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NEW VERSION OF PROPERTY (az)

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Abstract. In this paper we define new spectral properties (h), (gh), (ah) and (gah) as a continuation of [H. Zariouh, Property (gz) for bounded linear operators, Mat. Vesnik, 65 (2013), 94–103], which are variants to the properties (z), (gz) and (az) introduced in the mentioned paper. The purpose of the paper is to study the relationship between these properties and other Weyl-type theorems.

1. Introduction

This paper is a continuation of [9] where we introduced and studied the spectral properties (z) and (az). We investigate the spectral properties (h), (gh), (ah) and (gah) (see Definitions 2.1 and 2.8) for bounded linear operators as new versions of properties (z) and (az). The results obtained are summarized in the diagram presented at the end of this paper. For further definitions and symbols we refer the reader to [9], and we also refer to [3, 4, 5, 6, 10] for more details. In addition, we have the following usual notations that will be needed later.

$E(T)$: eigenvalues of T that are isolated in the spectrum $\sigma(T)$,
$E^0(T)$: eigenvalues of T of finite multiplicity that are isolated in $\sigma(T)$,
$E_a(T)$: eigenvalues of T that are isolated in the approximate point spectrum $\sigma_a(T)$,
$E_a^0(T)$: eigenvalues of T of finite multiplicity that are isolated in $\sigma_a(T)$,
$\Pi(T)$: poles of T,
$\Pi^0(T)$: poles of T of finite rank,
$\Pi_a(T)$: left poles of T,
$\Pi_a^0(T)$: left poles of T of finite rank,
$\sigma_{BW}(T)$: B-Weyl spectrum of T,
$\sigma_W(T)$: Weyl spectrum of T,
$\sigma_{SBF_+^-}(T)$: essential semi-B-Fredholm spectrum of T,
$\sigma_{SF_+^-}(T)$: Weyl essential approximate point spectrum of T,
$\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$,

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$\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$,
$\Delta_+(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T)$,
$\Delta_+^g(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T)$.

The inclusion of the following table [2, 5, 8, 9] which contains the meaning of some properties is motivated by having overview of the subject.

(z)	$\sigma(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$		(gz)	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$
(az)	$\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a^0(T)$		(gaz)	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$
(w)	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$		(gw)	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$
(b)	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi^0(T)$		(gb)	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$

2. Main results

Definition 2.1. A bounded linear operator $T \in L(X)$ is said to satisfy property (h) if $\Delta_+(T) = E^0(T)$ and is said to satisfy property (gh) if $\Delta_+^g(T) = E(T)$.

Example 2.2. Let L be the unilateral left shift operator defined on $\ell^2(\mathbf{N})$. It is well known that $\sigma(L) = D(0,1)$ is the closed unit disc in $\mathbf{C}$, $\sigma_{SF_+^-}(L) = D(0,1)$ is the closed unit disk in $\mathbf{C}$, $E(L) = E^0(L) = \emptyset$. Moreover, $\sigma_{SBF_+^-}(L) = D(0,1)$. Hence the properties (h) and (gh) are satisfied by L.

Proposition 2.3. *Let $T \in L(X)$. Then T satisfies property (h) $\Longleftrightarrow$ T satisfies property (w) and $\sigma(T) = \sigma_a(T)$.*

Proof. Assume that T satisfies property (h). If $\lambda \in \Delta_a(T)$, then $\lambda \in \Delta_+(T)$. Therefore $\lambda \in E^0(T)$ and $\Delta_a(T) \subset E^0(T)$. Now if $\lambda \in E^0(T)$, then $\lambda \in \sigma_a(T)$ and since T satisfies property (h), it follows that $\lambda \notin \sigma_{SF_+^-}(T)$. Thus $\lambda \in \Delta_a(T)$. Hence $\Delta_a(T) = E^0(T)$, i.e. T satisfies property (w). We then have $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$ and $\sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$. This implies that $\sigma_a(T) = \mathrm{iso}\,\sigma_a(T) \cup \sigma_{SF_+^-}(T)$ and $\sigma(T) = \mathrm{iso}\,\sigma_a(T) \cup \sigma_{SF_+^-}(T)$. So $\sigma(T) = \sigma_a(T)$. Conversely, assume that T satisfies property (w) and $\sigma(T) = \sigma_a(T)$. Then $\Delta_a(T) = E^0(T)$ and $\sigma(T) = \sigma_a(T)$. So $\Delta_+(T) = E^0(T)$, i.e. T satisfies property (h). ■

The next proposition gives a generalization of Proposition 2.3 to the context of B-Fredholm theory.

Proposition 2.4. *Let $T \in L(X)$. Then T satisfies property (gh) $\Longleftrightarrow$ T satisfies property (gw) and $\sigma(T) = \sigma_a(T)$.*

Proof. Assume that T satisfies property (gh). If $\lambda \in \Delta_a^g(T)$, then $\lambda \in \Delta_+^g(T)$. Thus $\lambda \in E(T)$ and $\Delta_a^g(T) \subset E(T)$. Now if $\lambda \in E(T)$, then $\lambda \in \sigma_a(T)$ and since T satisfies property (gh), $\lambda \notin \sigma_{SBF_+^-}(T)$ and so $\lambda \in \Delta_a^g(T)$. Hence $\Delta_a^g(T) =$

$E(T)$ and T satisfies property (gw). We then have $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$ and $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$. Thus $\sigma_a(T) = \text{iso}\sigma_a(T) \cup \sigma_{SBF_+^-}(T)$ and $\sigma(T) = \text{iso}\sigma_a(T) \cup \sigma_{SBF_+^-}(T)$ which entails that $\sigma(T) = \sigma_a(T)$. Conversely, assume that T satisfies property (gw) and $\sigma(T) = \sigma_a(T)$. Then $\Delta_a^g(T) = E(T)$ and $\sigma(T) = \sigma_a(T)$. So $\Delta_+^g(T) = E(T)$ and T satisfies property (gh). ∎

Remark 2.5. The equality $\sigma_a(T) = \sigma(T)$ which establishes a link between property (gh) [resp. property (h)] and property (gw) [resp. property (w)] is satisfied if T^* has the SVEP at every $\mu \notin \sigma_{SF_+^-}(T)$, which in turn implies that properties (w) and (h) are equivalent, and properties (gh) and (gw) are equivalent. Of course, in this case it is easily seen that $\sigma_W(T) = \sigma_{SF_+^-}(T)$. Thus if $\lambda \notin \sigma_a(T)$, then $T - \lambda I$ is injective and $\lambda \notin \sigma_W(T)$. Therefore $T - \lambda I$ is surjective, which implies that $\lambda \notin \sigma(T)$. Hence $\sigma(T) = \sigma_a(T)$.

From Proposition 2.3 and Proposition 2.4, if $T \in L(X)$ satisfies property (h) [resp. property (gh)] then it satisfies property (w) [resp. property (gw)]. However, the converses do not hold in general as seen by the following example.

Example 2.6. Let $T = R \oplus S$ defined on the Banach space $\ell^2(\mathbf{N}) \oplus \ell^2(\mathbf{N})$, where R is the unilateral right shift operator defined on $\ell^2(\mathbf{N})$ and S is defined on $\ell^2(\mathbf{N})$ by $S(x_1, x_2, x_3, \dots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \dots)$. Then $\sigma(T) = D(0,1)$ which is the closed unit disc in $\mathbf{C}$, $\sigma_a(T) = C(0,1) \cup \{0\}$ where $C(0,1)$ is the unit circle of $\mathbf{C}$, $\sigma_{SF_+^-}(T) = C(0,1) \cup \{0\}$, $E^0(T) = \emptyset$. Therefore $\Delta_a(T) = E^0(T)$ and T satisfies property (w). Moreover, we have $\sigma_{SBF_+^-}(T) = C(0,1) \cup \{0\}$ and $E(T) = \emptyset$. So $\Delta_a^g(T) = E(T)$ and T satisfies property (gw). But T does not satisfy either property (h) nor property (gh), since $\Delta_+(T) \neq E^0(T)$ and $\Delta_+^g(T) \neq E(T)$. Here $\sigma_a(T) \neq \sigma(T)$.

Theorem 2.7. *Let $T \in L(X)$. Then the following statements hold.*

(*i*) *T satisfies property (gh) $\Longleftrightarrow$ T satisfies property (gz).*

(*ii*) *T satisfies property (h) $\Longleftrightarrow$ T satisfies property (z).*

Proof. (*i*) If T satisfies (gh), that is $\Delta_+^g(T) = E(T)$, then from Proposition 2.4, we have $\sigma(T) = \sigma_a(T)$. So $E(T) = E_a(T)$. Therefore $\Delta_+^g(T) = E_a(T)$ and T satisfies property (gz). Conversely, if T satisfies property (gz) that is $\Delta_+^g(T) = E_a(T)$, then from [9, Theorem 2.4] we have $\sigma(T) = \sigma_a(T)$. So $E(T) = E_a(T)$. Thus $\Delta_+^g(T) = E(T)$, i.e. T satisfies property (gh).

(*ii*) Follows directly by Proposition 2.3 and [9, Theorem 2.4]. ∎

Definition 2.8. A bounded linear operator $T \in L(X)$ is said to satisfy property (ah) if $\Delta_+(T) = \Pi^0(T)$ and is said to satisfy property (gah) if $\Delta_+^g(T) = \Pi(T)$.

Example 2.9. Recall that the Volterra operator V on $L^2([0,1])$ is defined by $V(f)(t) = \int_0^t f(s)\,ds$, for $f \in L^2[0,1]$. It is well known that $\sigma(V) = \{0\}$,

$\sigma_{SF_+^-}(T) = \{0\}$, $\Pi^0(V) = \Pi(V) = \emptyset$ and $\sigma_{SBF_+^-}(V) = \{0\}$. Hence the Volterra operator is an example satisfying the two properties (ah) and (gah).

Theorem 2.10. *Let $T \in L(X)$. Then T satisfies property (gah) $\Longleftrightarrow$ T satisfies property (ah).*

Proof. Assume that T satisfies property (gah). If $\lambda \in \Delta_+(T)$, then $\lambda \in \Delta_+^g(T) = \Pi(T)$. Since $T - \lambda I$ is an upper semi-Fredholm operator, then $\alpha(T - \lambda I)$ is finite. Thus $\lambda \in \Pi^0(T)$ and $\Delta_+(T) \subset \Pi^0(T)$. As we always have that $\Delta_+(T) \supset \Pi^0(T)$, then $\Delta_+(T = \Pi^0(T)$ and T satisfies property (ah). Conversely, assume that T satisfies property (ah). Let $\lambda \in \Delta_+^g(T)$ be arbitrary. We can assume without loss of generality that $\lambda = 0$. Then T is an upper semi-B-Fredholm operator with $\mathrm{ind}(T) \leq 0$ and in particular is an operator of topological uniform descent [4]. From the punctured neighborhood theorem for semi-B-Fredholm operators [4, Corollary 3.2], there exists $\epsilon > 0$ such that if $0 < |\mu| < \epsilon$, then $T - \mu I$ is an upper semi-Fredholm operator with $\mathrm{ind}(T - \mu I) = \mathrm{ind}(T)$. Let $|\mu| < \epsilon$ and $\mu \notin \sigma(T)$. Then $a(T - \mu I) = \delta(T - \mu I) = 0$. The second possibility is that $\mu \in \sigma(T)$, then $\mu \in \sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi^0(T)$, since T satisfies property (ah). Thus $a(T - \mu I) = \delta(T - \mu I) < \infty$. In the two cases, we have $a(T - \mu I) = \delta(T - \mu I) < \infty$. By [7, Corollary 4.8] we then deduce that $a(T) = \delta(T) < \infty$ and since $0 \in \sigma(T)$ then $0 \in \Pi(T)$. Hence $\Delta_+^g(T) \subset \Pi(T)$, and since the opposite inclusion holds for every operator, it then follows that property (gah) is satisfied by T. ■

Proposition 2.11. *Let $T \in L(X)$. Then T satisfies property (gah) $\Longleftrightarrow$ T satisfies property (gb) and $\sigma(T) = \sigma_a(T)$.*

Proof. Assume that T satisfies property (gah). If $\lambda \in \Delta_a^g(T)$, then $\lambda \in \Delta_+^g(T)$. Therefore $\lambda \in \Pi(T)$ and $\Delta_a^g(T) \subset \Pi(T)$. As $\Delta_a^g(T) \supset \Pi(T)$, then $\Delta_a^g(T) = \Pi(T)$, i.e. T satisfies property (gb). We then have $\Delta_+^g(T) = \Pi(T)$, $\Delta_a^g(T) = \Pi(T)$. Hence $\sigma_a(T) = \mathrm{iso}\sigma_a(T) \cup \sigma_{SBF_+^-}(T)$ and $\sigma(T) = \mathrm{iso}\sigma_a(T) \cup \sigma_{SBF_+^-}(T)$. So $\sigma(T) = \sigma_a(T)$. Conversely, assume that T satisfies property (gb) and $\sigma(T) = \sigma_a(T)$. Then $\Delta_a^g(T) = \Pi(T)$ and $\sigma(T) = \sigma_a(T)$. So $\Delta_+^g(T) = \Pi(T)$, i.e. T satisfies property (gah). ■

Corollary 2.12. *Let $T \in L(X)$. Then T satisfies property (ah) $\Longleftrightarrow$ T satisfies property (b) and $\sigma(T) = \sigma_a(T)$.*

Proof. Assume that T satisfies property (ah), then from Theorem 2.10, T satisfies property (gah), which implies from Proposition 2.11 that T satisfies property (gb) and $\sigma(T) = \sigma_a(T)$. By [5, Theorem 2.3], T satisfies property (b). Conversely, assume that T satisfies property (b) and $\sigma(T) = \sigma_a(T)$. Then $\Delta_a(T) = \Pi^0(T)$ and $\sigma(T) = \sigma_a(T)$. So $\Delta_+(T) = \Pi^0(T)$, i.e. T satisfies property (ah). ■

Remark 2.13. 1) From Corollary 2.12, if $T \in L(X)$ satisfies (ah) then it satisfies property (b). But the converse is not true in general: for example, let $T = 0 \oplus R$ be defined on the Banach space $\ell^2(\mathbf{N}) \oplus \ell^2(\mathbf{N})$, where R is the unilateral right shift operator. Then $\sigma_a(T) = C(0,1) \cup \{0\}$, $\sigma(T) = D(0,1)$, $\Pi^0(T) = \emptyset$,

$\sigma_{SF_+^-}(T) = C(0,1) \cup \{0\}$. So $\Delta_a(T) = \Pi^0(T)$, i.e. T satisfies property (b). But it does not satisfy property (ah), since $\Delta_+(T) \neq \Pi^0(T)$. Here $\sigma_a(T) \neq \sigma(T)$.

2) From Proposition 2.11, if $T \in L(X)$ satisfies property (gah) then it satisfies property (gb). However, the converse does not hold in general. For this, if we consider the operator $T = R \oplus S$ defined as in Example 2.6 then T satisfies property (gb), because $\Delta_a^g(T) = \Pi(T) = \emptyset$. But it does not satisfy property (gah), because $\Delta_+^g(T) \neq \Pi(T)$. Here $\sigma_a(T) \neq \sigma(T)$.

THEOREM 2.14. *Let $T \in L(X)$. Then T satisfies property (ah) $\Longleftrightarrow$ T satisfies property (az).*

Proof. Assume that T satisfies property (ah), that is $\Delta_+(T) = \Pi^0(T)$. As the inclusion $\Pi^0(T) \subset \Pi_a^0(T)$ is always true then $\Delta_+(T) \subset \Pi_a^0(T)$. If $\lambda \in \Pi_a^0(T)$, then $T - \lambda I$ is upper semi-Fredholm operator with $\operatorname{ind}(T - \lambda I) \leq 0$ and $\lambda \in \sigma(T)$, so that $\lambda \in \Delta_+(T)$. Hence $\Delta_+(T) = \Pi_a^0(T)$, i.e. T satisfies property (az). Conversely, assume that T satisfies property (az), that is $\Delta_+(T) = \Pi_a^0(T)$. Let $\lambda \in \Delta_+(T)$, then $\lambda \in \operatorname{iso}\sigma_a(T)$ and this implies from [9, Theorem 3.2] that $\lambda \in \operatorname{iso}\sigma(T)$. Since $T - \lambda I$ is an upper semi-Fredholm of negative index, from [1, Theorem 3.77] we have $\lambda \in \Pi^0(T)$. Thus $\Delta_+(T) \subset \Pi^0(T)$. As $\Delta_+(T) \supset \Pi^0(T)$ holds for every operator then $\Delta_+(T) = \Pi^0(T)$, i.e. T satisfies property (ah). ■

The next corollary gives a similar result to Theorem 2.14 in the case of property (gah).

COROLLARY 2.15. *Let $T \in L(X)$. Then T satisfies property (gah) $\Longleftrightarrow$ T satisfies property (gaz).*

Proof. Assume that T satisfies property (gah), then T satisfies property (ah). By Theorem 2.14, T satisfies property (az). This is equivalent from [9, Corollary 3.5] to saying that T satisfies property (gaz). Conversely, assume that T satisfies property (gaz), then T satisfies property (az). This implies that T satisfies property (ah). Hence T satisfies property (gah). ■

$$
\begin{tikzcd}
(gz) \arrow[r, "{[9]}"] \arrow[d, Leftrightarrow, "2.7"] & (z) \arrow[r, "{[9]}"] \arrow[d, Leftrightarrow, "2.7"] & (az) \arrow[d, Leftrightarrow, "{[9]}"] & \\
(gh) \arrow[d, "2.4"] & (h) \arrow[d, "2.3"] & (gaz) \arrow[d, Leftrightarrow, "2.15"] & (gz) \arrow[l, "{[9]}"'] \\
(gw) \arrow[r, "{[2]}"'] & (w) & (gah) \arrow[r, "2.11"'] \arrow[d, Leftrightarrow, "2.10"] & (gb) \arrow[d, "{[5]}"] \\
& & (ah) \arrow[r, "2.12"'] & (b)
\end{tikzcd}
$$

As a conclusion, we give a summary of the results obtained in this paper. In the previous diagram, arrows signify implications between the properties studied in this paper and other Weyl type theorems. The numbers near the arrows are references to the results in the present paper (numbers without brackets) or to the bibliography therein (numbers in square brackets).

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