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CERTAIN CLASSES OF MULTIPLE GENERATING FUNCTIONS FOR SOME SETS OF POLYNOMIALS IN SEVERAL VARIABLES

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Abstract. In this paper some generating functions for some sets of polynomials in several variables are established. In these classes of generating functions an arbitrary sequence of multivariable functions is considered. The generating functions so derived are shown here to lead some known results of Raina, Raina and Bajpai and Zeitlin and are capable to provide as special cases, a large number of new summation formulas and generating functions for simpler sequences, extended polynomials and generalized Lauricella functions.

1. Introduction and results required

The generalized factorial function in terms of Pochhammer symbol [4, p. 22] is

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n=0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n=1,2,3,\dots, \end{cases} \tag{1}$$

$$(-n)_k = \frac{(-1)^k n!}{(n-k)!}. \tag{2}$$

The following generalizations of binomial expansions derivable from the Lagrange's expansions [4, p. 355, eqns. (5), (9)] are

$$\sum_{n=0}^{\infty} \binom{\alpha+(\beta+1)n}{n} t^n = \frac{(1+\omega)^{\alpha+1}}{(1-\beta\omega)}, \tag{3}$$

$$\sum_{n=0}^{\infty} \frac{\alpha}{\alpha+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} t^n = (1+\omega)^{\alpha}, \tag{4}$$

where

$$\binom{k}{n} = \frac{\Gamma(k+1)}{\Gamma(n+1)\Gamma(k-n+1)} \tag{5}$$

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and α, β are arbitrary complex numbers, ω is a function defined implicitly in terms of t given by

$$\omega = t(1+\omega)^{\beta+1}, \quad \omega(0) = 0. \tag{6}$$

Let the sequence ϕ_k $(k \geqslant 0)$ and ω be a function defined implicitly in terms of t by (6). Then for arbitrary complex parameters α, β and γ independent of n, we have [4, p. 363, eqn. (12)]

$$\sum_{n,k=0}^{\infty} \frac{\gamma}{\gamma+(\beta+1)(n+k)} \binom{\alpha+(\beta+1)(n+k)}{n} \phi_k t^{n+k} =$$
$$(1+\omega)^{\alpha} \sum_{n,k=0}^{\infty} \frac{\gamma}{\gamma+(\beta+1)k} \phi_k \omega^k (-1)^n \binom{\alpha-\gamma}{n} \binom{n+k+\gamma/(\beta+1)}{n}^{-1} \left(\frac{\omega}{1+\omega}\right)^n. \tag{7}$$

The following summation formulae include as special cases the above two results (3) and (4) [2, p. 3, eqn. (2.1)]

$$\sum_{n=0}^{\infty} \frac{\gamma(\delta+\mu n)}{\gamma+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} t^n =$$
$$\frac{\gamma\mu}{1+\beta} \sum_{n=0}^{\infty} \binom{\alpha+(\beta+1)n}{n} t^n + \left(\delta - \frac{\gamma\mu}{1+\beta}\right) \sum_{n=0}^{\infty} \frac{\gamma}{\gamma+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} t^n \tag{8}$$

and

$$\sum_{n=0}^{\infty} \frac{\gamma(\delta+\mu n)}{\gamma+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} t^n =$$
$$(1+\omega)^{\alpha} \left[\frac{\gamma\mu(1+\omega)}{(1+\beta)(1-\beta\omega)} + \left(\delta - \frac{\gamma\mu}{1+\beta}\right) \sum_{n=0}^{\infty} f_n^{(\alpha,\beta,\gamma)} \left(\frac{\omega}{1+\omega}\right)\right] \tag{9}$$

where the arbitrary parameters $\alpha, \beta, \gamma\delta$ and μ are independent of n, ω is given by (6) and

$$f_n^{(\alpha,\beta,\gamma)}(z) = (-1)^n \binom{\alpha-\gamma}{n} \binom{n+\gamma/(\beta+1)}{n}^{-1} z^n. \tag{10}$$

The generalization of summation formula (3) is also present in the literature [1, p. 525, eqn. (5.5)], i.e.

$$\sum_{n,k=0}^{\infty} \binom{\alpha+(\beta+1)(n+k)}{n} [a+b(n+k)]^k \frac{t^{n+k}}{k!} = \frac{e^{a\omega}(1+\omega)^{\alpha+1}}{1-\beta\omega-b\omega(1+\omega)} \tag{11}$$

where ω is given by

$$\omega = te^{b\omega}(1+\omega)^{\beta+1}. \tag{12}$$

The following series transformations are also required here [4, pp. 101–102, eqns. (6),(17)].

$$\sum_{n=0}^{\infty} \sum_{k_1,\ldots,k_r=0}^{M\leqslant n} \phi(k_1,\ldots,k_r;n) = \sum_{n=0}^{\infty} \sum_{k_1,\ldots,k_r=0}^{\infty} \phi(k_1,\ldots,k_r;n+M) \tag{13}$$

where $M = m_1k_1 + \cdots + m_rk_r$ and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}\beta(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\beta(k,n+mk). \tag{14}$$

In this paper Section 2 deals with the main generating functions presented in the form of three theorems. These theorems are proved with the help of the summation formulae (8) and (9). In Section 3 certain known special cases of the results established in Section 2 are derived. Some known and new generating functions involving extended sequences and generalized Lauricella functions are also obtained in this section.

2. Main generating functions

First generating function. Let $g(z_1,\dots,z_r)$ be a function of several complex variables $z_1, \dots, z_r$ defined by the formal series

$$g(z_1,\dots,z_r) = \sum_{k_1,\dots,k_r=0}^{\infty} C(k_1,\dots,k_r)\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!} \tag{15}$$

where the coefficients $C(k_1,\dots,k_r)$ ($k_j \geqslant 0$, $1 \leqslant j \leqslant r$) are arbitrary constants, real or complex.

A set of polynomials in r-complex variables $z_1,\dots,z_r$ is defined by [4, p. 459, eqn. (2)]:

$$\Omega_n^{(\alpha,\beta)}[\lambda_1,\dots,\lambda_r;m_1,\dots,m_r;z_1,\dots,z_r] = \sum_{k_1,\dots,k_r=0}^{M\leqslant n}\frac{(-n)_M[\alpha+(\beta+1)n+1]_L}{(\alpha+\beta n+1)_{L+M}}C(k_1,\dots,k_r)\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!} \tag{16}$$

where

$$L = \sum_{i=1}^{r}\lambda_ik_i, \quad M = \sum_{i=1}^{r}m_ik_i$$

α,β and $\lambda_1,\dots,\lambda_r$ are arbitrary complex numbers and $m_1,\dots,m_r$ are positive integers.

Theorem 1. *If $g(\cdot)$ and $\Omega_n^{(\alpha,\beta)}(\cdot)$ are defined by (15) and (16), respectively, then*

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{\delta+\mu n}{\gamma+(\beta+1)n}\binom{\alpha+(\beta+1)n}{n}\Omega_n^{(\alpha,\beta)}[\lambda_1,\dots,\lambda_r;m_1,\dots,m_r;z_1,\dots,z_r]t^n\\ &= (1+\omega)^{\alpha}\Bigg[\frac{\mu(1+\omega)}{(1+\beta)(1-\beta\omega)}g[z_1(-\omega)^{m_1}(1+\omega)^{\lambda_1},\dots,z_r(-\omega)^{m_r}(1+\omega)^{\lambda_r}]\\ &\quad+\left(\frac{\delta}{\gamma}-\frac{\mu}{\beta+1}\right)\sum_{n,k_1,\dots,k_r=0}^{\infty}C(k_1,\dots,k_r)\prod_{i=1}^{r}\left\{\frac{\left(z_i(-\omega)^{m_i}(1+\omega)^{\lambda_i}\right)^{k_i}}{k_i!}\right\}\times\\ &\quad\times\frac{(\gamma/(\beta+1))_M}{(1+n+\gamma/(\beta+1))_M}f_n^{(\alpha+L,\beta,\gamma)}\left(\frac{\omega}{1+\omega}\right)\Bigg]\end{aligned} \tag{17}$$

where ω is defined in (6) and provided that the series involved in (17) are absolutely convergent.

Second generating function. Let the multiple sequence of functions of several complex variables $z_1, \dots, z_r$ be defined in the following form [4, p. 485, eqn. (3.1)].

$$\Delta_{n_1,\dots,n_r;m_1,\dots,m_r}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}[\lambda_1,\dots,\lambda_r;z_1,\dots,z_r] = \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} C(k_1,\dots,k_r) \prod_{i=1}^{r} \binom{\alpha_i + (\beta_i+1)n_i + \lambda_i k_i}{n_i - m_i k_i} \frac{z_i^{k_i}}{k_i!}; \quad (18)$$

$n_i \in \{0,1,2,\dots\}$, $m_i \in \{1,2,3,\dots\}$, $i = 1,\dots,r$, where α_i, β_i, λ_i are complex parameters independent of $n_1,\dots,n_r$.

THEOREM 2. *If $g(\cdot)$ and $\Lambda_{n_1,\dots,n_r;m_1,\dots,m_r}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}[\cdot]$ are defined by (15) and (18), then*

$$\sum_{n_1,\dots,n_r=0}^{\infty} \prod_{i=1}^{r} \binom{\delta_i + \mu_i n_i}{\gamma_i + (\beta_i+1)n_i} \Delta_{n_1,\dots,n_r;m_1,\dots,m_r}^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r)}[\lambda_1,\dots,\lambda_r;z_1,\dots,z_r] \prod_{i=1}^{r} t_i^{n_i}$$
$$= \prod_{i=1}^{r} (1+\omega_i)^{\alpha_i} \left[\prod_{i=1}^{r} \left\{ \frac{\mu_i(1+\omega_i)}{(1+\beta_i)(1-\beta_i\omega_i)} \right\} g[z_1\omega_1^{m_1}(1+\omega_1)^{\lambda_1},\dots,z_r\omega_r^{m_r}(1+\omega_r)^{\lambda_r}] \right.$$
$$+ \prod_{i=1}^{r} \left(\frac{\delta_i}{\gamma_i} - \frac{\mu_i}{\beta_i+1} \right) \sum_{n_1,k_1,\dots,n_r,k_r=0}^{\infty} C(k_1,\dots,k_r) \times$$
$$\left. \times \prod_{i=1}^{r} \frac{\left((1+\omega_i)^{\lambda_i} z_i \omega_i^{m_i}\right)^{k_i}}{k_i!} \frac{(\gamma_i/(\beta_i+1))_{m_i k_i}}{(1+n_i+\gamma_i/(\beta_i+1))_{m_i k_i}} f_{n_i}^{(\alpha_i+\lambda_i k_i,\beta_i,\gamma_i)} \left(\frac{\omega_i}{1+\omega_i} \right) \right] \quad (19)$$

where $\omega_i = t_i(1+\omega_i)^{\beta_i+1}$, $i = 1,\dots,r$ and provided that the series involved in (19) are absolutely convergent.

Third generating function. Let the function of several complex variables be defined in terms of general multiple series [3, p. 122, eqn. (7)]

$$H_{n,k}^{(\alpha,\beta,a,b)}[(m_r),(n_r),(\lambda_r),(\mu_r);x_1,\dots,x_r,y_1,\dots y_r] = \sum_{k_1,l_1,\dots,k_r,l_r=0}^{M\leqslant n,\, N\leqslant k} \frac{(-n)_M(-k)_N[\alpha+1+(\beta+1)(n+k)]_r}{[1+\alpha+\beta(n+k)+k]_{T+M}} \times$$
$$\times [a+b(n+k)]^{-N} \Lambda(k_1,l_1,\dots,k_r,l_r) \prod_{i=1}^{r} [(x_i)^{k_i}(y_i)^{l_i}] \quad (20)$$

where $\Lambda(k_1,l_1,\dots,k_r,l_r)$ is any bounded sequence of real (or complex) numbers and $M = \sum_{i=1}^{r} m_i k_i$ (m_i is a positive integer), $N = \sum_{j=1}^{r} n_j l_j$ (l_j is a positive integer) and $T = \sum_{s=1}^{r}(\lambda_s k_s + \mu_s l_s)$ (λ_s and μ_s being arbitrary).

The symbol (λ_r) used in (20) condenses the array of r-parameters $\lambda_1,\dots,\lambda_r$ with similar interpretations for (m_r), (n_r) and (μ_r). Also let

$$h(x_1,\dots,x_r,y_1,\dots,y_r) = \sum_{k_1,l_1,\dots,k_r,l_r=0}^{\infty} \Lambda(k_1,l_1,\dots,k_r,l_r) \prod_{i=1}^{r} (x_i^{k_i} y_i^{l_i}). \quad (21)$$

Theorem 3. *If* $H_{n,k}^{(\alpha,\beta,a,b)}[\cdot]$ *and* $h(\cdot)$ *are defined by (20) and (21) then*

$$\sum_{n,k=0}^{\infty} \frac{\delta+\mu n}{\gamma+(\beta+1)(n+k)} \binom{\alpha+(\beta+1)(n+k)}{n} \times$$
$$\times H_{n,k}^{(\alpha,\beta,a,b)}[(m_r),(n_r),(\lambda_r),(\mu_r);x_1,\dots,x_r,y_1,\dots .y_r][a+b(n+k)]^k \frac{t^{n+k}}{k!}$$
$$= h[x_1(-\omega)^{m_1}(1+\omega)^{\lambda_1},\dots,x_r(-\omega)^{m_r}(1+\omega)^{\lambda_r},y_1(-\omega)^{n_1}(1+\omega)^{\mu_1},\dots,$$
$$y_r(-\omega)^{n_r}(1+\omega)^{\mu_r}] + \sum_{n,k=0}^{\infty} \sum_{k_1,l_1,\dots,k_r,l_r=0}^{\infty} \Lambda(k_1,l_1,\dots,k_r,l_r)(-\omega)^{M+N}(1+\omega)^T \times$$
$$\times \left[\frac{\delta}{\gamma} - \frac{(\gamma+(\beta+1)N)\mu}{\gamma(1+\beta)}\right] \frac{\omega^k[a+b(n+M+k+N)]^k}{k!} \times$$
$$\times \frac{\left(\frac{\gamma}{\beta+1}\right)_{M+N+k}}{\left(\frac{\gamma}{\beta+1}+n+1\right)_{M+N+k}} f_n^{(\alpha+T,\beta+\gamma)}\left(\frac{\omega}{1+\omega}\right) \prod_{s=1}^{r} \{x_s^{k_s} y_s^{l_s}\} \quad (22)$$

provided that the series involved in (22) are absolutely convergent.

Outline of proofs. To prove the assertion (17) of Theorem 1, we denote the L.H.S. of (17) by Δ_1, i.e.

$$\Delta_1 = \sum_{n=0}^{\infty} \frac{\delta+\mu n}{\gamma+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} \Omega_n^{(\alpha,\beta)}[\lambda_1,\dots,\lambda_r;m_1,\dots,m_r;z_1,\dots,z_r]t^n.$$

On using the definition of set of polynomials $\Omega_n^{\alpha,\beta}(\cdot)$ in (16) we have

$$\Delta_1 = \sum_{n=0}^{\infty} \frac{\delta+\mu n}{\gamma+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} \times$$
$$\times \sum_{k_1,\dots,k_r=0}^{M\leqslant n} \frac{(-n)_M(\alpha+(\beta+1)n+1)_L}{(\alpha+\beta n+1)_{L+M}} C(k_1,\dots,k_r) \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!} t^n.$$

Now on making series rearrangement in view of (13) and then on applying the results (2), (1) respectively, it gives

$$\Delta_1 = \sum_{k_1,\dots,k_r=0}^{\infty} \frac{C(k_1,\dots,k_r)}{\gamma+(\beta+1)n} \prod_{i=1}^{r} \left\{\frac{(z_i)^{k_i}(-t)^{m_i k_i}}{k_i!}\right\} \times$$
$$\times \left[\sum_{n=0}^{\infty} \frac{(\gamma+(\beta+1)M)(\delta+\mu(n+M))}{\gamma+(\beta+1)(n+M)} \binom{\alpha+(\beta+1)(n+M)+L}{n} t^n\right].$$

On interpreting the inner sum with the help of (9), we have

$$\Delta_1 = (1+\omega)^{\alpha} \left[\sum_{k_1,\dots,k_r=0}^{\infty} C(k_1,\dots,k_r) \prod_{i=1}^{r} \left\{\frac{(z_i)^{k_i}(-t)^{m_i k_i}}{k_i!}\right\} \times\right.$$
$$\times \frac{(1+\omega)^{(\beta+1)M+L+1}\mu[\delta+\mu(M+n)]}{(1+\beta)(1-\beta\omega)(\gamma+(\beta+1)(n+M))}$$

$$+\sum_{n,k_1,\ldots,k_r=0}^{\infty} C(k_1,\ldots,k_r)\prod_{i=1}^{r}\Big\{\frac{(z_i)^{k_i}(-t)^{m_ik_i}}{k_i!}\Big\}\frac{(1+\omega)^{(\beta+1)M+L}}{\gamma+(\beta+1)M}\times$$
$$\times\left((\delta+\mu M)-\frac{(\gamma+(\beta+1)M)\mu}{1+\beta}\right)f_n^{[\alpha+(\beta+1)M+L,\beta,\gamma+(\beta+1)M]}\left(\frac{\omega}{1+\omega}\right)\Bigg].$$

Now using the definition (10) , making slight simplification using (1) and (5) and then in view of (6) we have

$$\Delta_1=(1+\omega)^{\alpha}\Bigg[\sum_{k_1,\ldots,k_r=0}^{\infty} C(k_1,\ldots,k_r)\frac{\mu(1+\omega)^{L+1}}{(1+\beta)(1-\beta\omega)}\prod_{i=1}^{r}\Big\{\frac{(z_i)^{k_i}(-\omega)^{m_ik_i}}{k_i!}\Big\}$$
$$+\sum_{n,k_1,\ldots,k_r=0}^{\infty} C(k_1,\ldots,k_r)\prod_{i=1}^{r}\Big\{\frac{(z_i)^{k_i}(-\omega)^{m_ik_i}}{k_i!}\Big\}(1+\omega)^{L}\left(\frac{\delta}{\gamma}-\frac{\mu}{\beta+1}\right)\times$$
$$\times\frac{(\gamma/(\beta+1))_M}{(1+n+\gamma/(\beta+1))_M}f_n^{(\alpha+L,\beta,\gamma)}\left(\frac{\omega}{1+\omega}\right)\Bigg].$$

Now on interpreting the multiple series in view of (15), we at once arrive at the desired result in (17).

To prove the assertion (19) of Theorem 2,we denote the L.H.S. of (19) by Δ_2, i.e.

$$\Delta_2=\sum_{n_1,\ldots,n_r=0}^{\infty}\prod_{i=1}^{r}\left(\frac{\delta_i+\mu_in_i}{\gamma_i+(\beta_i+1)n_i}\right)\Delta_{n_1,\ldots,n_r;m_1,\ldots,m_r}^{(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r)}[\lambda_1,\ldots,\lambda_r;z_1,\ldots,z_r]\prod_{i=1}^{r}t_i^{n_i}.$$

On using the definition of multiple sequence $\Delta_{n_1,\ldots,n_r;m_1,\ldots,m_r}^{(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r)}(\cdot)$ in (18), we have

$$\Delta_2=\sum_{n_1,\ldots,n_r=0}^{\infty}\prod_{i=1}^{r}\left(\frac{\delta_i+\mu_in_i}{\gamma_i+(\beta_i+1)n_i}\right)\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}C(k_1,\ldots,k_r).$$

Now on making series rearrangement in view of (14), we have

$$\Delta_2=\sum_{k_1,\ldots,k_r=0}^{\infty}C(k_1,\ldots,k_r)\prod_{i=1}^{r}\Big\{\frac{(z_it_i^{m_i})^{k_i}}{(\gamma_i+(\beta_i+1)m_ik_i)k_i!}\Big\}\times$$
$$\times\sum_{n_1,\ldots,n_r=0}^{\infty}\prod_{i=1}^{r}\Bigg[\frac{(\gamma_i+(\beta_i+1)m_ik_i)}{\gamma_i+(\beta_i+1)m_ik_i+(\beta_i+1)n_i}\times$$
$$\times\binom{\alpha_i+(\beta_i+1)m_ik_i+\lambda_ik_i+(\beta_i+1)n_i}{n_i}\Bigg]$$

On interpreting the inner sum with the help of (9), we have

$$\Delta_2=\sum_{k_1,\ldots,k_r=0}^{\infty}C(k_1,\ldots,k_r)\prod_{i=1}^{r}\Big[\frac{\mu_i(z_it_i^{m_i}(1+\omega_i)^{(\beta_i+1)m_i})^{k_i}(1+\omega_i)^{1+\alpha_i+\lambda_ik_i}}{(1+\beta_i)(1-\beta_i\omega_i)k_i!}\Big]$$
$$+\sum_{k_1,\ldots,k_r=0}^{\infty}C(k_1,\ldots,k_r)\prod_{i=1}^{r}\Big[\frac{(z_it_i^{m_i}(1+\omega_i)^{(\beta_i+1)m_i})^{k_i}(1+\omega_i)^{\alpha_i+\lambda_ik_i}}{(\gamma_i+(\beta_i+1)m_ik_i)k_i!}\times$$
$$\times\Big\{(\delta_i+\mu_im_ik_i)-\frac{(\gamma_i+(\beta_i+1)m_ik_i)\mu_i}{1+\beta_i}\Big\}\times$$
$$\times\sum_{n_i=0}^{\infty}f_{n_i}^{(\alpha_i+(\beta_i+1)m_ik_i+\lambda_ik_i,\beta_i,\gamma_i+(\beta_i+1)m_ik_i)}\left(\frac{\omega_i}{1+\omega_i}\right).$$

Now using the definition in (10), making slight simplification using (1) and (5) and then in view of (6) and (10), we have

$$\Delta_2 = \sum_{k_1,\ldots,k_r=0}^{\infty} C(k_1,\ldots,k_r)\prod_{i=1}^{r}\Big[\frac{\mu_i(z_i\omega_i^{m_i})^{k_i}(1+\omega_i)^{1+\alpha_i+\lambda_i k_i}}{(1+\beta_i)(1-\beta_i\omega_i)k_i!}\Big]$$
$$+\sum_{k_1,\ldots,k_r=0}^{\infty} C(k_1,\ldots,k_r)\prod_{i=1}^{r}\Big[\frac{(z_i\omega_i^{m_i})^{k_i}(1+\omega_i)^{\alpha_i+\lambda_i k_i}}{k_i!}\left(\frac{\delta_i}{\gamma_i}-\frac{\mu_i}{\beta_i+1}\right)\times$$
$$\times\frac{(\gamma_i/(\beta_i+1))_{m_ik_i}}{(1+n_i+\gamma_i/(\beta_i+1))_{m_ik_i}}\sum_{n_i=0}^{\infty} f_{n_i}^{(\alpha_i+\lambda_ik_i,\beta_i,\gamma_i)}\left(\frac{\omega_i}{1+\omega_i}\right).$$

Now on interpreting the multiple series in view of (15) we at once arrive at the desired result in (19).

To prove the assertion (22) of Theorem 3, we denote the L.H.S. of (22) by Δ_3, i.e.

$$\Delta_3 = \sum_{n,k=0}^{\infty}\frac{\delta+\mu n}{\gamma+(\beta+1)(n+k)}\binom{\alpha+(\beta+1)(n+k)}{n}\times$$
$$\times H_{n,k}^{(\alpha,\beta,a,b)}[(m_r),(n_r),(\lambda_r),(\mu_r);x_1,\ldots,x_r,y_1,\ldots .y_r][a+b(n+k)]^k\frac{t^{n+k}}{k!}$$

On applying the definition of $H_{n,k}^{(\alpha,\beta,a,b)}[\cdot]$ given by (20), we have

$$\Delta_3 = \sum_{n,k=0}^{\infty}\frac{\delta+\mu n}{\gamma+(\beta+1)(n+k)}\binom{\alpha+(\beta+1)(n+k)}{n}\times$$
$$\times\left\{\sum_{k_1,l_1,\ldots,k_r,l_r=0}^{M\leqslant N,N\leqslant K}\frac{(-n)_M(-k)_N[\alpha+(\beta+1)(n+k)+1]_T}{[1+\alpha+\beta(n+k)+k]_{T+M}}\Lambda(k_1,l_1,\ldots,k_r,l_r)\times\right.$$
$$\left.\times[a+b(n+k)]^{-N+k}\prod_{i=1}^{r}(x_i^{k_i}y_i^{l_i})\right\}\frac{t^{n+k}}{k!}.$$

Now on using (13), we have

$$\Delta_3 = \sum_{k_1,l_1,\ldots,k_r,l_r=0}^{\infty}\frac{\Lambda(k_1,l_1,\ldots,k_r,l_r)\prod_{i=1}^{r}(x_i^{k_i}y_i^{l_i})(-1)^{M+N}}{\gamma+(\beta+1)(M+N)}\times$$
$$\times\Big[\sum_{n,k=0}^{\infty}\frac{(\gamma+(\beta+1)(M+N))(\delta+\mu(M+N))}{\gamma+(\beta+1)(n+M+N+k)}\times$$
$$\times\binom{\alpha+(\beta+1)(n+M+k+N)+T}{n}[a+b(n+k+M+N)]^k\frac{t^{n+k+M+N}}{k!}\Big]$$

On interpreting the inner sum with the help of (8), we have

$$\Delta_3 = \sum_{k_1,l_1,\ldots,k_r,l_r=0}^{\infty}\frac{\Lambda(k_1,l_1,\ldots,k_r,l_r)\prod_{i=1}^{r}(x_i^{k_i}y_i^{l_i})(-1)^{M+N}}{\gamma+(\beta+1)(M+N)}\times$$

$$\times\left[\sum_{n,k=0}^{\infty}\frac{(\gamma+(\beta+1)(M+N))\mu}{1+\beta}\binom{\alpha+(\beta+1)(n+M+k+N)+T}{n}\right.$$
$$\left.\times\frac{(a+b(n+M+k+N))^k}{k!}t^{n+M+k+N}\right]$$
$$+\sum_{k_1,l_1,\ldots,k_r,l_r=0}^{\infty}\frac{\Lambda(k_1,l_1,\ldots,k_r,l_r)\prod_{i=1}^{r}(x_i^{k_i}y_i^{l_i})(-1)^{M+N}}{\gamma+(\beta+1)(M+N)}\times$$
$$\times\left[\sum_{n,k=0}^{\infty}\left((\delta+\mu M)-\frac{(\gamma+(\beta+1)(M+N))\mu}{1+\beta}\right)\times\right.$$
$$\times\binom{\alpha+(\beta+1)(n+M+k+N)+T}{n}\times$$
$$\left.\times\frac{\gamma+(\beta+1)(N+M)}{\gamma+(\beta+1)(n+M+k+N)}\frac{[a+b(n+M+k+N)]^k t^{n+k+M+N}}{k!}\right].$$

Now making use of (11) in first term and (7) in second term we have

$$\Delta_3=\sum_{k_1,l_1,\ldots,k_r,l_r=0}^{\infty}\frac{\mu e^{a\omega}[-te^{b\omega}(1+\omega)^{\beta+1}]^{M+N}(1+\omega)^{\alpha+T+1}}{[1-\beta\omega-b\omega(1+\omega)](1+\beta)}\times$$
$$\times\Lambda(k_1,l_1,\ldots,k_r,l_r)\prod_{i=1}^{r}(x_i^{k_i}y_i^{l_i})$$
$$+\sum_{k_1,l_1,\ldots,k_r,l_r=0}^{\infty}\sum_{n,k=0}^{\infty}\left(\delta+\mu M-\frac{[\gamma+(\beta+1)(M+N)]\mu}{1+\beta}\right)\binom{\alpha+T-\gamma}{n}\times$$
$$\times\left(n+k+\frac{\gamma+(\beta+1)(M+N)}{\beta+1}\right)^{-1}\times$$
$$\times\frac{\Lambda(k_1,l_1,\ldots,k_r,l_r)(-1)^n(1+\omega)^T[-t(1+\omega)^{\beta+1}]^{M+N}}{\gamma+(\beta+1)(M+N+k)}\times$$
$$\times[a+b(b+M+k+N)]^k\omega^k\left(\frac{\omega}{1+\omega}\right)^n\prod_{i=1}^{r}(x_i^{k_i}y_i^{l_i}).$$

Now in view of (6) and (12) and using (1) and (5), we obtain

$$\Delta_3=\sum_{k_1,l_1,\ldots,k_r,l_r=0}^{\infty}\frac{\mu e^{a\omega}(-\omega)^{M+N}(1+\omega)^{T+\alpha+1}}{[1-b\omega-b\omega(1+\omega)](1+\beta)}\Lambda(k_1,l_1,\ldots,k_r,l_r)\prod_{i=1}^{r}(x_i^{k_i}y_i^{l_i})$$
$$+\sum_{k_1,l_1,\ldots,k_r,l_r=0}^{\infty}\sum_{n,k=0}^{\infty}\frac{\Lambda(k_1,l_1,\ldots,k_r,l_r)(-\omega)^{M+N}(1+\omega)^T\omega^k(-1)^n}{k!}\times$$
$$\times\left(\frac{\omega}{1+\omega}\right)^n[a+b(n+M+k+N)]^k\left[\frac{\delta}{\gamma}-\frac{(\gamma+(\beta+1)N)\mu}{\gamma(\beta+1)}\right]\times$$
$$\times\frac{(\gamma/(\beta+1))_{M+N+k}}{(n+1+\gamma/(\beta+1))_{M+N+k}}\binom{\alpha+T-\gamma}{n}\binom{n+1+\gamma/(\beta+1)}{n}^{-1}\prod_{i=1}^{r}(x_i^{k_i}y_i^{l_i}).$$

Finally, interpreting the multiple series in view of (21) and then with the help of (10), we at once arrive at the desired result in (22).

3. Particular cases

The main results involve various parameters, and also the arbitrary sequences, therefore, by appropriately selecting these parameters, (and sequences), one can deduce several results from the main theorems. To illustrate, we deduce here the following examples from the main results.

If we take $r = 1$, $\lambda = h$ and $C(k) \to k!\,A_k$ then Theorem 1 reduces to the known result [2, p. 6, eqn. (4.2)].

If in Theorem 1, we take $\delta \to \gamma$, $\mu = 0$, all $\lambda_i \to 0$ for $i = 1,\dots,r$ and we select the arbitrary sequence where $\Omega k_1,\dots,k_r$ is known sequence [4, p. 64, eqn. (19)], then

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\gamma}{\gamma+(\beta+1)n}\binom{\alpha+(\beta+1)n}{n}\times\\
&\quad\times F^{A+1;B';\dots;B^{(r)}}_{C+1;D';\dots'D^{(r)}}\left[\begin{matrix}[(a);\,\theta',\dots,\theta^{(r)}], & [-n;\,m_1,\dots,m_r],\\ [(c);\,\psi',\dots,\psi^{(r)}], & [\alpha+\beta n+1;\,m_1,\dots,m_r],\end{matrix}\right.\\
&\qquad\qquad\left.\begin{matrix}[(b');\,\phi']; & \dots; & [(b^{(r)};)\phi^{(r)}]:\\ [(d');\,\delta']; & \dots; & [(d^{(r)}),\delta^{(r)}]:\end{matrix}\; z_1,\dots,z_r\right] t^n\\
&= (1+\omega)^{\alpha}\sum_{n=0}^{\infty}(-1)^n\binom{\alpha-\gamma}{n}\binom{n+\gamma/(\beta+1)}{n}^{-1}\left(\frac{\omega}{1+\omega}\right)^n\times\\
&\quad\times F^{A+1;B';\dots;B^{(r)}}_{C+1;D';\dots'D^{(r)}}\left[\begin{matrix}[(a);\,\theta',\dots,\theta^{(r)}], & [\gamma/(\beta+1);\,m_1,\dots,m_r],\\ [(c);\,\psi',\dots,\psi^{(r)}], & [1+n+\gamma/(\beta+1);\,m_1,\dots,m_r],\end{matrix}\right.\\
&\qquad\qquad\left.\begin{matrix}[(b');\,\phi']; & \dots; & [(b^{(r)};)\phi^{(r)}]:\\ [(d');\,\delta']; & \dots; & [(d^{(r)}),\delta^{(r)}]:\end{matrix}\; z_1(-\omega_1)^{m_1},\dots,z_r(-\omega_r)^{m_r}\right]
\end{aligned}\tag{23}$$

If we take $r = 1$, $\lambda_i = 0$ for all $i = 1,\dots,r$ and $C(k) \to k!\,A_k$ then Theorem 2 reduces to the known result [2, p. 4, eqn. (2.5)].

If we take $r = 1$, $\lambda_i = 0$ for all $i = 1,\dots,r$ and $C(k) \to k!\,A_k$, $\delta = \gamma$, $\mu = 0$ and $z_1 = 1$ then Theorem 2 reduces to the known generating function [5, p. 410, Theorem 3].

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