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ON THE RINGS ON TORSION-FREE GROUPS

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Abstract. The typeset of a torsion-free group is one of the important concepts in the theory of abelian groups. We use the typeset of an abelian group to study the rings that exist over such groups. Moreover, we consider the types of rational groups belonging to an independent set of a group and obtain some results about their relation with the rings over the group.

1. Introduction

All groups considered in this paper are abelian, with addition as the group operation. In the present paper we focus on the rings over torsion-free groups of rank three and the related problems which are the main ideas of some papers such as [4, 6, 7].

At first we consider this question: If all proper subgroups of a torsion-free group are nil, then the whole group is nil? And in Theorem 3.2, we answer this question under some conditions for torsion-free groups with rank less than or equal two. Moreover, in Theorem 3.3, we find a property for the types of rank one subgroups which yields the existence of a special ring over the group. In the sequel we have some outcomes which discuss about the relation between elements with maximal types and the nilpotency of a group with rank three.

Finally, in Theorems 3.8 and 3.11 we deal with the types of rational groups belonging to an independent set of a group and their relation with the rings that exist over the group and its homomorphic images.

2. Notations and preliminaries

All groups considered in this paper are torsion-free and abelian, with addition as the group operation. Terminology and notation will mostly follow [5]. For a torsion-free group A and a prime p, the $p-$height of $x \in A$ denoted by $h_p^A(x)$, is the largest integer k such that p^k divides x in A; if no such maximal integer exists, we

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set $h_p^A(x)=\infty$. Now let $p_1, p_2, \dots$ be an increasing sequence of all primes. Then the sequence

$$\chi_A(x) = (h_{p_1}^A(x), h_{p_2}^A(x), \dots, h_{p_n}^A(x), \dots),$$

is said to be the height-sequence of x. We omit the subscript A if no ambiguity arises. For any two height-sequences $\chi = (k_1, k_2, \dots, k_n, \dots)$ and $\mu = (l_1, l_2, \dots, l_n, \dots)$ we set $\chi \geq \mu$ if $k_n \geq l_n$ for all n. Moreover, χ and μ will be considered equivalent if $\sum_n |k_n - l_n|$ is finite [we set $\infty - \infty = 0$]. An equivalence class of height-sequences is called a type. If $\chi(x)$ belongs to the type $\boldsymbol{t}$, then we say that x is of type $\boldsymbol{t}$. By the typeset of a torsion-free group A we mean the partially ordered set of types, i.e.,

$$T(A) = \{t(x) \mid 0 \neq x \in A\}.$$

For two types $t_1 = [(l_1, l_2, \dots)]$ and $t_2 = [(k_1, k_2, \dots)]$ we set

$$t_1 \cap t_2 = [(\min\{l_1, k_1\}, \min\{l_2, k_2\}, \dots)]$$

and

$$t_1 t_2 = [(l_1 + k_1, l_2 + k_2, \dots)].$$

A function $\mu : A \times A \longrightarrow A$ is called a multiplication on A if it satisfies

$$\mu(a, b + c) = \mu(a, b) + \mu(a, c),$$
$$\mu(b + c, a) = \mu(b, a) + \mu(c, a)$$

for all $a, b, c \in A$. Every ring R on A gives rise to a multiplication μ, namely, $\mu(a, b) = ab$, and this correspondence between rings structures and multiplications on A is bijective. For two arbitrary multiplications μ and ν on A, we set

$$(\mu + \nu)(a, b) = \mu(a, b) + \nu(a, b),$$

for all $a, b \in A$. Then under this rule of composition, the multiplications on A forms an abelian group, the group of multiplications on A, that is denoted by $\mathrm{Mult}(A)$. A finite rank torsion-free group A is completely decomposable if A is the direct sum of rank one groups and an arbitrary torsion-free group is nil group if the zero ring is the only ring on it. Moreover, a torsion-free group A is of field type if there exists a ring R on A with $\mathbb{Q} \otimes R$ a field, where $\mathbb{Q}$ is the field of rational numbers and the tensor product is taken over the integers. Finally, a ring R is said to be a nil ring if for any element $a \in R$ there exists an integer n such that $a^n = 0$ and R is nilpotent, if $R^k = 0$ for some integer k. Also the ring R is called periodic if for each $x \in R$ the set $\{x, x^2, x^3, \dots\}$ is finite.

3. Main results

At first we have a theorem which contains our efforts to answer the question:

If all proper subgroups of a torsion-free group are nil, then under which conditions the whole group is nil?

Lemma 3.1 [4, Corollary 2.1.3] *Let $A = \bigoplus_{i\in I} A_i$ be a completely decomposable torsion-free group with $r(A_i) = 1$ for each $i \in I$. Then A is nil if and only if $t(A_i)t(A_j) \nleq t(A_k)$ for all $i, j, k \in I$.*

Theorem 3.2. *Let A be a group of rank ≤ 2 which $|\,\mathrm{T}(A)| \leq 3$ and A be a completely decomposable group when $|\,\mathrm{T}(A)| = 2$. If all proper subgroups of A are nil, then A is nil.*

Proof. At first note that the group of integers, $\mathbb{Z}$, does not satisfy in the hypothesis of the Theorem; i.e., $\mathbb{Z}$ has many proper subgroups which are non-nil and so the type of any rank one group which all its proper subgroups are nil, is greater than $t(\mathbb{Z})$.

Now let A be a rank one group with $t(A) = [(k_i)_{i\in I}]$ and all proper subgroups of A be nil. If A is non-nil, then $t^2(A) = t(A)$ and so $k_i = 0$ or ∞ for almost i. We know B is a subgroup of A if and only if $t(B) \leq t(A)$ and so in this case, because the type of A is greater than $t(\mathbb{Z})$, it is easy to choose an idempotent type $t' < t$ and $B \leq A$ with $t(B) = t'$. But such B is a non-nil subgroup of A which yields a contradiction and this completes the firs part of the proof.

Now consider the case $r(A) = 2$. If $|\,\mathrm{T}(A)| = 2$, then by hypothesis of the Theorem, A is a completely decomposable group. Write $A = A_1 \oplus A_2$. Now A_1 and A_2 are nil, because all proper subgroups of A are nil. Therefore $t_1 = t(A_1)$ and $t_2 = t(A_2)$ are not idempotent, which means almost all of their components are finite. Moreover, by $|\,\mathrm{T}(A)| = 2$, without loss of generality, we could assume that $t_1 < t_2$. Hence $t_1t_2 > t_1, t_2$ and by Lemma 3.1, A is a nil group.

Finally, let A be a rank two group with $|\,\mathrm{T}(A)| = 1$ or 3 and all proper subgroups of A are nil. Suppose that A be non-nil:

If $|\,\mathrm{T}(A)| = 1$, then the type of A is idempotent and the first step of proof is applied.

If $\mathrm{T}(A) = \{t_0, t_1, t_2\}$ such that $t_0 < t_1$ and $t_0 < t_2$. Let $\{x, y\}$ be a maximal independent set of A such that $t(x) = t_1$ and $t(y) = t_2$. Then we have one of the following cases:

(i) If t_1, t_2 are comparable, then any ring on A satisfies $x^2 = ax, y^2 = by, xy = yx = 0$ for some rational numbers a, b;

(ii) If $t_1^2 = t_1$ and $t_2^2 \neq t_2$, then any multiplication on A satisfies $x^2 = ax, y^2 = xy = yx = 0$ for some rational number a.

(iii) If $t_1^2 = t_1$ and $t_2^2 = t_2$, then $x^2 = ax, y^2 = by, xy = yx = 0$ for some rational numbers a, b which are not both zero.

Therefore in all cases we could obtain some proper subgroup $B = \langle x\rangle_*$ or $\langle y\rangle_*$ such that B is not nil and the proof completes. ■

Theorem 3.3 *Let $A = A_1 \oplus A_2$ be a torsion-free group of rank three such that $r(A_2) = 2$. If there exists a pure subgroup B of A_2 with $t(B) = t(A_1)$ and $t(\frac{A}{A_1\oplus B})^2 \leq t(A_1 \oplus B)$, then A supports a non-trivial periodic ring.*

Proof. Let $R_1 \leq R_2$ be two rational groups such that $1 \in R_1$, and

$$t(R_1) = t(\frac{A}{A_1 \oplus B}) \ , \ t(R_2) = t(A_1 \oplus B).$$

By hypothesis in the theorem we have $R_1^2 = \{rr' \colon \mid r, r' \in R_1\} \subseteq R_2$. Now let $a \in A$ be such that

$$\frac{A}{A_1 \oplus B} = R_1(a + (A_1 \oplus B))$$

and $a' \in A_1, b \in B$ such that $A_1 \oplus B = R_2 a' + R_2 b$. If $x_1, x_2 \in A$, then $n_i x_i = m_i a + a_i + b_i$ with $\frac{m_i}{n_i} \in R_1$ and $a_i + b_i \in A_1 \oplus B$ for $i = 1, 2$. Define

$$x_1 x_2 = \frac{m_1 m_2}{n_1 n_2}(a' + b),$$

and it is easy to see that this multiplication yields a periodic ring on A such that $x_1 x_2 x_3 = 0$ for all $x_1, x_2, x_3 \in A$. ∎

Proposition 3.4. [3, Proposition 2] *Let A be a torsion-free group and $a_1, a_2 \in A$ such that $t(a_1)$ and $t(a_2)$ are not idempotent. Then for all multiplications on A, either $t(a_1 a_2) > t(a_1)$ or $t(a_1 a_2) > t(a_2)$.*

Theorem 3.5. *Let A be a torsion-free group of rank three. If A is non-nil, then $\mathrm{T}(A)$ does not contain maximal elements $t(x_i), i = 1, 2, 3$, with $t(x_i)$ non-idempotent and $\{x_1, x_2, x_3\}$ independent.*

Proof. Let $t_1 = t(x_1), t_2 = t(x_2), t_3 = t(x_3)$ and $\{x_1, x_2, x_3\}$ be an independent set of A. Now by Proposition 3.4, $t(x_i x_j) > t(x_i)$ or $t(x_i x_j) > t(x_j)$. Hence by maximality of t_i and t_j we obtain $x_i x_j = 0$ for all $1 \leq i, j \leq 3$.

Moreover $x_i^2 = 0$ because $t_i^2 \neq t_i$ and t_i is a maximal element in $\mathrm{T}(A)$. But x_1, x_2, x_3 are independent, hence for all $x, y \in A$ there exist $n, m \in \mathbb{Z} - \{0\}$ and $n_i, m_i \in \mathbb{Z}, (i = 1, 2, 3)$, such that

$$nx = \sum_{i=1}^{3} n_i x_i, \quad my = \sum_{i=1}^{3} m_i x_i.$$

This implies $nmxy = \sum_{i,j=1}^{3} n_i m_j x_i x_j$, hence $xy = 0$ contradicting the fact that A is non-nil. ∎

Remark 3.6. Similar result as Theorem 3.5 holds for every torsion-free groups of finite rank, i.e., let A be a non-nil torsion-free abelian group of rank n, then $T(A)$ does not contain maximal elements $t(x_i), i = 1, 2, \ldots, n$, with $t^2(x_i) \neq t(x_i)$ and $\{x_1, x_2, \ldots, x_n\}$ a maximal independent subset of A. The proof of this statement is easy and similar to the proof of Theorem 3.5.

Theorem 3.7. *Let A be a torsion-free group of rank three and $t(x_1), t(x_2)$ and $t(x_3)$ be maximal elements in $T(A)$ such that $\{x_1, x_2, x_3\}$ be an independent set of A. If A is non-nil, then $T(A)$ contains no another maximal element.*

Proof. The proof is similar to a part of the proof of Lemma 2.1.5, in [4]. Let $A_i = \langle x_i \rangle_*$ and $t(x_i) = t_i, i = 1, 2, 3$. Then $t(A_i) = t_i$ and $t(A_i \otimes A_j) = t_i t_j > t_i, t_j$, for all $i \neq j \in \{1, 2, 3\}$. Now the sequence

$$0 \longrightarrow A_i \oplus A_j \longrightarrow A \longrightarrow \frac{A}{A_i \oplus A_j} \longrightarrow 0$$

is exact, for all $i \neq j \in \{1, 2, 3\}$. This implies that

$$0 \longrightarrow (A_i \oplus A_j) \otimes A_k \longrightarrow A \otimes A_k \longrightarrow (\frac{A}{A_i \oplus A_j}) \otimes A_k \longrightarrow 0$$

is exact, for all $i \neq j \neq k \in \{1, 2, 3\}$. Therefore we obtain the following exact sequence:

$$0 \to \mathrm{Hom}((\frac{A}{A_i \oplus A_j}) \otimes A_k, A) \to \mathrm{Hom}(A \otimes A_k, A) \to \mathrm{Hom}((A_i \oplus A_j) \otimes A_k, A).$$

But $\mathrm{Hom}((A_i \oplus A_j) \otimes A_k, A) \cong \mathrm{Hom}(A_i \otimes A_k, A) \oplus \mathrm{Hom}(A_j \otimes A_k, A)$ and $\mathrm{Hom}(A_i \otimes A_k, A) = 0$, because if $0 \neq x \in A$ with $t(x) \geq t(A_i \otimes A_k) = t_i t_k$, then we must have $t(x) > t_i, t_k$ that is a contradiction. Similarly, $\mathrm{Hom}(A_j \otimes A_k, A) = 0$ and so

$$\mathrm{Hom}((\frac{A}{A_i \oplus A_j}) \otimes A_k, A) \cong \mathrm{Hom}(A \otimes A_k, A) \cong \mathrm{Hom}(A_k, \mathrm{End}(A)).$$

Now if $t_4 = t(x_4)$ is another maximal element in $T(A)$, then $mx_4 = n_1 x_1 + n_2 x_2 + n_3 x_3$ for some $0 \neq m \in \mathbb{Z}$ and $n_1, n_2, n_3 \in \mathbb{Z}$ with at least both of them non-zero. Let for example $n_1, n_3 \neq 0$. Hence $mx_4 + (A_1 \oplus A_2) = n_3 x_3 + (A_1 \oplus A_2)$, and $t(x_4 + (A_1 \oplus A_2)) = t(x_3 + (A_1 \oplus A_2))$. But $t(x_4 + (A_1 \oplus A_2)) > t(x_4) = t_4$, for otherwise $t_4 = t(x_4 + (A_1 \oplus A_2)) = t(x_3 + (A_1 \oplus A_2)) \geq t_3$ and by maximality of t_3 we have $t_4 = t_3$, a contradiction. So $t(x_4 + (A_1 \oplus A_2)) > t_4$ and hence $\mathrm{Hom}((\frac{A}{A_1 \oplus A_2}) \otimes A_3, A) = 0$, because $A/(A_1 \oplus A_2)$ is a rank one group and $x_4 + (A_1 \oplus A_2)$ is a non-zero element of this group, which means $t(A/(A_1 \oplus A_2)) = t(x_4 + (A_1 \oplus A_2)) > t_4$. So $\frac{A}{A_1 \oplus A_2} \otimes A_3$ is a rank one group in which:

$$t' = t(\frac{A}{A_1 \oplus A_2} \otimes A_3) > t_4 t_3 > t_4, t_3.$$

Now if $(0 \neq)\varphi \in \mathrm{Hom}((\frac{A}{A_1 \oplus A_2}) \otimes A_3, A)$, then $\varphi((\frac{A}{A_1 \oplus A_2}) \otimes A_3)$ is a rank one subgroup of A and

$$t(\varphi((\frac{A}{A_1 \oplus A_2}) \otimes A_3)) \geq t' > t_4, t_3,$$

but t_3, t_4 are maximal elements of $\mathrm{T}(A)$ and A has no element of type greater than t_3 or t_4 and so $\mathrm{Hom}((\frac{A}{A_1 \oplus A_2}) \otimes A_3, A)$ must be equal to zero.

Now by putting $i = 1, j = 2, k = 3$ in above sequences, we have $\mathrm{Hom}(A_3, \mathrm{End}(A)) = 0$. Similarly $\mathrm{Hom}(A_2, \mathrm{End}(A)) = \mathrm{Hom}(A_1, \mathrm{End}(A)) = 0$. Now let $a \in A$ and$\varphi \in \mathrm{Hom}(A, \mathrm{End}(A))$. Then there exists an integer $n \neq 0$ and $m_1, m_2, m_3 \in \mathbb{Z}$ such that $na = m_1 x_1 + m_2 x_2 + m_3 x_3$. So $n\varphi(a) =$

$m_1\varphi(x_1) + m_2\varphi(x_2) + m_3\varphi(x_3) = 0$. But $\mathrm{End}(A)$ is torsion-free and therefore $\varphi(a) = 0$, i.e.,

$$\mathrm{Hom}(A, \mathrm{End}(A)) = \mathrm{Mult}(A) = 0,$$

which means A is a nil group. This yields a contradiction and so t_1, t_2, t_3 are the only maximal elements in $T(A)$. ■

Now let A be a torsion-free group of rank three and $\{x, y, z\}$ be a maximal independent set of A. Each element $a \in A$ has a unique representation $a = \alpha x + \beta y + \gamma z$, where α, β, γ are rational numbers. Let

$$\begin{aligned} U &= \{\alpha \in \mathbb{Q} \mid \alpha x + \beta y + \gamma z \in A \text{ for some } \beta, \gamma \in \mathbb{Q}\}, \quad U_0 = \{\alpha \in \mathbb{Q} \mid \alpha x \in A\};\\ V &= \{\beta \in \mathbb{Q} \mid \alpha x + \beta y + \gamma z \in A \text{ for some } \alpha, \gamma \in \mathbb{Q}\}, \quad V_0 = \{\beta \in \mathbb{Q} \mid \beta y \in A\};\\ W &= \{\gamma \in \mathbb{Q} \mid \alpha x + \beta y + \gamma z \in A \text{ for some } \alpha, \beta \in \mathbb{Q}\}, \quad W_0 = \{\gamma \in \mathbb{Q} \mid \gamma z \in A\}. \end{aligned}$$

The rank one groups U, U_0, V, V_0, W, W_0 are called the rational groups belonging to x, y and z respectively. Note that $\langle x\rangle_* \cong U_0 \subseteq U$, $\langle y\rangle_* \cong V_0 \subseteq V$, $\langle z\rangle_* \cong W_0 \subseteq W$.

Now in this part we deal with the types of rational groups belonging to an independent set of a group and the rings that there exist over group and its homomorphic images.

THEOREM 3.8. *Let $A = A_1 \oplus A_2$ be a group of rank three such that $r(A_1) = 1$ and A_1, A_2 be nil groups. Suppose that $\{x, y\}$ be a maximal independent set of A_2 and $z \in A_1$. If U, U_0, V, V_0, W, W_0 be groups belonging to x, y and z such that $t(W_0) > t(U), t(V)$, then any ring on A is nil.*

Proof. First note that $t(z) = t(\langle z\rangle_*) = t(W_0) = t(A_1)$. If $z^2 = ux + vy + wz$ for some $u \in U, v \in V, w \in W$, then for any non-zero $\beta \in W_0$ we have:

$$\beta z^2 = \beta ux + \beta vy + \beta wz.$$

This implies $\beta u \in U$ that is impossible unless $u = 0$, (because $t(W_0) > t(U)$). Similarly $v = 0$ and therefore $z^2 = wz$. But it holds only if $w = 0$. In fact, if $0 \neq w$, then $t^2(z) = t(z)$ which is a contradiction, hence $z^2 = 0$. Moreover, if $x^2 = \alpha x + \beta y + \gamma z$ and $\alpha \neq 0$ or $\beta \neq 0$, then this contradicts the hypotheses that A_2 is nil. Therefore, $x^2 = \gamma z$. Similarly y^2, xy, yx are some rational multiples of z. Now let $xz = rx + sy + tz$ for some $r \in U, s \in V, t \in W$. Then for any $\beta \in W_0$ we have:

$$x(\beta z) = r\beta x + s\beta y + t\beta z.$$

But $r\beta = 0$ and $s\beta = 0$, because $t(W_0) > t(U), t(V)$. Hence $xz = sz$. Similarly $zx = \delta z, yz = \gamma z, yz = \eta z$ for some $\delta, \gamma, \eta \in \mathbb{Q}$. This implies that any multiplication on A must be nil. ■

Now we review some concepts which are needed in Theorem 3.11.

Let A and B be groups. A is quasi-isomorphic to B ($A \sim B$) if and only if there exist subgroups $A' \subseteq A, B' \subseteq B$ such that $A' \cong B'$ and A/A' and B/B' are of bounded order. In [7], it is shown that if A is a torsion-free group of finite rank, then $A \sim A_1 \oplus \cdots A_l$ with each A_i strongly indecomposable in the sense that A_i is not quasi-isomorphic to a direct sum of non-zero torsion-free groups. This is called quasi-decomposition of A and A_i is called the strongly indecomposable component of the decomposition.

Theorem 3.9. [7, Theorem 2.1] *Let A be a torsion-free group of finite rank. Any ring on A is nilpotent iff A has no strongly indecomposable component of field type.*

Remark 3.10. A is the additive group of a ring R with $\mathbb{Q} \otimes R$ a field, i.e., A is of field type, then A is homogeneous of idempotent type.

Theorem 3.11. *Let A be a torsion-free group of rank three. Then any ring on any torsion-free homomorphic image of A is nilpotent iff for any independent set $\{x, y, z\}$ of A, the rank one groups U, V, W are of non-idempotent type.*

Proof. Let A be such that any ring on any its torsion-free homomorphic image be nilpotent and $\{x, y, z\}$ be any maximal independent set of A. Now U, V, W are the homomorphic images of A with homomorphisms φ, ψ, η given by:

$$\varphi(\alpha x + \beta y + \gamma z) = \alpha, \ \psi(\alpha x + \beta y + \gamma z) = \beta, \ \eta(\alpha x + \beta y + \gamma z) = \gamma$$

therefore U, V, W are rank one of non-idempotent type.

Conversely, let $\varphi : A \longrightarrow T$ be an epimorphism, with T torsion-free and T supports a ring which is not nilpotent.

If $r(T) = 1$, choose $0 \neq t \in T$ and $a \in A$ with $\varphi(a) = t$. Then for any maximal independent set x, y of $\ker\varphi$ we have $\{a, x, y\}$ is a maximal independent set of A with rank one subgroups U, U_0, V, V_0, W, W_0 such that $U \cong T$. So $t(U) = t(T)$ is idempotent.

If $r(T) = 2$, by Theorem 3.9, T is of field type or $T \sim M \oplus N$ where M is a rank one group of field type. In the first case Theorem 5 of [1] says that we can find independent elements $t_1, t_2 \in T$ such that U', V' belonging to t_1, t_2 are of idempotent type. In the second case, it is easy to construct a maximal independent set $\{t_1, t_2\} \subset T$ such that the rank one group U' belonging to t_1 is of idempotent type. So if we choose $x, y \in A$ such that $\varphi(x) = t_1, \varphi(y) = t_2$ then $\{x, y\}$ are rationally independent and for any $0 \neq z \in \ker\varphi$, the set $\{x, y, z\}$ is a maximal independent set of A such that if U, U_0, V, V_0, W, W_0 are rank one groups belonging to x, y, z then $U \cong U', V \cong V'$. Then at least one of U and V is of idempotent type.

If $r(T) = 3$, then $A \cong T$. So by Theorem 3.9, A is of field type or it has a strongly indecomposable component of field type. In the first case using Remark 3.10, for all independent set $\{x, y, z\}$ in A we have U, V, W are of idempotent type.

In the second case if A has a strongly indecomposable component of rank one and of field type then we have $A \sim B \oplus C; r(B) = 1$ and B is of field type. So we could choose a maximal independent set $\{x, y, z\}$ in A such that the rank one group U belonging to x is of idempotent type.

Moreover, if A contains a rank two strongly indecomposable component of field type then the first part of the proof in case $r(T) = 2$ is applied there and this completes the proof. ■

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REFERENCES

[**1**] R.A. Beaumont, R.J. Wisner, *Rings with additive group which is a torsion-free group of rank two*, Acta. Sci. Math. (Szeged) **20** (1959), 105–116.

[**2**] S. Feigelstock, *On the nilstufe of homogeneous groups*, Acta Sci. Math. (Szeged) **36** (1974), 27–28.

[**3**] S. Feigelstock, *On the type set of groups and nilpotence*, Comment. Univ. Sancti Pauli **25** (1976), 159–165.

[**4**] S. Feigelstock, *The Additive Group of Rings,* Pitman Research Notes, Boston-London, 1983.

[**5**] L. Fuchs, *Infinite Abelian Groups,* Vol. 2, Academic Press, New York–London, 1973.

[**6**] D.R. Jacket, *The nil-degree of a torsion-free abelian group*, Comm. Math. Univ. Carolinae **21** (1980), 393–406.

[**7**] W.J. Wickless, *Abelian groups which admits only nilpotent multiplications*, Pacific. J. Math. **40** (1972), 251–259.

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