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# CAUCHY OPERATOR ON BERGMAN SPACE OF HARMONIC FUNCTIONS ON UNIT DISC

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**Abstract.** We find the exact asymptotic behaviour of singular values of the operator $CP_h$, where $C$ is the integral Cauchy's operator and $P_h$ integral operator with the kernel

$$K(z,\zeta)=\frac{\left(1-|z|^2|\zeta|^2\right)^2}{\pi|1-z\overline{\zeta}|^4}-\frac{2}{\pi}\,\frac{|z|^2|\zeta|^2}{|1-z\overline{\zeta}|^2}.$$

## 1. Introduction

Let $D$ be the unit disc in $C$ and let $dA$ denote Lebesgue measure on $D$. By $L_a^2(D)$ $\left(L_h^2(D)\right)$ we denote the space of all analytic (harmonic) functions $f$ on $D$ with finite norm

$$\left(\int_D |f|^2\,dA\right)^{1/2}=\|f\|<\infty.$$

It is well known that $L_a^2(D)$ and $L_h^2(D)$ are closed subspaces of $L^2(D)$. By $P$ $(P_h)$ we denote the orthogonal projection of $L^2(D)$ onto $L_a^2(D)$ $\left(L_h^2(D)\right)$. With $\langle\cdot,\cdot\rangle$ we denote the inner product on $L^2(D)$.

It is known that $P_h$ is an integral operator on $L^2(D)$ with the kernel

$$K(z,\zeta)=\frac{\left(1-|z|^2|\zeta|^2\right)^2}{\pi|1-z\overline{\zeta}|^4}-\frac{2}{\pi}\,\frac{|z|^2|\zeta|^2}{|1-z\overline{\zeta}|^2}.$$

By $C$ we denote the operator acting on $L^2(D)$ in the following way:

$$Cf(z)=-\frac{1}{\pi}\int_D\frac{f(\zeta)}{\zeta-z}\,dA(\zeta)\quad\text{(Cauchy's operator)}.$$

For a compact operator $T$, let $s_n(T)$ denote the eigenvalues of the operator $(T^*T)^{1/2}$ arranged in non-decreasing order ([5]).

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By $\mathcal{N}_t(T) = \sum_{s_n(T)\geq t} 1,\ t > 0$ we denote the singular values distribution function of $T$. The notation $a_n \sim b_n$ ($a_n \asymp b_n$) means

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \quad \left(0 < c_1 \leq \frac{a_n}{b_n} \leq c_2 < \infty\right),$$

where $c_1, c_2$ do not depend on $n$.

The authors of [1] and [2] have determined the norm and singular values of $C$ on the space $L^2(D)$. It is known, [4], that $s_n(C) \sim \frac{1}{\sqrt{n}}$ $(n \to \infty)$. It was proved in [3] that the restriction of $C$ on $L^2_a(D)$ accelerates the descending of its singular values, i.e.,

$$s_n\left(C|_{L^2_a(D)}\right) = s_n(CP) \asymp \frac{1}{n}.$$

The exact asymptotic behaviour of the singular values of operator $PC$ was given in [4] (for an arbitrary domain), implying

$$s_n(CP) \sim \frac{1}{n}.$$

In this paper we find the exact asymptotic behaviour of singular values of the operator $CP_h$.

## 2. Result

Theorem. *The following asymptotic formula*

$$s_n\left(C|_{L^2_h(D)}\right) = s_n(CP_h) \sim \frac{\sqrt{2}+1}{n}, \quad n \to \infty,$$

*holds.*

*Proof.* The kernel $H_0(\cdot,\cdot)$ of the operator $CP_h$ is given by

$$H_0(z,\zeta) = -\frac{1}{\pi}\int_D \frac{K(t,\zeta)}{t-z}\,dA(t),$$

i.e.,

$$CP_h f(z) = \int_D H_0(z,\zeta) f(\zeta)\,dA(\zeta).$$

The kernel $H_0$ can be represented as

$$H_0(z,\zeta) = -\frac{1}{\pi^2} A(z,\zeta) + \frac{2}{\pi^2} B(z,\zeta), \tag{1}$$

where

$$A(z,\zeta) = \int_D \frac{(1-|t|^2|\zeta|^2)^2}{(t-z)(1-\bar{t}\zeta)^2(1-t\bar{\zeta})^2}\,dA(t),$$

$$B(z,\zeta) = \int_D \frac{|t|^2|\zeta|^2}{(t-z)(1-\bar{t}\zeta)(1-t\bar{\zeta})}\,dA(t).$$

The functions $A$ and $B$ can be determined explicitly using Cauchy-Green formula ([6], p. 42):

$$-\frac{1}{\pi}\int_D \frac{\partial f}{\partial\overline{\zeta}}\frac{dA(\zeta)}{\zeta - z} = f(z) - \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Since

$$\begin{aligned} A(z,\zeta) &= \int_D \frac{\left(1 - t\overline{\zeta} + t\overline{\zeta}(1-\overline{t}\zeta)\right)^2}{(t-z)(1-\overline{t}\zeta)^2(1-t\overline{\zeta})^2}\,dA(t) \\ &= \int_D \frac{dA(t)}{(t-z)(1-\overline{t}\zeta)^2} + \int_D \frac{t^2\overline{\zeta}^2}{(t-z)(1-t\overline{\zeta})^2}\,dA(t) \\ &\quad + \int_D \frac{2t\overline{\zeta}}{(t-z)(1-\overline{t}\zeta)(1-t\overline{\zeta})}\,dA(t), \end{aligned}$$

we obtain

$$\begin{aligned} -\frac{1}{\pi}A(z,\zeta) &= -\frac{1}{\pi}\int_D \frac{dA(t)}{(t-z)(1-\overline{t}\zeta)^2} \\ &\quad -\frac{1}{\pi}\int_D \frac{t^2\,dA(t)}{(t-z)(1-t\overline{\zeta})^2}\cdot\overline{\zeta}^2 + 2\overline{\zeta}\left(-\frac{1}{\pi}\int_D \frac{t\,dA(t)}{(t-z)(1-\overline{t}\zeta)(1-t\overline{\zeta})}\right). \end{aligned} \tag{2}$$

From Cauchy-Green formula, it follows

$$\begin{aligned} -\frac{1}{\pi}\int_D \frac{dA(t)}{(t-z)(1-\overline{t}\zeta)^2} &= \frac{1}{\zeta}(1-\overline{z}\zeta)^{-1} - \frac{1}{\zeta}, \\ -\frac{1}{\pi}\int_D \frac{dA(t)}{(t-z)(1-t\overline{\zeta})^2} &= \frac{z}{(1-z\overline{\zeta})^2}(|z|^2-1). \end{aligned}$$

Hence, from (2), we get

$$\begin{aligned} -\frac{1}{\pi}A(z,\zeta) &= \frac{1}{\zeta}(1-\overline{z}\zeta)^{-1} - \frac{1}{\zeta} + \frac{z\overline{\zeta}^2}{(1-z\overline{\zeta})^2}(|z|^2-1) \\ &\quad + 2\overline{\zeta}\left(-\frac{1}{\pi}\int_D \frac{t\,dA(t)}{(t-z)(1-\overline{t}\zeta)(1-t\overline{\zeta})}\right). \end{aligned} \tag{3}$$

Since, $-\frac{1}{\pi}B(z,\zeta) = -\frac{1}{\pi}\int_D \frac{t\overline{\zeta}(\overline{t}\zeta - 1 + 1)\,dA(t)}{(t-z)(1-\overline{t}\zeta)(1-t\overline{\zeta})}$, we get

$$\begin{aligned} -\frac{1}{\pi}B(z,\zeta) &= -\overline{\zeta}\left(-\frac{1}{\pi}\int_D \frac{t\,dA(t)}{(t-z)(1-t\overline{\zeta})}\right) \\ &\quad + \overline{\zeta}\left(-\frac{1}{\pi}\int_D \frac{t\,dA(t)}{(t-z)(1-\overline{t}\zeta)(1-t\overline{\zeta})}\right). \end{aligned} \tag{4}$$

It follows from (1), (3) and (4) that

$$\begin{aligned} H_0(z,\zeta) &= \frac{1}{\pi}\left(\frac{1}{\zeta}(1-\overline{z}\zeta)^{-1} - \frac{1}{\zeta}\right) + \frac{1}{\pi}\frac{z\overline{\zeta}^2}{(1-z\overline{\zeta})}(|z|^2-1) \\ &\quad + \frac{2\overline{\zeta}}{\pi}\left(-\frac{1}{\pi}\int_D \frac{t\,dA(t)}{(t-z)(1-t\overline{\zeta})}\right). \end{aligned}$$

Applying Cauchy-Green formula again, we obtain

$$H_0(z,\zeta) = \frac{1}{\pi}\left(\frac{1}{\zeta}(1-\overline{z}\zeta)^{-1} - \frac{1}{\zeta}\right) + \frac{1}{\pi}\frac{z\overline{\zeta}^2}{(1-z\overline{\zeta})}(|z|^2-1) + \frac{2\overline{\zeta}}{\pi}\frac{|z|^2-1}{1-z\overline{\zeta}}. \quad (5)$$

Let $P, Q, R : L^2(D) \to L^2(D)$ be linear operators defined by

$$Pf(z) = \frac{1}{\pi}\int_D (\sum_{n=1}^{\infty} \overline{z}^n \zeta^{n-1}) f(\zeta)\, dA(\zeta),$$
$$Qf(z) = \frac{1}{\pi}\int_D (\sum_{n=1}^{\infty} (n+2)(|z|^2-1) z^n \overline{\zeta}^{n+1}) f(\zeta)\, dA(\zeta),$$
$$Rf(z) = \frac{2}{\pi}\int_D \overline{\zeta} f(\zeta)\, dA(\zeta)\cdot(|z|^2-1).$$

Then, it follows from (5) that $CP_h = P + Q + R$. Since $P^*Q = Q^*P = 0$ and $QP^* = PQ^* = 0$, we obtain

$$\mathcal{N}_t(P+Q) = \mathcal{N}_t(P) + \mathcal{N}_t(Q). \quad (6)$$

Since

$$Pf = \sum_{n=1}^{\infty}\langle f, \overline{e}_{n-1}\rangle \overline{e_n(z)} \cdot \frac{1}{\sqrt{n(n+1)}},$$

where $e_n(z) = \sqrt{\frac{n+1}{\pi}} z^n$, $n = 0, 1, \ldots$, we obtain

$$s_n(P) = \frac{1}{\sqrt{n(n+1)}} \sim \frac{1}{n},$$

and so

$$\lim_{t\to 0+} t\mathcal{N}_t(P) = 1. \quad (7)$$

Consider the sequence $f_n(z) = \frac{1}{\sqrt{2\pi}}(|z|^2-1)z^n\sqrt{(n+1)(n+2)(n+3)}$, $n \geq 1$. The system $(f_n)_{n=1}^{\infty}$ is orthogonal on $L^2(D)$.

Notice that

$$Qf(z) = \sum_{n=1}^{\infty} \frac{\sqrt{2}}{\sqrt{(n+1)(n+3)}}\langle f, e_{n+1}\rangle f_n(z);$$

hence we have

$$s_n(Q) = \frac{\sqrt{2}}{\sqrt{(n+1)(n+3)}} \sim \frac{\sqrt{2}}{n}$$

and so

$$\lim_{t\to 0+} t\mathcal{N}_t(Q) = \sqrt{2}. \quad (8)$$

It follows from (6), (7) and (8) that

$$\lim_{t\to 0+} t\mathcal{N}_t(P+Q) = \sqrt{2}+1.$$

Putting $t = s_n(P+Q)$ in the previous equality, we obtain

$$s_n(P+Q) \sim \frac{\sqrt{2}+1}{n}, \quad n \to \infty. \tag{9}$$

Since the rank of $R$ is one, according to Ky-Fan theorem ([5], p. 52), it follows from (9) that

$$s_n(CP_h) \sim \frac{1+\sqrt{2}}{n}, \quad n \to \infty. \quad \blacksquare$$

CONJECTURE. *For arbitrary bounded, simple connected domain $\Omega \subset C$ having analytic boundary,*

$$\lim_{n\to\infty} n s_n(CP_h^{\Omega}) = d$$

*holds.*

Here, $P_h^{\Omega}$ denotes Bergman projection on $L_h^2(\Omega)$ ($L_h^2(\Omega)$ is the space of harmonic functions on $\Omega$), and the constant $d$ depends on $\Omega$. There are some indications that $d = \frac{1+\sqrt{2}}{2\pi}|\partial\Omega|$, where $|\partial\Omega|$ denotes the length of the boundary of $\Omega$.

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