

## EIGENVALUES OF CERTAIN SPARSE MATRICES

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**Abstract.** We introduce a special class of matrices of arbitrary size, which we call C-matrices. We compute some exact values of the eigenvalues of the C-matrices.

### 1. Introduction

We study square matrices of the order (size)  $n$ . These matrices appeared in [1] as the authors studied the linear algebraic properties of magic squares. The scope of [1] was the computation of the dimension of some linear spaces generated by pandiagonal magic squares. On the other hand, Tamimi and Al-Ashhab studied these squares in [9]. Our target in this article is to study the eigenvalues of these matrices. In fact, these matrices have a systematic structure for all orders. This structure reveals some symmetry. However, the matrices are not symmetric in the usual sense. Therefore, the eigenvalues are not necessarily real.

In the literature, there are many papers concerned with the techniques of approximation of the value the eigenvalues for the sparse matrices. These techniques use iteration methods and other numerical ideas. Nakatsukasa considered in [8] some tridiagonal matrices  $A$ ,  $B$ , as well as the generalized eigenvalue problem: Find the value of  $\lambda$  such that  $Ax = \lambda Bx$  for some  $x \neq 0$ . He estimated the values by using techniques similar to the classical Gerschgorin's Circle Theorem. Morgan and Scott (see [7]) presented algorithms for numerically computing the smallest eigenvalue, which are implemented on a sparse diagonal matrix of order 1000.

In this paper we are rather interested in computing the exact value of the eigenvalues. Our approach is based on using the definition of the eigenvalues. In addition, We use the techniques of linear algebra for the computation of the exact value of the determinant.

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*2010 AMS Subject Classification:* 15A18, 11B83.

*Keywords and phrases:* Matrices; eigenvalues; finite sequence.

## 2. Definitions

We classify the following two types of C-matrices according to the value of  $n$ . Actually, the definition of the C-matrix of an even order differs from the definition of the C-matrix of an odd order. In the latter case, there is no a nonzero element in the middle row except the middle entry. We present now the matrices which we wish to study.

DEFINITION 1. Let  $n = 2k$ . The C-matrix is the square matrix  $(a_{ij})$  of order  $n$  such that the nonzero elements off the main diagonal are  $a_{ij} = 1$  for  $j = 2i$ , and if  $1 \leq i \leq k$ ; and  $a_{ij} = a_{n+1-i, n+1-j}$ , if  $k < i \leq n$ .

For example, the C-matrix of order 4 is

$$\begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Definition 1 means that the C-matrix of an even order has  $-2$  on the main diagonal. Another definition of C-matrices is set to the matrices of an odd order. In this case, we add a row in the middle of the matrix, which is consistent with the main diagonal.

DEFINITION 2. Let  $n = 2k + 1$ . We define the C-matrix as the square matrix  $(a_{ij})$  of order  $n$  such that the nonzero elements off the main diagonal are  $a_{ij} = 1$  for  $j = 2i$ , and if  $1 \leq i \leq k$ ; and  $a_{ij} = a_{n+1-i, n+1-j}$ , if  $k + 1 < i \leq n$ .

For example, the C-matrix of order 3 is

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

Definition 2 means that the C-matrix of odd order has  $-2$  on the main diagonal. However, the number 1 does not appear in the middle row. We note that the C-matrices are strictly diagonal dominant. Therefore, they are invertible.

## 3. Main results

According to Gerschgorin's Theorem (see [6]), the eigenvalues of the C-matrices lie in the unit circle with center  $(-2, 0)$  in the complex plane. We show that the real bounds of the circle  $-1$  and  $-3$  are indeed eigenvalues in many cases. Actually, we will notice that the absolute value of the eigenvalues, which we have computed using the computer, are between the numbers 1 and 3.

We start with results, which tell us something about the existence of an eigenvalue for all orders. We prove first results for matrices of an even order. In the rest of this paper,  $N$  denotes the set of positive integers.

LEMMA 3. *If  $k \in N$  and  $A$  is a C-matrix of order  $2k$ , then  $-1$  is an eigenvalue of  $A$ .*

*Proof.* Let  $I$  be the identity matrix of order  $2k$ . Now, the matrix  $A + I$  has the values  $-1$  and  $1$  in all nonzero entries. According to the structure of  $A$ , we find in each column of  $A + I$  exactly one entry  $1$  and one entry  $-1$ . Hence, the sum of all rows of the matrix  $A + I$  is the zero row, which means that its determinant is zero. ■

For C-matrices of an odd order, we prove the following result.

LEMMA 4. *If  $k \in N$  and  $A$  is a C-matrix of order  $2k + 1$ , then  $-2$  is an eigenvalue of  $A$ .*

*Proof.* The matrix  $A + 2I$  has the zero row as the middle row, i.e., the row  $k + 1$  is a zero row. Hence, we are done. ■

The eigenvalue  $-2$  is in some cases the unique eigenvalue. This is illustrated in the following proposition.

PROPOSITION 5. *If  $l$  is a nonnegative integer and  $A$  is a C-matrix of order  $n = 2^l - 1$ , then the characteristic polynomial of  $A$  has the form  $(\lambda + 2)^n$ .*

*Proof.* We already know that  $-2$  is an eigenvalue of  $A$ . We need to show that  $-2$  is the unique eigenvalue of  $A$ . We make an indirect proof. Suppose that there exists an eigenvalue  $\beta$  for  $A$ , which satisfies  $\beta \neq -2$ . Let us denote with  $(x_1, x_2, \dots, x_n)^t$  an arbitrary  $n$ -dimensional vector. Further, let  $\alpha = \beta + 2$  and  $k = 2^{l-1}$ . Then, we obtain the relation for the multiplication of matrices

$$(\beta I - A) * \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \dots \\ d_n \end{pmatrix},$$

where  $d_1 = \alpha x_1 + x_2$ ,  $d_2 = \alpha x_2 + x_4$ ,  $\dots$ ,  $d_{k-1} = \alpha x_{k-1} + x_{2k-2}$ ,  $d_k = \alpha x_k$ ,  $d_{k+1} = \alpha x_{k+1} + x_2$ ,  $d_{k+2} = \alpha x_{k+2} + x_4$ ,  $\dots$ ,  $d_n = \alpha x_n + x_{n-1}$ .

Now, since  $\alpha \neq 0$ , we deduce that the solution of the equation  $(\beta I - A) * (x_1, x_2, \dots, x_n)^t = 0$  is only the trivial solution, i.e., the kernel of  $(\beta I - A)$  is  $\{0\}$ . This is a contradiction. ■

Next, we give more information about the eigenvalues of C-matrices of an even order. The following result is an improvement of the estimation, which we obtain using Gerschgorin's theorem, regarding the location of the real eigenvalues.

PROPOSITION 6. *Let  $k \in N$ , and  $A$  be a C-matrix of order  $2k$ . If  $\gamma$  is a real eigenvalue of  $A$ , then  $\gamma = -1$ , or  $\gamma = -3$ .*

*Proof.* We make an indirect proof. Suppose that there exists a real eigenvalue

$\gamma$  for  $A$ , which satisfies  $|\gamma+2| \neq 1$ . The matrix  $(\gamma I - A)$  has the following structure

$$\begin{bmatrix} \gamma+2 & 1 & 0 & 0 & \dots & 0 & & 0 & 0 \\ 0 & \gamma+2 & 0 & 1 & \dots & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & \gamma+2 & \dots & 0 & 0 \\ & & & & \dots & & & & \\ 0 & 0 & 0 & 0 & \dots & 0 & & 1 & \gamma+2 \end{bmatrix}$$

We consider two cases:

*Case 1:* Assume  $|\gamma+2| > 1$ . The matrix  $(\gamma I - A)$  is strictly diagonal dominant. Hence, it is invertible, and this is a contradiction.

*Case 2:* Assume  $|\gamma+2| < 1$ . We rearrange the columns of  $(\gamma I - A)$  in the following order:

$$C_2, C_4, \dots, C_{2k}, C_1, C_3, \dots, C_{n-1},$$

where  $C_i$  denotes the  $i$ -th column. The matrix in the new order is strictly diagonal dominant. Hence, it is invertible, and its determinant is nonzero. Thus, the determinant of  $(\gamma I - A)$  is nonzero. This is a contradiction ■

In some cases,  $-3$  is not an eigenvalue of a C-matrix. However, we can show sometimes that it is indeed an eigenvalue.

**PROPOSITION 7.** *If  $l$  is a nonnegative integer and  $A$  is a C-matrix of order  $n = 2 + 6l$ , then  $-3$  is a real eigenvalue of  $A$ .*

*Proof.* We show in this case that two rows of  $(A + 3I)$  are identical. In fact, the two rows having the number  $4l + 2$  and  $2l + 1$  are identical. We explain this using the definition of the C-matrix. The main diagonal of  $(A + 3I)$  consists of one's. The  $(2l + 1)$ -th row has 1 in the entries  $(2l + 1, 2l + 1)$  and  $(2l + 1, 2(2l + 1))$ , since  $2l + 1 < 1 + 3l$ . On the other side, the  $(4l + 2)$ -th row has 1 in the entries  $(4l + 2, 4l + 2)$  and  $(4l + 2, 2(4l + 2 - (1 + 3l)) - 1) = (4l + 2, 2l + 1)$ . ■

We can extend the idea behind the proof of Propositions 6 and 7 for other C-matrices of an even order such as matrices, where the sum of several rows is identical to the sum of another set of rows. For example, let  $A$  be the C-matrix of order 4. We obtain

$$A + 3I = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The sum of the first and fourth rows is the same as the sum of both the second and third rows. If the order is 10, then the rows ranking 1, 3, 4, 5, and 9 in the matrix  $A + 3I$  sum up to a row of one's, while the rows ranking 2, 6, 7, 8, and 10 sum up to the the same row. We write the finite sequence

$$(1, 2, 4, 8, 5, 10, 9, 7, 3, 6, 1).$$

This sequence illustrates how to choose the rank of the rows, which sum up to the same row. The odd-numbered entries in the sequence form the class 1, 3,

4, 5, 9, while the even numbered entries in the sequence form the class 2, 6, 7, 8, 10. In this sequence, we start with 1, which indicates the first row of the matrix  $A + 3I$ . In this row, we have the number 1 located in the entries (1,1) and (1,2). The second element in the sequence, which is 2, indicates that the number 1 occurs in the first row in the second entry. It also means that we move to the second row. Now, in the second row of the matrix, we find 1 in the entry (2,2). In the same time, we have the number 1 in the entry (2,4). Remember that the third element in the sequence is 4. If we go to the fourth row we find 1 in the entry (4,4) and (4,8). The eighth row has 1 located in the entry (8,5). The fifth row has 1 located in the entry (5,10). Hence, we shall move to the tenth row. This row in turn indicates that we shall move to the ninth row. We continue tracing the nondiagonal entry, where we find 1 until we reach the sixth row, which takes us back to the first row.

We can repeat this procedure for any C-matrix of an even order, and obtain a finite sequence of numbers starting and ending with 1. The sequence indicates the location of the number 1 (the nonzero element) of the matrix  $A + 3I$  for several rows. If the number of the different rows, whose rank involved in the sequence, is even, then we obtain two classes. The sum of all rows in both classes shall be identical. This means that  $-3$  is an eigenvalue of the C-matrix. Formally, we compute the sequence  $Q_k$  for any even number  $n = 2m$  as follows  $Q_0 = 1, Q_k = 2Q_{k-1}$ , if  $Q_{k-1} \leq m$ ;  $Q_k = 2Q_{k-1} - (2m + 1)$ , if  $Q_{k-1} > m$ , for  $k = 1, 2, \dots$

The C-matrix of order 6 has the eigenvalues  $\frac{1}{2}i\sqrt{3} \pm \frac{3}{2}, \frac{1}{2}i\sqrt{3} \pm \frac{5}{2}, -1, -1$ . Therefore, some C-matrices of an even order do not have  $-3$  as an eigenvalue. If the finite sequence starting and ending with 1 has an even number of elements, then we can not deduce any information about  $-3$ . For example, for the matrix of order 6, we obtain the sequence (1, 2, 4, 1), but  $-3$  is an eigenvalue of the C-matrix of order 366, although its sequence has 184 elements.

LEMMA 8. *If  $l$  is an even integer such that  $l > 3$  and  $A$  is a C-matrix of order  $n = 2^l - 2$ , then  $-3$  is an eigenvalue of  $A$ .*

*Proof.* When we compute the elements of the finite sequence, we start with 1, 2, 4 until we reach  $2^{l-1}$ , since  $2^{l-2} < \frac{n}{2}$ . Now, because  $2^{l-1} = \frac{n}{2} + 1$ , the next element, according to the definition, is  $2(\frac{n}{2} + 1) - (n + 1)$ . Thus, we obtain 1 as the next element, and we find the sequence consisting of different elements. The number of elements of the sequence is  $l + 1$  which is odd. ■

PROPOSITION 9. *If  $l$  is an integer such that  $l > 1$  and  $A$  is a C-matrix of order  $n = 2^l$ , then  $-3$  is an eigenvalue of  $A$ .*

*Proof.* When we compute the elements of the finite sequence, we start with 1, 2, 4 until we reach  $2^l$ . This element is  $Q_l$ . The next element (i.e.,  $Q_{l+1}$ ), according to our procedure, is  $2n - (n + 1) = 2^l - 1 = Q_l - 1$ . The element  $Q_{l+2} = 2(2^l - 1) - (n + 1) = Q_{l+1} - 2$ . This goes on in the following manner  $Q_{l+j} = Q_{l+j-1} - 2^{j-1}$ . When the index becomes  $2l - 1$ , we reach the value  $n + 1$ , i.e.,  $Q_{2l-1} = n + 1$ . Thus, we form a sequence having  $2l + 1$  different elements. ■

PROPOSITION 10. *If  $l$  is a positive integer and  $A$  is a C-matrix of order  $n = 4l + 1$ , then  $-1$  is an eigenvalue of  $A$ .*

*Proof.* The order of the matrix  $A + I$  is odd. Hence, we have one entry equals 1 and one entry equals  $-1$  in each row of this matrix except the middle row. We recall from the definition that the nondiagonal entries of  $A$  (in this case,  $A$  has the order  $4l + 1$ ) is  $a_{ij} = 1$  for  $j = 2i$ , and if  $i \leq 2l$ ; and  $a_{ij} = 1$  for  $j = 2(i - 2l - 1)$ , and if  $i > 2l + 1$ . Hence, the nonzero entries occur only in the even ordered cells of each row. Next,  $2l + 1$  is odd so that the middle column - namely the column with rank  $2l + 1$  - has one nonzero entry. If we sum up all columns of  $A + I$  except the middle column, then we obtain a zero vector. Hence, the determinant of  $A + I$  is zero, and we are done. ■

### 3.1. Computational results

Using the computer, we have calculated the eigenvalues of the C-matrices of even order up to the order 4780. We found that the following values of the order, for which  $-3$  is not an eigenvalue:

6, 22, 30, 46, 48, 70, 72, 78, 88, 102, 126, 150, 160, 166, 190, 198, 216, 222, 232, 238, 262, 270, 310, 328, 336, 342, 358, 430, 438, 496, 510, 552, 600, 622, 630, 712, 720, 880, 888, 910, 918, 936, 960, 1056, 1102, 1288, 1392, 1432, 1456, 1518, 1560, 1678, 1800, 1896, 2046, 2088, 2142, 2200, 2262, 2350, 2358, 2592, 2686, 2758, 2920, 3016, 3190, 3390, 3478, 3472, 3576, 3936, 4056, 4176, 4206, 4512, 4576, 4680.

In the previous section, we used the idea of the existence of a finite sequence starting and ending with 1. We cannot right now give a proof for the finiteness of the sequence in all cases. In other words, given an even number  $n = 2m$ , define the sequence  $Q_k^m$  as follows  $Q_0^m = 1$ ;  $Q_k^m = 2Q_{k-1}^m$ , if  $Q_{k-1}^m \leq m$ ;  $Q_k^m = 2Q_{k-1}^m - (2m + 1)$ , if  $Q_{k-1}^m > m$ ; for  $k = 1, 2, \dots$ .

Does an integer  $N(m) > 1$  exist such that  $Q_N^m = 1$ , and all the elements  $Q_0^m, \dots, Q_N^m$  are distinct?

The answer of this question helps determining whether  $-3$  is an eigenvalue of the C-matrix or not. More precisely, using a similar reasoning to that in the proof of Proposition 9, we can show that if a C-matrix of the order  $n = 2m$  has the following properties:

- 1) the integer  $N(m)$  exists, and
- 2)  $N(m)$  is odd,

then the C-matrix includes the value  $-3$  in its spectrum. However, there are C-matrices of an even order with  $-3$  as an eigenvalue, but do not have the properties 1) and 2). For example, the C-matrix of order 462 has  $-3$  as an eigenvalue, although the value of  $N(231)$  is even. In fact, the finite sequence for  $n = 462$  is

(1, 2, 4, 8, 16, 32, 64, 128, 256, 49, 98, 196, 392, 321, 179, 358, 253, 43, 86, 172, 344, 225, 450, 437, 411, 359, 255, 47, 94, 188, 376, 289, 115, 230, 460, 457, 451, 439, 415, 367, 271, 79, 158, 316, 169, 338, 213, 426, 389, 315, 167, 334, 205, 410,

357, 251, 39, 78, 156, 312, 161, 322, 181, 362, 261, 59, 118, 236, 9, 18, 36, 72, 144, 288, 113, 226, 452, 441, 419, 375, 287, 111, 222, 444, 425, 387, 311, 159, 318, 173, 346, 229, 458, 453, 443, 423, 383, 303, 143, 286, 109, 218, 436, 409, 355, 247, 31, 62, 124, 248, 33, 66, 132, 264, 65, 130, 260, 57, 114, 228, 456, 449, 435, 407, 351, 239, 15, 30, 60, 120, 240, 17, 34, 68, 136, 272, 81, 162, 324, 185, 370, 277, 91, 182, 364, 265, 67, 134, 268, 73, 146, 292, 121, 242, 21, 42, 84, 168, 336, 209, 418, 373, 283, 103, 206, 412, 361, 259, 55, 110, 220, 440, 417, 371, 279, 95, 190, 380, 297, 131, 262, 61, 122, 244, 25, 50, 100, 200, 400, 337, 211, 422, 381, 299, 135, 270, 77, 154, 308, 153, 306, 149, 298, 133, 266, 69, 138, 276, 89, 178, 356, 249, 35, 70, 140, 280, 97, 194, 388, 313, 163, 326, 189, 378, 293, 123, 246, 29, 58, 116, 232, 1).

We note that  $N = 230$ . Using the computer, we find that for each of the values  $n = 4, \dots, 450000$  there exists an integer  $M < n$  such that  $Q_M = 1$ . Hence, we do not have any doubts that the existence of the finite sequence  $Q_k$  is always possible. This motivates us to make the following conjecture.

CONJECTURE 11. For each natural value of  $n$ , there exists an integer  $M < n$  such that  $Q_M = 1$ .

Using the computer, we find that for each of the following odd values of  $n < 5001$  the C-matrix does not have  $-3$  as an eigenvalue:

3, 7, 13, 15, 27, 31, 45, 55, 61, 63, 91, 93, 97, 111, 123, 127, 141, 145, 157, 177, 183, 187, 195, 205, 223, 247, 253, 255, 283, 291, 301, 315, 321, 333, 355, 367, 375, 381, 391, 397, 411, 433, 445, 447, 465, 477, 495, 507, 511, 525, 541, 567, 583, 603, 621, 631, 643, 657, 667, 673, 685, 711, 717, 735, 751, 763, 783, 795, 823, 861, 867, 877, 891, 895, 931, 955, 991, 993, 1015, 1021, 1023, 1051, 1083, 1105, 1135, 1167, 1201, 1207, 1243, 1245, 1261, 1263, 1287, 1315, 1335, 1347, 1371, 1423, 1425, 1435, 1441, 1453, 1471, 1503, 1527, 1567, 1591, 1647, 1723, 1735, 1755, 1761, 1777, 1783, 1791, 1821, 1837, 1863, 1873, 1911, 1921, 1983, 1987, 2031, 2043, 2047, 2103, 2113, 2167, 2205, 2211, 2271, 2335, 2403, 2415, 2487, 2491, 2523, 2527, 2575, 2577, 2631, 2671, 2695, 2743, 2785, 2847, 2851, 2865, 2871, 2883, 2907, 2913, 2943, 3007, 3037, 3055, 3121, 3135, 3183, 3261, 3295, 3357, 3447, 3471, 3511, 3523, 3555, 3567, 3583, 3601, 3643, 3675, 3727, 3747, 3793, 3823, 3843, 3967, 3975, 4063, 4087, 4093, 4095, 4177, 4207, 4227, 4285, 4335, 4401, 4411, 4423, 4525, 4543, 4671, 4701, 4717, 4807, 4831, 4975, 4983.

On the other side, we know that all C-matrices of odd order  $n$  such that  $n = 2^l - 1$  cannot have  $-1$  as an eigenvalue. In fact, all C-matrices of odd order  $n$  such that  $n < 5001$  have  $-1$  as an eigenvalue, except as expected for : 3, 7, 15, 163, 127, 255, 511, 1023, 2047, 4095.

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(received 06.07.2011; in revised form 04.01.2012; available online 01.05.2012)

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