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THE χ^2 FUZZY NUMBERS DEFINED BY A MODULUS

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This paper is dedicated to my beloved teacher Dr. Professor Umakanta Misra on occasion of his sixtieth anniversary.

Abstract. In this paper, we introduce the χ^2 fuzzy numbers defined by a modulus, study some of their properties and inclusion results.

1. Introduction

Throughout the paper, w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate-wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [7]. Later on, they were investigated by Hardy [14], Moricz [25], Moricz and Rhoades [26], Basarir and Solankan [4], Tripathy [46], Turkmenoglu [49], and many others.

Let us define the following sets of double sequences:

$$\begin{aligned}
\mathcal{M}_u(t) &:= \{(x_{mn}) \in w^2 : \sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\},\\
\mathcal{C}_p(t) &:= \{(x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn} - \lambda|^{t_{mn}} = 1 \text{ for some } \lambda \in \mathbb{C}\},\\
\mathcal{C}_{0p}(t) &:= \{(x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\},\\
\mathcal{L}_u(t) &:= \{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\},\\
\mathcal{C}_{bp}(t) &:= \mathcal{C}_p(t) \cap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \cap \mathcal{M}_u(t);
\end{aligned}$$

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where $t=(t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m,n\in\mathbb{N}$ and $p-\lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn}=1$ for all $m,n\in\mathbb{N}$; $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{0p}$, $\mathcal{L}_u$, $\mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively.

Now, we may summarize the knowledge given in some documents related to the double sequence spaces. Gökhan and Colak [12,13] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t)$, $\mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-,\beta-,\gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [52] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [29] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Next, Mursaleen [30] and Mursaleen and Edely [31] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequence $x=(x_{jk})$ into one whose core is a subset of the M-core of x. More recently, Altay and Basar [2] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the α-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(\vartheta)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [8] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [44] have studied the space $\chi^2_M(p,q,u)$ of double sequences and gave some inclusion relations.

Spaces of strongly summable sequences were discussed by Kuttner [22], Maddox [24], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [27] as an extension of the definition of strongly Cesàro summable sequences. Connor [10] further extended this definition to a definition of strong A-summability with respect to a modulus where $A=(a_{n,k})$ is a nonnegative regular matrix and established some connections between strong A-summability, strong A-summability with respect to a modulus, and A-statistical convergence. In [41] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [15–18], and [43] the four dimensional matrix transformation $(Ax)_{k,\ell}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a^{mn}_{k\ell}x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a,b,\geq 0$ and $0<p<1$, we have

$$(a+b)^p\leq a^p+b^p. \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty}x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn}=\sum_{i,j=1}^{m,n}x_{ij}(m,n\in\mathbb{N})$ (see[1]).

A sequence $x=(x_{mn})$is said to be double analytic if $\sup_{mn}|x_{mn}|^{1/m+n}<\infty$.

The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{$all finite sequences$\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m, n \in \mathbb{N}$, where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.

Orlicz [39] used the idea of Orlicz function to construct the space (L^M). Lindenstrauss and Tzafriri [23] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (1 \leq p < \infty)$. Subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [40], Mursaleen et al. [28] and [32–34], Bektas and Altin [6], Tripathy et al. [47,48], Rao and Subramanian [9], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [19].

Recalling [39] and [19], an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [35] and further discussed by Ruckle [42] and Maddox [24], and many others.

An Orlicz function M is said to satisfy the Δ_2- condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)(u \geq 0)$. The Δ_2-condition is equivalent to $M(\ell u) \leq K\ell M(u)$, for all values of u and for $\ell > 1$.

Lindenstrauss and Tzafriri [23] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \{x \in w : \sum_{k=1}^{\infty} M(\tfrac{|x_k|}{\rho}) < \infty, \text{ for some } \rho > 0\},$$

The space ℓ_M with the norm

$$\|x\| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M(\tfrac{|x_k|}{\rho}) \leq 1\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p (1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

If X is a sequence space, we give the following definitions:

(i) $X' =$ the continuous dual of X;

(ii) $X^\alpha = \{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X\};$

(iii) $X^\beta = \{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn}$ is convegent, for each $x \in X\}$;

(iv) $X^\gamma = \{a = (a_{mn}) : \sup_{mn\geq 1} |\sum_{m,n=1}^{M,N} a_{mn}x_{mn}| < \infty,$ for each $x \in X\}$;

(v) let X be an FK-space $\supset \phi$; then $X^f = \{f(\Im_{mn}) : f \in X^{'}\}$;

(vi) $X^\delta = \{a = (a_{mn}) : \sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty,$ for each $x \in X\}$;

X^α, X^β, X^γ, X^δ are called α- (or Köthe-Toeplitz) dual of X, β- (or generalized-Köthe-Toeplitz) dual of X, γ-dual of X, δ-dual of X respectively. X^α is defined by Gupta and Kamptan [20]. It is clear that $x^\alpha \subset X^\beta$ and $X^\alpha \subset X^\gamma$, but $X^\beta \subset X^\gamma$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [21] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here c, c_0 and ℓ_∞ denote the classes of convergent,null and bounded scalar valued single sequences respectively. The difference space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay in [5] and in the case $0 < p < 1$ by Altay and BaŞar in [3]. The spaces $c(\Delta)$, $c_0(\Delta)$, $\ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k\geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = (\sum_{k=1}^{\infty} |x_k|^p)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\},$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

2. Definitions and preliminaries

Throughout the paper, a double sequence is denoted by $\langle X_{mn} \rangle$, a double infinite array of fuzzy real numbers.

Let D denote the set of all closed and bounded intervals $X = [a_1, a_2]$ on the real line $\mathbb{R}$. For $X = [a_1, a_2] \in D$ and $Y = [b_1, b_2] \in D$, define

$$d(X, Y) = \max(|a_1 - b_1|, |a_2 - b_2|)$$

It is known that (D, d) is a complete metric space.

A fuzzy real number X is a fuzzy set on $\mathbb{R}$, that is, a mapping $X : \mathbb{R} \to I(= [0, 1])$ associating each real number t with its grade of membership $X(t)$. The $\alpha-$ level set $[X]^\alpha$, of the fuzzy real number X, for $0 < \alpha \leq 1$, defined by

$$[X]^\alpha = \{t \in \mathbb{R} : X(t) \geq \alpha\}.$$

The 0-level set is the closure of the strong 0-cut that is, $cl\{t \in \mathbb{R} : X(t) > 0\}$.

A fuzzy real number X is called convex if $X(t) \geq X(s) \wedge X(r) = \min\{X(s), X(r)\}$, where $s < t < r$. If there exists $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$ then, the fuzzy real number X is called normal.

A fuzzy real number X is said to be upper-semi continuous if, for each $\epsilon > 0, X^{-1}([0, a + \epsilon))$ is open in the usual topology of $\mathbb{R}$ for all $a \in I$. The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by $L(\mathbb{R})$.

The absolute value, $|X|$ of $X \in L(\mathbb{R})$ is defined by

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0 \end{cases}$$

Let $\bar{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ be defined by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d([X]^{\alpha}, [Y]^{\alpha}).$$

Then, $\bar{d}$ defines a metric on $L(\mathbb{R})$ and it is well-known that $(L(\mathbb{R}), \bar{d})$ is a complete metric space. A sequence $\langle X_m \rangle \subset L(\mathbb{R})$ of fuzzy real numbers is said to be null to the fuzzy real number 0, such that $\bar{d}(X_m, \bar{0}) = 0$.

A double sequence $\langle X_{mn} \rangle$ of fuzzy real numbers is said to be chi in Pringsheim's sense to a fuzzy number 0 if $\lim_{m,n\to\infty}((m+n)!X_{mn})^{1/m+n} = 0$. A double sequence $\langle X_{mn} \rangle$ is said to chi regularly if it converges in the Prinsheim's sense and the following limts zero:

$$\lim_{m\to\infty}((m+n)!X_{mn})^{1/m+n} = 0 \text{ for each } n \in \mathbb{N},$$

and

$$\lim_{n\to\infty}((m+n)!X_{mn})^{1/m+n} = 0 \text{ for each } m \in \mathbb{N}.$$

A fuzzy real-valued double sequence space E^F is said to be solid if $\langle Y_{mn} \rangle \in E^F$ whenever $\langle X_{mn} \rangle \in E^F$ and $|Y_{mn}| \leq |X_{mn}|$ for all $m, n \in \mathbb{N}$.

Let $K = \{(m_i, n_i) : i \in \mathbb{N}; m_1 < m_2 < m_3 \cdots \text{ and } n_1 < n_2 < n_3 < \cdots\} \subseteq \mathbb{N} \times \mathbb{N}$ and E^F be a double sequence space. A K-step space of E^F is a sequence space $\lambda_K^E = \{\langle X_{m_i n_i} \rangle \in w^{2F} : \langle X_{mn} \rangle \in E^F\}$.

A canonical pre-image of a sequence $\langle X_{m_i n_i} \rangle \in E^F$ is a sequence $\langle Y_{mn} \rangle$ defined as follows:

$$Y_{mn} = \begin{cases} X_{mn}, & \text{if } (m, n) \in K, \\ \bar{0}, & \text{otherwise }. \end{cases}$$

A canonical pre-image of a step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E.

A sequence E^F is said to be monotone if E^F contains the canonical pre-images of all its step spaces. A sequence E^F is said to be symmetric if $\langle X_{\pi_{(m)},\pi_{(n)}} \rangle \in E^F$ whenever $\langle X_{mn} \rangle \in E^F$, where π is a permutation of $\mathbb{N}$. A fuzzy real-valued sequence space E^F is said to be convergent free if $\langle Y_{mn} \rangle \in E^F$ whenever $\langle X_{mn} \rangle \in E^F$ and $X_{mn} = \bar{0}$ implies $Y_{mn} = \bar{0}$.

We define the following classes of sequences:

$\Lambda_f^{2F} = \{\langle X_{mn}\rangle : \sup_{mn} f(\bar{d}(X_{mn}^{1/m+n}, \bar{0})) < \infty, X_{mn} \in L(\mathbb{R})\}$.

$\chi_f^{2F} = \{\langle X_{mn}\rangle : \lim_{mn\to\infty} f(\bar{d}(((m+n)!X_{mn})^{1/m+n}, \bar{0})) = 0\}$.

Also, we define the classes of sequences $\chi_f^{2F^R}$ as follows:

A sequence $\langle X_{mn}\rangle \in \chi_f^{2F^R}$ if $\langle x_{mn}\rangle \in \chi_f^{2F}$ and the following limits hold

$$\lim_{m\to\infty} f(\bar{d}(((m+n)!X_{mn})^{1/m+n}, \bar{0})) = 0 \text{ for each } n \in \mathbb{N}.$$

$$\lim_{n\to\infty} f(\bar{d}(((m+n)!X_{mn})^{1/m+n}, \bar{0})) = 0 \text{ for each } m \in \mathbb{N}.$$

2.1. Definition. A modulus function was introduced by Nakano [35]. We recall that a modulus f is a function from $[0,\infty) \to [0,\infty)$, such that

(1) $f(x) = 0$ if and only if $x = 0$

2) $f(x+y) \le f(x) + f(y)$, for all $x \ge 0$, $y \ge 0$,

(3) f is increasing,

(4) f is continuous from the right at 0. Since $|f(x) - f(y)| \le f(|x-y|)$, it follows from here that f is continuous on $[0,\infty)$.

2.2. Definition. Let $A = (a_{k,\ell}^{mn})$ denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the k,ℓ-th term to Ax is as follows:

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn};$$

such transformation is said to be nonnegative if $a_{k\ell}^{mn}$ is nonnegative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman [45] and Toeplitz [50]. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which they both added an adiditional assumption of boundedness. This assumption was made because a double sequence which is P-convergent is not necessarily bounded.

3. Main results

3.1. Theorem. *Let* $N_1 = \min\{n_0 : \sup_{mn\ge n_0} f(\bar{d}(((m+n)!(X_{mn} - Y_{mn}))^{1/m+n}, \bar{0}))^{P_{mn}} < \infty\}$, $N_2 = \min\{n_0 : \sup_{mn\ge n_0} P_{mn} < \infty\}$ *and* $N = \max(N_1, N_2)$.

(*i*) $\langle X_{mn}\rangle \in \chi_{f_p}^{2F^R}$ *is a paranormed space with*

$$g(X) = \lim_{N\to\infty} \sup_{mn\ge N} f(\bar{d}(((m+n)!(X_{mn} - Y_{mn}))^{1/m+n}, \bar{0}))^{P_{mn}/M} \tag{3.1}$$

if and only if $\mu > 0$, *where* $\mu = \lim_{N\to\infty} \inf_{mn\ge N} P_{mn}$ *and* $M = \max(1, \sup_{mn\ge N} P_{mn})$.

(ii) $\chi_{f_p}^{2F^R}$ *is complete with the paranorm* (3.1).

Proof. (i) Necesity. Let $\chi_{f_p}^{2F^R}$ be a paranormed space with (3.1) and suppose that $\mu = 0$. Then $\alpha = inf_{mn \geq N} P_{mn} = 0$ for all $N \in \mathbb{N}$ and $g\langle \lambda X \rangle = \lim_{N\to\infty} \sup_{mn\geq N} |\lambda|^{P_{mn/M}} = 1$ for all $\lambda \in (0, 1]$, where $X = \langle \alpha \rangle \in \chi_{f_p}^{2F^R}$ whence $\lambda \to 0$ does not imply $\lambda X \to \theta$, when X is fixed. But this contradicts to (3.1) to be a paranorm.

Sufficiency. Let $\mu > 0$. It is trivial that $g(\theta) = 0, g(-X) = g(X)$ and $g\langle X + Y, \bar{0}\rangle \leq g\langle X, \bar{0}\rangle + g\langle Y, \bar{0}\rangle$. Since $\mu > 0$ there exists a positive number β such that $P_{mn} > \beta$ for sufficiently large positive integer m, n. Hence for any $\lambda \in \mathbb{C}$, we may write $|\lambda|^{P_{mn}} \leq \max(|\lambda|^M, |\lambda|^\beta)$ for sufficiently large positive integers $m, n \geq N$. Therefore, we obtain $g\langle \lambda X, \bar{0}\rangle \leq \max(|\lambda|, |\lambda|^{\beta/M}) g\langle X\rangle$. Using this, one can prove that $\lambda X \to \theta$, whenever X is fixed and $\lambda \to 0$ or $\lambda \to 0$ and $X \to \theta$, or λ is fixed and $X \to \theta$.

(ii) Let $\langle X^{k\ell}\rangle$ be a Cauchy sequence in $\chi_{f_p}^{2F^R}$, where $X^{k\ell} = \langle X_{mn}^{k\ell}\rangle_{m,n\in\mathbb{N}}$. Then for every $\epsilon > 0$ $(0 < \epsilon < 1)$ there exists a positive integer s_0 such that

$$g\langle X^{k\ell} - X^{rt}\rangle = \lim_{N\to\infty} \sup_{mn\geq N} f(\bar{d}(((m+n)!(X_{mn}^{k\ell} - X_{mn}^{rt}))^{1/m+n}, \bar{0}))^{P_{mn}/M} < \frac{\epsilon}{2} \tag{3.2}$$

for all $k, \ell, r, t > s_0$. By (3.2) there exists a positive integer n_0 such that

$$\sup_{mn\geq N} f(\bar{d}(((m+n)!(X_{mn}^{k\ell} - X_{mn}^{rt}))^{1/m+n}, \bar{0}))^{P_{mn}/M} < \frac{\epsilon}{2} \tag{3.3}$$

for all $k, \ell, r, t > s_0$ and for $N > n_0$. Hence we obtain

$$f(\bar{d}(((m+n)!(X_{mn}^{k\ell} - X_{mn}^{rt}))^{1/m+n}, \bar{0}))^{P_{mn}/M} < \frac{\epsilon}{2} < 1 \tag{3.4}$$

so that

$$\begin{aligned} &f(\bar{d}(((m+n)!(X_{mn}^{k\ell} - X_{mn}^{rt}))^{1/m+n}, \bar{0})) \\ &\qquad < f(\bar{d}(((m+n)!(X_{mn}^{k\ell} - X_{mn}^{rt}))^{1/m+n}, \bar{0}))^{P_{mn}/M} < \frac{\epsilon}{2} \end{aligned} \tag{3.5}$$

for all $k, \ell, r, t > s_0$. This implies that $\langle X_{mn}^{k\ell}\rangle_{k\ell\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for each fixed $m, n \geq n_0$. Hence the sequence $\langle X_{mn}^{k\ell}\rangle_{k\ell\in\mathbb{N}}$ is convergent to X_{mn} say,

$$\lim_{k\ell\to\infty} X_{mn}^{k\ell} = X_{mn} \text{ for each fixed } m, n > n_0. \tag{3.6}$$

Getting X_{mn}, we define $X = \langle X_{mn}\rangle$. From (3.2) we obtain

$$g\langle X^{k\ell} - X\rangle = \lim_{N\to\infty} \sup_{mn\geq N} f(\bar{d}(((m+n)!(X_{mn}^{k\ell} - X_{mn}))^{1/m+n}, \bar{0}))^{P_{mn}/M} < \frac{\epsilon}{2} \tag{3.7}$$

as $r, t \to \infty$, for all $k, \ell, r, t > s_0$. By (3.6). This implies that $\lim_{k\ell\to\infty} X^{k\ell} = X$.

Now we show that $X = \langle X_{mn}\rangle \in \chi_{f_p}^{2F^R}$. Since $X^{k\ell} \in \chi_{f_p}^{2F^R}$ for each $(k, 1) \in N \times N$ for every $\epsilon > 0 (0 < \epsilon < 1)$ there exists a positive integer $n_1 \in N$ such that

$$f(\bar{d}(((m+n)!X_{mn})^{1/m+n}, \bar{0}))^{P_{mn}/M} < \frac{\epsilon}{2} \text{ for every } m, n > n_1. \tag{3.8}$$

By (3.6), (3.7) and (1.1) we obtain $f(\bar{d}(((m+n)!(X_{mn}))^{1/m+n},\bar{0}))^{P_{mn}/M} \leq f(\bar{d}(((m+n)!(X^{k\ell}_{mn}))^{1/m+n},\bar{0}))^{P_{mn}/M} + f(\bar{d}(((m+n)!(X^{k\ell}_{mn}-X_{mn}))^{1/m+n},\bar{0}))^{P_{mn}/M} < \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$ for $k,\ell > max(s_0,s_1)$ and $m,n > max(n_0,n_1)$. This implies that $X \in \chi^{2F^R}_{f_p}$. ■

3.2. Proposition. *The class of sequences Λ^{2F}_f is symmetric but the classes of sequences χ^{2F}_f and $\chi^{2F^R}_f$ are not symmetric.*

Proof. Obviously the class of sequences Λ^{2F}_f is symmetric. For the other classes of sequences consider the following example

Example. Consider the class of sequences χ^{2F}_f. Let $f(X)=X$ and consider the sequence $\langle X_{mn}\rangle$ be defined by

$$X_{1n}(t)=\begin{cases}\frac{(-t+1)^{1+n}}{(1+n)!}, & \text{for } t=-1,\\ \frac{(t-1)^{1+n}}{(1+n)!}, & \text{for } t=1,\\ 0, & \text{otherwise.}\end{cases}$$

and for $m>1$,

$$X_{mn}(t)=\begin{cases}\frac{(t+2)^{m+n}}{(m+n)!}, & \text{for } t=-2,\\ \frac{(-t-1)^{m+n}}{(m+n)!}, & \text{for } t=-1,\\ 0, & \text{otherwise.}\end{cases}$$

Let $\langle Y_{mn}\rangle$ be a rearrangement of $\langle X_{mn}\rangle$ defined by

$$Y_{nn}(t)=\begin{cases}\frac{(-t+1)^{2n}}{(2n)!}, & \text{for } t=-1,\\ \frac{(t-1)^{2n}}{(2n)!}, & \text{for } t=1,\\ 0, & \text{otherwise.}\end{cases}$$

and for $m\neq n$,

$$Y_{mn}(t)=\begin{cases}\frac{(t+2)^{m+n}}{(m+n)!}, & \text{for } t=-2,\\ \frac{(-t-1)^{m+n}}{(m+n)!}, & \text{for } t=-1,\\ 0, & \text{otherwise.}\end{cases}$$

Then, $\langle X_{mn}\rangle \in \chi^{2F}_f$ but $\langle Y_{mn}\rangle \notin \chi^{2F}_f$. Hence, χ^{2F}_p is not symmetric. Similarly other sequence also not symmetric. ■

3.3. Proposition. *The classes of sequences $\Lambda^{2F}_f, \chi^{2F}_f$ and $\chi^{2F^R}_f$ are solid.*

Proof. Consider the class of sequences χ^{2F}_f. Let $\langle X_{mn}\rangle$ and $\langle Y_{mn}\rangle \in \chi^{2F}_f$ be such that $\bar{d}(((m+n)!Y_{mn})^{1/m+n},\bar{0}) \leq \bar{d}(((m+n)!X_{mn})^{1/m+n},\bar{0})$. As f is non-decreasing, we have $\lim_{mn\to\infty} f(\bar{d}(((m+n)!Y_{mn})^{1/m+n},\bar{0})) \leq \lim_{mn\to\infty} f(\bar{d}(((m+n)!X_{mn})^{1/m+n},\bar{0}))$. Hence, the class of sequence χ^{2F}_f is solid. Similarly it can be shown that the other classes of sequences are also solid. ■

3.4. Proposition. *The classes of sequences χ_f^{2F} and $\chi_f^{2F^R}$ are not monotone and hence not solid.*

Proof. The result follows from the following example.

Example. Consider the class of sequences χ_f^{2F} and $f(X) = X$. Let $J = \{(m,n) : m \geq n\} \subseteq N \times N$. Let $\langle X_{mn} \rangle$ be defined by

$$X_{mn}(t) = \begin{cases} \frac{(t+3)^{m+n}}{(m+n)!}, & \text{for } -3 < t \leq -2, \\ \frac{(mt)^{m+n}}{(3m-1)^{m+n}(m+n)!} + \frac{(3m)^{m+n}}{(3m-1)^{m+n}(m+n)!}, & \text{for } -2 \leq t \leq -1 + \frac{1}{m}, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

for all $m, n \in N$. Then $\langle X_{mn} \rangle \in \chi_f^{2F}$. Let $\langle Y_{mn} \rangle$ be the canonical pre-image of $\langle X_{mn} \rangle_J$ for the subsequence J of $N \times N$. Then

$$Y_{mn} = \begin{cases} X_{mn}, & \text{for } (m,n) \in J, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

Then, $\langle Y_{mn} \rangle \notin \chi_f^{2F}$. Hence χ_f^{2F} is not monotone. Similarly, it can be shown that the other classes of sequences are also not monotone. Hence, the classes of sequences χ_f^{2F} and $\chi_f^{2F^R}$ are not solid. ∎

3.5. Proposition. *(i)* $\chi_{f_1}^{2F} \cap \chi_{f_2}^{2F} \subseteq \chi_{f_1+f_2}^{2F}$, *(ii)* $\chi_{f_1}^{2F^R} \cap \chi_{f_2}^{2F^R} \subseteq \chi_{f_1+f_2}^{2F^R}$.

Proof. It is easy, so it is omitted. ∎

3.6. Proposition. *Let f and f_1 be two modulus functions, then, (i) $\chi_{f_1}^{2F} \subseteq \chi_{f \circ f_1}^{2F}$. (ii) $\chi_{f_1}^{2F^R} \subseteq \chi_{f \circ f_1}^{2F^R}$. (iii) $\Lambda_{f_1}^{2F} \subseteq \Lambda_{f \circ f_1}^{2F}$.*

Proof. We prove the result for the case $\chi_{f_1}^{2F} \subseteq \chi_{f \circ f_1}^{2F}$, the other cases similar. Let $\epsilon > 0$ be given. As f is continuous and non-decreasing, so there exists $\eta > 0$, such that $f(\eta) = \epsilon$. Let $\langle X_{mn} \rangle \in \chi_{f_1}^{2F}$. Then, there exist $m_0, n_0 \in \mathbb{N}$, such that

$$f_1(\bar{d}(((m+n)!X_{mn})^{1/m+n}, \bar{0})) < \eta, \text{ for all } m \geq m_0, n \geq n_0,$$

$$\Rightarrow f \circ f_1(\bar{d}(((m+n)!X_{mn})^{1/m+n}, \bar{0})) < \epsilon, \text{ for all } m \geq m_0, n \geq n_0.$$

Hence, $\langle X_{mn} \rangle \in \chi_{f \circ f_1}^{2F}$. Thus, $\chi_{f_1}^{2F} \subseteq \chi_{f \circ f_1}^{2F}$. ∎

3.7. Proposition. *(i)* $\chi_f^{2F} \subseteq \Lambda_f^{2F}$. *(ii)* $\chi_f^{2F^R} \subseteq \Lambda_f^{2F}$. *The inclusion are strict.*

Proof. The inclusion (i) $\chi_f^{2F} \subseteq \Lambda_f^{2F}$ (ii) $\chi_f^{2F^R} \subseteq \Lambda_f^{2F}$ is obvious. For establishing that the inclusions are proper, consider the following example.

Example. We prove the result for the case $\chi_f^{2F} \subseteq \Lambda_f^{2F}$, the other case similar. Let $f(X) = X$. Let the sequence $\langle X_{mn} \rangle$ be defined by for $m > n$,

$$X_{mn}(t) = \begin{cases} \frac{(mt-m-1)^{m+n}(m-1)^{-(m+n)}}{(m+n)!}, & \text{for } 1 + \frac{1}{m} \leq t \leq 2, \\ \frac{(3-t)^{m+n}}{(m+n)!}, & \text{for } 2 < t \leq 3, \\ 0, & \text{otherwise.} \end{cases}$$

and for $m < n$

$$X_{mn}(t) = \begin{cases} \frac{(mt-1)^{m+n}(m-1)^{-(m+n)}}{(m+n)!}, & \text{for } \frac{1}{m} \leq t \leq 1, \\ \frac{(-t+2)^{m+n}}{(m+n)!}, & \text{for } 1 \leq t \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\langle X_{mn} \rangle \in \Lambda_f^{2F}$ but $\langle X_{mn} \rangle \notin \chi_f^{2F}$. ■

3.8. Proposition. *The classes of sequences* $\Lambda_f^{2F}, \chi_f^{2F}$ *and* $\chi_f^{2F^R}$ *are not convergent free.*

Proof. The result follows from the following example.

Example. Consider the classes of sequences χ_f^{2F}. Let $f(X) = X$ and consider the sequence $\langle X_{mn} \rangle$ defined by $((1+n)!X_{1n})^{1/1+n} = \bar{0}$, and for other values,

$$X_{mn}(t) = \begin{cases} \frac{1^{m+n}}{(m+n)!}, & \text{for } 0 \leq t \leq 1, \\ \frac{(-mt)^{m+n}(m+1)^{-(m+n)} + (2m+1)^{m+n}(1+m)^{-(m+n)}}{(m+n)!}, & \text{for } 1 < t \leq 2 + \frac{1}{m}, \\ 0, & \text{otherwise.} \end{cases}$$

Let the sequence $\langle Y_{mn} \rangle$ be defined by $((1+n)!Y_{1n})^{1/1+n} = \bar{0}$, and for other values,

$$Y_{mn}(t) = \begin{cases} \frac{1^{m+n}}{(m+n)!}, & \text{for } 0 \leq t \leq 1, \\ \frac{(m-t)^{m+n}(m-1)^{-(m+n)}}{(m+n)!}, & \text{for } 1 < t \leq m, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\langle X_{mn} \rangle \in \chi_f^{2F}$ but $\langle Y_{mn} \rangle \notin \chi_f^{2F}$. Hence, the classes of sequences χ_f^{2F} is not convergent free. Similarly, the other spaces are also not convergent free. ■

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