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SENSITIVITY ANALYSIS IN MULTI-PARAMETRIC STRICTLY CONVEX QUADRATIC OPTIMIZATION

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Abstract. In this paper, we study multi-parametric sensitivity analysis for support set and optimal partition invariancy with simultaneous perturbations in the right-hand-side of constraints and the Linear Term of the objective function of the quadratic programming. We show that the invariancy regions are convex polyhedral sets and we describe the set of admissible parameters by the basis vectors of the lineality space and the extreme directions of the defined cone over appropriate problems, and compare them with the linear optimization case.

1. Introduction

Sensitivity analysis and parametric programming, in particular, multi-parametric programming are still in focus of research [3, 5, 6]. In practice, numerical results are subject to errors and the exact solution of the problem under consideration is not known. The results obtained by numerical methods although are approximations of the solutions of the problem but they could be considered as the exact solutions of a corresponding perturbed problem. This is a motivation for sensitivity analysis. Usually perturbations occur in the right-hand-side (RHS) of the constraints and/or in the linear term of the objective function. If perturbation in the data happens with identical parameter, the problem is called a single-parametric optimization problem. Quadratic optimization problems have been solved parametrically [11, 17] and the method is based on simplex method. Karmarkar [12] introduced a method which solves linear optimization problems in polynomial time which is known as interior point method led to reconsider sensitivity analysis for linear optimization [1, 6, 8, 16], quadratic optimization [6]. The concept of optimal partition introduced originally [7] has been extended to quadratic optimization [2]. The optimal partition sensitivity analysis for quadratic optimization problems have been studied when perturbation occurs in the right-hand-side (RHS) or in the linear term of the objective function where the authors investigated the behavior of the optimal value function along with its characteristics [6]. A different view

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of sensitivity analysis named support set sensitivity analysis has been introduced by Koltai and Terlaky [13]. Hladik [10] studied support set and optimal partition invariancy sensitivity analysis for multi-parameter linear optimization. However, there are some differences in support set and optimal partition invariancy sensitivity analysis in linear optimization and in convex quadratic optimization problems.

In this paper, we study these differences and show how to determine the invariancy regions for convex quadratic optimization in view of critical regions which are polyhedral sets. Also we point out the cases in which the support set and optimal partition invariancy sensitivity analysis of convex quadratic optimization problems are the same as support set and optimal partition invariancy sensitivity analysis of linear optimization problems.

Let us consider the primal problem

$$\begin{aligned} \min \mathbf{c}^T\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \\ s.t: \ \mathbf{A}\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq 0, \end{aligned} \tag{QP}$$

and its Wolfe dual

$$\begin{aligned} \max \mathbf{b}^T\mathbf{y} - \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \\ s.t: \ \mathbf{A}^T\mathbf{y} + \mathbf{s} - \mathbf{Q}\mathbf{x} = \mathbf{c} \\ \mathbf{s} \geq 0, \end{aligned} \tag{QD}$$

where $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is a symmetric positive semidefinite matrix, $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$ are fixed data and $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$ are unknown vectors. We denote the feasible solution set of the primal and dual problems by

$$\begin{aligned} \mathcal{QP} &= \{\mathbf{x} : \ \mathbf{A}\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq 0\}, \\ \mathcal{QD} &= \{(\mathbf{y}, \mathbf{s}) : \ \mathbf{A}^T\mathbf{y} + \mathbf{s} - \mathbf{Q}\mathbf{x} = \mathbf{c}, \ \mathbf{s} \geq 0\}, \end{aligned}$$

respectively. Feasible solutions $\mathbf{x} \in \mathcal{QP}$ and $(\mathbf{y}, \mathbf{s}) \in \mathcal{QD}$ are optimal if and only if $\mathbf{x}^T\mathbf{s} = 0$ [4]. Also let $\mathcal{QP}^*$ and $\mathcal{QD}^*$ denote the corresponding optimal solution sets. Then for any $\mathbf{x} \in \mathcal{QP}^*$ and $(\mathbf{y}, \mathbf{s}) \in \mathcal{QD}^*$ we have

$$x_i s_i = 0, \ i = 1, 2, \ldots, n.$$

The support set of a nonnegative vector $\mathbf{x}$ is defined by

$$\sigma(\mathbf{x}) = \{i : \ x_i > 0\}.$$

The index set $\{1, 2, \ldots, n\}$ can be partitioned into subsets

$$\begin{aligned} \mathcal{B} &= \{i : \ x_i > 0 \ \text{ for some } \mathbf{x} \in \mathcal{QP}^*\}, \\ \mathcal{N} &= \{i : \ s_i > 0 \ \text{ for some } (\mathbf{y}, \mathbf{s}) \in \mathcal{QD}^*\}, \\ \mathcal{T} &= \{1, 2, \ldots, n\} \setminus (\mathcal{B} \cup \mathcal{N}) \\ &= \{i : \ x_i = s_i = 0 \ \text{ for all } \mathbf{x} \in \mathcal{QP}^* \ \text{ and } (\mathbf{x}, y, s) \in \mathcal{QD}^*\}. \end{aligned}$$

This partition is known as the optimal partition of the index set $\{1, 2, \ldots, n\}$ for problems (QP) and (QD), and is denoted by $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T})$. The uniqueness of the optimal partition follows from the convexity of the optimal solution sets $\mathcal{QP}^*$ and $\mathcal{QD}^*$. A *maximally complementary solution* $(\mathbf{x}, \mathbf{y}, \mathbf{s})$ is a pair of primal-dual optimal solutions of QP and QD for which

$$\begin{aligned} x_i > 0 &\text{ if and only if } i \in \mathcal{B}, \\ s_i > 0 &\text{ if and only if } i \in \mathcal{N}. \end{aligned}$$

The existence of maximally complementary solution is a consequence of the convexity of the optimal solution sets $\mathcal{QP}^*$ and $\mathcal{QD}^*$ [14]. Knowing a maximally complementary solution, one can easily determine the optimal partition as well. If $\mathcal{T} = \emptyset$ in an optimal partition, then any maximally complementary solution is strictly complementary. It is worth to mention that we have $\sigma(\mathbf{x}^*) \subseteq \mathcal{B}$ and $\sigma(\mathbf{s}^*) \subseteq \mathcal{N}$ for any pair of primal-dual optimal solutions $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{s}^*)$.

Let λ and ϵ be k-dimensional vectors of parameters. We consider the parametric primal problem in the general form

$$\begin{aligned} \min \mathbf{c}^T(\epsilon)\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \\ s.t:\ \ \mathbf{A}\mathbf{x} = \mathbf{b}(\lambda) \\ \mathbf{x} \geq 0, \end{aligned} \tag{QPP}$$

and its Wolfe dual

$$\begin{aligned} \max \mathbf{b}^T(\lambda)\mathbf{y} - \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \\ s.t:\ \ \mathbf{A}^T\mathbf{y} + \mathbf{s} - \mathbf{Q}\mathbf{x} = \mathbf{c}(\epsilon) \\ \mathbf{s} \geq 0. \end{aligned} \tag{QDP}$$

Let $\mathbf{x}^*$ and $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{s}^*)$ be the optimal solutions of (QP) and (QD) respectively. The corresponding optimal partition is denoted by $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T})$. Let us define two kinds of invariancy sensitivity analysis as follows.

Support set invariancy: Sensitivity analysis aims to identify the range of the parameters variations, in which the perturbed problem has an optimal solution with the same support set as of the unperturbed problem i.e. $\sigma(\mathbf{x}) = \sigma(\mathbf{x}^*)$. Note that the given optimal solution is not necessarily a basic feasible solution.

Optimal partition invariancy: We want to find (λ, ϵ) such that the perturbed problem has a the same optimal partition as the unperturbed problems i.e. we are interested in finding the region where

$$\{(\lambda, \epsilon) : \pi(\lambda, \epsilon) = \pi\}.$$

The corresponding sets of sensitivity analysis are denoted by $\Upsilon_P(\mathbf{x}^*)$ and Υ_π respectively which are referred to as critical regions.

Now we try to extend support set and optimal partition invariancy sensitivity analysis to multi-parametric case when the perturbations occur independently and simultaneously. Thus some definitions and theorems on polyhedron are quoted from [15].

Definition 1. A set $\mathbf{C} \subseteq \mathbb{R}^n$ is a polyhedron if and only if there exist an $m \times n$ matrix $\mathbf{H}$ and a vector $\mathbf{h}$ of m real numbers such that $\mathbf{C} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Hx} \leq \mathbf{h}\}$ where $0 < m < \infty$.

Definition 2. The lineality space of $\mathbf{C}$ is defined as

$$L_C = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{Hx} = 0\}.$$

Clearly, $0 \in L_C$ and we have $L_C = \{0\}$ if and only if $r(\mathbf{H}) = n$. Let

$$L_C^{\perp} = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{y}^T\mathbf{x} = 0 \ \text{ for all } \mathbf{y} \in L_C\},$$

be the orthogonal complement of L_C in $\mathbb{R}^n$. Thus, we have $dimL_C = n - r(\mathbf{H})$ and $dimL_C^{\perp} = r(\mathbf{H})$ and $L_C = \{0\}$ if and only if $L_C^{\perp} = \mathbb{R}^n$. Let $\mathbf{G}$ be the matrix of the rows vectors that form a basis for L_C. Then $\mathbf{G}$ has $n - r(\mathbf{H})$ rows and n columns $r(\mathbf{G}) = n - r(\mathbf{H})$ and $L_C^{\perp} = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{Gx} = 0\}$.

Definition 3. Let $\mathbf{S} \subseteq \mathbb{R}^n$ be any set. Then the set

$$\{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{x} = \sum_{i=1}^{t} \mu_i \mathbf{x}^i, \ \sum_{i=1}^{t} \mu_i = 1, \ \mu_i \geq 0, \ \mathbf{x}^i \in \mathbf{S}, 0 \leq t < \infty\},$$

is the convex hull, or conv($\mathbf{S}$) of $\mathbf{S}$ and the set

$$\{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{x} = \sum_{i=1}^{t} \mu_i \mathbf{x}^i, \ \mu_i \geq 0, \ \mathbf{x}^i \in \mathbf{S}, 0 \leq t < \infty\},$$

is the conical hull of $\mathbf{S}$, or cone($\mathbf{S}$).

Theorem 4. *Let $\mathbf{C}$ be a polyhedron, L_C its lineality space and $\mathbf{C}^0 = \mathbf{C} \cap L_C^{\perp}$. Denote by $\mathbf{S} = \{\mathbf{x}^1, \dots, \mathbf{x}^q\}$ the extreme points and by $\mathbf{T} = \{\mathbf{y}^1, \dots, \mathbf{y}^r\}$ the extreme directions of $\mathbf{C}^0$. Then $\mathbf{C}^0 = conv(\mathbf{S}) + cone(\mathbf{T})$ and $\mathbf{C} = L_C + conv(\mathbf{S}) + cone(\mathbf{T})$.*

2. Invariancy regions

To identify the sets $\Upsilon_P(\mathbf{x}^*)$ and Υ_π, a computational method is introduced in this section.

2.1. Support set invariancy.

Let $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{s}^*)$ be a primal-dual optimal solution of (QP) and (QD) with $P = \sigma(\mathbf{x}^*)$ and $Z = \{1, 2, \dots, n\} \setminus P$. Consider the partition (P, Z) of the index set $\{1, 2, \dots, n\}$ for matrices $\mathbf{A}$, $\mathbf{Q}$ and the vectors $\mathbf{x}, \mathbf{c}$ and $\mathbf{s}$ as follows

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_{PP} & \mathbf{Q}_{PZ} \\ \mathbf{Q}_{PZ}^T & \mathbf{Q}_{ZZ} \end{pmatrix}, \qquad \mathbf{A} = (\, \mathbf{A}_P \quad \mathbf{A}_Z \,),$$
$$\mathbf{c} = \begin{pmatrix} \mathbf{c}_P \\ \mathbf{c}_Z \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{x}_P \\ \mathbf{x}_Z \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} \mathbf{s}_P \\ \mathbf{s}_Z \end{pmatrix}. \tag{1}$$

We want to identify the set $\Upsilon_P(\mathbf{x}^*)$.

THEOREM 5. *Let* $(\mathbf{h}_i^P, \mathbf{h}_i)$, $i \in I$ *be a basis of the lineality space*

$$L = \{(\mathbf{v}, \mathbf{u}) : \mathbf{A}_P^T \mathbf{v} - \mathbf{Q}_{PP}\mathbf{u}_P - \mathbf{Q}_{PZ}\mathbf{u}_Z = 0,\ \mathbf{A}\mathbf{u} = 0,\ \mathbf{u}_Z = 0\},$$

and let $(\mathbf{g}_j^P, \mathbf{g}_j)$, $\forall j \in J$ *be all extreme directions of the convex polyhedron cone*

$$S = \{(\mathbf{v}, \mathbf{u}) : \mathbf{A}_P^T \mathbf{v} - \mathbf{Q}_{PP}\mathbf{u}_P - \mathbf{Q}_{PZ}\mathbf{u}_Z \geq 0,\ \mathbf{A}\mathbf{u} = 0,\ \mathbf{u}_Z \geq 0\} \cap L^{\perp}.$$

Then

$$\Upsilon_P(\mathbf{x}^*) = \{(\lambda, \epsilon) : \mathbf{b}(\lambda)^T \mathbf{h}_i^P + \mathbf{c}(\epsilon)^T \mathbf{h}_i = 0,\ \forall i \in I,\ \mathbf{b}(\lambda)^T \mathbf{g}_j^P + \mathbf{c}(\epsilon)^T \mathbf{g}_j > 0,\ \forall j \in J\}.$$

Proof. First we identify the set of (λ, ϵ) such that support set of the given solution remains invariant i.e.

$$\begin{aligned}
\Upsilon_P(\mathbf{x}^*) &= \{(\lambda, \epsilon) : \exists(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{QPP}^* \times \mathcal{QDP}^* \ \text{ with } \sigma(\mathbf{x}) = P\} \\
&= \{(\lambda, \epsilon) : \mathbf{A}\mathbf{x} = \mathbf{b}(\lambda),\ \mathbf{A}^T\mathbf{y} + \mathbf{s} - \mathbf{Q}\mathbf{x} = \mathbf{c}(\epsilon), \\
&\qquad \mathbf{x}, \mathbf{s} \geq 0,\ \mathbf{x}^T\mathbf{s} = 0,\ \sigma(\mathbf{x}) = P\} \\
&= \{(\lambda, \epsilon) : \mathbf{A}_P\mathbf{x}_P = \mathbf{b}(\lambda), \mathbf{x}_P > 0,\ \mathbf{A}_P^T\mathbf{y} - \mathbf{Q}_{PP}\mathbf{x}_P = \mathbf{c}_P(\epsilon), \\
&\qquad \mathbf{A}_Z^T\mathbf{y} - (\mathbf{Q}_{PZ})^T\mathbf{x}_P \leq \mathbf{c}_Z(\epsilon)\}.
\end{aligned}$$

Therefore, it is sufficient to determine the set of λ and ϵ for which the system

$$\begin{aligned}
\mathbf{A}_P\mathbf{x}_P &= \mathbf{b}(\lambda) \\
\mathbf{A}_P^T\mathbf{y} - \mathbf{Q}_{PP}\mathbf{x}_P &= \mathbf{c}_P(\epsilon) \\
\mathbf{A}_Z^T\mathbf{y} - (\mathbf{Q}_{PZ})^T\mathbf{x}_P &\leq \mathbf{c}_Z(\epsilon) \\
\mathbf{x}_P &> 0,
\end{aligned} \tag{2}$$

can be solved. But the system (2) is solvable if and only if the corresponding problem

$$\begin{aligned}
&\max 0^T\mathbf{y} + 0^T\mathbf{x}_P \\
&s.t: \ \mathbf{A}_P\mathbf{x}_P = \mathbf{b}(\lambda) \\
&\mathbf{A}_P^T\mathbf{y} - \mathbf{Q}_{PP}\mathbf{x}_P = \mathbf{c}_P(\epsilon) \\
&\mathbf{A}_Z^T\mathbf{y} - (\mathbf{Q}_{PZ})^T\mathbf{x}_P \leq \mathbf{c}_Z(\epsilon) \\
&\mathbf{x}_P \leq -\eta,
\end{aligned} \tag{3}$$

has an optimal solution for sufficiently small $\eta > 0$. From duality theory in linear programming it is equivalent to the optimality of the following dual problem

$$\begin{aligned}
&\min\ \mathbf{b}(\lambda)^T\mathbf{v} + \mathbf{c}(\epsilon)^T\mathbf{u} - \eta^T\mathbf{w}_P \\
&s.t: \ \mathbf{A}_P^T\mathbf{v} - \mathbf{Q}_{PP}\mathbf{u}_P - \mathbf{Q}_{PZ}\mathbf{u}_Z - \mathbf{w}_P = 0 \\
&\mathbf{A}\mathbf{u} = 0 \\
&\mathbf{u}_Z \geq 0 \\
&\mathbf{w}_P \geq 0.
\end{aligned} \tag{4}$$

On the other hand the problem (4) is equivalent to

$$\begin{aligned} \min \big(\mathbf{b}(\lambda) - \mathbf{A}_P\eta\big)^T\mathbf{v} + \big(\mathbf{c}(\epsilon)^T + \eta^T[\mathbf{Q}_{PP}, \mathbf{Q}_{PZ}]\big)\mathbf{u} \\ s.t:\ \mathbf{A}_P^T\mathbf{v} - \mathbf{Q}_{PP}\mathbf{u}_P - \mathbf{Q}_{PZ}\mathbf{u}_Z \geq 0 \\ \mathbf{A}\mathbf{u} = 0 \\ \mathbf{u}_Z \geq 0. \end{aligned} \tag{5}$$

Now let L denote the lineality space of the problem (5) i.e.

$$L = \{(\mathbf{v}, \mathbf{u}) : \mathbf{A}_P^T\mathbf{v} - \mathbf{Q}_{PP}\mathbf{u}_P - \mathbf{Q}_{PZ}\mathbf{u}_Z = 0,\ \mathbf{A}\mathbf{u} = 0,\ \mathbf{u}_Z = 0\}.$$

Let $(\mathbf{h}_i^P, \mathbf{h}_i),\ i \in I$ denote the vectors of basis of L, and $(\mathbf{g}_j^P, \mathbf{g}_j),\ j \in J$ denote the extreme directions of $S = \{(\mathbf{v}, \mathbf{u}) : \mathbf{A}_P^T\mathbf{v} - \mathbf{Q}_{PP}\mathbf{u}_P - \mathbf{Q}_{PZ}\mathbf{u}_Z \geq 0,\ \mathbf{A}\mathbf{u} = 0,\ \mathbf{u}_Z \geq 0\} \cap L^\perp$. Any solution of problem (5) can be written as

$$\begin{aligned} (\mathbf{v}, \mathbf{u}) &= \sum_{i\in I}\mu_i(\mathbf{h}_i^P, \mathbf{h}_i) + \sum_{j\in J}\mu_j^{'}(\mathbf{g}_j^P, \mathbf{g}_j) \\ &= \Big(\sum_{i\in I}\mu_i\mathbf{h}_i^P + \sum_{j\in J}\mu_j^{'}\mathbf{g}_j^P,\ \sum_{i\in I}\mu_i\mathbf{h}_i + \sum_{j\in J}\mu_j^{'}\mathbf{g}\Big),\ \ \mu_j^{'} \geq 0,\ \forall j \in J, \end{aligned} \tag{6}$$

by Theorem 1. From weak duality Theorem, we have

$$\begin{aligned} (\mathbf{b}(\lambda) - \mathbf{A}_P\eta)^T(\sum_{i\in I}\mu_i\mathbf{h}_i^P + \sum_{j\in J}\mu_j^{'}\mathbf{g}_j^P) \\ + \big(\mathbf{c}(\epsilon)^T + \eta^T[\mathbf{Q}_{PP}, \mathbf{Q}_{PZ}]\big)\big(\sum_{i\in I}\mu_i\mathbf{h}_i + \sum_{j\in J}\mu_j^{'}\mathbf{g}_j)\big) \geq 0, \end{aligned}$$

which holds if and only if

$$\begin{cases} \mathbf{b}(\lambda) - \mathbf{A}_P\eta)^T\mathbf{h}_i^P + \big(\mathbf{c}(\epsilon)^T + \eta^T[\mathbf{Q}_{PP}, \mathbf{Q}_{PZ}]\big)\mathbf{h}_i = 0, & \forall i \in I, \\ (\mathbf{b}(\lambda) - \mathbf{A}_P\eta)^T\mathbf{g}_j^P + \big(\mathbf{c}(\epsilon)^T + \eta^T[\mathbf{Q}_{PP},\ \ \mathbf{Q}_{PZ}]\big)\mathbf{g}_j \geq 0, & \forall j \in J. \end{cases} \tag{7}$$

Since $\mathbf{A}_P^T\mathbf{h}_i^P - [\mathbf{Q}_{PP}, \mathbf{Q}_{PZ}]\mathbf{h}_i = 0$ for all $i \in I$ and $0 \neq -\mathbf{A}_P^T\mathbf{g}_j^P + [\mathbf{Q}_{PP}, \mathbf{Q}_{PZ}]\mathbf{g}_j \leq 0$ for all $j \in J$, then from (7) we get

$$\begin{cases} \mathbf{b}(\lambda)^T\mathbf{h}_i^P + \mathbf{c}(\epsilon)^T\mathbf{h}_i = 0, & \forall i \in I, \\ \mathbf{b}(\lambda)^T\mathbf{g}_j^P + \mathbf{c}(\epsilon)^T\mathbf{g}_j > 0, & \forall j \in J. \end{cases} \tag{8}$$

Therefore, (8) describes the set $\Upsilon_P(\mathbf{x}^*)$. ■

REMARK 6. If $Q = 0$, then the problems (QPP) and (QDP) reduce to linear optimization problems. In this case, we have

$$\Upsilon_P(\mathbf{x}^*) = \{\lambda : \mathbf{A}_P\mathbf{x}_P = \mathbf{b}(\lambda), \mathbf{x}_P > 0\} \times \{\epsilon : \mathbf{A}_P^T\mathbf{y} = \mathbf{c}_P(\epsilon), \mathbf{A}_Z^T\mathbf{y} \leq \mathbf{c}_Z(\epsilon)\},$$

and the relation (7) reduces to

$$\begin{cases} (\mathbf{b}(\lambda) - \mathbf{A}_P\eta)^T\mathbf{h}_i^P = 0,\ \forall i \in I, \\ (\mathbf{b}(\lambda) - \mathbf{A}_P\eta)^T\mathbf{g}_j^P \geq 0,\ \forall j \in J, \\ \mathbf{c}(\epsilon)^T\mathbf{h}_i = 0,\ \forall i \in I, \\ \mathbf{c}(\epsilon)^T\mathbf{g}_j \geq 0,\ \forall j \in J. \end{cases} \tag{9}$$

Therefore

$$\Upsilon_P(\mathbf{x}^*) = \{\lambda : \mathbf{b}(\lambda)^T \mathbf{h}_i^P = 0,\ \forall i \in I,\ \mathbf{b}(\lambda)^T \mathbf{g}_j^P > 0,\ \forall j \in J\} \\ \times \{\epsilon : \mathbf{c}(\epsilon)^T \mathbf{h}_i = 0,\ \forall i \in I,\ \mathbf{c}(\epsilon)^T \mathbf{g}_j \geq 0,\ \forall j \in J\},$$

where $\mathbf{h}_i^P,\ i \in I$, and $\mathbf{g}_j^P,\ j \in J$, are the basis vectors

$$L_1 = \{v : \mathbf{A}_p^T v = 0\},$$

and the extreme directions

$$S_1 = \{v : \mathbf{A}_p^T v \geq 0\} \cap L_1^{\perp},$$

and also $\mathbf{h}_i,\ i \in I$, and $\mathbf{g}_j,\ j \in J$, are the basis vectors

$$L_2 = \{u : \mathbf{A}u = 0,\ u_z = 0\},$$

and the extreme directions

$$S_2 = \{u : \mathbf{A}u = 0,\ u_z \geq 0\} \cap L_2^{\perp},$$

respectively.

Remark 7. Let $Q = 0$ and $\epsilon = 0$. In this case, the set of optimal solutions of dual problem is invariant [16]. Therefore, we will have

$$\Upsilon_P(\mathbf{x}^*) = \{\lambda : \mathbf{A}_P \mathbf{x}_P = \mathbf{b}(\lambda), \mathbf{x}_P > 0\}.$$

Thus,

$$\Upsilon_P(\mathbf{x}^*) = \{\lambda : \mathbf{b}(\lambda)^T \mathbf{h}_i^P = 0,\ \forall i \in I,\ \mathbf{b}(\lambda)^T \mathbf{g}_j^P > 0,\ \forall j \in J\},$$

where $\mathbf{h}_i^P,\ i \in I$, and $\mathbf{g}_j^P,\ j \in J$ are defined as in Remark 6. One can see that $\Upsilon_P(\mathbf{x}^*)$ is the same as Theorem 5 in [10].

Remark 8. Let $Q = 0$ and $\lambda = 0$. In this case, the set of optimal solutions of primal problem is invariant [16]. Therefore, we will have

$$\Upsilon_P(\mathbf{x}^*) = \{\epsilon : \mathbf{A}_P^T \mathbf{y} = \mathbf{c}_P(\epsilon), \mathbf{A}_Z^T \mathbf{y} \leq \mathbf{c}_Z(\epsilon)\}.$$

Thus,

$$\Upsilon_P(\mathbf{x}^*) = \{\epsilon : \mathbf{c}(\epsilon)^T \mathbf{h}_i = 0,\ \forall i \in I,\ \mathbf{c}(\epsilon)^T \mathbf{g}_j \geq 0,\ \forall j \in J\},$$

where $\mathbf{h}_i,\ i \in I$, and $\mathbf{g}_j,\ j \in J$ are defined as in Remark 6, and note that $\Upsilon_P(\mathbf{x}^*)$ is as Theorem 1 in [10].

Example 1. Consider the problem

$$\begin{aligned} &\min x_1^2 + 2x_1x_2 + x_2^2 \\ s.t:\ & x_1 + x_2 - x_3 = 2 \\ & x_2 + x_4 = 2 \\ & -x_1 + x_2 + x_5 = 2 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

Let $\mathbf{b}(\lambda) = (2+\lambda, 2-\lambda, 2-2\lambda)$. It is easy to verify that optimal partition for $\lambda = 0$ is as follows

$$\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T}) = (\{1,2,4,5\}, \{3\}, \emptyset),$$

and $\mathbf{x}^* = (2,0,0,2,4)$ is an optimal solution with $\sigma(\mathbf{x}^*) = \{1,4,5\}$. Using Theorem 5, we specify $\Upsilon_P(\mathbf{x}^*)$. The lineality space and convex polyhedron cone are

$$\begin{aligned} L = \{(\mathbf{v},\mathbf{u}) : & v_1 - v_3 - 2u_1 - 2u_2 = 0, u_1 + u_2 - u_3 = 0,\ u_2 + u_4 = 0, \\ & -u_1 + u_2 + u_5 = 0,\ v_2 = v_3 = u_2 = u_3 = 0\}, \\ S = \{(\mathbf{v},\mathbf{u}) : & v_1 - v_3 - 2u_1 - 2u_2 \geq 0,\ u_1 + u_2 - u_3 = 0,\ u_2 + u_4 = 0, \\ & -u_1 + u_2 + u_5 = 0,\ u_2, u_3, v_2, v_3 \geq 0\}. \end{aligned}$$

Since $L = \{0\}$, thus there is no basis for the lineality space and the extreme directions of the set S are as follows

$$\begin{aligned} &g_1^P = (1,0,0)^T, \quad g_1 = (-1,1,0,-1,-2)^T, \quad g_3^P = (1,0,1)^T, \\ &g_2^P = (0,1,0)^T, \quad g_2 = (0,0,0,0,0)^T, \qquad\quad g_3 = (0,0,0,0,0)^T. \end{aligned}$$

Hence, we have

$$\Upsilon_P(\mathbf{x}^*) = \{\lambda :\ 2+\lambda > 0,\ 2-\lambda > 0\} = (-2,2).$$

The region is matched with the region obtained for single-parametric case [9].

3. Optimal partition invariancy

Let $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T})$ be an optimal partition of the primal-dual problems (QP) and (QD). We consider partition $(\mathcal{B}, \mathcal{N}, \mathcal{T})$ of the index set $\{1, 2, \ldots, n\}$ for matrices $\mathbf{A}$, $\mathbf{Q}$ and vectors $\mathbf{x}, \mathbf{c}$ and $\mathbf{s}$ as follows.

$$\begin{aligned} \mathbf{Q} = \begin{pmatrix} \mathbf{Q}_{\mathcal{B}\mathcal{B}} & \mathbf{Q}_{\mathcal{B}\mathcal{N}} & \mathbf{Q}_{\mathcal{B}\mathcal{T}} \\ \mathbf{Q}_{\mathcal{B}\mathcal{N}}^T & \mathbf{Q}_{\mathcal{N}\mathcal{N}} & \mathbf{Q}_{\mathcal{N}\mathcal{T}} \\ \mathbf{Q}_{\mathcal{B}\mathcal{T}}^T & \mathbf{Q}_{\mathcal{N}\mathcal{T}}^T & \mathbf{Q}_{\mathcal{T}\mathcal{T}} \end{pmatrix} \qquad \mathbf{A} = (\, \mathbf{A}_{\mathcal{B}} \quad \mathbf{A}_{\mathcal{N}} \quad \mathbf{A}_{\mathcal{T}} \,) \\ \mathbf{c} = \begin{pmatrix} \mathbf{c}_{\mathcal{B}} \\ \mathbf{c}_{\mathcal{N}} \\ \mathbf{c}_{\mathcal{T}} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{x}_{\mathcal{B}} \\ \mathbf{x}_{\mathcal{N}} \\ \mathbf{x}_{\mathcal{T}} \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} \mathbf{s}_{\mathcal{B}} \\ \mathbf{s}_{\mathcal{N}} \\ \mathbf{s}_{\mathcal{T}} \end{pmatrix}. \end{aligned} \tag{10}$$

The following theorem describes the set Υ_π.

THEOREM 9. *Let $(\mathbf{h}_i^{\mathcal{B}}, \mathbf{h}_i)$, $i \in I$ be a basis of the lineality space*

$$L = \{(\mathbf{u},\mathbf{w}) : \mathbf{A}_{\mathcal{B}}^T\mathbf{u} - \left[\mathbf{Q}_{\mathcal{B}\mathcal{B}}^T, \mathbf{Q}_{\mathcal{B}\mathcal{T}}, \mathbf{Q}_{\mathcal{B}\mathcal{N}}\right] \begin{pmatrix} \mathbf{w}_{\mathcal{B}} \\ \mathbf{w}_{\mathcal{T}} \\ \mathbf{w}_{\mathcal{N}} \end{pmatrix} = 0,\ \mathbf{A}\mathbf{w} = 0,\ \mathbf{w}_{\mathcal{N}} = 0\},$$

and let $(\mathbf{g}_j^{\mathcal{B}}, \mathbf{g}_j)$, $\forall j \in J$ be all extreme directions of the convex polyhedron cone

$$S = \{(\mathbf{u},\mathbf{w}) : \mathbf{A}_{\mathcal{B}}^T\mathbf{u} - \left[\mathbf{Q}_{\mathcal{B}\mathcal{B}}^T, \mathbf{Q}_{\mathcal{B}\mathcal{T}}, \mathbf{Q}_{\mathcal{B}\mathcal{N}}\right] \begin{pmatrix} \mathbf{w}_{\mathcal{B}} \\ \mathbf{w}_{\mathcal{T}} \\ \mathbf{w}_{\mathcal{N}} \end{pmatrix} \geq 0,\ \mathbf{A}\mathbf{w} = 0,\ \mathbf{w}_{\mathcal{N}} \geq 0\} \cap L^{\perp}.$$

Then

$$\Upsilon_\pi = \{(\lambda,\epsilon) : \mathbf{b}(\lambda)^T\mathbf{h}_i^{\mathcal{B}} + \mathbf{c}(\epsilon)^T\mathbf{h}_i = 0,\ \forall i \in I,\ \mathbf{b}(\lambda)^T\mathbf{g}_j^{\mathcal{B}} + \mathbf{c}(\epsilon)^T\mathbf{g}_j > 0,\ \forall j \in J\}.$$

Proof. First we identify the set of (λ,ϵ) such that given optimal partition remains invariant, i.e.,

$$\begin{aligned}\Upsilon_\pi &= \{(\lambda,\epsilon) : \pi(\lambda,\epsilon) = \pi\}\\ &= \{(\lambda,\epsilon) : \mathbf{A}\mathbf{x} = \mathbf{b}(\lambda),\ \mathbf{A}^T\mathbf{y} + \mathbf{s} - \mathbf{Q}\mathbf{x} = \mathbf{c}(\epsilon),\\ &\qquad \mathbf{x}_{\mathcal{B}} > 0, \mathbf{s}_{\mathcal{N}} > 0,\ \mathbf{x}_{\mathcal{N}\cup\mathcal{T}} = \mathbf{s}_{\mathcal{B}\cup\mathcal{T}} = 0\}\\ &= \{(\lambda,\epsilon) : \mathbf{A}_{\mathcal{B}}\mathbf{x}_{\mathcal{B}} = \mathbf{b}(\lambda), \mathbf{x}_{\mathcal{B}} > 0,\ \mathbf{A}_{\mathcal{B}}^T\mathbf{y} - \mathbf{Q}_{\mathcal{B}\mathcal{B}}\mathbf{x}_{\mathcal{B}} = \mathbf{c}_{\mathcal{B}}(\epsilon),\\ &\qquad \mathbf{A}_{\mathcal{N}}^T\mathbf{y} - \mathbf{Q}_{\mathcal{B}\mathcal{N}}^T\mathbf{x}_{\mathcal{B}} < \mathbf{c}_{\mathcal{N}}(\epsilon),\ \mathbf{A}_{\mathcal{T}}^T\mathbf{y} - \mathbf{Q}_{\mathcal{B}\mathcal{T}}^T\mathbf{x}_{\mathcal{B}} = \mathbf{c}_{\mathcal{T}}(\epsilon)\}.\end{aligned}$$

Therefore, it is sufficient to determine the set of λ and ϵ for which the system

$$\begin{aligned}\mathbf{A}_{\mathcal{B}}\mathbf{x}_{\mathcal{B}} &= \mathbf{b}(\lambda)\\ \mathbf{A}_{\mathcal{B}}^T\mathbf{y} - \mathbf{Q}_{\mathcal{B}\mathcal{B}}\mathbf{x}_{\mathcal{B}} &= \mathbf{c}_{\mathcal{B}}(\epsilon)\\ \mathbf{A}_{\mathcal{N}}^T\mathbf{y} - \mathbf{Q}_{\mathcal{B}\mathcal{N}}^T\mathbf{x}_{\mathcal{B}} &< \mathbf{c}_{\mathcal{N}}(\epsilon)\\ \mathbf{A}_{\mathcal{T}}^T\mathbf{y} - \mathbf{Q}_{\mathcal{B}\mathcal{T}}^T\mathbf{x}_{\mathcal{B}} &= \mathbf{c}_{\mathcal{T}}(\epsilon)\\ \mathbf{x}_{\mathcal{B}} &> 0,\end{aligned} \tag{11}$$

can be solved. The system (11) is solvable if and only if the corresponding problem

$$\begin{aligned}&\max 0^T\mathbf{y} + 0^T\mathbf{x}_P\\ s.t:\ \ &\mathbf{A}_{\mathcal{B}}\mathbf{x}_{\mathcal{B}} = \mathbf{b}(\lambda)\\ &\mathbf{A}_{\mathcal{B}}^T\mathbf{y} - \mathbf{Q}_{\mathcal{B}\mathcal{B}}\mathbf{x}_{\mathcal{B}} = \mathbf{c}_{\mathcal{B}}(\epsilon)\\ &\mathbf{A}_{\mathcal{N}}^T\mathbf{y} - \mathbf{Q}_{\mathcal{B}\mathcal{N}}^T\mathbf{x}_{\mathcal{B}} \le \mathbf{c}_{\mathcal{N}}(\epsilon) - \eta\\ &\mathbf{A}_{\mathcal{T}}^T\mathbf{y} - \mathbf{Q}_{\mathcal{B}\mathcal{T}}^T\mathbf{x}_{\mathcal{B}} = \mathbf{c}_{\mathcal{T}}(\epsilon)\\ &\mathbf{x}_{\mathcal{B}} \ge \zeta,\end{aligned} \tag{12}$$

has an optimal solution for sufficiently small vectors $\eta > 0$ and $\zeta > 0$. From duality theory in linear programming it is equivalent to the optimality of the following dual problem

$$\begin{aligned}\min\ &\mathbf{b}(\lambda)^T\mathbf{u} + \mathbf{c}_{\mathcal{B}}(\epsilon)^T\mathbf{w}_{\mathcal{B}} + \mathbf{c}_{\mathcal{T}}(\epsilon)^T\mathbf{w}_{\mathcal{T}} + (\mathbf{c}_{\mathcal{N}}(\epsilon) - \eta)^T\mathbf{w}_{\mathcal{N}} - \zeta^T\mathbf{v}\\ s.t:\ \ &\mathbf{A}_{\mathcal{B}}^T\mathbf{u} - \mathbf{Q}_{\mathcal{B}\mathcal{B}}^T\mathbf{w}_{\mathcal{B}} - \mathbf{Q}_{\mathcal{B}\mathcal{T}}\mathbf{w}_{\mathcal{T}} - \mathbf{Q}_{\mathcal{B}\mathcal{N}}\mathbf{w}_{\mathcal{N}} - \mathbf{v} = 0\\ &\mathbf{A}\mathbf{w} = 0\\ &\mathbf{w}_{\mathcal{N}} \ge 0\\ &\mathbf{v} \ge 0.\end{aligned} \tag{13}$$

On the other hand the problem (13) is equivalent to

$$\begin{aligned} \min \big(\mathbf{b}(\lambda) - \mathbf{A}_{\mathcal{B}}\zeta\big)^T \mathbf{u} + \mathbf{c}(\epsilon)^T \mathbf{w} - \eta^T \mathbf{w}_{\mathcal{N}} + \zeta^T [\mathbf{Q}_{\mathcal{B}\mathcal{B}}^T, \mathbf{Q}_{\mathcal{B}\mathcal{T}}, \mathbf{Q}_{\mathcal{B}\mathcal{N}}] \begin{pmatrix} \mathbf{w}_{\mathcal{B}} \\ \mathbf{w}_{\mathcal{T}} \\ \mathbf{w}_{\mathcal{N}} \end{pmatrix} \\ s.t: \ \mathbf{A}_{\mathcal{B}}^T \mathbf{u} - \mathbf{Q}_{\mathcal{B}\mathcal{B}}^T \mathbf{w}_{\mathcal{B}} - \mathbf{Q}_{\mathcal{B}\mathcal{T}} \mathbf{w}_{\mathcal{T}} - \mathbf{Q}_{\mathcal{B}\mathcal{N}} \mathbf{w}_{\mathcal{N}} \geq 0 \\ \mathbf{A}\mathbf{w} = 0 \\ \mathbf{w}_{\mathcal{N}} \geq 0. \end{aligned} \tag{14}$$

Now let L denote the lineality space of the problem (14) i.e.

$$L = \{(\mathbf{u}, \mathbf{w}) : \mathbf{A}_{\mathcal{B}}^T \mathbf{u} - \big[\mathbf{Q}_{\mathcal{B}\mathcal{B}}^T, \mathbf{Q}_{\mathcal{B}\mathcal{T}}, \mathbf{Q}_{\mathcal{B}\mathcal{N}}\big] \begin{pmatrix} \mathbf{w}_{\mathcal{B}} \\ \mathbf{w}_{\mathcal{T}} \\ \mathbf{w}_{\mathcal{N}} \end{pmatrix} = 0, \ \mathbf{A}\mathbf{w} = 0, \ \mathbf{w}_{\mathcal{N}} = 0\}.$$

Let $(\mathbf{h}_i^{\mathcal{B}}, \mathbf{h}_i), \ i \in I$ denote the vectors of basis of L, and $(\mathbf{g}_j^{\mathcal{B}}, \mathbf{g}_j), \ j \in J$ denote the extreme directions of

$$S = \{(\mathbf{u}, \mathbf{w}) : \mathbf{A}_{\mathcal{B}}^T \mathbf{u} - \big[\mathbf{Q}_{\mathcal{B}\mathcal{B}}^T, \mathbf{Q}_{\mathcal{B}\mathcal{T}}, \mathbf{Q}_{\mathcal{B}\mathcal{N}}\big] \begin{pmatrix} \mathbf{w}_{\mathcal{B}} \\ \mathbf{w}_{\mathcal{T}} \\ \mathbf{w}_{\mathcal{N}} \end{pmatrix} \geq 0, \ \mathbf{A}\mathbf{w} = 0, \ \mathbf{w}_{\mathcal{N}} \geq 0\} \cap L^{\perp}.$$

By Theorem 4, any solution of problem (14) can be written as

$$\begin{aligned} (\mathbf{u}, \mathbf{w}) &= \sum_{i \in I} \mu_i (\mathbf{h}_i^{\mathcal{B}}, \mathbf{h}_i) + \sum_{j \in J} \mu_j^{'} (\mathbf{g}_j^{\mathcal{B}}, \mathbf{g}_j) \\ &= \Big(\sum_{i \in I} \mu_i \mathbf{h}_i^{\mathcal{B}} + \sum_{j \in J} \mu_j^{'} \mathbf{g}_j^{\mathcal{B}}, \ \sum_{i \in I} \mu_i \mathbf{h}_i + \sum_{j \in J} \mu_j^{'} \mathbf{g} \Big), \ \mu_j^{'} \geq 0, \ \forall j \in J. \end{aligned}$$

From weak duality Theorem, we have

$$\begin{aligned} (\mathbf{b}(\lambda) - \mathbf{A}_{\mathcal{B}}\zeta)^T (\sum_{i \in I} \mu_i \mathbf{h}_i^{\mathcal{B}} + \sum_{j \in J} \mu_j^{'} \mathbf{g}_j^{\mathcal{B}}) + \big(\mathbf{c}(\epsilon)^T + \zeta^T [\mathbf{Q}_{\mathcal{B}\mathcal{B}}^T, \mathbf{Q}_{\mathcal{B}\mathcal{T}}, \mathbf{Q}_{\mathcal{B}\mathcal{N}}]\big) \times \\ \times \big(\sum_{i \in I} \mu_i \mathbf{h}_i + \sum_{j \in J} \mu_j^{'} \mathbf{g}_j)\big) - \eta^T \big(\sum_{i \in I} \mu_i \mathbf{h}_i + \sum_{j \in J} \mu_j^{'} \mathbf{g}_j)\big)_{\mathcal{N}} \geq 0. \end{aligned}$$

The above relation holds if and only if

$$\begin{cases} (\mathbf{b}(\lambda) - \mathbf{A}_{\mathcal{B}}\zeta)^T \mathbf{h}_i^{\mathcal{B}} + \big(\mathbf{c}(\epsilon)^T + \zeta^T [\mathbf{Q}_{\mathcal{B}\mathcal{B}}^T, \mathbf{Q}_{\mathcal{B}\mathcal{T}}, \mathbf{Q}_{\mathcal{B}\mathcal{N}}]\big) \mathbf{h}_i - \eta^T (\mathbf{h}_i)_{\mathcal{N}} = 0, \ \forall i \in I, \\ (\mathbf{b}(\lambda) - \mathbf{A}_{\mathcal{B}}\zeta)^T \mathbf{g}_j^{\mathcal{B}} + \big(\mathbf{c}(\epsilon)^T + \zeta^T [\mathbf{Q}_{\mathcal{B}\mathcal{B}}^T, \mathbf{Q}_{\mathcal{B}\mathcal{T}}, \mathbf{Q}_{\mathcal{B}\mathcal{N}}]\big) \mathbf{g}_j - \eta^T (\mathbf{g}_i)_{\mathcal{N}} \geq 0, \ \forall j \in J, \end{cases} \tag{15}$$

Since $\mathbf{A}_{\mathcal{B}}^T \mathbf{h}_i^{\mathcal{B}} - [\mathbf{Q}_{\mathcal{B}\mathcal{B}}^T, \mathbf{Q}_{\mathcal{B}\mathcal{T}}, \mathbf{Q}_{\mathcal{B}\mathcal{N}}] \mathbf{h}_i = 0$, $0 \neq -\mathbf{A}_{\mathcal{B}}^T \mathbf{g}_j^{\mathcal{B}} + [\mathbf{Q}_{\mathcal{B}\mathcal{B}}^T, \mathbf{Q}_{\mathcal{B}\mathcal{T}}, \mathbf{Q}_{\mathcal{B}\mathcal{N}}] \mathbf{g}_j \leq 0$, $\eta^T (\mathbf{h}_i)_{\mathcal{N}} = 0$ and $\eta^T (\mathbf{g}_j)_{\mathcal{N}} > 0$ from (15) we get

$$\begin{cases} \mathbf{b}(\lambda)^T \mathbf{h}_i^{\mathcal{B}} + \mathbf{c}(\epsilon)^T \mathbf{h}_i = 0, \ \forall i \in I, \\ \mathbf{b}(\lambda)^T \mathbf{g}_j^{\mathcal{B}} + \mathbf{c}(\epsilon)^T \mathbf{g}_j > 0, \ \forall j \in J. \end{cases} \tag{16}$$

Therefore, (16) describes the set Υ_{π}. ■

REMARK 10. If the given pair of primal-dual optimal solution $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{s}^*)$ is strictly complementary, then $\sigma(\mathbf{x}^*) = \mathcal{P} = \mathcal{B}$, $Z = \mathcal{N}$ and $\mathcal{T} = \varnothing$. Therefore we have

$$\begin{aligned}\Upsilon_{\mathcal{B}}(\mathbf{x}^*) = \{(\lambda,\epsilon) : \mathbf{A}_{\mathcal{B}}\mathbf{x}_{\mathcal{B}} = \mathbf{b}(\lambda), \mathbf{x}_{\mathcal{B}} > 0,\ \mathbf{A}_{\mathcal{B}}^T\mathbf{y} - \mathbf{Q}_{\mathcal{B}\mathcal{B}}\mathbf{x}_{\mathcal{B}} = \mathbf{c}_{\mathcal{B}}(\epsilon),\\ \mathbf{A}_{\mathcal{N}}^T\mathbf{y} - (\mathbf{Q}_{\mathcal{B}\mathcal{N}})^T\mathbf{x}_{\mathcal{B}} \le \mathbf{c}_{\mathcal{N}}(\epsilon)\},\\ \Upsilon_{\pi} = \{(\lambda,\epsilon) : \mathbf{A}_{\mathcal{B}}\mathbf{x}_{\mathcal{B}} = \mathbf{b}(\lambda), \mathbf{x}_{\mathcal{B}} > 0,\ \mathbf{A}_{\mathcal{B}}^T\mathbf{y} - \mathbf{Q}_{\mathcal{B}\mathcal{B}}\mathbf{x}_{\mathcal{B}} = \mathbf{c}_{\mathcal{B}}(\epsilon),\\ \mathbf{A}_{\mathcal{N}}^T\mathbf{y} - (\mathbf{Q}_{\mathcal{B}\mathcal{N}})^T\mathbf{x}_{\mathcal{B}} < \mathbf{c}_{\mathcal{N}}(\epsilon)\}.\end{aligned}$$

These show that $\Upsilon_\pi \subseteq \Upsilon_{\mathcal{B}}(\mathbf{x}^*)$; that is, the optimal partition invariancy region is a subset of the support set invariancy region when the given optimal solution is a strictly complementary solution.

REMARK 11. If $Q = 0$, then $\mathcal{T} = \varnothing$ and we will have

$$\Upsilon_\pi = \{\lambda : \mathbf{A}_B\mathbf{x}_B = \mathbf{b}(\lambda), \mathbf{x}_B > 0\} \times \{\epsilon : \mathbf{A}_B^T\mathbf{y} = \mathbf{c}_B(\epsilon),\ \mathbf{A}_N^T\mathbf{y} < \mathbf{c}_N(\epsilon)\},$$

so

$$\begin{aligned}\Upsilon_\pi = \{\lambda : \mathbf{b}(\lambda)^T\mathbf{h}_i^{\mathcal{B}} = 0,\ \forall i \in I,\ \mathbf{b}(\lambda)^T\mathbf{g}_j^{\mathcal{B}} > 0,\ \forall j \in J\} \times\\ \times \{\epsilon : \mathbf{c}(\epsilon)^T\mathbf{h}_i = 0,\ \forall i \in I,\ \mathbf{c}(\epsilon)^T\mathbf{g}_j > 0,\ \forall j \in J\},\end{aligned}$$

where $\mathbf{h}_i^{\mathcal{B}},\ i \in I$, and $\mathbf{g}_j^{\mathcal{B}},\ j \in J$, are the basis vectors

$$L_3 = \{u : \mathbf{A}_B^T u = 0\},$$

and the extreme directions

$$S_3 = \{u : \mathbf{A}_B^T u \ge 0\} \cap L_3^\perp,$$

and also $\mathbf{h}_i,\ i \in I$, and $\mathbf{g}_j,\ j \in J$, are the basis vectors

$$L_4 = \{w : \mathbf{A}w = 0,\ w_N = 0\},$$

and the extreme directions

$$S_4 = \{w : \mathbf{A}w = 0,\ w_N \ge 0\} \cap L_4^\perp,$$

respectively.

REMARK 12. If $Q = 0$ and $\lambda = 0$, then we have

$$\Upsilon_\pi = \{\epsilon : \mathbf{c}(\epsilon)^T\mathbf{h}_i = 0,\ \forall i \in I,\ \mathbf{c}(\epsilon)^T\mathbf{g}_j > 0,\ \forall j \in J\},$$

where $\mathbf{h}_i$ and $\mathbf{g}_j$ are defined as in Remark 11. Note that Υ_π is the same as Theorem 3 in [10].

REMARK 13. If $Q = 0$ and $\epsilon = 0$, then

$$\Upsilon_\pi = \{\lambda : \mathbf{b}(\lambda)^T\mathbf{h}_i^{\mathcal{B}} = 0,\ \forall i \in I,\ \mathbf{b}(\lambda)^T\mathbf{g}_j^{\mathcal{B}} > 0,\ \forall j \in J\},$$

where $\mathbf{h}_i^{\mathcal{B}}$ and $\mathbf{g}_j^{\mathcal{B}}$ are defined as in Remark 11. It is obvious that Υ_π is the same as Theorem 7 in [10].

EXAMPLE 2. Consider the Example 1. Let $\mathbf{b}(\lambda) = (2 + \lambda_1 - 2\lambda_2, 2 - 3\lambda_1 + \lambda_2, 2 - 2\lambda_1 - 3\lambda_2)^T$ and $\mathbf{c}(\epsilon) = (\epsilon_1 + 2\epsilon_2, 3\epsilon_1 - 5\epsilon_2, 0, 0, 0)^T$. We specify Υ_π, by using Theorem 9. Basis for the lineality space

$$L = \{(\mathbf{u}, \mathbf{w}) : u_1 - u_3 - 2w_1 - 2w_2 = 0, u_1 + u_2 + u_3 - 2w_1 - 2w_2 = 0, w_1 + w_2 = 0, \\ w_2 + w_4 = 0, -w_1 + w_2 + w_5 = 0, u_2 = u_3 = w_3 = 0\}$$

is $\mathbf{h}_i^{\mathcal{B}} = (0, 0, 0)^T$, and $\mathbf{h}_i = (1, -1, 0, 1, 2)^T$. The extreme directions of the polyhedron

$$S = \{(\mathbf{u}, \mathbf{w}) : u_1 - u_3 - 2w_1 - 2w_2 \geq 0, u_1 + u_2 + u_3 - 2w_1 - 2w_2 \geq 0, w_1 + w_2 = 0, \\ w_2 + w_4 = 0, -w_1 + w_2 + w_5 = 0, u_2, u_3, w_3 \geq 0\} \cap \{(\mathbf{u}, \mathbf{w}) : w_1 - w_2 + w_4 + 2w_5 = 0\},$$

are as follows

$$\mathbf{g}_1^{\mathcal{B}} = (1, 0, 0)^T, \ \mathbf{g}_2^{\mathcal{B}} = (0, 1, 0)^T, \ \mathbf{g}_3^{\mathcal{B}} = (1, 0, 1)^T, \\ \mathbf{g}_1 = \mathbf{g}_2 = \mathbf{g}_3 = (0, 0, 0, 0, 0)^T.$$

Therefore, we get

$$\Upsilon_\pi = \{(\lambda, \epsilon) : -2\epsilon_1 + 7\epsilon_2 = 0, \ 2 + \lambda_1 - 2\lambda_2 > 0, \ 2 - 3\lambda_1 + \lambda_2 > 0, \ 4 - \lambda_1 - 5\lambda_2 > 0\}.$$

REMARK 14. If $\epsilon_1 = \lambda_1, \epsilon_2 = \lambda_2$, then the invariancy region is

$$\lambda_2 = \frac{2}{7}\lambda_1, \ -\frac{14}{3} \leq \lambda_1 < \frac{14}{19}.$$

REMARK 15. If $\mathbf{b}(\lambda) = (2 + \lambda, 2 - \lambda, 2 - 2\lambda)^T$ and $\epsilon = 0$, then $\Upsilon_\pi = (-2, 2)$. The region is matched with the region obtained for single-parametric case [9].

4. Conclusion

We studied multi-parametric sensitivity analysis for quadratic optimization in view of support set and optimal partition invariancy. The resulting critical regions are determined by linear equality and linear inequalities or strict inequalities which represent polyhedral set. We stated them for linear optimization with simultaneously perturbations in the right-hand-side of the constraints and the objective coefficients, and compared them with independent perturbations [10]. Our results are extension of the results linear optimization in [10].

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