbb{N}} \mathcal{V}_{m_k}$ covers O_1. It follows that $\bigcap_{k\in\mathbb{N}} W_k = \emptyset$, and so ONE wins.‘■

Of course the Hurewicz property implies the Menger property. Thus by Theorem 2 we also have in Theorem 4 the equivalence that TWO has a winning strategy if, and only if, TWO has a winning tactic. Evidently, a proof of (1) $\Rightarrow$ (2) which does not invoke the game-theoretic equivalence can be given.

Corollary 5. *The Banach-Mazur game is determined in separable metric spaces with the Hurewicz property.*

It is well known that the product of Baire spaces need not be a Baire space again.

Corollary 6. *Let X be a separable metric space with the Hurewicz property. If X and Y are Baire, then $X \times Y$ is a Baire space.*

Corollary 7. *Let X be a separable metric space with the Hurewicz property. If X is a Baire space, then all powers of X have the Baire property, even in the box topology.*

However, when X is a separable metric space which has the Baire property and the Hurewicz property, X^2 need not have the Hurewicz property. To see this, let C be the Cantor set in $\mathbb{R}$. Then $Y = \mathbb{R} \setminus C$ is σ-compact and Baire. Let $Z \subset C$ be a set with the Hurewicz property in the inherited topology, but for which $Z \times Z$ does not have the Hurewicz property. The Continuum Hypothesis can be used to find such a subset of the Cantor set - (see the remark following Theorem 2.11 of [7]). Put $X = Y \bigcup Z$. Then X is a Baire space and has the Hurewicz property. But the closed subset $Z \times Z$ of $X \times X$ does not have the Hurewicz property, and so $X \times X$ does not have the Hurewicz property.

Note that X also is not σ-compact. From Theorem 4 we can conclude that a separable metric space T which is Baire and has the Hurewicz property contains a dense subset which is σ-compact: But we cannot conclude that T is σ-compact.

3. The game MB(X) and selection principles.

The game MB(X) is played like BM(X), except that now ONE wins if $\bigcap_{n\in\mathbb{N}} B_n \neq \emptyset$, and TWO wins otherwise.

The relationship between player TWO of BM(X) and player ONE of MB(X) is as follows: If TWO has a winning strategy F in BM(X), then ONE has a winning strategy in MB(X): ONE of MB(X) simply pretends to be TWO of BM(X) and uses F as strategy, and assumes ONE of BM(X) started the game with the move $O_1 = X$. If ONE has a winning strategy G in MB(X), then the nonempty open set $U = G(X) \subseteq X$ is such that TWO has a winning strategy in BM(U): TWO now simply pretends to be ONE of MB(X) and uses G to respond to moves of the opponent. This suggests that the results on BM(X) above have analogues for MB(X). This is the topic of this section.

There is one caveat in applying the ideas above to transfer information from BM(X) to MB(X): If one wants to use the Theorem 2 in Theorem 8 below, in the proof of (1) ⇒ (2), we would use ONE's strategy in MB(X) as a strategy for TWO in BM(F(X)), where F(X) is an open subset of X. Then from Theorem 2 we could conclude that D is dense in F(X), and thus nonempty. But this application of Theorem 2 would require that the open set F(X) has the Menger property. Unfortunately, the Menger property is not open hereditary. But in T_3 spaces it is possible to circumvent this point:

THEOREM 8. *For X a T_3-space with $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ the following are equivalent:*

(1) *ONE has a winning strategy in* MB*(X).*

(2) *$D = \{x \in X : x$ has a neighborhood with compact closure$\}$ is dense in some nonempty open set.*

(3) *$D = \{x \in X : x$ has a neighborhood with compact closure$\}$ is nonempty.*

Proof. It is clear that (2) ⇒ (3). The proof that (3) ⇒ (1) does not require that X has the property $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ and uses a standard argument. We prove (1) ⇒ (2): Let F be a winning strategy for ONE of MB(X). Choose a nonempty open set U with $\overline{U} \subset F(X)$. Then $\overline{U}$ inherits the property $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ from X. Now TWO has a winning strategy G in $\mathsf{BM}(\overline{U})$. By Theorem 2 the set $E = \{x \in \overline{U} : x$ has a neighborhood with compact closure$\}$ is dense in $\overline{U}$, and (2) follows. ■

THEOREM 9. (Oxtoby) *For a topological space X the following are equivalent:*

(1) *TWO has a winning strategy in* MB*(X).*

(2) *X is first category in itself.*

THEOREM 10. *Let X be a separable metric space with the Hurewicz property. Then the following are equivalent:*

(1) *X is not first category.*

(2) *$D = \{x \in X : x$ has a neighborhood with compact closure$\}$ is nonempty.*

(3) *ONE has a winning strategy in* MB*(X).*

Proof. The equivalence of (2) and (3) is in Theorem 8. It is clear from Theorem 9 that (3) ⇒ (1). To prove (1) ⇒ (2), prove the contrapositive by showing that if $D = \emptyset$, then TWO has a winning strategy in MB(X). The ideas are as in the proof of Theorem 4. ■

It follows that $\mathsf{MB}(X)$ is determined in separable metric spaces with the Hurewicz property. It follows that if a subset of the real line has the Hurewicz property but does not contain any perfect set, then it is perfectly meager (since their intersection with any perfect subset of the real line has the Hurewicz property). This gives an alternative proof of Theorem 5.5 of [7].

REFERENCES

[**1**] L. Babinkostova, *When does the Haver property imply selective screenability?*, Topology Appl. **154** (2007), 1971–1979.

[**2**] J. Chaber and R. Pol, *A remark on Fremlin-Miller theorem concerning the Menger property and Michael concentrated sets*, manuscript of 10.10.2002.

[**3**] G. Debs, *Strategies gagnantes dans certains jeux topologiques*, Fundamenta Math. **126** (1985), 93–105.

[**4**] F. Galvin and R. Telgársky, *Stationary strategies in topological games*, Topology Appl. **22** (1986), 51–69.

[**5**] W. Hurewicz, *Über eine Verallgemeinerung des Borelschen Theorems*, Math. Z. **24** (1925), 401–425.

[**6**] W. Hurewicz, *Über Folgen stetiger Funktionen*, Fundamenta Math. **9** (1927), 193–204.

[**7**] W. Just, A.W. Miller, M. Scheepers and P.J. Szeptycki, *Combinatorics of open covers II*, Topology Appl. **73** (1996), 241–266.

[**8**] P.S. Kenderov and J.P. Revalski, *The Banach-Mazur game and generic existence of solutions to optimization problems*, Proc. Amer. Math. Soc. **118** (1993), 911–917.

[**9**] Lj.D.R. Kočinac and M. Scheepers, *Combinatorics of open covers (VII): Groupability*, Fundamenta Math., **179** (2003), 131–155.

[**10**] E.A. Michael, *Paracompactness and the Lindelöf property in finite and countable Cartesian products*, Compositio Math. **23** (1971), 199–244.

[**11**] A.W. Miller and D.H. Fremlin, *On some properties of Hurewicz, Menger and Rothberger*, Fundamenta Math. **129** (1988), 17–33.

[**12**] J.C. Oxtoby, *The Banach-Mazur game and the Banach category theorem*, in: Contributions to the Theory of Games, Vol. III, Annals of Mathematics Studies **39** (1957), 159–163.

[**13**] E. and R. Pol, *On metric spaces with the Haver property which are Menger spaces*, preprint.

[**14**] E. and R. Pol, *A metric space with the Haver property whose square fails this property*, Proc. Amer. Math. Soc. **137** (2009), 745–750.

[**15**] R. Telgársky, *Topological games: On the 50th anniversary of the Banach-Mazur game*, Rocky Mountain J. Math. **17**:2 (1987), 227–276.

[**16**] B. Tsaban and L. Zdomsky, *Scales, fields, and a problem of Hurewicz*, J. Europ. Math. Soc. **10** (2008), 837–866.

[**17**] S. Willard, *General Topology*, Addison Wesley Publ. Co., 1970.

(received 01.05.2008, in revised form 20.05.2009)

Department of Mathematics, Boise State University, Boise, Idaho 83725 USA

E-mail: marion@math.boisestate.edu