МАТЕМАТИЧКИ ВЕСНИК
61, 3 (2009), 227–233
September 2009

UDK 517.977
оригинални научни рад
research paper

COMPACTNESS AND WEAK COMPACTNESS OF ELEMENTARY OPERATORS ON $B(l^2)$ INDUCED BY COMPOSITION OPERATORS ON l^2

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Abstract. In this paper we have given simple proofs of some range inclusion results of elementary operators on $B(l^2)$ induced by composition operators on l^2. By using these results we have characterized compact and weakly compact elementary operators on $B(l^2)$ induced by composition operators on l^2.

1. Introduction

Definition 1.1. Let $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ be n-tuples of elements in an algebra $\mathcal{A}$. The elementary operator $E_{a,b}$ on $\mathcal{A}$ into itself associated with a and b is defined by $E_{a,b}(x) = a_1xb_1 + a_2xb_2 + \cdots + a_nxb_n$.

We denote by $M_{a,b}$ the elementary multiplication operator defined by $M_{a,b}(x) = axb$, $x \in \mathcal{A}$, $V_{a,b}(x) = axb - bxa$ for all $x \in \mathcal{A}$. For a fixed $a \in \mathcal{A}$, inner derivation δ_a is defined by $\delta_a(x) = ax - xa$. For fixed $a, b \in \mathcal{A}$, generalized derivation $\delta_{a,b}$ is defined by $\delta_{a,b}(x) = ax - xb$ for all $x \in \mathcal{A}$.

It is clear that δ_a and $\delta_{a,b}$ are elementary operators of length 2.

Definition 1.2. Let X and Y be normed linear spaces and S be the closed unit ball in X. A linear operator $T : X \to Y$ is

(i) a finite rank operator if dimension of the range of T is finite.

(ii) a compact operator if the closure of T(S) is compact in Y.

(iii) a weakly compact operator if T(S) is weakly compact in Y.

Definition 1.3. A Banach space X is said to have the approximation property if for every compact subset C of X and for every $\epsilon > 0$ there exists a finite rank operator $T \in B(X)$ such that $\|Tx - x\| < \epsilon$ for each $x \in C$.

AMS Subject Classification: 47B33, 47B47.

Keywords and phrases: Compactness; composition operators; elementary operators; thin operators.

Research work is supported by CSIR(award no.9/13(951)/2000-EMR-1).

Since every Banach space with a Schauder basis has the approximation property [1], a separable Hilbert space has approximation property.

Definition 1.4. Let l^2 be the Hilbert space of all square summable sequences of complex numbers under the standard inner product on it and ϕ be a function on $\mathbb{N}$ into itself. We denote by χ_n, characteristic function of $\{n\}$. Let $A_n = \phi^{-1}(n)$ and let $\overline{\overline{A_n}}$ denote the number of elements in A_n. The composition operator C_ϕ on l^2 is defined by $C_\phi(f) = f \circ \phi$ for all $f \in l^2$.

A necessary and sufficient condition that a function ϕ on $\mathbb{N}$ into itself induces a composition operator on l^2 is the set $\left\{\overline{\overline{A_n}} : n \in \mathbb{N}\right\}$ is bounded, see [12].

In the direction of compactness of elementary operators, first study was done by Vala [15] in 1964. He proved that "On $B(X)$ where X is a Banach space the mapping $T \mapsto ATB$ is compact if and only if A and B are compact operators". Vala defined an element a of a normed algebra $\mathcal{A}$ as compact if the mapping $x \mapsto axa$ is compact. By using this notion of compactness K.Ylinen [16] proved that compact elements of C^*-algebra $\mathcal{A}$ form a closed two sided ideal which is the closure of the finite elements of $\mathcal{A}$, i.e. those elements a, for which the map $x \mapsto axa$ is a finite rank operator. Akemann and Wright [3] obtained the necessary and sufficient condition for a C^*-algebra to admit a nonzero compact or weakly compact derivation. In 1977, Y.Ho [7] proved that derivation induced by non-scalars in $B(H)$ is non-compact. In 1979, Fong and Sourour [5] characterized the compactness of elementary operators on $B(H)$ where H is a separable Hilbert space. Precisely they showed that "An elementary operator on $B(H)$ is compact if and only if it has a representation $E(X) = \sum_{i=1}^{n} A_i X B_i$, where each A_i and each B_i is compact".

In the same paper they conjectured that there is no nonzero compact elementary operator on Calkin algebra, which was independently affirmed by Apostal and Fialkow [2], B. Magajna [9] and by M. Mathieu [8]. M. Mathieu generalized above results on C^*-algebra. Saksman and Tylli [13] studied compact and weakly compact elementary operators for a large class of Banach spaces. Now we state some known results which are useful in our context.

Theorem 1.1. [3, Theorem 3.1] *Let δ be a derivation on $B(H)$. The following are equivalent:*

(i) δ is weakly compact.

(ii) The range of δ is contained in $K(H)$.

(iii) $\delta = \delta_T$ with $T \in K(H)$.

Theorem 1.2. [8, Proposition 3.2] *Let $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ be n-tuples of elements in $B(H)$ and $E_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i$. If the set $\{B_1, B_2, \ldots, B_n\}$ is linearly independent modulo $K(H)$, then the following are equivalent:*

(a) $E_{A,B}$ is weakly compact.

(b) $A_i \in K(H)$ for all $1 \leq i \leq n$.

THEOREM 1.3. [8, Corollary 3.9] *A non-zero elementary operator on a prime C^*-algebra $\mathcal{A}$ is compact if and only if there are linearly independent subsets $\{A_1, A_2, \dots, A_n\}$ and $\{B_1, B_2, \dots, B_n\}$ in $K(\mathcal{A})$ such that $E(X) = \sum_{i=1}^{n} A_i X B_i$. Here $K(\mathcal{A})$ is the ideal of all compact elements in $\mathcal{A}$.*

Now we state a result due to E. Saksman.

THEOREM 1.4. [11, Proposition 5] *Let X be a reflexive Banach space with approximation property. Assume that A and B are n-tuples of operators on X. Then the elementary operator $E_{A,B}$ on $B(X)$ is weakly compact if and only if $E_{A,B}(X) \subseteq K(X)$.*

Now we state some results about composition operators on l^2, which are useful in our context.

THEOREM 1.5. [6] *Let C_ϕ and C_ψ be two composition operators on l^2. Then $C_\phi - C_\psi$ is a finite rank operator if and only if $\phi(n) = \psi(n)$ for all but finitely many $n \in \mathbb{N}$.*

THEOREM 1.5. [6] *The difference of two composition operators $C_\phi - C_\psi$ is compact if and only if $C_\phi - C_\psi$ is a finite rank operator.*

2. Main Results

In this section we shall characterize compact and weakly compact elementary operators on $B(l^2)$ induced by composition operators on l^2.

THEOREM 2.1. *Let $C_\phi = (C_{\phi_1}, C_{\phi_2}, \dots, C_{\phi_n})$ and $C_\psi = (C_{\psi_1}, C_{\psi_2}, \dots, C_{\psi_n})$ be n-tuples of composition operators on l^2. The elementary operator $E_{C_\phi, C_\psi}(X) = \sum_{i=1}^{n} C_{\phi_i} X C_{\psi_i}$ is never weakly compact, hence never compact.*

First we shall prove a lemma.

LEMMA 2.1. *Sum of a finite number of composition operators on l^2 is not compact.*

Proof. Let $C_{\phi_1}, C_{\phi_2}, \dots, C_{\phi_n}$ be the composition operators on l^2 and let $M = \{n_i : \phi_1^{-1}(n_i) \text{ is nonempty}\}$. Clearly M is an infinite subset of $\mathbb{N}$ and $\{\chi_{n_i}\}_{n_i \in M}$ is a weakly convergent sequence of orthonormal vectors in l^2. We have

$$(C_{\phi_1} + C_{\phi_2} + \cdots + C_{\phi_k})(\chi_{n_i}) = \chi_{\phi_1^{-1}(n_i)} + \cdots + \chi_{\phi_k^{-1}(n_i)}.$$

It follows that

$$\|(C_{\phi_1} + \cdots + C_{\phi_k})(\chi_{n_i})\|^2 = \|\chi_{\phi_1^{-1}(n_i)} + \cdots + \chi_{\phi_k^{-1}(n_i)}\|^2 \geq \overline{\overline{\phi^{-1}(n_i)}} \geq 1$$

for $n_i \in M$. Therefore $\{(C_{\phi_1} + C_{\phi_2} + \cdots + C_{\phi_k})(\chi_{n_i})\}_{n_i \in M}$ does not converge strongly to zero in l^2. Hence $(C_{\phi_1} + C_{\phi_2} + \cdots + C_{\phi_k})$ is not compact. ∎

Proof of Theorem 2.1. We have $E_{C_\phi,C_\psi}(I) = C_{\phi_1}C_{\psi_1} + \cdots + C_{\phi_n}C_{\psi_n}$. Due to the fact that composition of two composition operators is a composition operator, by above lemma we get $E_{C_\phi,C_\psi}(I) \notin K(l^2)$. Since l^2 has approximation property, E_{C_ϕ,C_ψ} is not weakly compact by Theorem 1.4. Hence E_{C_ϕ,C_ψ} is not compact. ∎

Now we give simple proofs of some range inclusion results on elementary operators induced by composition operators on l^2. Here recall that an operator $T \in B(H)$ of the form scalar plus compact is called thin.

THEOREM 2.2. *Let δ_{C_ϕ} be an inner derivation on $B(l^2)$ defined by $\delta_{C_\phi}(X) = C_\phi X - XC_\phi$. Then*

(i) If C_ϕ is a thin composition operator then $R(\delta_{C_\phi}) \subseteq F(l^2)$.

(ii) If C_ϕ is not a thin composition operator on l^2 then $R(\delta_{C_\phi}) \nsubseteq K(l^2)$.

Proof. (i) Let C_ϕ be a thin composition operator on l^2. From Theorem 1.5 it follows that $C_\phi = I + F_\phi$, where F_ϕ is a finite rank operator on l^2. Now

$$\begin{aligned}\delta_{C_\phi}(X) &= C_\phi X - XC_\phi = (I + F_\phi)X - X(I + F_\phi)\\ &= F_\phi X - XF_\phi \in F(l^2), \text{ for each } X \in B(l^2).\end{aligned}$$

Thus $R(\delta_{C_\phi}) \subseteq F(l^2)$.

(ii) Suppose C_ϕ is not a thin operator. Let M_w be a multiplication operator on l^2 defined by $M_w(f) = \sum_{j=1}^{\infty} w_j f(j)\chi_j$ for each $f \in l^2$, where w is a weight function with $w_j\{0, 1\}$, and we will define the sequence w_j later. We shall show that $C_\phi M_w^* - M_w^* C_\phi \notin K(l^2)$.

Now $(C_\phi M_w^* - M_w^* C_\phi)^* = -(C_\phi^* M_w - M_w C_\phi^*)$. We have

$$\begin{aligned}(C_\phi^* M_w - M_w C_\phi^*)(\chi_j) &= C_\phi^* M_w(\chi_j) - M_w C_\phi^*(\chi_j) = C_\phi^*(w_j\chi_j) - M_w(\chi_{\phi(j)})\\ &= w_j\chi_{\phi(j)} - w_{\phi(j)}\chi_{\phi(j)} = (w_j - w_{\phi(j)})\chi_{\phi(j)}\end{aligned}$$

Since C_ϕ is not thin, $M = \{n \in \mathbb{N} : \phi(j) \neq j\}$ is an infinite subset of $\mathbb{N}$ by Theorem (1.5).

For some $n_1 \in M$, define $w_{n_1} = 1$ and $w_{\phi(n_1)} = 0$, suppose $\phi(n_1) = m_1$. Now there is $n_2 \in M - (\{n_1\} \cup \phi^{-1}(m_1))$. Define $w_{n_2} = 1$ and $w_{\phi(n_2)} = 0$, suppose $\phi(n_2) = m_2$. Similarly there is an $n_3 \in M - (\{n_1, n_2\} \cup (\bigcup_{i=1}^{2} \phi^{-1}(n_i)))$.

Define $w_{n_3} = 1$ and $w_{\phi(n_3)} = 0$; suppose $\phi(n_3) = m_3$. In this way inductively we can get $n_k \in M - (\{n_1, n_2, \ldots, n_k\} \cup (\bigcup_{i=1}^{k-1} \phi^{-1}(n_i))$.

Define $w_{n_k} = 1$ and $w_{\phi(n_k)} = 0$; suppose $\phi(n_k) = m_k$. Define $w_j = 0$ for $j \in \mathbb{N} - (\{m_1, m_2, \ldots\ldots, \} \cup (\{n_1, n_2, \ldots\ldots, \})$. Thus $w_j - w_{\phi(j)} = 1$ for infinitely many $j \in \mathbb{N}$. Let $M_1 = \{j \in M : w_j - w_{\phi(j)} = 1\}$. Clearly M_1 is an infinite subset of $\mathbb{N}$. Now we have $\|(C_\phi^* M_w - M_w C_\phi^*)(\chi_j)\| \geq 1$ for all $j \in M_1$. It follows that $C_\phi^* M_w - M_w C_\phi^*$ is not compact and so $C_\phi M_w^* - M_w^* C_\phi$ is not compact. Hence $R(\delta_{C_\phi}) \nsubseteq K(l^2)$. ∎

COROLLARY 2.1. *$R(\delta_{C_\phi}) \subseteq K(l^2)$ if and only if $R(\delta_{C_\phi}) \subseteq F(l^2)$ if and only if C_ϕ is thin.*

THEOREM 2.3. *Let C_ϕ and C_ψ be two composition operators on l^2 and δ_{C_ϕ,C_ψ} be the generalized derivation on $B(l^2)$ defined by $\delta_{C_\phi,C_\psi} = C_\phi X - XC_\psi$. Then $R(\delta_{C_\phi,C_\psi}) \subset F(l^2)$ if and only if C_ϕ and C_ψ are thin operators.*

Proof. Let C_ϕ and C_ψ be two thin composition operators on l^2. Then $C_\phi = I + F_\phi$ and $C_\psi = I + F_\psi$ for some finite rank operator F_ϕ and F_ψ. We get $\delta_{C_\phi,C_\psi} = C_\phi X - XC_\psi \in F(l^2)$, for all $X \in B(l^2)$. Thus $R(\delta_{C_\phi,C_\psi}) \subseteq F(l^2)$.

Conversely, suppose $R(\delta_{C_\phi,C_\psi}) \in F(l^2)$ i.e. $C_\phi X - XC_\psi \in F(l^2)$ for all $X \in B(l^2)$. In particular $\delta_{C_\phi,C_\psi}(I) = C_\phi - C_\psi \in F(l^2)$ i.e. $C_\phi - C_\psi = F, F \in F(l^2)$. It follows that $\delta_{C_\phi}(X) \in F(l^2)$ for all $X \in B(l^2)$ which implies that C_ϕ is thin by Corollary 2.1. Therefore $C_\psi = C_\phi - F$ is also thin. Thus both C_ϕ and C_ψ are thin operators on l^2. ∎

By Corollary 2.1 and the above Theorem, we have the following corollary.

COROLLARY 2.2. *$R(\delta_{C_\phi,C_\psi}) \subseteq K(l^2)$ if and only if C_ϕ and C_ψ are thin.*

EXAMPLE 2.1. Let $A = 2I + K$ and $B = I + K$, $K \in K(l^2)$ be two thin operators. $\delta_{A,B}(I) = (2I + K)I - (I + K) = I \notin K(l^2)$.

This shows that Theorem 2.3 may not be true for general thin operators.

THEOREM 2.4. *Let C_ϕ and C_ψ be two composition operators on l^2 and V_{C_ϕ,C_ψ} be an elementary operator on $B(l^2)$ defined by $V_{C_\phi,C_\psi}(X) = C_\phi X C_\psi - C_\psi X C_\phi$. Then $R(V_{C_\phi,C_\psi}) \subseteq F(l^2)$ if and only if $C_\phi - C_\psi$ is a finite rank operator.*

Proof. We have $V_{C_\phi,C_\psi}(X) = C_\phi X C_\psi - C_\psi X C_\phi$. Suppose $C_\phi - C_\psi = F$, where F is a finite rank operator on l^2. Then $V_{C_\phi,C_\psi}(X) = FXC_\psi - C_\phi XF \in F(l^2)$ for all $X \in B(l^2)$. Thus $R(V_{C_\phi,C_\psi}) \subseteq F(l^2)$.

Conversely, suppose $C_\phi - C_\psi$ is not a finite rank operator, i.e. $\phi(n) \neq \psi(n)$ for infinitely many $n \in \mathbb{N}$, by Theorem 1.5.. Let M_w be a multiplication operator on l^2 defined by $M_w(f) = \sum_{j=1}^{\infty} w_j f(j)\chi_j$, where w is a weight function with $w_j\{0,1\}$, and we will define the sequence w_j later. We shall show that $C_\phi^* M_w C_\psi^* - C_\psi^* M_w C_\psi^* \notin K(l^2)$.

$$\begin{aligned}(C_\phi^* M_w C_\psi^* - C_\psi^* M_w C_\phi^*)(\chi_k) &= (C_\phi^* M_w C_\psi^*)(\chi_k) - (C_\psi^* M_w C_\phi^*)(\chi_k)\\ &= C_\phi^* M_w(\chi_{\psi(k)}) - C_\psi^* M_w(\chi_{\phi(k)}) = C_\phi^*(w_{\psi(k)}\chi_{\psi(k)}) - C_\psi^*(w_{\phi(k)}\chi_{\phi(k)})\\ &= w_{\psi(k)}\chi_{(\phi\circ\psi)(k)} - w_{\phi(k)}\chi_{(\psi\circ\phi)(k)}.\end{aligned}$$

Now

$$\begin{aligned}&\|(C_\phi^* M_w C_\psi^* - C_\psi^* M_w C_\phi^*)(\chi_k)\|^2\\ &\quad = |w_{\psi(k)}|^2 + |w_{\phi(k)}|^2 - (w_{\psi(k)}\overline{w}_{\phi(k)} + w_{\phi(k)}\overline{w}_{\psi(k)})\langle \chi_{(\phi\circ\psi)(k)}, \chi_{(\psi\circ\phi)(k)}\rangle.\end{aligned}$$

If $\phi\circ\psi(k) \neq \psi\circ\phi(k)$, then

$$\|(C_\phi^* M_w C_\psi^* - C_\psi^* M_w C_\phi^*)(\chi_k)\|^2 = |w_{\psi(k)}|^2 + |w_{\phi(k)}|^2. \tag{1}$$

If $\phi \circ \psi(k) = \psi \circ \phi(k)$, then

$$\|(C_\phi^* M_w C_\psi^* - C_\psi^* M_w C_\phi^*)(\chi_k)\|^2 = |w_{\phi(k)} - w_{\psi(k)}|^2. \tag{2}$$

Now $M = \{n \in \mathbb{N} : \phi(n) \neq \psi(n)\}$ is an infinite subset of $\mathbb{N}$. For some $n_1 \in M$, define $w_{\phi(n_1)} = 1$ and $w_{\psi(n_1)} = 0$, suppose $\phi(n_1) = l_1$ and $\psi(n_1) = m_1$. Now there is some $n_2 \in M - (\phi^{-1}(l_1) \cup \phi^{-1}(m_1) \cup \psi^{-1}(l_1) \cup \psi^{-1}(m_1))$. Define $w_{\phi(n_2)} = 1$ and $w_{\psi(n_2)} = 0$, suppose $\phi(n_2) = l_2$ and $\psi(n_2) = m_2$. Now there is some

$$n_3 \in M - (\textstyle\bigcup_{i=1}^2 \phi^{-1}(l_i)) \cup (\bigcup_{i=1}^2 \phi^{-1}(m_i)) \cup (\bigcup_{i=1}^2 \psi^{-1}(l_i)) \cup (\bigcup_{i=1}^2 \psi^{-1}(m_i)).$$

Define $w_{\phi(n_3)} = 1$ and $w_{\psi(n_3)} = 0$, suppose $\phi(n_3) = l_3$ and $\psi(n_3) = m_3$.

In this way inductively we can find

$$n_k \in M - (\textstyle\bigcup_{i=1}^{k-1} \phi^{-1}(l_i)) \cup (\bigcup_{i=1}^{k-1} \phi^{-1}(m_i)) \cup (\bigcup_{i=1}^{k-1} \psi^{-1}(l_i)) \cup (\bigcup_{i=1}^{k-1} \psi^{-1}(m_i)).$$

Define $w_n = 0$ for $n \in \mathbb{N} - (\{l_i : i \in \mathbb{N}\}) \cup \{m_i : i \in \mathbb{N}\})$. Clearly $w_{\phi(n)} - w_{\psi(n)} = 1$ for infinitely many $n \in \mathbb{N}$, and so $M_1 = \{n \in M : w_{\phi(n)} - w_{\psi(n)} = 1\}$ is an infinite subset of M.

Now for $n \in M_1$, by equations (1) and (2), we have

$$\|(C_\phi^* M_w C_\psi^* - C_\psi^* M_w C_\phi^*)(\chi_n)\|^2 \geq 1,$$

which implies that $C_\phi^* M_w C_\psi^* - C_\psi^* M_w C_\phi^*$ and so $C_\phi^* M_w C_\psi^* - C_\psi^* M_w C_\phi^*$ is not compact on l^2.

Thus $R(V_{C_\phi, C_\psi}) \nsubseteq F(l^2)$. Hence the proof. ∎

As a consequence of the proof of Theorem 2.4, we have the following corollary.

Corollary 2.3. *$R(V_{C_\phi, C_\psi}) \subseteq K(l^2)$ if and only if $C_\phi - C_\psi$ is compact.*

In view of Theorem 1.4 and Corollaries 2.1, 2.2 and 2.3 we have the following characterization of weakly compact elementary operators on l^2.

Theorem 2.5. *Let C_ϕ and C_ψ be two composition operators on l^2. Then*

(i) δ_{C_ϕ} is weakly compact if and only if C_ϕ is a thin operator on l^2.

(ii) δ_{C_ϕ, C_ϕ} is weakly compact if and only if C_ϕ and C_ψ are thin operators on l^2.

(iii) V_{C_ϕ, C_ψ} is weakly compact if and only if $C_\phi - C_\psi$ is a compact operator on l^2.

Acknowledgements. 1. The author is grateful to Prof. Nand Lal for his helpful suggestions and discussions.

2. The author is grateful to the referee for his helpful suggestions.

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(received 02.07.2008, in revised form 14.04.2009)

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