МАТЕМАТИЧКИ ВЕСНИК
65, 2 (2013), 271–284
June 2013

оригинални научни рад
research paper

FIXED POINT TECHNIQUES AND STABILITY IN NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH VARIABLE DELAYS

Abdelouaheb Ardjouni and Ahcene Djoudi

Abstract. In this paper we use fixed point techniques to obtain asymptotic stability results of the zero solution of a nonlinear neutral differential equation with variable delays. This investigation uses new conditions which allow the coefficient functions to change sign and do not require the boundedness of delays. An asymptotic stability theorem with a necessary and sufficient condition is proved. The obtained results improve and extend those due to Burton, Zhang, Raffoul, Jin and Luo, Ardjouni and Djoudi, and Djoudi and Khemis. Two examples are also given to illustrate this work.

1. Introduction

For more than 100 years, stability properties of ordinary, functional, partial differential equation have been mainly investigated by the ultimate Lyapunov direct method. But the method has encountered serious obstacles and a number of problems remain unsolved. In recent years, investigators such as Burton, Furumochi, Zhang and others began a project in the idea to overcome some of these difficulties. Particularly, Burton and Furumochi considered, in a series of papers (see [3–10]), specific examples and challenging problems for stability using Lyapunov's method and have presented solutions by means of various fixed point techniques. In the same time, they pointed out that the fixed point method have other significant advantages over Lyapunov's method. The former asks conditions of averaging nature while the latter usually asks pointwise conditions (see [1–14,16]).

Having in mind the above ideas, we consider, here, the following nonlinear neutral differential equation with variable delays

$$x'(t) = -a(t)x(t-\tau_1(t))+b(t)G(x(t-\tau_2(t)))+c(t)x'(t-\tau_2(t))Q'(x(t-\tau_2(t))), \quad (1.1)$$

with the initial condition

$$x(t) = \psi(t) \text{ for } t \in [m(t_0), t_0],$$

2010 AMS Subject Classification: 34K20, 34K30, 34K40

Keywords and phrases: Fixed points; stability; neutral differential equation; integral equation; variable delays

where $\psi \in C([m(t_0), t_0], \mathbb{R})$ and for each $t_0 \geq 0$,

$$m_j(t_0) = \inf\{t - \tau_j(t),\ t \geq t_0\},\ m(t_0) = \min\{m_j(t_0),\ j = 1, 2\}.$$

Here $C(S_1, S_2)$ denotes the set of all continuous functions $\varphi : S_1 \to S_2$ with the supremum norm $\|\cdot\|$. Throughout this paper we assume that $a, b \in C(\mathbb{R}^+, \mathbb{R})$, $c \in C^1(\mathbb{R}^+, \mathbb{R})$ and $\tau_1, \tau_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $t - \tau_1(t) \to \infty$ and $t - \tau_2(t) \to \infty$ as $t \to \infty$. The functions Q and G are locally Lipschitz continuous. That is, there are positive constants L_1 and L_2 so that if $|x|, |y| \leq L$ for some positive constant L then

$$|Q(x) - Q(y)| \leq L_1 \|x - y\| \text{ and } Q(0) = 0, \tag{1.2}$$

and

$$|G(x) - G(y)| \leq L_2 \|x - y\| \text{ and } G(0) = 0. \tag{1.3}$$

Less general forms of equation (1.1) have been previously investigated by many authors. For example, Burton in [5], and Zhang in [16] have studied the equation

$$x'(t) = -a(t)x(t - \tau_1(t)), \tag{1.4}$$

and proved the following.

Theorem A. [5] *Suppose that $\tau_1(t) = \tau$ and there exists a constant $\alpha < 1$ such that*

$$\int_{t-\tau}^{t} |a(s+\tau)|\, ds + \int_0^t |a(s+\tau)| e^{-\int_s^t a(u+\tau)\, du} \left(\int_{s-\tau}^{s} |a(u+\tau)|\, du \right) ds \leq \alpha, \tag{1.5}$$

for all $t \geq 0$ and $\int_0^\infty a(s) ds = \infty$. Then, for every continuous initial function $\psi : [-\tau, 0] \to \mathbb{R}$, the solution $x(t) = x(t, 0, \psi)$ of (1.4) *is bounded and tends to zero as $t \to \infty$.*

Theorem B. [16] *Suppose that τ_1 is differentiable, the inverse function g of $t - \tau_1(t)$ exists, and there exists a constant $\alpha \in (0, 1)$ such that for $t \geq 0$, $\liminf_{t\to\infty} \int_0^t a(g(s))\, ds > -\infty$ and*

$$\int_{t-\tau_1(t)}^{t} |a(g(s))|\, ds + \int_0^t e^{-\int_s^t a(g(u))\, du} |a(s)||\tau_1'(s)|\, ds$$
$$+ \int_0^t e^{-\int_s^t a(g(u))\, du} |a(g(s))| \left(\int_{s-\tau_1(s)}^{s} |a(g(u))|\, du \right) ds \leq \alpha. \tag{1.6}$$

Then the zero solution of (1.4) *is asymptotically stable if and only if $\int_0^t a(g(s))\, ds \to \infty$, as $t \to \infty$.*

Obviously, Theorem B improves Theorem A. On the other hand, Raffoul in [14], Jin and Luo in [13], and the authors in [1] considered the following linear neutral differential equation

$$x'(t) = -a(t)x(t) - b(t)x(t - \tau_2(t)) + c(t)x'(t - \tau_2(t)), \tag{1.7}$$

and obtained the following.

THEOREM C [14] *Let τ_2 be twice differentiable and $\tau_2'(t) \neq 1$ for all $t \in \mathbb{R}^+$. Suppose that there exists a constant $\alpha \in (0,1)$ such that for $t \geq 0$, $\int_0^t a(s)\,ds \to \infty$* as *$t \to \infty$, and*

$$|\frac{c(t)}{1-\tau_2'(t)}| + \int_0^t e^{-\int_s^t a(u)\,du}|b(s) + r_1(s)|\,ds \leq \alpha, \tag{1.8}$$

where $r_1(t) = \dfrac{[c(t)a(t) + c'(t)](1-\tau_2'(t)) + c(t)\tau_2''(t)}{(1-\tau_2'(t))^2}$. Then every solution $x(t) = x(t,0,\psi)$ of (1.7) *with a small continuous initial function ψ is bounded and tends to zero as $t \to \infty$.*

THEOREM D. [13] *Let τ_2 be twice differentiable and $\tau_2'(t) \neq 1$ for all $t \in \mathbb{R}^+$. Suppose that there exists a constant $\alpha \in (0,1)$ and a function $h \in C(\mathbb{R}^+, \mathbb{R})$ such that for $t \geq 0$, $\liminf_{t\to\infty} \int_0^t h(s)ds > -\infty$,* and

$$\begin{aligned}
&|\frac{c(t)}{1-\tau_2'(t)}| + \int_{t-\tau_2(t)}^t |h(s) - a(s)|\,ds \\
&\quad + \int_0^t e^{-\int_s^t h(u)\,du}| - b(s) + [h(s-\tau_2(s)) - a(s-\tau_2(s))] - r_2(s)|\,ds \\
&\qquad + \int_0^t e^{-\int_s^t h(u)\,du}|h(s)|(\int_{s-\tau_2(s)}^s |h(u) - a(u)|\,du)\,ds \leq \alpha,
\end{aligned} \tag{1.9}$$

where $r_2(t) = \dfrac{[c(t)h(t) + c'(t)](1-\tau_2'(t)) + c(t)\tau_2''(t)}{(1-\tau_2'(t))^2}$. Then the zero solution of (1.7) *is asymptotically stable if and only if $\int_0^t h(s)\,ds \to \infty$* as *$t \to \infty$.*

THEOREM E. [1] *Suppose that $\tau_2$is twice differentiable and $\tau_2'(t) \neq 1$for all $t \in \mathbb{R}^+$,and there exist continuous functions $h_j : [m_j(t_0), \infty) \to \mathbb{R}$for $j = 1,2$and a constant $\alpha \in (0,1)$such that for $t \geq 0$*

$$\liminf_{t\to\infty} \int_0^t H(s)\,ds > -\infty,$$

and

$$\begin{aligned}
&|\frac{c(t)}{1-\tau_2'(t)}| + \int_{t-\tau_2(t)}^t |h_2(s)|\,ds \\
&+ \int_0^t e^{-\int_s^t H(u)\,du}(| - a(s) + h_1(s)| + | - b(s) + h_2(s-\tau_2(s))(1-\tau_2'(s)) - r(s)|)\,ds \\
&\qquad + \int_0^t e^{-\int_s^t H(u)\,du}|H(s)|(\int_{s-\tau_2(s)}^s |h_2(u)|\,du)\,ds \leq \alpha,
\end{aligned} \tag{1.10}$$

where $H(t) = \sum_{j=1}^2 h_j(t)$and $r(t) = \dfrac{[c(t)H(t) + c'(t)](1-\tau_2'(t)) + c(t)\tau_2''(t)}{(1-\tau_2'(t))^2}$. Then the zero solution of (1.7)*is asymptotically stable if and only if*

$$\int_0^t H(s)\,ds \to \infty \text{ as } t \to \infty.$$

Obviously, Theorem E improves Theorems C and D. On the other hand, the second author with Khemis in [11], considered the following nonlinear neutral differential equation

$$x'(t) = -a(t)x(t) + b(t)G(x(t-\tau_2(t))) + c(t)x'(t-\tau_2(t)), \tag{1.11}$$

and obtained the following.

Theorem F. [14] *Suppose* (1.3)*holds with* $L_2 = 1$. *Let* τ_2 *be twice differentiable and* $\tau_2'(t) \neq 1$ *for all* $t \in \mathbb{R}^+$. *Suppose that there exists a constant* $\alpha \in (0,1)$ *such that for* $t \geq 0$, $\int_0^t a(s)\,ds \to \infty$ as $t \to \infty$, *and*

$$|\frac{c(t)}{1-\tau_2'(t)}| + \int_0^t e^{-\int_s^t a(u)\,du}(|b(s)| + |r_1(s)|)\,ds \leq \alpha, \tag{1.12}$$

where r_1*is as in Theorem* C. *Then every solution* $x(t) = x(t,0,\psi)$ *of* (1.11) *with a small continuous initial function* ψ *is bounded and tends to zero as* $t \to \infty$.

Also, the second author with Khemis in [11] considered the following nonlinear neutral differential equation

$$x'(t) = -a(t)x(t) - b(t)x^2(t-\tau_2(t)) + c(t)x'(t-\tau_2(t))x(t-\tau_2(t)), \tag{1.13}$$

and obtained the following.

Theorem G. [11] *Let* τ_2*be twice differentiable and* $\tau_2'(t) \neq 1$*for all* $t \in \mathbb{R}^+$.*Suppose that there exists a constant* $\alpha \in (0,1)$ *such that for* $t \geq 0$, $\int_0^t a(s)\,ds \to \infty$ *as* $t \to \infty$, *and*

$$L\{|\frac{c(t)}{1-\tau_2'(t)}| + \int_0^t e^{-\int_s^t a(u)\,du}|r_1(s) + 2b(s)|\,ds\} \leq \alpha, \tag{1.14}$$

where r_1*is as in Theorem* C. *Then every solution* $x(t) = x(t,0,\psi)$ *of* (1.13) *with a small continuous initial function* ψ *is bounded and tends to zero as* $t \to \infty$.

Our purpose here is to give, by using a fixed point approach, asymptotic stability results of the zero solution of the nonlinear neutral differential equation with variable delays (1.1). We provide, what we think, minimal conditions to reach these objectives for a such general equation. An asymptotic stability theorem with a necessary and sufficient condition is proved. It is worth pointing out that our results do not ask for a fixed sign on the coefficient functions nor do they need boundedness of delays. We end by giving two examples to illustrate our work. The results presented in this paper improve and generalize the main results in [1, 5, 11, 13, 14, 16].

2. Main results

For each $(t_0, \psi) \in \mathbb{R}^+ \times C([m(t_0), t_0], \mathbb{R})$, a solution of (1.1) through (t_0, ψ) is a continuous function $x : [m(t_0), t_0 + \alpha) \to \mathbb{R}$ for some positive constant $\alpha > 0$ such that x satisfies (1.1) on $[t_0, t_0 + \alpha)$ and $x = \psi$ on $[m(t_0), t_0]$. We denote such a solution by $x(t) = x(t, t_0, \psi)$. For each $(t_0, \psi) \in \mathbb{R}^+ \times C([m(t_0), t_0], \mathbb{R})$, there exists a unique solution $x(t) = x(t, t_0, \psi)$ of (1.1) defined on $[t_0, \infty)$. For fixed t_0, we define $\|\psi\| = \max\{|\psi(t)| : m(t_0) \le t \le t_0\}$. Stability definitions may be found in [3], for example.

Our aim here is to improve and generalize Theorems A–G to (1.1).

THEOREM 1. *Suppose* (1.2) *and* (1.3) *hold. Let* τ_1 *be differentiable and* τ_2 *be twice differentiable with* $\tau_2'(t) \neq 1$ *for all* $t \in \mathbb{R}^+$. *Suppose that there exist continuous functions* $h_j : [m_j(t_0), \infty) \to \mathbb{R}$ *for* $j = 1, 2$ *and a constant* $\alpha \in (0, 1)$ *such that for* $t \ge 0$

$$\liminf_{t\to\infty} \int_0^t H(s)ds > -\infty, \tag{2.1}$$

and

$$\begin{aligned} L_1 \left| \frac{c(t)}{1-\tau_2'(t)} \right| &+ \sum_{j=1}^{2} \int_{t-\tau_j(t)}^{t} |h_j(s)|\, ds \\ &+ \int_0^t e^{-\int_s^t H(u)\,du} \{|-a(s) + h_1(s-\tau_1(s))(1-\tau_1'(s))| \\ &+ |h_2(s-\tau_2(s))(1-\tau_2'(s))| + L_1|r(s)| + L_2|b(s)|\}\, ds \\ &\quad + \sum_{j=1}^{2} \int_0^t e^{-\int_s^t H(u)\,du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\, du \right) ds \le \alpha, \end{aligned} \tag{2.2}$$

where $H(t) = \sum_{j=1}^{2} h_j(t)$ *and* $r(t) = \dfrac{[c(t)H(t) + c'(t)](1-\tau_2'(t)) + c(t)\tau_2''(t)}{(1-\tau_2'(t))^2}$. *Then the zero solution of* (1.1) *is asymptotically stable if and only if*

$$\int_0^t H(s)\, ds \to \infty \text{ as } t \to \infty. \tag{2.3}$$

Proof. First, suppose that (2.3) holds. For each $t_0 \ge 0$, we set

$$K = \sup_{t\ge 0} \{e^{-\int_0^t H(s)\,ds}\}. \tag{2.4}$$

Let $\psi \in C([m(t_0), t_0], \mathbb{R})$ be fixed and define

$S_\psi = \{\varphi \in C([m(t_0), \infty), \mathbb{R}) : \varphi(t) \to 0 \text{ as } t \to \infty,\ \varphi(t) = \psi(t) \text{ for } t \in [m(t_0), t_0]\}$.

Then S_ψ is a complete metric space with metric $\rho(x, y) = \sup_{t \ge t_0}\{|x(t) - y(t)|\}$.

Multiply both sides of (1.1) by $e^{\int_{t_0}^t H(u)\,du}$ and then integrate from t_0 to t to obtain

$$\begin{aligned} x(t) = \psi(t_0) e^{-\int_{t_0}^t H(u)\,du} &+ \sum_{j=1}^{2} \int_{t_0}^{t} e^{-\int_s^t H(u)\,du} h_j(s) x(s)\, ds \\ &+ \int_{t_0}^{t} e^{-\int_s^t H(u)\,du} \{-a(s)x(s-\tau_1(s)) \\ &+ c(s)x'(s-\tau_2(s))Q'(x(s-\tau_2(s))) + b(s)G(x(s-\tau_2(s)))\}\, ds. \end{aligned}$$

Performing an integration by parts, we have

$$
\begin{aligned}
x(t) &= \Big(\psi(t_0) - \frac{c(t_0)}{1-\tau_2'(t_0)} Q(\psi(t_0-\tau_2(t_0)))\Big) e^{-\int_{t_0}^t H(u)\,du} \\
&\quad + \frac{c(t)}{1-\tau_2'(t)} Q(x(t-\tau_2(t))) + \sum_{j=1}^{2} \int_{t_0}^{t} e^{-\int_s^t H(u)\,du} d\Big(\int_{s-\tau_j(s)}^{s} h_j(u)x(u)\,du\Big) \\
&\qquad + \sum_{j=1}^{2} \int_{t_0}^{t} e^{-\int_s^t H(u)\,du} h_j(s-\tau_j(s))(1-\tau_j'(s))x(s-\tau_j(s))\,ds \\
&+ \int_{t_0}^{t} e^{-\int_s^t H(u)\,du} \{-a(s)x(s-\tau_1(s)) - r(s)Q(x(s-\tau_2(s))) + b(s)G(x(s-\tau_2(s)))\}\,ds \\
&= \Big(\psi(t_0) - \frac{c(t_0)}{1-\tau_2'(t_0)} Q(\psi(t_0-\tau_2(t_0)))\Big) - \sum_{j=1}^{2} \int_{t_0-\tau_j(t_0)}^{t_0} h_j(s)\psi(s)\,ds e^{-\int_{t_0}^t H(u)\,du} \\
&\qquad + \frac{c(t)}{1-\tau_2'(t)} Q(x(t-\tau_2(t))) + \sum_{j=1}^{2} \int_{t-\tau_j(t)}^{t} h_j(s)x(s)\,ds \\
&\quad + \int_{t_0}^{t} e^{-\int_s^t H(u)\,du} \{(-a(s) + h_1(s-\tau_1(s))(1-\tau_1'(s)))x(s-\tau_1(s)) \\
&\qquad + h_2(s-\tau_2(s))(1-\tau_2'(s))x(s-\tau_2(s))\}\,ds \\
&\quad + \int_{t_0}^{t} e^{-\int_s^t H(u)\,du} \{-r(s)Q(x(s-\tau_2(s))) + b(s)G(x(s-\tau_2(s)))\}\,ds \\
&\qquad - \sum_{j=1}^{2} \int_{t_0}^{t} e^{-\int_s^t H(u)\,du} H(s)\Big(\int_{s-\tau_j(s)}^{s} h_j(u)x(u)\,du\Big)\,ds. \qquad (2.5)
\end{aligned}
$$

Use (2.5) to define the operator $P : S_\psi \to S_\psi$ by $(P\varphi)(t) = \psi(t)$ for $t \in [m(t_0), t_0]$ and

$$
\begin{aligned}
(P\varphi)(t) &= \Big(\psi(t_0) - \frac{c(t_0)}{1-\tau_2'(t_0)} Q(\psi(t_0-\tau_2(t_0))) - \sum_{j=1}^{2} \int_{t_0-\tau_j(t_0)}^{t_0} h_j(s)\psi(s)\,ds\Big) \times \\
&\quad \times e^{-\int_{t_0}^t H(u)\,du} + \frac{c(t)}{1-\tau_2'(t)} Q(\varphi(t-\tau_2(t))) + \sum_{j=1}^{2} \int_{t-\tau_j(t)}^{t} h_j(s)\varphi(s)\,ds \\
&\quad + \int_{t_0}^{t} e^{-\int_s^t H(u)\,du} \{(-a(s) + h_1(s-\tau_1(s))(1-\tau_1'(s)))\varphi(s-\tau_1(s)) \\
&\qquad + h_2(s-\tau_2(s))(1-\tau_2'(s))\varphi(s-\tau_2(s))\}\,ds \\
&\quad + \int_{t_0}^{t} e^{-\int_s^t H(u)\,du} \{-r(s)Q(\varphi(s-\tau_2(s))) + b(s)G(x(s-\tau_2(s)))\}\,ds \\
&\qquad - \sum_{j=1}^{2} \int_{t_0}^{t} e^{-\int_s^t H(u)\,du} H(s)\Big(\int_{s-\tau_j(s)}^{s} h_j(u)\varphi(u)\,du\Big)\,ds. \qquad (2.6)
\end{aligned}
$$

for $t \geq t_0$. It is clear that $(P\varphi) \in C([m(t_0), \infty), \mathbb{R})$. We now show that $(P\varphi)(t) \to 0$ as $t \to \infty$. Since $\varphi(t) \to 0$ and $t - \tau_j(t) \to \infty$ as $t \to \infty$, for each $\varepsilon > 0$, there exists a $T_1 > t_0$ such that $s \geq T_1$ implies that $|x(s - \tau_j(s))| < \varepsilon$ for $j = 1, 2$. Thus, for $t \geq T_1$, the last term I_6 in (2.6) satisfies

$$\begin{aligned}
|I_6| &= \left|\sum_{j=1}^{2} \int_{t_0}^{t} e^{-\int_s^t H(u)\,du} H(s) \left(\int_{s-\tau_j(s)}^{s} h_j(u)\varphi(u)\,du \right) ds \right| \\
&\leq \sum_{j=1}^{2} \int_{t_0}^{T_1} e^{-\int_s^t H(u)\,du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)||\varphi(u)|\,du \right) ds \\
&\quad + \sum_{j=1}^{2} \int_{T_1}^{t} e^{-\int_s^t H(u)\,du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)||\varphi(u)|\,du \right) ds \\
&\leq \sup_{\sigma \geq m(t_0)} |\varphi(\sigma)| \sum_{j=1}^{2} \int_{t_0}^{T_1} e^{-\int_s^t H(u)\,du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\,du \right) ds \\
&\quad + \varepsilon \sum_{j=1}^{2} \int_{T_1}^{t} e^{-\int_s^t H(u)\,du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\,du \right) ds.
\end{aligned}$$

By (2.3), there exists $T_2 > T_1$ such that $t \geq T_2$ implies

$$\begin{aligned}
&\sup_{\sigma \geq m(t_0)} |\varphi(\sigma)| \sum_{j=1}^{2} \int_{t_0}^{T_1} e^{-\int_s^t H(u)\,du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\,du \right) ds \\
&\qquad = \sup_{\sigma \geq m(t_0)} |\varphi(\sigma)| e^{-\int_{T_1}^{t} H(u)\,du} \sum_{j=1}^{2} \int_{t_0}^{T_1} e^{-\int_s^{T_1} H(u)\,du} |H(s)| \times \\
&\qquad\qquad \times \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\,du \right) ds < \varepsilon.
\end{aligned}$$

Apply (2.2) to obtain $|I_6| < \varepsilon + \alpha\epsilon < 2\varepsilon$. Thus, $I_6 \to 0$ as $t \to \infty$. Similarly, we can show that the rest of the terms in (2.6) approach zero as $t \to \infty$. This yiel ds $(P\varphi)(t) \to 0$ as $t \to \infty$, and hence $P\varphi \in S_\psi$. Also, by (2.2), P is a contraction mapping with contraction constant α. By the contraction mapping principle (Smart [15, *p.* 2]), P has a unique fixed point x in S_ψ which is a solution of (1.1) with $x(t) = \psi(t)$ on $[m(t_0), t_0]$ and $x(t) = x(t, t_0, \psi) \to 0$ as $t \to \infty$.

To obtain the asymptotic stability, we need to show that the zero solution of (1.1) is stable. Let $\varepsilon > 0$ be given and choose $\delta > 0$ ($\delta < \varepsilon$) satisfying $2\delta K e^{\int_0^{t_0} H(u)\,du} + \alpha\varepsilon < \varepsilon$. If $x(t) = x(t, t_0, \psi)$ is a solution of (1.1) with $\|\psi\| < \delta$, then $x(t) = (Px)(t)$ defined in (2.6). We claim that $|x(t)| < \varepsilon$ for all $t \geq t_0$. Notice that $|x(s)| < \varepsilon$ on $[m(t_0), t_0]$. If there exists $t^* > t_0$ such that $|x(t^*)| = \varepsilon$ and $|x(s)| < \varepsilon$ for $m(t_0) \leq s < t^*$, then it follows from (2.6) that

$$|x(t^*)| \leq \|\psi\| \left(1 + L_1 \left| \frac{c(t_0)}{1 - \tau_2'(t_0)} \right| + \sum_{j=1}^{2} \int_{t_0-\tau_j(t_0)}^{t_0} |h_j(s)|\,ds \right) e^{-\int_{t_0}^{t^*} H(u)\,du}$$

$$
\begin{aligned}
&+ \epsilon L_1 \left| \frac{c(t^*)}{1-\tau_2'(t^*)} \right| + \epsilon \sum_{j=1}^{2} \int_{t^*-\tau_j(t^*)}^{t^*} |h_j(s)|\, ds \\
&+ \epsilon \int_{t_0}^{t^*} e^{-\int_s^{t^*} H(u)\,du} \{ |-a(s) + h_1(s-\tau_1(s))(1-\tau_1'(s))| \\
&+ |h_2(s-\tau_2(s))(1-\tau_2'(s))| + L_1|r(s)| + L_2|b(s)| \}\, ds \\
&+ \epsilon \sum_{j=1}^{2} \int_{t_0}^{t^*} e^{-\int_s^{t^*} H(u)\,du} |H(s)| \left(\int_{s-\tau_j(s)}^{s} |h_j(u)|\, du \right) ds \\
\leq\ & 2\delta K e^{\int_0^{t_0} H(u)\,du} + \alpha\varepsilon < \epsilon,
\end{aligned}
$$

which contradicts the definition of t^*. Thus, $|x(t)| < \varepsilon$ for all $t \geq t_0$, and the zero solution of (1.1) is stable. This shows that the zero solution of (1.1) is asymptotically stable if (2.3) holds.

Conversely, suppose (2.3) fails. Then by (2.1) there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_0^{t_n} H(u)\,du = l$ for some $l \in \mathbb{R}$. We may also choose a positive constant J satisfying

$$
-J \leq \int_0^{t_n} H(u)\,du \leq J,
$$

for all $n \geq 1$. To simplify our expressions, we define

$$
\begin{aligned}
\omega(s) = & |-a(s) + h_1(s-\tau_1(s))(1-\tau_1'(s))| + |h_2(s-\tau_2(s))(1-\tau_2'(s))| \\
& + L_1|r(s)| + L_2|b(s)| + |H(s)| \sum_{j=1}^{2} \int_{s-\tau_j(s)}^{s} |h_j(u)|\, du,
\end{aligned}
$$

for all $s \geq 0$. By (2.2), we have

$$
\int_0^{t_n} e^{-\int_s^{t_n} H(u)\,du} \omega(s)\, ds \leq \alpha.
$$

This yields

$$
\int_0^{t_n} e^{\int_0^{s} H(u)\,du} \omega(s)\, ds \leq \alpha e^{\int_0^{t_n} H(u)\,du} \leq J.
$$

The sequence $\{\int_0^{t_n} e^{\int_0^s H(u)\,du} \omega(s)\, ds\}$ is bounded, so there exists a convergent subsequence. For brevity of notation, we may assume that

$$
\lim_{n\to\infty} \int_0^{t_n} e^{\int_0^{s} H(u)\,du} \omega(s)\, ds = \gamma,
$$

for some $\gamma \in \mathbb{R}^+$ and choose a positive integer m so large that

$$
\int_{t_m}^{t_n} e^{\int_0^{s} H(u)\,du} \omega(s)\, ds < \delta_0/4K,
$$

for all $n \geq m$, where $\delta_0 > 0$ satisfies $2\delta_0 K e^J + \alpha \leq 1$.

By (2.1), K in (2.4) is well defined. We now consider the solution $x(t) = x(t, t_m, \psi)$ of (1.1) with $\psi(t_m) = \delta_0$ and $|\psi(s)| \le \delta_0$ for $s \le t_m$. We may choose ψ so that $|x(t)| \le 1$ for $t \ge t_m$ and

$$\psi(t_m) - \frac{c(t_m)}{1-\tau_2'(t_m)} Q(\psi(t_m - \tau_2(t_m))) - \sum_{j=1}^{2} \int_{t_m-\tau_j(t_m)}^{t_m} h_j(s)\psi(s)\,ds \ge \frac{1}{2}\delta_0.$$

It follows from (2.6) with $x(t) = (Px)(t)$ that for $n \ge m$

$$\begin{aligned}
&\left| x(t_n) - \frac{c(t_n)}{1-\tau_2'(t_n)} Q(x(t_n - \tau_2(t_n))) - \sum_{j=1}^{2} \int_{t_n-\tau_j(t_n)}^{t_n} h_j(s)x(s)\,ds \right| \\
&\quad \ge \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n} H(u)\,du} - \int_{t_m}^{t_n} e^{-\int_s^{t_n} H(u)\,du}\omega(s)\,ds \\
&\quad = \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n} H(u)\,du} - e^{-\int_0^{t_n} H(u)\,du} \int_{t_m}^{t_n} e^{\int_0^s H(u)\,du}\omega(s)\,ds \\
&\quad = e^{-\int_{t_m}^{t_n} H(u)\,du} \left(\frac{1}{2}\delta_0 - e^{-\int_0^{t_m} H(u)\,du} \int_{t_m}^{t_n} e^{\int_0^s H(u)\,du}\omega(s)\,ds \right) \\
&\quad \ge e^{-\int_{t_m}^{t_n} H(u)\,du} \left(\frac{1}{2}\delta_0 - K \int_{t_m}^{t_n} e^{\int_0^s H(u)\,du}\omega(s)\,ds \right) \\
&\quad \ge \frac{1}{4}\delta_0 e^{-\int_{t_m}^{t_n} H(u)\,du} \ge \frac{1}{4}\delta_0 e^{-2J} > 0.
\end{aligned} \tag{2.7}$$

On the other hand, if the zero solution of (1.1) is asymptotically stable, then $x(t) = x(t, t_m, \psi) \to 0$ as $t \to \infty$. Since $t_n - \tau_j(t_n) \to \infty$ as $n \to \infty$ and (2.2) holds, we have

$$x(t_n) - \frac{c(t_n)}{1-\tau_2'(t_n)} Q(x(t_n - \tau_2(t_n))) - \sum_{j=1}^{2} \int_{t_n-\tau_j(t_n)}^{t_n} h_j(s)x(s)\,ds \to 0 \text{ as } n \to \infty,$$

which contradicts (2.7). Hence condition (2.3) is necessary for the asymptotic stability of the zero solution of (1.1). The proof is complete. ∎

Remark 1. It follows from the first part of the proof of Theorem 1 that the zero solution of (1.1) is stable under (2.1) and (2.2). Moreover, Theorem 1 still holds if (2.2) is satisfied for $t \ge t_\sigma$ for some $t_\sigma \in \mathbb{R}^+$.

For the special case $b = 0$ and $c = 0$, we get

Corollary 1. *Let τ_1 be differentiable, and suppose that there exist continuous function $h_1 : [m_1(t_0), \infty) \to \mathbb{R}$ for and a constant $\alpha \in (0,1)$ such that for $t \ge 0$*

$$\liminf_{t\to\infty} \int_0^t h_1(s)\,ds > -\infty,$$

and

$$\int_{t-\tau_1(t)}^{t} |h_1(s)|\,ds + \int_0^t e^{-\int_s^t h_1(u)\,du} |-a(s) + h_1(s-\tau_1(s))(1-\tau_1'(s))|\,ds + \int_0^t e^{-\int_s^t h_1(u)\,du}|h_1(s)| \left(\int_{s-\tau_1(s)}^{s} |h_1(u)|\,du \right) ds \le \alpha. \tag{2.8}$$

Then the zero solution of (1.4) *is asymptotically stable if and only if*

$$\int_0^t h_1(s)\,ds \to \infty \ \textit{as}\ t \to \infty.$$

Remark 2. When $\tau_1(s) = \tau$, a constant, $h_1(s) = a(s+\tau)$, Corollary 1 contains Theorem A. When $h_1(s) = a(g(s))$, where $g(s)$ is the inverse function of $s - \tau_1(s)$, Corollary 1 reduces to Theorem B.

Remark 3. When $\tau_1 = 0$, $G(x) = -x$ and $Q(x) = x$, Theorem E is a corollary of Theorem 1.

For the special case $\tau_1 = 0$ and $Q(x) = x$, we get

Corollary 2. *Suppose* (1.3) *holds with* $L_2 = 1$. *Let* τ_2 *be twice differentiable and* $\tau_2'(t) \neq 1$ *for all* $t \in \mathbb{R}^+$. *Suppose that there exist continuous functions* $h_j : [m_j(t_0), \infty) \to \mathbb{R}$ *for* $j = 1, 2$ *and a constant* $\alpha \in (0, 1)$ *such that for* $t \geq 0$

$$\liminf_{t\to\infty} \int_0^t H(s)\,ds > -\infty,$$

and

$$\begin{aligned}&\left|\frac{c(t)}{1-\tau_2'(t)}\right| + \int_{t-\tau_2(t)}^t |h_2(s)|\,ds \\ &+ \int_0^t e^{-\int_s^t H(u)\,du}\{|-a(s)+h_1(s)| + |h_2(s-\tau_2(s))(1-\tau_2'(s)) - r(s)| + |b(s)|\}\,ds \\ &\qquad + \int_0^t e^{-\int_s^t H(u)\,du}|H(s)|\left(\int_{s-\tau_2(s)}^s |h_2(u)|\,du\right) ds \leq \alpha, \qquad (2.9)\end{aligned}$$

where $H(t) = \sum_{j=1}^2 h_j(t)$ *and* $r(t) = \dfrac{[c(t)H(t) + c'(t)](1-\tau_2'(t)) + c(t)\tau_2''(t)}{(1-\tau_2'(t))^2}$. *Then the zero solution of* (1.11) *is asymptotically stable if and only if*

$$\int_0^t H(s)\,ds \to \infty \ \textit{as}\ t \to \infty.$$

For the special case $\tau_1 = 0$, $G(x) = -x^2$ and $Q(x) = \frac{1}{2}x^2$, we get

Corollary 3. *Let* τ_2 *be twice differentiable with* $\tau_2'(t) \neq 1$ *for all* $t \in \mathbb{R}^+$. *Suppose that there exist continuous functions* $h_j : [m_j(t_0), \infty) \to \mathbb{R}$ *for* $j = 1, 2$ *and a constant* $\alpha \in (0, 1)$ *such that for* $t \geq 0$

$$\liminf_{t\to\infty} \int_0^t H(s)\,ds > -\infty,$$

and

$$L\left\{\left|\frac{c(t)}{1-\tau_2'(t)}\right| + \int_0^t e^{-\int_s^t H(u)\,du}|2b(s) + r(s)|\,ds\right\} + \int_{t-\tau_2(t)}^t |h_2(s)|\,ds$$

$$+\int_0^t e^{-\int_s^t H(u)\,du}\{|-a(s)+h_1(s)|+|h_2(s-\tau_2(s))(1-\tau_2'(s))|\}\,ds$$
$$+\int_0^t e^{-\int_s^t H(u)\,du}|H(s)|\Big(\int_{s-\tau_2(s)}^s |h_2(u)|\,du\Big)\,ds \le \alpha, \quad (2.10)$$

where $H(t)=\sum_{j=1}^2 h_j(t)$ *and* $r(t)=\dfrac{[c(t)H(t)+c'(t)](1-\tau_2'(t))+c(t)\tau_2''(t)}{(1-\tau_2'(t))^2}$. *Then the zero solution of* (1.13) *is asymptotically stable if and only if*

$$\int_0^t H(s)\,ds \to \infty \ \text{as}\ t\to\infty.$$

Remark 4. When $h_1(s)=a(s)$ and $h_2(s)=0$, then Corollaries 2 and 3 contain Theorems F and G, respectively.

3. Two examples

In this section, we give two examples to illustrate the applications of Corollary 2 and Theorem 1.

Example 1. Consider the following nonlinear neutral differential equation

$$x'(t)=-a(t)x(t)+b(t)G(x(t-\tau_2(t)))+c(t)x'(t-\tau_2(t)), \quad (3.1)$$

where $\tau_2(t)=0.066t$, $a(t)=1/(t+1)$, $b(t)=0.55/(t+1)$, $c(t)=0.32$ and $G(x)=\sin x$. Then the zero solution of (3.1) is asymptotically stable.

Proof. Choosing $h_1(t)=1/(t+1)$ and $h_2(t)=0.25/(t+1)$ in Corollary 2, we have $H(t)=1.25/(t+1)$,

$$|\frac{c(t)}{1-\tau_2'(t)}|=\frac{0.32}{0.934}<0.3427,$$
$$\int_{t-\tau_2(t)}^t |h_2(s)|\,ds=\int_{0.934t}^t \frac{0.25}{s+1}\,ds=0.25\ln\Big(\frac{t+1}{0.934t+1}\Big)<0.0171,$$

$$\int_0^t e^{-\int_s^t H(u)\,du}|H(s)|\Big(\int_{s-\tau_2(s)}^s |h_2(u)|\,du\Big)\,ds$$
$$<\int_0^t e^{-\int_s^t (1.25/(u+1))\,du}\frac{1.25}{s+1}\cdot 0.0171<0.0171,$$

and

$$\int_0^t e^{-\int_s^t H(u)\,du}\{|-a(s)+h_1(s)|+|h_2(s-\tau_2(s))(1-\tau_2'(s))-r(s)|+|b(s)|\}\,ds$$
$$=\int_0^t e^{-\int_s^t (1.25/(u+1))\,du}\left(\left|\frac{0.25\times 0.934}{0.934s+1}-\frac{1.25\times 0.32}{0.934(s+1)}\right|+\frac{0.55}{s+1}\right)\,ds$$
$$<\frac{0.32}{0.934}-\frac{0.25}{1.25}+\frac{0.55}{1.25}<0.5827.$$

It is easy to see that all the conditions of Corollary 2 hold for $\alpha = 0.3427+0.0171+0.5827+0.0171 = 0.9596 < 1$. Thus, Corollary 2 implies that the zero solution of (3.1) is asymptotically stable.

However, Theorem F cannot be used to verify that the zero solution of (3.1) is asymptotically stable. Obviously,

$$\left|\frac{c(t)}{1-\tau_2'(t)}\right| + \int_0^t e^{-\int_s^t a(u)\,du}(|b(s)|+|r_1(s)|)\,ds = \frac{0.32(2t+1)}{0.934(t+1)} + \frac{0.55t}{t+1}.$$

Thus, we have

$$\limsup_{t\geq 0}\left\{\left|\frac{c(t)}{1-\tau_2'(t)}\right| + \int_0^t e^{-\int_s^t a(u)\,du}(|b(s)|+|r_1(s)|)\,ds\right\} = \frac{0.64}{0.934} + 0.55 \simeq 1.2352.$$

In addition, the left-hand side of the following inequality is increasing in $t > 0$, then there exists some $t_0 > 0$ such that for $t \geq t_0$,

$$\left|\frac{c(t)}{1-\tau_2'(t)}\right| + \int_0^t e^{-\int_s^t a(u)\,du}(|b(s)|+|r_1(s)|)\,ds > 1.23.$$

This implies that condition (1.12) does not hold. Thus, Theorem F cannot be applied to equation (3.1). ■

Example 2. Consider the following nonlinear neutral differential equation

$$x'(t) = -a(t)x(t-\tau_1(t))+b(t)G(x(t-\tau_2(t)))+c(t)x'(t-\tau_2(t))Q'(x(t-\tau_2(t))), \quad (3.2)$$

where $\tau_1(t) = 0.068$, $\tau_2(t) = 0.074t$, $a(t) = 0.932/(0.932t+1)$, $b(t) = 0.082/(t+1)$, $c(t) = 0.44$, $Q(x) = 0.52(1-\cos(x))$, $G(x) = 1.22\sin(x)$. Then the zero solution of (3.2) is asymptotically stable.

Proof. Choosing $h_1(t) = 1/(t+1)$ and $h_2(t) = 0.31/(t+1)$ in Theorem 1, we have $H(t) = 1.31/(t+1)$,

$$L_1 = 0.52,\ L_2 = 1.22,$$
$$L_1\left|\frac{c(t)}{1-\tau_2'(t)}\right| = 0.52 \times \frac{0.44}{0.926} < 0.2471,$$

$$\begin{aligned}\sum_{j=1}^{2}\int_{t-\tau_j(t)}^{t}|h_j(s)|\,ds &= \int_{0.932t}^{t}\frac{1}{s+1}\,ds + \int_{0.926t}^{t}\frac{0.31}{s+1}\,ds\\ &= \ln\Big(\frac{t+1}{0.932t+1}\Big) + 0.31\ln\Big(\frac{t+1}{0.926t+1}\Big) < 0.0943,\end{aligned}$$

$$\begin{aligned}&\sum_{j=1}^{2}\int_0^t e^{-\int_s^t H(u)\,du}|H(s)|\Big(\int_{s-\tau_j(s)}^{s}|h_j(u)|\,du\Big)\,ds\\ &\qquad < \int_0^t e^{-\int_s^t (1.31/(u+1))\,du}\frac{1.31}{s+1} \times 0.0943 < 0.0943,\end{aligned}$$

$$\int_0^t e^{-\int_s^t H(u)\,du} |-a(s) + h_1(s - \tau_1(s))(1 - \tau_1'(s))| \, ds = 0,$$

and

$$\begin{aligned}
&\int_0^t e^{-\int_s^t H(u)\,du} \{|h_2(s - \tau_2(s))(1 - \tau_2'(s))| + L_1|r(s)| + L_2|b(s)|\} \, ds \\
&= \int_0^t e^{-\int_s^t (1.31/(u+1))\,du} \Big(\frac{0.31 \times 0.926}{0.926s + 1} + \frac{0.52 \times 1.31 \times 0.44}{0.926(s+1)} + \frac{1.22 \times 0.082}{s+1} \Big) \, ds \\
&< \frac{0.31}{1.31} + \frac{0.52 \times 0.44}{0.926} + \frac{1.22 \times 0.082}{1.31} < 0.5601.
\end{aligned}$$

It is easy to see that all the conditions of Theorem 1 hold for $\alpha = 0.2471 + 0.0943 + 0.5601 + 0.0943 = 0.9958 < 1$. Thus, Theorem 1 implies that the zero solution of (3.2) is asymptotically stable. ∎

Acknowledgement. The author would like to thank anonymous referees on suggestions to improve this text.

REFERENCES

[**1**] A. Ardjouni, A. Djoudi, *Fixed points and stability in linear neutral differential equations with variable delays*, Nonlinear Anal. **74** (2011), 2062–2070.

[**2**] T.A. Burton, *Volterra Integral and Differential Equations*, Academic Press, New York, 1983.

[**3**] T.A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*, Dover Publications, New York, 2006.

[**4**] T.A. Burton, *Liapunov functionals, fixed points, and stability by Krasnoselskii's theorem*, Nonlinear Studies **9** (2001), 181–190.

[**5**] T.A. Burton, *Stability by fixed point theory or Liapunov's theory: A comparison*, Fixed Point Theory **4** (2003), 15–32.

[**6**] T. A. Burton, *Fixed points and stability of a nonconvolution equation*, Proc. Amer. Math. Soc. **132** (2004), 3679–3687.

[**7**] T.A. Burton, T. Furumochi, *A note on stability by Schauder's theorem*, Funkc. Ekvacioj **44** (2001), 73–82.

[**8**] T.A. Burton, T. Furumochi, *Fixed points and problems in stability theory for ordinary and functional differential equations*, Dyn. Syst. Appl. **10** (2001), 89–116.

[**9**] T.A. Burton, T. Furumochi, *Asymptotic behavior of solutions of functional differential equations by fixed point theorems*, Dyn. Syst. Appl. **11** (2002), 499–519.

[**10**] T.A. Burton, T. Furumochi, *Krasnoselskii's fixed point theorem and stability*, Nonlinear Anal. **49** (2002), 445–454.

[**11**] A. Djoudi, R. Khemis, *Fixed point techniques and stability for neutral nonlinear differential equations with unbounded delays*, Georgian Math. J. **13** (2006), 25–34.

[**12**] C.H. Jin, J.W. Luo, *Stability in functional differential equations established using fixed point theory*, Nonlinear Anal. 68 (2008) 3307 3315.

[**13**] C.H. Jin, J.W. Luo, *Fixed points and stability in neutral differential equations with variable delays*, Proc. Amer. Math. Soc. 136 (2008), 909–918.

[**14**] Y.N. Raffoul, *Stability in neutral nonlinear differential equations with functional delays using fixed-point theory*, Math. Comput. Modelling **40** (2004), 691–700.

[**15**] D.R. Smart, *Fixed Point Theorems*, Cambridge Tracts in Mathematics, No. 66. Cambridge University Press, London-New York, 1974.

[**16**] B. Zhang, *Fixed points and stability in differential equations with variable delays*, Nonlinear Anal. **63** (2005), e233–e242.

(received 27.07.2011; in revised form 24.11.2011; available online 01.01.2012)

Laboratory of Applied Mathematics, University of Annaba, Department of Mathematics, P.O.Box 12, Annaba 23000, Algeria.

E-mail: abd_ardjouni@yahoo.fr, adjoudi@yahoo.com