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A GENERALIZED OPERATOR INVOLVING THE q-HYPERGEOMETRIC FUNCTION

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Abstract. Motivated by the familiar q-hypergeometric functions, we introduce here a new general operator. By this operator, we define a subclass of analytic function. The class generalizes well known classes of starlike and convex functions. The integral means inequalities of this class are investigated. Also, we consider p-γ-neighborhood for functions in this class. Our result contains some interesting corollaries as its special cases.

1. Introduction

Let $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $\mathcal{A}$ denote the class of functions normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U}$ and satisfy the condition $f(0) = f'(0) - 1 = 0$.

We say that a function $f \in \mathcal{A}$ is starlike of order δ and belongs to the class $S^*(\delta)$, if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta \qquad (z \in \mathcal{U};\ 0 \le \delta < 1)\,.$$

The class C of convex functions of order δ is a subclass of $\mathcal{A}$ where the functions $f \in \mathcal{A}$ satisfy the inequality

$$\Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > \delta \qquad (z \in \mathcal{U};\ 0 \le \delta < 1)\,.$$

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For functions $F \in \mathcal{A}$ given by $F(z) = z + \sum_{n=2}^{\infty} b_n z^n$ and $G \in \mathcal{A}$, given by $G(z) = z + \sum_{n=2}^{\infty} c_n z^n$, we define the Hadamard product (or convolution) of F and G by

$$(F * G)(z) = z + \sum_{n=2}^{\infty} b_n c_n z^n \ (z \in \mathcal{U}).$$

Let g and h be analytic functions in the unit disk $\mathcal{U}$. The function g is subordinate to h, written as $g \prec h$, if g is univalent, $g(0) = h(0)$ and $g(\mathcal{U}) \subset h(\mathcal{U})$.

In general, given two functions g and h which are analytic in $\mathcal{U}$, the function g is said to be subordinate to h if there exists a function w analytic in $\mathcal{U}$ with

$$w(0) = 0 \ \text{ and } \ |w(z)| < 1 \quad (z \in \mathcal{U})$$

such that $g(z) = h(w(z))$ $(z \in \mathcal{U})$.

A $q-$hypergeometric series is a power series in one complex variable z with power series coefficients which depend, apart from q, on r complex upper parameters $a_1, a_2, \dots, a_r$ and s complex lower parameters $b_1, b_2, \dots, b_s$ as follows (see [14, p. 4, Eq. (1.2.22)]):

$$\begin{aligned} {}_r\phi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z) &= {}_r\phi_s \begin{bmatrix} a_1, \dots, a_r & \\ & ; q, z \\ b_1, \dots, b_s & \end{bmatrix} \\ &= \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n, \end{aligned} \tag{1.2}$$

with $\binom{n}{2} = n(n-1)/2$, where $q \neq 0$ when $r > s + 1$.

Here $(a, q)_n$ is the q-shifted factorial defined by

$$(a; q)_n = \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)(1-aq^2) \cdots (1-aq^{n-1}), & n \in \mathbb{N}, \end{cases}$$

where $\mathbb{N}$ denotes the set of all positive integers. It is easy to see that

$$\lim_{q \to 1^-} {}_r\phi_s(q^{a_1}, \dots, q^{a_r}; q^{b_1}, \dots, q^{b_s}; q, (q-1)^{1+s-r} z) = {}_rF_s(a_1, \dots, a_r; b_1, \dots, b_s; z),$$

where ${}_rF_s(a_1, \dots, a_r; b_1, \dots, b_s; z)$ is the well known generalized hypergeometric function defined by (for $a_1, \dots, a_r, b_1, \dots, b_s$, $z \in \mathbb{C}$)

$$\begin{aligned} {}_rF_s(a_1, \dots, a_r; b_1, \dots, b_s; z) &= {}_rF_s \begin{bmatrix} a_1, \dots, a_r; & \\ & ; z \\ b_1, \dots, b_s & \end{bmatrix} \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} \frac{z^n}{n!} \end{aligned}$$

$(r \leq s+1,\ r, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$, where $b_j \neq 0, -1, -2, \dots,$ $(j = 1, 2, \dots, s)$ and $(\mu)_n$ is the Pochhammer symbol defined by

$$(\mu)_n = \begin{cases} 1, & \text{if } n = 0 \text{ and } \mu \in \mathbb{C} \setminus \{0\}, \\ \mu(\mu+1) \cdots (\mu+n-1), & \text{if } n \in \mathbb{N} \text{ and } \mu \in \mathbb{C}. \end{cases}$$

An important property that we will use is the convergence criteria of the q-hypergeometric series defined in (1.2) depending on the values of q, r and s. Since

$$\frac{u_{n+1}}{u_n} = \frac{(1-a_1q^n)(1-a_2q^n)\cdots(1-a_rq^n)}{(1-q^{n+1})(1-b_1q^n)\cdots(1-b_sq^n)}(-q^n)^{1+s-r}z,$$

where u_n denotes the terms of the series (1.2) containing z, then by the ratio test, we conclude that (see [14]), if $0<|q|<1$, the ${}_r\phi_s$ series converges absolutely for all z if $r\le s$ and for $|z|<1$ if $r=s+1$. If $|q|>1$ and $|z|<|b_1\cdots b_sq|/|a_1\cdots a_r|$, then also ${}_r\phi_s$ converges absolutely. The series ${}_r\phi_s$ diverges for $z\neq 0$ when $0<|q|<1$ and $r>s+1$, and when $|q|>1$ and $|z|>|b_1\cdots b_sq|/|a_1\cdots a_r|$, unless it terminates. As is customary the ${}_r\phi_s$ notation is also used for the sums of this series inside the circle of convergence and for their analytic continuations (called q-hypergeometric function) outside the circle of convergence.

In 1908, Jackson reintroduced and started a systematic study of the q-difference operator [4, 14]:

$$D_qh(x) = \frac{h(qx)-h(x)}{(q-1)x}, \quad q\neq 1,\ x\neq 0, \tag{1.3}$$

which is now sometimes referred to as Euler-Jackson, Jackson q-difference operator, or simply the q-derivative. Observe that

$$D_qz^n = \frac{1-q^n}{1-q}z^{n-1} \text{ and } \lim_{q\to 1} D_qh(z) = h'(z),$$

where $h'(z)$ is the ordinary derivative.

The formulas for the q-derivative D_q of a product and a quotient of two functions are

$$D_q(h(z)g(z)) = h(qz)D_qg(z) + g(z)D_qh(z),$$
$$D_q\left(\frac{h(z)}{g(z)}\right) = \frac{g(z)D_qh(z)-h(z)D_qg(z)}{g(qz)g(z)},\ g(qz)g(z)\neq 0.$$

For more properties of D_q see [9, 15].

Now for $z\in\mathcal{U}$, $0<|q|<1$, and $r=s+1$, the q-hypergeometric function defined in (1.2) takes the form

$${}_r\Phi_s(a_1,\dots,a_r;b_1,\cdots,b_s;q,z) = \sum_{n=0}^{\infty}\frac{(a_1,q)_n\cdots(a_r,q)_n}{(q,q)_n(b_1,q)_n\cdots(b_r,q)_n}z^n,$$

which converges absolutely in the open unit disk $\mathcal{U}$. Let

$$\begin{aligned} m(a_1,\dots,a_r;b_1,\dots,b_s;q,z) &= z\,{}_r\Phi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,z) \\ &= z + \sum_{n=2}^{\infty}\frac{(a_1,q)_{n-1}\cdots(a_r,q)_{n-1}}{(q,q)_{n-1}(b_1,q)_{n-1}\cdots(b_r,q)_{n-1}}z^n. \end{aligned}$$

We define for $f\in\mathcal{A}$, an operator $\mathcal{M}_s^r(a_1,\dots,a_r;b_1,\dots,b_s;q)$ by the Hadamard product

$$\mathcal{M}_s^r(a_1,\dots,a_r;b_1,\dots,b_s;q)f(z) = m(a_1,\dots,a_r;b_1,\dots,b_s;q,z)*f(z). \tag{1.4}$$

So, for a function of the form (1.1) and from (1.4),

$$\mathcal{M}_s^r(a_1,\ldots,a_r;b_1,\ldots,b_s;q)f(z)=z+\sum_{n=2}^{\infty}\Upsilon_n a_n z^n, \tag{1.5}$$

where, for convenience,

$$\Upsilon_n=\frac{(a_1,q)_{n-1}\cdots(a_r,q)_{n-1}}{(q,q)_{n-1}(b_1,q)_{n-1}\cdots(b_s,q)_{n-1}}.$$

For brevity, we write,

$$\mathcal{M}_s^r[a_i;b_j;\,q]f(z)=\mathcal{M}_s^r(a_1,\ldots,a_r;b_1,\ldots,b_s;q)f(z),$$
$$(i=1,2,\ldots r,\ j=1,2,\ldots s).$$

Remark 1.1. When $a_i=q^{\alpha_i}$, $b_j=q^{\beta_j},\alpha_i,\ \beta_j\in\mathbb{C},\ \beta_j\neq 0,-1,\ldots;$ $(i=1,2,\ldots r,\ j=1,2,\ldots s)$ and $q\to 1$, we obtain the Dziok-Srivastava linear operator [8] (for $r=s+1$), so that it includes (as its special cases) various other linear operators introduced and studied by Ruscheweyh [22], Carlson-Shaffer [6] and Bernardi-Libera-Livingston operators [5, 19, 21]. Some of relations for the general operator (1.5) are discussed in the next lemma.

Lemma 1.1. *For $f\in\mathcal{A}$, we have*

(i) $\mathcal{M}_0^1[q;\_;q]f(z)=f(z)$,

(ii) $\mathcal{M}_0^1[q^2;\_;q]f(z)=zD_qf(z)$, *and* $\lim_{q\to1}\mathcal{M}_0^1[q^2,\_;q]f(z)=zf'(z)$, *where* D_q *is the q-derivative defined in* (1.3).

Now using $\mathcal{M}_s^r[a_i;b_j;\,q]f$, we define the following subclass of analytic functions.

Definition 1.1. Given $0<\alpha\le 1$ and $0\le\beta\le 1$ and functions

$$\Phi(z)=z+\sum_{n=2}^{\infty}\lambda_n z^n,\quad \Psi(z)=z+\sum_{n=2}^{\infty}\mu_n z^n,$$

analytic in $\mathcal{U}$ such that $\lambda_n\ge 0$, $\mu_n\ge 0$, $\lambda_n>\mu_n$, $(n\ge 2)$, we say that $f\in\mathcal{A}$ is in $\mathcal{M}_{r,s}\left(a_i,b_j,q,\Phi,\Psi,\alpha,\beta\right)$ if $f(z)*\Psi(z)\neq 0$ and

$$\left|\frac{\mathcal{M}_s^r\left(a_i,b_j,q\right)\left(f*\Phi\right)(z)}{\mathcal{M}_s^r\left(a_i,b_j,q\right)\left(f*\Psi\right)(z)}-1\right|<\alpha\left|\beta\frac{\mathcal{M}_s^r\left(a_i,b_j,q\right)\left(f*\Phi\right)(z)}{\mathcal{M}_s^r\left(a_i,b_j,q\right)\left(f*\Psi\right)(z)}+1\right|,$$

where $\mathcal{M}_s^r[a_i,b_j;\,q]f(z)$ is given by (1.5). We further let

$$\mathcal{M}_{\mathcal{T}r,s}\left(a_i,b_j,q,\Phi,\Psi,\alpha,\beta\right)=\mathcal{M}_{r,s}\left(a_i,b_j,q,\Phi,\Psi,\alpha,\beta\right)\cap\mathcal{T},$$

where

$$\mathcal{T}=\left\{f\in\mathcal{A}:f(z)=z-\sum_{n=2}^{\infty}|a_n|\,z^n,\ z\in\mathcal{U}\right\}, \tag{1.6}$$

a subclass of $\mathcal{A}$ being introduced and studied by Silverman [25].

By suitably specializing the values of r, s, a_1, $a_2, \ldots, a_r$, b_1, $b_2, \ldots, b_s$, q, Φ, Ψ, α and β, the class $\mathcal{M}_{\mathcal{T}\,r,s}\,(a_i, b_j, q, \Phi, \Psi, \alpha, \beta)$ leads to various subclasses. As illustrations, we present some examples:

EXAMPLE 1. For $r = 1$, $s = 0$ and $a_1 = q$, we have

$$\begin{aligned}\mathcal{M}_{\mathcal{T}\,1,\,0}\,(q, \_, q, \Phi, \Psi, \alpha, \beta) &= D_{\mathcal{T}}(\Phi, \Psi, \alpha, \beta)\\ &= \left\{ f \in \mathcal{T} : \left| \frac{(f*\Phi)\,(z)}{(f*\Psi)\,(z)} - 1 \right| < \alpha \left| \beta \frac{(f*\Phi)\,(z)}{(f*\Psi)\,(z)} + 1 \right| \right\},\end{aligned}$$

where $D_{\mathcal{T}}(\Phi, \Psi, \alpha, \beta)$ was introduced and studied by Darus [7].

EXAMPLE 2. For $r = 1$, $s = 0$, $a_1 = q$, $\alpha = \frac{1-\delta}{2(1-\nu)}, \beta = 0$,

$$\mathcal{M}_{\mathcal{T}\,1,0}\left(q, \_, q, \Phi, \Psi, \frac{1-\delta}{2\,(1-\nu)}, 0\right) = \left\{ f \in \mathcal{T} : \left| \frac{(f*\Phi)\,(z)}{(f*\Psi)\,(z)} - 1 \right| < \frac{1-\delta}{2\,(1-\nu)} \right\},$$

which implies the class $\mathcal{B}_{\mathcal{T}}\,(\Phi, \Psi, \delta, \nu)$, introduced and studied by Frasin [10], Frasin and Darus [11, 12].

EXAMPLE 3. For $r = 1$, $s = 0$, $a_1 = q$, $\Phi(z) = \frac{z}{(1-z)^2}$, $\Psi(z) = \frac{z}{1-z}$, we get

$$\begin{aligned}\mathcal{M}_{\mathcal{T}\,1,\,0}\left(q, \_, q, \frac{z}{(1-z)^2}, \frac{z}{1-z}, \alpha, \beta\right) &= M_{\mathcal{T}}(\alpha, \beta)\\ &= \left\{ f \in \mathcal{T} : \left| \frac{zf'(z)}{f(z)} - 1 \right| < \alpha \left| \beta \frac{zf'(z)}{f(z)} + 1 \right| \right\},\end{aligned}$$

where $\mathcal{M}(\alpha, \beta)$ was introduced by Lakshminarasimhan [18].

EXAMPLE 4. For $r = 1$, $s = 0$, $a_1 = q$, $\alpha = 1 - \delta$, $\beta = 0$,

$$\mathcal{M}_{\mathcal{T}\,1,\,0}\,(q, \_, q, \Phi, \Psi, 1-\delta, 0) = D_{\mathcal{T}}(\Phi, \Psi, \delta) = \left\{ f \in \mathcal{T} : \left| \frac{(f*\Phi)\,(z)}{(f*\Psi)\,(z)} - 1 \right| < 1 - \delta \right\},$$

where $D_{\mathcal{T}}(\Phi, \Psi, \delta)$ was introduced by Juneja et.al [16]. In particular, for $r = 1$, $s = 0$, $a_1 = q$, $\Phi(z) = \frac{z}{(1-z)^2}$, $\Psi(z) = \frac{z}{1-z}, \alpha = 1 - \delta, \beta = 0$,

$$\mathcal{M}_{\mathcal{T}\,1,0}\left(q, \_, q, \frac{z}{(1-z)^2}, \frac{z}{1-z}, 1-\delta, 0\right) = S^*_{\mathcal{T}}(\delta) = \left\{ f \in \mathcal{T} : \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \delta \right\}$$

and for $r = 1$, $s = 0$, $a_1 = q$, $\Phi(z) = \frac{z+z^2}{(1-z)^3}$, $\Psi(z) = \frac{z}{(1-z)^2}$, $\alpha = 1 - \delta$, $\beta = 0$ we get

$$\mathcal{M}_{\mathcal{T}\,1,0}\left(q, \_, q, \frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, 1-\delta, 0\right) = C_{\mathcal{T}}(\delta) = \left\{ f \in \mathcal{T} : \left| \frac{zf''(z)}{f'(z)} \right| < 1 - \delta \right\},$$

where $S^*_{\mathcal{T}}(\delta)$ and $C_{\mathcal{T}}(\delta)$ denote the subfamilies of $\mathcal{T}$ that are starlike of order δ and convex of order δ which were studied by Silverman [25].

EXAMPLE 5. For $r = 1$, $s = 0$, $a_1 = q^2$ and $q \to 1$, we get

$$\mathcal{M}_{\mathcal{T}\,1,0}\left(q^2, \_, 1, \Phi, \Psi, \alpha, \beta\right) = \left| \frac{(f*\Phi)'\,(z)}{(f*\Psi)'\,(z)} - 1 \right| < \alpha \left| \beta \frac{(f*\Phi)'\,(z)}{(f*\Psi)'\,(z)} + 1 \right|.$$

Example 6. For $r = 1$, $s = 0$, $a_1 = q^2$, $q \to 1$, $\Phi(z) = \frac{z}{(1-z)^2}$, $\Psi(z) = \frac{z}{1-z}$, we get

$$\mathcal{M}_{\mathcal{T}\,1,0}\Big(q^2, \_, 1, \frac{z}{(1-z)^2}, \frac{z}{1-z}, \alpha, \beta\Big) = \left\{ f \in \mathcal{T} : \left| \frac{zf''(z)}{f'(z)} \right| < \alpha \left| \beta \left(\frac{zf''(z)}{f'(z)} + 1 \right) + 1 \right| \right\}.$$

Example 7. For $r = 1$, $s = 0$, $a_1 = q^2$, $q \to 1$, $\Phi(z) = \frac{z+z^2}{(1-z)^3}$, $\Psi(z) = \frac{z}{(1-z)^2}$, we get

$$\mathcal{M}_{\mathcal{T}\,1,\,0}\left(q^2, \_, 1, \frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, \alpha, \beta\right) = \left\{ f \in \mathcal{T} : \left| \frac{z\,(zf'''(z) + 2f''(z))}{zf''(z) + f'(z)} \right| < \alpha \left| \beta \left(\frac{z\,(zf'''(z) + 2f''(z))}{zf''(z) + f'(z)} + 1 \right) + 1 \right| \right\}.$$

Making use of the similar arguments as Darus [7], we get the following necessary and sufficient condition of the class $\mathcal{M}_{\mathcal{T}\,r,s}\,(a_i, b_j, q, \Phi, \Psi, \alpha, \beta)$.

A function $f \in \mathcal{M}_{\mathcal{T}\,r,s}\,(a_i, b_j, q, \Phi, \Psi, \alpha, \beta)$ if, and only if,

$$\sum_{n=2}^{\infty} \frac{[(1+\alpha\beta)\,\lambda_n - (1-\alpha)\,\mu_n]}{\alpha(\beta+1)} \,|\Upsilon_n|\,|a_n| \le 1, \quad 0 < \alpha \le 1, \;\; 0 \le \beta \le 1. \tag{1.7}$$

The result is sharp with the extremal functions

$$f_n(z) = z - \frac{\alpha(\beta+1)}{\sigma(\alpha,\beta,n)} z^n, \quad n \ge 2,$$

where $\sigma(\alpha,\beta,n) = [(1+\alpha\beta)\,\lambda_n - (1-\alpha)\,\mu_n]\,|\Upsilon_n|$.

In [25], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal for the family $\mathcal{T}$. He applied this function to resolve his integral means inequality settled in [26], that

$$\int_0^{2\pi} \left| f(re^{i\theta}) \right|^{\eta} d\theta \le \int_0^{2\pi} \left| f_2(re^{i\theta}) \right|^{\eta} d\theta,$$

for all $f \in \mathcal{T}$, $\eta > 0$ and $0 < r < 1$. Silverman [27] also proved his conjecture for the subclasses $S^*_{\mathcal{T}}(\delta)$ and $C_{\mathcal{T}}(\delta)$ of $\mathcal{T}$.

In this paper, we prove Silverman's conjecture for the functions in the family $\mathcal{M}_{\mathcal{T}\,r,s}\,(a_i, b_j, q, \Phi, \Psi, \alpha, \beta)$. By taking appropriate choices of the parameters r, s, a_1, $a_2, \ldots, a_r$, b_1, $b_2, \ldots, b_s$, q, Φ, Ψ, α and β, we obtain the integral means inequalities for several known as well as new subclasses of convex and starlike functions in $\mathcal{U}$. In fact, these results also settle the Silverman's conjecture for several other subclasses of $\mathcal{T}$. Also we consider the p-γ-neighborhood for function $f(z)$ belongs to this class. For other results dealing with integral means inequalities and neighborhoods of certain subclasses of analytic functions, see [1, 2, 3, 17, 24].

2. Integral means inequalities

Following the work of Littlewood [20], we obtain integral means inequalities for the functions in the family $\mathcal{M}_{\mathcal{T}r,s}(a_i, b_j, q, \Phi, \Psi, \alpha, \beta)$.

LEMMA 2.1. [20] *If the functions f and g are analytic in $\mathcal{U}$ with $g \prec f$, then for $\eta > 0$ and $0 < r < 1$,*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^{\eta} d\theta.$$

Applying (1.7) and Lemma 2.1, we prove the following result.

THEOREM 2.1. *Let $f \in \mathcal{M}_{\mathcal{T}r,s}(a_i, b_j, q, \Phi, \Psi, \alpha, \beta)$, $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $\{\sigma(\alpha, \beta, n)\}_{n=2}^{\infty}$ be a non-decreasing sequence and $f_2(z)$ be defined by*

$$f_2(z) = z - \frac{\alpha(\beta+1)}{\sigma(\alpha,\beta,2)} z^2,$$

where

$$\sigma(\alpha,\beta,2) = \left[(1+\alpha\beta)\lambda_2 - (1-\alpha)\mu_2\right]|\Upsilon_2| \tag{2.1}$$

and Υ_2 is given by

$$\Upsilon_2 = \frac{(1-a_1)\cdots(1-a_r)}{(1-q)(1-b_1)\cdots(1-b_s)}.$$

Then for $z = re^{\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^{\eta} d\theta \leq \int_0^{2\pi} |f_2(z)|^{\eta} d\theta. \tag{2.2}$$

Proof. For a function f of the form (1.6), the inequality (2.2) is equivalent to

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} |a_n| z^{n-1}\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|1 - \frac{\alpha(\beta+1)}{\sigma(\alpha,\beta,2)} z\right|^{\eta} d\theta.$$

By Lemma 2.1, it suffices to show that

$$\sum_{n=2}^{\infty} |a_n| z^{n-1} \prec \frac{\alpha(\beta+1)}{\sigma(\alpha,\beta,2)} z. \tag{2.3}$$

Setting $\sum_{n=2}^{\infty} a_n z^{n-1} = \frac{\alpha(\beta+1)}{\sigma(\alpha,\beta,2)} w(z)$, from (2.3) and (1.7), we obtain

$$|w(z)| = \left|\sum_{n=2}^{\infty} \frac{\sigma(\alpha,\beta,2)}{\alpha(\beta+1)} a_n z^{n-1}\right| \leq |z| \sum_{n=2}^{\infty} \frac{\sigma(\alpha,\beta,n)}{\alpha(\beta+1)} a_n \leq |z| < 1.$$

By the definition of subordination, we have (2.3). This completes the proof. ∎

In the view of Examples 1 to 7, we state the following corollaries.

Corollary 2.1. *Let* $f \in \mathcal{M}_{\mathcal{T}\,1,\,0}\,(q, \_, q, \Phi, \Psi, \alpha, \beta) = D_{\mathcal{T}}(\Phi, \Psi, \alpha, \beta)$, $0 < \alpha \le 1$, $0 \le \beta \le 1$ *and* $f_2(z)$ *be defined by*

$$f_2(z) = z - \frac{\alpha(\beta+1)}{(1+\alpha\beta)\,\lambda_2 - (1-\alpha)\,\mu_2} z^2.$$

Then for $z = re^{\theta}$, $0 < r < 1$, *we have*

$$\int_0^{2\pi} |f(z)|^{\eta} d\theta \le \int_0^{2\pi} |f_2(z)|^{\eta} d\theta. \tag{2.4}$$

Corollary 2.2. *Let* $f \in \mathcal{M}_{\mathcal{T}\,1,0}\left(q, \_, q, \Phi, \Psi, \frac{1-\delta}{2(1-\nu)}, 0\right)$, *and* $f_2(z)$ *be defined by*

$$f_2(z) = z - \frac{1-\delta}{2\,(1-\nu)\,\lambda_2 - (1+\delta-2\nu)\,\mu_2} z^2.$$

Then for $z = re^{\theta}$, $0 < r < 1$, (2.4) *holds true.*

Corollary 2.3. *Let* $f \in \mathcal{M}_{\mathcal{T}\,1,\,0}\left(q, \_, q, \frac{z}{(1-z)^2}, \frac{z}{1-z}, \alpha, \beta\right) = \mathcal{M}_{\mathcal{T}}(\alpha, \beta)$, $0 < \alpha \le 1$, $0 \le \beta \le 1$ *and* $f_2(z)$ *be defined by*

$$f_2(z) = z - \frac{\alpha(\beta+1)}{\alpha(2\beta+1)+1} z^2.$$

Then for $z = re^{\theta}, 0 < r < 1$, (2.4) *holds true.*

Corollary 2.4. *Let* $\mathcal{M}_{\mathcal{T}\,1,\,0}\,(q, \_, q, \Phi, \Psi, 1-\delta, 0) = D_{\mathcal{T}}(\Phi, \Psi, \delta)$ *and* $f_2(z)$ *be defined by*

$$f_2(z) = z - \frac{1-\delta}{\lambda_2 - \delta\mu_2} z^2.$$

Then for $z = re^{\theta}, 0 < r < 1$, (2.4) *holds true.*

Corollary 2.5. *Let* $f \in \mathcal{M}_{\mathcal{T}\,1,\,0}\left(q, \_, q, \frac{z}{(1-z)^2}, \frac{z}{1-z}, 1-\delta, 0\right) = S^*_{\mathcal{T}}(\delta)$ *and* $f_2(z)$ *be defined by*

$$f_2(z) = z - \frac{1-\delta}{2-\delta} z^2.$$

Then for $z = re^{\theta}, 0 < r < 1$, (2.4) *holds true.*

Corollary 2.6. *Let* $f \in \mathcal{M}_{\mathcal{T}\,1,\,0}\left(q, \_, q, \frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, 1-\delta, 0\right) = C_{\mathcal{T}}(\delta)$ *and* $f_2(z)$ *be defined by*

$$f_2(z) = z - \frac{1-\delta}{2\,(2-\delta)} z^2.$$

Then for $z = re^{\theta}, 0 < r < 1$, (2.4) *holds true.*

Corollary 2.7. *Let* $f \in \mathcal{M}_{\mathcal{T}\,1,\,0}\left(q^2, \_, 1, \Phi, \Psi, \alpha, \beta\right)$ *and* $f_2(z)$ *be defined by*

$$f_2(z) = z - \frac{\alpha(\beta+1)}{2\left[(1+\alpha\beta)\,\lambda_2 - (1-\alpha)\,\mu_2\right]} z^2.$$

Then for $z = re^{\theta}, 0 < r < 1$, (2.4) *holds true.*

Corollary 2.8. *Let* $f \in \mathcal{M}_{\mathcal{T}\,1,0}\left(q^2, \_, 1, \frac{z}{(1-z)^2}, \frac{z}{1-z}, \alpha, \beta\right)$ *and* $f_2(z)$ *be defined by*

$$f_2(z) = z - \frac{\alpha(\beta+1)}{2\left[\alpha(2\beta+1)+1\right]} z^2.$$

Then for $z = re^{\theta}$, $0 < r < 1$, (2.4) *holds true.*

Corollary 2.9. *Let* $f \in \mathcal{M}_{\mathcal{T}\,1,0}\left(q^2, \_, 1, \frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, \alpha, \beta\right)$ *and* $f_2(z)$ *be defined by*

$$f_2(z) = z - \frac{\alpha(\beta+1)}{4\left[\alpha(2\beta+1)+1\right]} z^2.$$

Then for $z = re^{\theta}$, $0 < r < 1$, (2.4) *holds true.*

Remark 2.1. If we take $\delta = 0$ in $S^*_{\mathcal{T}}(\delta)$ of Corollary 2.5 and $C_{\mathcal{T}}(\delta)$ of Corollary 2.6, we get the integral means results obtained by Silverman [27].

Remark 2.2. With the help of Remark 1.1 and by suitably specializing the various parameters involved in Theorem 2.1, we can state the corresponding results for the subclasses defined in Examples 1 to 7 and also for many relatively more familiar function classes.

3. Neighborhoods of the class $\mathcal{M}_{\mathcal{T}\,r,s}\left(a_i, b_j, q, \Phi, \Psi, \alpha, \beta\right)$.

For $f \in \mathcal{T}$ of the form (1.6), and $\gamma \geq 0$, Frasin and Darus [13] investigated the p-γ-neighborhood of f as the following

$$M^p_\gamma(f) = \left\{ g \in \mathcal{T} :\ g(z) = z - \sum_{n=2}^{\infty} b_n z^n,\ \sum_{n=2}^{\infty} n^{p+1} \left|a_n - b_n\right| \leq \gamma \right\}, \tag{3.1}$$

where p is a fixed positive integer. It follows from (3.1), that if $e(z) = z$, then

$$M^p_\gamma(e) = \left\{ g \in \mathcal{T} :\ g(z) = z - \sum_{n=2}^{\infty} b_n z^n,\ \sum_{n=2}^{\infty} n^{p+1} \left|b_n\right| \leq \gamma \right\}.$$

We observe that $M^0_\gamma(f) \equiv N_\gamma(f)$, $M^1_\gamma(f) \equiv M_\gamma(f)$, where $N_\gamma(f)$ is called a γ-neighborhood of f introduced by Ruscheweyh [23] and $M_\gamma(f)$ was defined by Silverman [28].

Now, we obtain p-γ-neighborhood for function in the class $\mathcal{M}_{\mathcal{T}\,r,s}\left(a_i, b_j, q, \Phi, \Psi, \alpha, \beta\right)$.

Theorem 3.1. *If* $\left\{\sigma(\alpha, \beta, n)/n^{p+1}\right\}_{n=2}^{\infty}$ *is a non-decreasing sequence, then* $\mathcal{M}_{\mathcal{T}\,r,s}\left(a_i, b_j, q, \Phi, \Psi, \alpha, \beta\right) \subset M^p_\gamma(e)$, *where*

$$\gamma = \frac{2^{p+1}\alpha(\beta+1)}{\sigma(\alpha, \beta, 2)},$$

and $\sigma(\alpha, \beta, 2)$ *is defined as in* (2.1).

Proof. It follows from (1.7) that if $f \in \mathcal{M}_{\mathcal{T}r,s}(a_i, b_j, q, \Phi, \Psi, \alpha, \beta)$, then

$$\sum_{n=2}^{\infty} n^{p+1} |a_n| \leq \frac{2^{p+1}\alpha(\beta+1)}{\sigma(\alpha, \beta, 2)}.$$

This gives that $\mathcal{M}_{\mathcal{T}r,s}(a_i, b_j, q, \Phi, \Psi, \alpha, \beta) \subset M_\gamma^p(e)$. ∎

By taking different choices of r, s, a_1, $a_2, \ldots, a_r$, b_1, $b_2, \ldots, b_s$, q, Φ, Ψ, α and β in the above theorem, we can state the following neighborhood results for various subclasses studied earlier by several researchers.

In view of the Examples 1 to 7 in Section 1 and Theorem 3.1, we have the following corollaries for the classes defined in these examples.

Corollary 3.1. *$D_{\mathcal{T}}(\Phi, \Psi, \alpha, \beta) \subset M_\gamma^p(e)$, where*

$$\gamma = \frac{2^{p+1}\alpha(\beta+1)}{[(1+\alpha\beta)\lambda_2 - (1-\alpha)\mu_2]}.$$

Corollary 3.2. *$\mathcal{M}_{\mathcal{T}1,0}\left(q, \_, q, \Phi, \Psi, \frac{1-\delta}{2(1-\nu)}, 0\right) \subset M_\gamma^p(e)$, where*

$$\gamma = \frac{2^{p+1}(1-\delta)}{2(1-\nu)\lambda_2 - (1+\delta-2\nu)\mu_2}.$$

Corollary 3.3. *$M_{\mathcal{T}}(\alpha, \beta) \subset M_\gamma^p(e)$, where $\gamma = \dfrac{2^{p+1}\alpha(\beta+1)}{[\alpha(2\beta+1)+(2\beta-1)]}$.*

Corollary 3.4. *$D_{\mathcal{T}}(\Phi, \Psi, \delta) \subset M_\gamma^p(e)$, where $\gamma = \dfrac{2^{p+1}(1-\delta)}{\lambda_2 - \delta\mu_2}$.*

Corollary 3.5. *$S_{\mathcal{T}}^*(\delta) \subset M_\gamma^p(e)$, where $\gamma = \dfrac{2^{p+1}(1-\delta)}{(2-\delta)}$.*

Corollary 3.6. *$C_{\mathcal{T}}(\delta) \subset M_\gamma^p(e)$, where $\gamma = \dfrac{2^{p+1}(1-\delta)}{2(2-\delta)}$.*

Corollary 3.7. *$\mathcal{M}_{\mathcal{T}1,0}\left(q^2, \_, 1, \Phi, \Psi, \alpha, \beta\right) \subset M_\gamma^p(e)$, where*

$$\gamma = \frac{2^{p+1}\alpha(\beta+1)}{2[(1+\alpha\beta)\lambda_2 - (1-\alpha)\mu_2]}.$$

Corollary 3.8. *$\mathcal{M}_{\mathcal{T}1,0}\left(q^2, \_, 1, \frac{z}{(1-z)^2}, \frac{z}{1-z}, \alpha, \beta\right) \subset M_\gamma^p(e)$, where*

$$\gamma = \frac{2^{p+1}\alpha(\beta+1)}{2(1+2\alpha\beta+\alpha)}.$$

Corollary 3.9. $\mathcal{M}_{\mathcal{T}1,0}\left(q^2, \_, 1, \frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, \alpha, \beta\right) \subset M_\gamma^p(e)$, *where*

$$\gamma = \frac{2^{p+1}\alpha(\beta+1)}{4\,(1+2\alpha\beta+\alpha)}.$$

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