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SOME CHARACTERIZATIONS OF SPACES WITH LOCALLY COUNTABLE NETWORKS

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Abstract. In this paper, we give some characterizations of spaces with locally countable network and some characterizations of sn-symmetric (or Cauchy sn-symmetric) spaces with locally countable sn-networks by compact images (or π-images) of locally separable metric spaces.

1. Introduction

One of the central problems in general topology is to establish relationships between various topological spaces and metric spaces by means of various maps. Many kinds of characterizations have been obtained by means of certain networks (see [2, 3, 9, 16], for example). In [3], T.V. An and L.Q. Tuyen proved some characterizations of spaces with countable networks and some characterizations of sn-symmetric (or Cauchy sn-symmetric) spaces with countable sn-networks by compact images (or π-images) of separable metric spaces. Furthermore, in [18], L.Q. Tuyen proved that a regular space with a locally countable sn-network (resp., weak base) if and only if it is a compact-covering (resp., compact-covering quotient) compact and ss-image of a metric space, if and only if it is a sequentially-quotient (resp., quotient) π and ss-image of a metric space.

In this paper, we give some characterizations of spaces with locally countable networks and some characterizations of sn-symmetric (or Cauchy sn-symmetric) spaces with locally countable sn-networks by compact images (or π-images) of locally separable metric spaces. As an application of this result, we give some characterizations of symmetric (or Cauchy symmetric) spaces with locally countable weak bases by compact images (or π-images) of locally separable metric spaces.

Throughout this paper, all spaces are assumed to be Hausdorff, all maps are continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers. Let $\mathcal{P}$ be a family of subsets of X, and $f : X \longrightarrow Y$ be a map, we denote $\bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}$, and $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$. For a sequence $\{x_n\}$ converging to x and $P \subset X$, we say

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that $\{x_n\}$ is *eventually* in P if $\{x\}\bigcup\{x_n : n \geq m\} \subset P$ for some $m \in \mathbb{N}$, and $\{x_n\}$ is *frequently* in P if some subsequence of $\{x_n\}$ is eventually in P.

Definition 1.1. Let X be a space and P be a subset of X.

(1) P is a *sequential neighborhood* of x in X, if each sequence S converging to x is eventually in P.
(2) P is a *sequentially open* subset of X, if P is a sequential neighborhood of x in X for all $x \in P$.

Definition 1.2. Let $\mathcal{P}$ be a collection of subsets of a space X and $x \in X$. Then,

(1) $\mathcal{P}$ is a *network at* x in X [16], if $x \in P$ for every $P \in \mathcal{P}$, and if $x \in U$ with U open in X, there exists $P \in \mathcal{P}$ such that $x \in P \subset U$.
(2) $\mathcal{P}$ is a *network* for X [16], if $\{P \in \mathcal{P} : x \in P\}$ is a network at x in X for all $x \in X$.
(3) $\mathcal{P}$ is a *cs***-network* for X [17], if for each sequence S converging to a point $x \in U$ with U open in X, S is frequently in $P \subset U$ for some $P \in \mathcal{P}$.
(4) $\mathcal{P}$ is a *cs-network* for X [17], if each sequence S converging to a point $x \in U$ with U open in X, S is eventually in $P \subset U$ for some $P \in \mathcal{P}$.
(5) $\mathcal{P}$ is *point-countable*, if each point $x \in X$ belongs to only countably many members of $\mathcal{P}$.
(6) $\mathcal{P}$ is *locally countable*, if for each $x \in X$, there exists a neighborhood V of x such that V meets only countably many members of $\mathcal{P}$.
(7) $\mathcal{P}$ is *star-countable* [14], if each $P \in \mathcal{P}$ meets only countably many members of $\mathcal{P}$.

Definition 1.3. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a family of subsets of a space X satisfying that, for every $x \in X$, $\mathcal{P}_x$ is a network at x in X, and if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

(1) $\mathcal{P}$ is a *weak base* for X [4], if whenever $G \subset X$ satisfying for every $x \in G$, there exists $P \in \mathcal{P}_x$ with $P \subset G$, then G open in X. Here, $\mathcal{P}_x$ is a *weak neighborhood base* at x in X.
(2) $\mathcal{P}$ is an *sn-network* for X [11], if each member of $\mathcal{P}_x$ is a sequential neighborhood of x for all $x \in X$. Here, $\mathcal{P}_x$ is an *sn-network* at x in X.

Remark 1.5.

(1) weak bases $\Longrightarrow$ *sn*-networks.
(2) In a sequential space, weak bases $\Longleftrightarrow$ *sn*-networks.

Definition 1.5. Let $f : X \longrightarrow Y$ be a map.

(1) f is a *weak-open* map [19], if there exists a weak base $\mathcal{B} = \bigcup\{\mathcal{B}_y : y \in Y\}$ for Y, and for every $y \in Y$, there exists $x_y \in f^{-1}(y)$ such that for each open neighborhood U of x_y, $B \subset f(U)$ for some $B \in \mathcal{B}_y$.
(2) f is a *1-sequence-covering* map [11], if for each $y \in Y$, there exists $x_y \in f^{-1}(y)$ such that each sequence converging to y in Y is an image of some sequence converging to x_y in X.
(3) f is a *sequence-covering* map [15], if for every convergent sequence S in Y, there exists a convergent sequence L in X such that $f(L) = S$.

(4) f is a *compact-covering* map [13], if for each compact subset K of Y, there exists a compact subset L of X such that $f(L) = K$.
(5) f is a *pseudo-sequence-covering* map [9], if for each convergent sequence S in Y, there exists a compact subset K of X such that $f(K) = S$. Note that a pseudo-sequence-covering map is a *sequence-covering* map in the sense of [8].
(6) f is a *sequentially-quotient* map [5], if for each convergent sequence S in Y, there exists a convergent sequence L in X such that $f(L)$ is a subsequence of S.
(7) f is a *quotient* map [18], if whenever $U \subset Y$, U open in Y if and only if $f^{-1}(U)$ open in X.
(8) f is an *ss-map* [10], if for each $y \in Y$, there exists a neighborhood U of y such that $f^{-1}(U)$ is separable in X.
(9) f is a *compact* map [17], if $f^{-1}(y)$ is compact in X for all $y \in Y$.
(10) f is a *π-map* [4], if for every $y \in Y$ and for every neighborhood U of y in Y, $d\big(f^{-1}(y), X - f^{-1}(U)\big) > 0$, where X is a metric space with a metric d.

Remark 1.6. Let $f : X \longrightarrow Y$ be a map. Then
(1) 1-sequence-covering maps $\Longrightarrow$ sequence-covering maps $\Longrightarrow$ pseudo-sequence-covering maps $\Longrightarrow$ sequentially-quotient maps.
(2) compact-covering maps $\Longrightarrow$ pseudo-sequence-covering maps.
(3) weak-open maps $\Longrightarrow$ quotient maps.
(4) compact maps $\Longrightarrow$ π-maps, if X is metric.

Definition 1.7. Let d be a d-function on a space X.
(1) For each $x \in X$ and $n \in \mathbb{N}$, let $S_n(x) = \{y \in X : d(x, y) < 1/n\}$.
(2) X is *sn-symmetric* (resp., *symmetric*) [7], if $\{S_n(x) : n \in \mathbb{N}\}$ is an sn-network (resp., weak base) at x in X for all $x \in X$.
(3) X is *Cauchy sn-symmetric* [3] (resp., *Cauchy symmetric* [7], if it is sn-symmetric (resp., symmetric) and every convergent sequence in X is d-Cauchy.

Remark 1.8.
(1) symmetric spaces $\Longleftrightarrow$ sequential and sn-symmetric spaces.
(2) Cauchy symmetric spaces $\Longleftrightarrow$ sequential and Cauchy sn-symmetric spaces.

For some undefined or related concepts, we refer the reader to [9] and [17].

2. Main results

Theorem 2.1. *The following are equivalent for a space X.*
(1) *X has a locally countable network;*
(2) *X is a compact and ss-image of a locally separable metric space;*
(3) *X is a π and ss-image of a locally separable metric space;*
(4) *X is an ss-image of a locally separable metric space.*

Proof. (1) $\Longrightarrow$ (2). Let $\mathcal{P}$ be a locally countable network for X. Then for each $x \in X$, there exists an open neighborhood V_x of x such that V_x meets only countably many members of $\mathcal{P}$. Let

$$\mathcal{Q} = \{P \in \mathcal{P} : P \subset V_x \text{ for some } x \in X\}.$$

Then $\mathcal{Q}$ is a locally countable and star-countable network for X. By Lemma 2.1 in [14], $\mathcal{Q} = \bigcup_{\alpha\in\Lambda} \mathcal{Q}_\alpha$, where each $\mathcal{Q}_\alpha$ is a countable subfamily of $\mathcal{Q}$ and $(\bigcup \mathcal{Q}_\alpha) \cap (\bigcup \mathcal{Q}_\beta) = \emptyset$ for all $\alpha \neq \beta$. For each $\alpha \in \Lambda$, let $X_\alpha = \bigcup \mathcal{Q}_\alpha$. Then $X = \bigcup_{\alpha\in\Lambda} X_\alpha$ and each X_α has a countable network. It follows from Theorem 2.10 in [3] that for each $\alpha \in \Lambda$, there exists a compact map $f_\alpha : M_\alpha \longrightarrow X_\alpha$, where M_α is a separable metric space. Now, we put

$$M = \bigoplus_{\alpha\in\Lambda} M_\alpha,\ Z = \bigoplus_{\alpha\in\Lambda} P_\alpha,\ f = \bigoplus_{\alpha\in\Lambda} f_\alpha : M \longrightarrow Z.$$

Then M is a locally separable metric space. Define $h : Z \longrightarrow X$ is a natural map, and $g = h \circ f$. Then

Claim 1. g is a compact map. Let $x \in X$. Then there exists unique a $\alpha \in \Lambda$ such that $x \in X_\alpha$. Since $g^{-1}(x) = f^{-1}\big(h^{-1}(x)\big) = f_\alpha^{-1}(x)$, it implies that $g^{-1}(x)$ is a compact subset in M_α. Thus, $g^{-1}(x)$ is a compact subset in M. Therefore, g is a compact map.

Claim 2. g is an ss-map. Let $x \in X$. Since V_x meets only countably many members of $\mathcal{P}$, the family $\Delta = \{\alpha \in \Lambda : V_x \cap X_\alpha \neq \emptyset\}$ is countable. On the other hand, since

$$g^{-1}(V_x) = f^{-1}\Big(h^{-1}(V_x)\Big) \subset f^{-1}\Big(\bigoplus_{\alpha\in\Delta} P_\alpha\Big) = \bigoplus_{\alpha\in\Delta} M_\alpha,$$

it follows that g is an ss-map.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4). It is obvious.

(4) $\Longrightarrow$ (1). Let $f : M \longrightarrow X$ be an ss-map, where M is a locally separable metric space, and $\mathcal{B}$ be a point-countable base for M. If we put $\mathcal{P} = f(\mathcal{B})$, then $\mathcal{P}$ is a locally countable network for X. Therefore, (1) holds. ∎

Theorem 2.2. *The following are equivalent for a space X.*

(1) *X is an sn-symmetric space with a locally countable cs^*-network;*
(2) *X is an sn-symmetric space with a locally countable sn-network;*
(3) *X is a pseudo-sequence-covering compact and ss-image of a locally separable metric space;*
(4) *X is a sequentially-quotient π and ss-image of a locally separable metric space.*

Proof. (1) $\Longrightarrow$ (2). By Theorem 2.1 in [18].

(2) $\Longrightarrow$ (3). Let $\mathcal{P}$ be a locally countable sn-network for an sn-symmetric space X. By using again notations and arguments as in the proof (1) $\Longrightarrow$ (2) of Theorem 2.1, it implies that $X = \bigcup_{\alpha\in\Lambda} X_\alpha$, where each X_α is sequentially open and it has a countable sn-network. Since X is an sn-symmetric space, each X_α is an sn-symmetric subspace has a countable sn-network. It follows from Theorem 2.4 in [3] that for each $\alpha \in \Lambda$, there exists a pseudo-sequence-covering and compact map $f_\alpha : M_\alpha \longrightarrow X_\alpha$, where M_α is a separable metric space. By using again notations and arguments as in the proof (1) $\Longrightarrow$ (2) of Theorem 2.1, it suffices to prove that g is a pseudo-sequence-covering map.

Let $\{x_n\}$ be a sequence converging to x in X. Then $x \in X_\alpha$ for some $\alpha \in \Lambda$. On the other hand, since X_α is sequentially open, there exists $m \in \mathbb{N}$ such that $\{x\} \bigcup \{x_n : n \geq m\} \subset X_\alpha$. Furthermore, since f_α is a pseudo-sequence-covering map and $\{x_n : n \geq m\}$ is a sequence converging to x in X_α, there exists a compact subset L_α in M_α such that $f_\alpha(L_\alpha) = \{x\} \bigcup \{x_n : n \geq m\}$. For each $i < m$, we take $z_i \in M$ such that $g(z_i) = x_i$ and put $L = \{z_i : i < m\} \bigcup L_\alpha$. Then L is a compact subset in M and $g(L) = \{x\} \bigcup \{x_n : n \in \mathbb{N}\}$. Therefore, g is a pseudo-sequence-covering map.

(3) $\Longrightarrow$ (4). It is obvious.

(4) $\Longrightarrow$ (1). Let X be a sequentially-quotient π and ss-image of a locally separable metric space. Then X has a locally countable cs^*-network by Theorem 2.1 in [18]. Furthermore, it follows from Corollary 2.6 in [6] and Lemma 2.1(1) in [3] that X is an sn-symmetric space. Therefore, (1) holds. ∎

Corollary 2.3. *The following are equivalent for a space X.*

(1) *X is a symmetric space with a locally countable cs^*-network;*

(2) *X is a symmetric space with a locally countable weak base;*

(3) *X is a pseudo-sequence-covering quotient compact and ss-image of a locally separable metric space;*

(4) *X is a quotient π and ss-image of a locally separable metric space.*

Remark 2.4. By Remark 2.9 in [3], it follows that

(1) "sn-symmetric" (resp., "symmetric") cannot be omitted in Theorem 2.2 (resp., Corollary 2.3).

(2) Spaces with locally countable sn-networks (resp., weak bases) $\not\Longrightarrow$ sn-symmetric (resp., symmetric) spaces.

Theorem 2.5. *The following are equivalent for a space X.*

(1) *X is a Cauchy sn-symmetric space with a locally countable cs^*-network;*

(2) *X is a Cauchy sn-symmetric space with a locally countable sn-network;*

(3) *X is a 1-sequence-covering compact-covering compact and ss-image of a locally separable metric space;*

(4) *X is a sequence-covering π and ss-image of a locally separable metric space.*

Proof. (1) $\Longrightarrow$ (2). By Theorem 2.1 in [18].

(2) $\Longrightarrow$ (3). Let $\mathcal{P}$ be a locally countable sn-network for a Cauchy sn-symmetric space X. By using again notations and arguments as in the proof (1) $\Longrightarrow$ (2) of Theorem 2.1, it implies that $X = \bigcup_{\alpha \in \Lambda} X_\alpha$, where each X_α is sequentially open and it has a countable sn-network. Since X is a Cauchy sn-symmetric space, each X_α is a Cauchy sn-symmetric subspace has a countable sn-network. It follows from Theorem 2.7 in [3] that for each $\alpha \in \Lambda$, there exists a 1-sequence-covering compact-covering and compact map $f_\alpha : M_\alpha \longrightarrow X_\alpha$, where M_α is a separable metric space. By using again notations and arguments as in the proof (1) $\Longrightarrow$ (2) of Theorem 2.1, it suffices to prove that g is a 1-sequence-covering and compact-covering map.

Claim 1. g is a 1-sequence-covering map. Let $x \in X$. Then $x \in X_\alpha$ for some $\alpha \in \Lambda$. Since f_α is a 1-sequence-covering map, there exists $z_x \in f_\alpha^{-1}(x)$ such that each sequence converging to z_x in X_α is an image of some sequence converging to z_x in M_α.

Now, let $\{x_n\}$ be a sequence converging to x in X. Then $x \in X_\alpha$ for some $\alpha \in \Lambda$. On the other hand, since X_α is sequentially open, there exists $m \in \mathbb{N}$ such that $\{x\} \bigcup \{x_n : n \geq m\} \subset X_\alpha$. This implies that $\{x_n : n \geq m\}$ is a sequence converging to x in X_α. Furthermore, since f_α is a 1-sequence-covering map, there exists a sequence $\{z_n : n \geq m\} \subset M_\alpha$ such that $\{z_n\}$ converges to z_x in M_α and $f_\alpha(z_n) = x_n$ for all $n \geq m$. For each $i < m$, we take $z_i \in g^{-1}(x_i)$. Then $\{z_n\}$ is a sequence converging to z_x in M and $z_n \in g^{-1}(x_n)$ for all $n \in \mathbb{N}$. Therefore, g is a 1-sequence-covering map.

Claim 2. g is a compact-covering map. Let K be a compact subset of X. Since X has a locally countable sn-network, K is metrizable. On the other hand, since each X_α is sequentially open in X and $X_\alpha \cap X_\beta = \emptyset$ for all $\alpha \neq \beta$, it implies that the family $\Gamma = \{\alpha \in \Lambda : K \cap X_\alpha \neq \emptyset\}$ is finite. For each $\alpha \in \Gamma$, we put $K_\alpha = K \cap X_\alpha$. Since K is metrizable and each X_α is sequentially open, K_α is a compact subset in X_α for all $\alpha \in \Gamma$. Furthermore, since each f_α is a compact-covering map, it implies that for each $\alpha \in \Gamma$, there a compact subset L_α in M_α such that $f_\alpha(L_\alpha) = K_\alpha$ for all $\alpha \in \Gamma$. If we put $L = \bigoplus_{\alpha \in \Gamma} L_\alpha$, then L is a compact subset in M and $g(L) = K$. Therefore, g is a compact-covering map.

(3) $\Longrightarrow$ (4). It is obvious.

(4) $\Longrightarrow$ (1). Let X be a sequence-covering π and ss-image of a locally separable metric space. Then X has a locally countable cs^*-network by Theorem 2.1 in [18]. Furthermore, it follows from Proposition 16(3b) in [9] and Lemma 2.1(2) in [3] that X is a Cauchy sn-symmetric space. Therefore, (1) holds. ∎

By Theorem 2.5 and Corollary 2.8 in [1], the following corollary holds.

COROLLARY 2.6. *The following are equivalent for a space X.*

(1) *X is a Cauchy symmetric space with a locally countable cs^*-network;*
(2) *X is a Cauchy symmetric space with a locally countable weak base;*
(3) *X is a weak-open compact-covering compact and ss-image of a locally separable metric space;*
(4) *X is a weak-open π and ss-image of a locally separable metric space.*

REMARK 2.7. By Remark 2.9 in [3], it implies that "Cauchy sn-symmetric" (resp., "Cauchy symmetric") cannot be omitted in Theorem 2.5 (resp., Corollary 2.6).

Finally, we pose the following question.

QUESTION 2.8. Let X be an sn-symmetric space with a countable (resp., locally countable) sn-network. Is X a compact-covering and compact image of a separable metric space (resp., compact-covering compact and ss-image of a metric space)?

Remark 2.9. Recently, L.Q. Tuyen has showed that a regular space with a locally countable sn-network is a compact-covering compact and ss-image of a locally separable metric space (see, [18, Theorem 2.1]), and Y. Ge showed that a regular space with a countable sn-network is a compact-covering and compact image of a separable metric space (see, [6, Proposition 3.9(2)]). This follows that if X is a regular space, then the above question is affirmative.

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REFERENCES

[**1**] T.V. An, L.Q. Tuyen, *Further properties of 1-sequence-covering maps*, Comment. Math. Univ. Carolin. **49** (2008), 477–484.
[**2**] T.V. An, L.Q. Tuyen, *On an affirmative answer to S. Lin's problem*, Topology Appl. **158** (2011), 1567–1570
[**3**] T.V. An, L.Q. Tuyen, *On π-images of separable metric spaces and a problem of Shou Lin*, Mat. Vesnik, to appear.
[**4**] A.V. Arhangelskii, *Mappings and spaces*, Russian Math. Surveys **21** (1966), 115–162.
[**5**] J.R. Boone, F. Siwiec, *Sequentially quotient mappings*, Czech. Math. J. **26** (1976), 174–182.
[**6**] Y. Ge, *On π-images of metric spaces*, Acta Math. APN. **22** (2006), 209–215.
[**7**] Y. Ge, S. Lin, *g-metrizable spaces and the images of semi-metric spaces*, Czech. Math. J. **57**(132) (2007), 1141–1149.
[**8**] G. Gruenhage, E. Michael, Y. Tanaka, *Spaces determined by point-countable covers*, Pacific J. Math. **113** (1984), 303–332.
[**9**] Y. Ikeda, C. Liu, Y. Tanaka, *Quotient compact images of metric spaces, and related matters*, Topology Appl. **122** (2002), 237–252.
[**10**] S. Lin, *On a generalization of Michael's theorem*, Northeast. Math. J. **4** (1988), 162–168.
[**11**] S. Lin, *On sequence-covering s-mappings*, Adv. Math. **25** (1996), 548–551 (in Chinese).
[**12**] S. Lin, P. Yan, *Sequence-covering maps of metric spaces*, Topology Appl. **109** (2001), 301–314.
[**13**] E. Michael, *$\aleph_0$-spaces*, J. Math. Mech. **15** (1966), 983–1002.
[**14**] M. Sakai, *On spaces with a star-countable k-networks*, Houston J. Math. **23** (1997), 45–56.
[**15**] F. Siwiec, *Sequence-covering and countably bi-quotient mappings*, General Topology Appl. **1** (1971), 143–153.
[**16**] Y. Tanaka, *Theory of k-networks* II, Q. and A. General Topology **19** (2001), 27–46.
[**17**] Y. Tanaka, Y. Ge, *Around quotient compact images of metric spaces, and symmetric spaces*, Houston J. Math. **32** (2006), 99–117.
[**18**] L.Q. Tuyen, *A new characterization of spaces with locally countable sn-networks*, Mat. Vesnik, to appear.
[**19**] S. Xia, *Characterizations of certain g-first countable spaces*, Adv. Math. **29** (2000), 61–64.

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