МАТЕМАТИЧКИ ВЕСНИК
67, 3 (2015), 155–165
September 2015

оригинални научни рад
research paper

B-FREDHOLM SPECTRA AND RIESZ PERTURBATIONS

M. Berkani and H. Zariouh

Abstract. Let T be a bounded linear Banach space operator and let Q be a quasinilpotent one commuting with T. The main purpose of the paper is to show that we do not have $\sigma_*(T+Q) = \sigma_*(T)$ where $\sigma_* \in \{\sigma_D, \sigma_{LD}\}$, contrary to what has been announced in the proof of Lemma 3.5 from M. Amouch, Polaroid operators with SVEP and perturbations of property (gw), Mediterr. J. Math. 6 (2009), 461–470, where $\sigma_D(T)$ is the Drazin spectrum of T and $\sigma_{LD}(T)$ its left Drazin spectrum. However, under the additional hypothesis iso $\sigma_{ub}(T) = \emptyset$, the mentioned equality holds. Moreover, we study the preservation of various spectra originating from B-Fredholm theory under perturbations by Riesz operators.

1. Introduction

Recently, we have defined and studied several properties (generalized or not) in connection with Weyl-Browder type theorems, see [9,11] and when we have been interested in the study of their perturbations, see [10,13], it was necessary to consider some crucial open questions related to the ideas developed in the papers cited above; these questions are based essentially on the stability of spectra originating from B-Fredholm theory under perturbations by commuting nilpotent operators, see [12,13], and very recently they have been answered affirmatively in [21]. More precisely, it has been proved that these spectra are stable under commuting power finite rank perturbations.

Our essential aim in this paper is to show that, generally, various spectra originating from B-Fredholm theory are not preserved under commuting quasinilpotent perturbations, contrary to what has been announced in [2, Lemma 3.5], in [19, Theorem 3.15] and in the proof of [19, Theorem 3.16]. Furthermore, we study the stability of these spectra under commuting Riesz perturbations, and we show in particular that if T is a bounded Banach space operator satisfying iso $\sigma_{SF_+^-}(T) = \emptyset$ and if R is a Riesz one commuting with T then $\sigma_*(T+R) = \sigma_*(T)$; where $\sigma_* \in \{\sigma_{BW}, \sigma_{SBF_+^-}\}$.

Preliminarily, we give some definitions that will be needed later. Let $L(X)$ denote the Banach algebra of all bounded linear operators acting on a complex

2010 Mathematics Subject Classification: 47A53, 47A10, 47A11
Keywords and phrases: B-Fredholm spectrum; Riesz perturbations.

Banach space X. For $T \in L(X)$, let T^*, $\mathcal{N}(T)$, $n(T)$, $\mathcal{R}(T)$, $d(T)$, $\sigma(T)$ and $\sigma_a(T)$ denote respectively the dual, the null space, the nullity, the range, the defect, the spectrum and the approximate point spectrum of T. B-Fredholm operators were introduced in [4] as a natural generalization of Fredholm operators, and have been extensively studied in [4,5,8]. For a bounded linear operator T and a nonnegative integer n define $T_{[n]}$ to be the restriction of T to $\mathcal{R}(T^n)$ viewed as a map from $\mathcal{R}(T^n)$ into $\mathcal{R}(T^n)$ (in particular $T_{[0]} = T$). If for some integer n the range space $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is an upper (resp. a lower) semi-Fredholm operator, then T is called an *upper (resp. a lower) semi-B-Fredholm* operator. In this case the *index* $\mathrm{ind}(T)$ of T is defined as the index of the semi-Fredholm operator $T_{[n]}$, see [4,8]. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a *B-Fredholm* operator, see [4]. Recall that an operator $T \in L(X)$ is called *upper semi-Fredholm* if $\mathcal{R}(T)$ is closed and $n(T) < \infty$ and called *lower semi-Fredholm* if $d(T) < \infty$. If both $n(T)$ and $d(T)$ are finite, then T is called a *Fredholm* operator. T is called a *Weyl* operator if it is Fredholm of index 0. The *Weyl spectrum* $\sigma_W(T)$ of T is defined by $\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Weyl operator}\}$, and the *essential spectrum* $\sigma_e(T)$ of T is defined by $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Fredholm operator}\}$. Similarly the *B-Weyl spectrum* $\sigma_{BW}(T)$ and *B-Fredholm spectrum* $\sigma_{BF}(T)$ of T are defined.

Let $SF_+(X)$ be the class of all upper semi-Fredholm operators and $SF_+^-(X) = \{T \in SF_+(X) : \mathrm{ind}(T) \leq 0\}$. The *upper semi-Weyl spectrum* $\sigma_{SF_+^-}(T)$ of T is defined by $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(X)\}$. Similarly the *upper semi-B-Weyl spectrum* $\sigma_{SBF_+^-}(T)$ of T is defined.

Recall that the *ascent* $a(T)$, of an operator T, is defined by $a(T) = \inf\{n \in \mathbb{N} : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$ and the *descent* $\delta(T)$ of T, is defined by $\delta(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$, with $\inf \emptyset = \infty$. An operator $T \in L(X)$ is called *Browder* if it is Fredholm of finite ascent and descent, and is called *upper semi-Browder* if it is upper semi-Fredholm of finite ascent. The *upper semi-Browder spectrum* $\sigma_{ub}(T)$ of T is defined by $\sigma_{ub}(T) = \{\lambda \in \mathbf{C} : T - \lambda I \text{ is not upper semi-Browder}\}$, and the *Browder spectrum* $\sigma_b(T)$ of T is defined by $\sigma_b(T) = \{\lambda \in \mathbf{C} : T - \lambda I \text{ is not Browder}\}$.

According to [16], a complex number $\lambda \in \sigma(T)$ is a *pole* of the resolvent of T if $T - \lambda I$ has a finite ascent and finite descent, and in this case they are equal. Let $\Pi(T)$ denote the set of all poles of T; the *Drazin spectrum* of T is defined as $\sigma_D(T) = \sigma(T) \setminus \Pi(T)$. Following [7], a complex number $\lambda \in \sigma_a(T)$ is a *left pole* of T if $a(T - \lambda I) < \infty$ and $R(T^{a(T-\lambda I)+1})$ is closed. Let $\Pi_a(T)$ denote the set of all left poles of T; the *left Drazin spectrum* of T is defined as $\sigma_{LD}(T) = \sigma_a(T) \setminus \Pi_a(T)$.

An operator $T \in L(X)$ is said to have the *single valued extension property* at $\mu_0 \in \mathbb{C}$ (abbreviated SVEP at μ_0), if for every open neighborhood $\mathcal{U}$ of μ_0, the only analytic function $f : \mathcal{U} \to X$ which satisfies the equation $(T - \mu I)f(\mu) = 0$ for all $\mu \in \mathcal{U}$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every $\mu \in \mathbb{C}$ (see [17] for more details about this concept). Hereafter $\mathrm{iso}\, A$ denotes isolated points of a given subset A of $\mathbf{C}$.

2. Stability under Riesz perturbations

We recall from [14] that an operator $R \in L(X)$ is said to be *Riesz* if $R - \mu I$ is Fredholm for every non-zero complex μ, that is, $\pi(R)$ is quasinilpotent in the Calkin algebra $C(X) = L(X)/K(X)$ where $K(X)$ is the ideal of compact operators in $L(X)$ and π is the canonical mapping of $L(X)$ into $C(X)$. Of course compact and quasinilpotent are particular cases of Riesz operators. Now, we start the present section by some remarks about [2, Lemma 3.5] where it was established that if $T \in L(X)$ has SVEP and if $Q \in \text{Ł}(X)$ is a quasinilpotent operator commuting with T, then:

(i) $\sigma_{BW}(T+Q) = \sigma_{BW}(T) = \sigma_D(T+Q) = \sigma_D(T)$,

(ii) $\sigma_{SBF_+^-}(T+Q) = \sigma_{SBF_+^-}(T) = \sigma_{LD}(T+Q) = \sigma_{LD}(T)$,

(iii) $\sigma_{BW}((T+Q)^*) = \sigma_{BW}(T^*) = \sigma_D((T+Q)^*) = \sigma_D(T^*)$,

(iv) $\sigma_{SBF_+^-}((T+Q)^*) = \sigma_{SBF_+^-}(T^*) = \sigma_{LD}((T+Q)^*) = \sigma_{LD}(T^*)$.

However, its proof is incorrect, since it is based on the fact that $\sigma_D(T+Q) = \sigma_D(T)$ and $\sigma_{LD}(T+Q) = \sigma_{LD}(T)$. But this is not always true as we can see in Example 2.1 below. Note that the first equality of upper semi-B-Weyl spectra of statement (ii) above was also proved in [19, Theorem 3.15] for every operator $T \in L(X)$ commuting with Q. But this is also not true in general as we can see in the same exmple.

Example 2.1. Let $X = \ell^2(\mathbb{N})$, and let $B = \{e_i \mid e_i = (\delta_i^j)_{j\in\mathbb{N}}, i \in \mathbb{N}\}$ be the canonical basis of $\ell^2(\mathbb{N})$. Let E be the subspace of $\ell^2(\mathbb{N})$ generated by the set $\{e_i \mid 1 \le i \le n\}$. Let P be the orthogonal projection on E. Let S be the quasinilpotent operator defined on $\ell^2(\mathbb{N})$, by $S(x_1, x_2, x_3, \dots) = (x_2/2, x_3/3, \dots)$ for all $x = (x_1, x_2, x_3, \dots) \in \ell^2(\mathbb{N})$.

Consider the operator T defined on $X \oplus X$, by $T = 0 \oplus P$. Then T has SVEP and $\sigma_{BW}(T) = \sigma_{BF}(T) = \sigma_D(T) = \sigma_{SBF_+^-}(T) = \sigma_{LD}(T) = \emptyset$. Let $Q \in L(X \oplus X)$ the operator defined by $Q = S \oplus 0$. Then Q is a quasinilpotent operator of infinite ascent, since S is of infinite ascent, satisfying $QT = TQ = 0$. But $\sigma_{BW}(T+Q) = \sigma_{BF}(T+Q) = \sigma_D(T+Q) = \sigma_{SBF_+^-}(T+Q) = \sigma_{LD}(T+Q) = \{0\}$.

For the statements (iii) and (iv), the adjoint S^* of the operator S defined above is given by $S^*(x_1, x_2, x_3, \dots) = (0, x_1/2, x_2/3, x_3/4, \dots)$, and S^* is of infinite descent. Since $T^* = T$, we have: $\sigma_{BW}(T^*) = \sigma_{BF}(T^*) = \sigma_D(T^*) = \sigma_{SBF_+^-}(T^*) = \sigma_{LD}(T^*) = \emptyset$. But $\sigma_{BW}((T+Q)^*) = \sigma_{BF}((T+Q)^*) = \sigma_D((T+Q)^*) = \sigma_{SBF_+^-}((T+Q)^*) = \sigma_{LD}((T+Q)^*) = \{0\}$.

Before giving the correct versions (see Corollary 2.8 and Proposition 2.4 below) of [2, Lemma 3.5] and [19, Theorem 3.15], we need the following comments on B-Fredholm spectra and some auxiliary lemmas. Obviously, for every $T \in L(X)$ we know that $\sigma_{BW}(T) \subset \sigma_W(T)$, $\sigma_{SBF_+^-}(T) \subset \sigma_{SF_+^-}(T)$ and $\sigma_{BF}(T) \subset \sigma_e(T)$, but generally these inclusions are proper. Indeed, let $T = 0 \oplus R$ be defined on the Banach space $\ell^2(\mathbf{N}) \oplus \ell^2(\mathbf{N})$, where R is the right shift operator on $\ell^2(\mathbf{N})$. Then $\sigma_{SBF_+^-}(T) = C(0,1) \subsetneq \sigma_{SF_+^-}(T) = C(0,1) \cup \{0\}$, where $C(0,1)$ is the unit circle of $\mathbf{C}$. On the other hand, if we consider the operator V on $\ell^2(\mathbf{N})$ defined by $V(x_1, x_2, \dots) = (0, x_1/2, 0, 0, \dots)$, then $\sigma_{BF}(V) = \sigma_{BW}(V) = \emptyset \subsetneq \sigma_e(V) = \sigma_W(V) = \{0\}$.

So it is naturally to ask the following question: what are the defect sets $\sigma_W(T) \setminus \sigma_{BW}(T)$, $\sigma_{SF_+^-}(T) \setminus \sigma_{SBF_+^-}(T)$ and $\sigma_e(T) \setminus \sigma_{BF}(T)$? The next lemma answers this question.

LEMMA 2.2. *Let $T \in L(X)$. The following statements hold.*

(i) $\sigma_{SF_+^-}(T) = \sigma_{SBF_+^-}(T) \cup \operatorname{iso} \sigma_{SF_+^-}(T)$. *In particular, if* $\operatorname{iso} \sigma_{SF_+^-}(T) \subset \sigma_{SBF_+^-}(T)$ *then* $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$ *and* $\sigma_{BW}(T) = \sigma_W(T)$.

(ii) $\sigma_W(T) = \sigma_{BW}(T) \cup \operatorname{iso} \sigma_W(T)$ *and* $\sigma_e(T) = \sigma_{BF}(T) \cup \operatorname{iso} \sigma_e(T)$.

Proof. In order to prove the first statement and let $\lambda \in \sigma_{SF_+^-}(T) \setminus \sigma_{SBF_+^-}(T)$. Then $T - \lambda I$ is a semi-B-Fredholm operator. From the punctured neighborhood theorem for semi-B-Fredholm operators [8, Corollary 3.2], there exists $\epsilon > 0$ such that if $0 < |\mu| < \epsilon$. Then $T - \lambda I - \mu I$ is an upper semi-Fredholm operator and $\operatorname{ind}(T - \lambda I - \mu I) = \operatorname{ind}(T - \lambda I)$. Thus for every scalar z such that $0 < |z - \lambda| < \epsilon$, we have that $T - \lambda I - (z - \lambda)I = T - zI$ is an upper semi-Fredholm operator with $\operatorname{ind}(T - zI) \leq 0$. This implies that $D(\lambda, \epsilon) \setminus \{\lambda\} \cap \sigma_{SF_+^-}(T) = \emptyset$, and as $\lambda \in \sigma_{SF_+^-}(T)$, and then $\lambda \in \operatorname{iso} \sigma_{SF_+^-}(T)$. Hence $\sigma_{SF_+^-}(T) \subset \sigma_{SBF_+^-}(T) \cup \operatorname{iso} \sigma_{SF_+^-}(T)$ and since the opposite inclusion is always true, it then follows that $\sigma_{SF_+^-}(T) = \sigma_{SBF_+^-}(T) \cup \operatorname{iso} \sigma_{SF_+^-}(T)$. In particular, if $\operatorname{iso} \sigma_{SF_+^-}(T) \subset \sigma_{SBF_+^-}(T)$, then $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$.

In order to show the second equality, let $\mu \notin \sigma_{BW}(T)$ be arbitrary. Then $\mu \notin \sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$. Thus $\mu \notin \sigma_W(T)$. Hence $\sigma_{BW}(T) \supset \sigma_W(T)$ and so $\sigma_{BW}(T) = \sigma_W(T)$.

The second statement is obtained by the same arguments used in the proof of the first. ■

REMARK 2.3. We know from [6, Lemma 2.4] that if $T \in L(X)$ with $n(T) < \infty$, then T is semi-B-Fredholm (resp. B-Fredholm) $\Longleftrightarrow$ T is semi-Fredholm (resp. Fredholm). Using this fact, we have immediately $\sigma_{SF_+^-}(T) = \sigma_{SBF_+^-}(T) \cup \Omega(T)$, $\sigma_W(T) = \sigma_{BW}(T) \cup \Omega(T)$ and $\sigma_e(T) = \sigma_{BF}(T) \cup \Omega(T)$, where $\Omega(T) = \{\lambda \in \mathbf{C} : n(T - \lambda I) = \infty\}$.

The next proposition gives the correct version of [19, Theorem 3.15] and the correct version to what has been announced in the proof of [19, Theorem 3.16] where it was affirmed that the B-Weyl spectrum is preserved under commuting quasinilpotent perturbations. Observe that the operator T defined in Example 2.1 satisfies $\operatorname{iso} \sigma_{SF_+^-}(T) = \{0\}$, $\operatorname{iso} \sigma_W(T) = \{0\}$ and $\operatorname{iso} \sigma_e(T) = \{0\}$.

Proposition 2.4. *Let $T \in L(X)$ and let $R \in L(X)$ be a Riesz operator which commutes with T. The following statements hold.*

(i) *If* $\operatorname{iso}\sigma_W(T) = \emptyset$ *then* $\sigma_{BW}(T+R) = \sigma_{BW}(T)$. *Moreover, if* $\operatorname{iso}\sigma_{SF_+^-}(T) = \emptyset$ *then* $\sigma_{SBF_+^-}(T+R) = \sigma_{SBF_+^-}(T)$ *and* $\sigma_{BW}(T+R) = \sigma_{BW}(T)$.

(ii) *If* $\operatorname{iso}\sigma_e(T) = \emptyset$ *then* $\sigma_{BF}(T+R) = \sigma_{BF}(T)$.

Proof. Case 1. R is of power finite rank. From [21, Corollary 2.3] and [21, Theorem 2.8], we have $\sigma_*(T+R) = \sigma_*(T)$ for every operator T which commutes with R; where $\sigma_* \in \{\sigma_{BW}, \sigma_{BF}, \sigma_{SBF_+^-}\}$.

Case 2. R is not of power finite rank.

(i) As R is Riesz and commutes with T then from [20, Proposition 5] we know that $\sigma_W(T+R) = \sigma_W(T)$. Since $\operatorname{iso}\sigma_W(T) = \emptyset$ then from Lemma 2.2, $\sigma_{BW}(T+R) = \sigma_W(T+R) = \sigma_W(T) = \sigma_{BW}(T)$. Moreover, if $\operatorname{iso}\sigma_{SF_+^-}(T) = \emptyset$, as R is Riesz and commutes with T then from [20, Proposition 5], we have $\sigma_{SF_+^-}(T+R) = \sigma_{SF_+^-}(T)$. Again Lemma 2.2 implies that $\sigma_{SBF_+^-}(T+R) = \sigma_{SF_+^-}(T+R) = \sigma_{SF_+^-}(T) = \sigma_{SBF_+^-}(T)$.

Let us show the second equality. For this, let $\lambda \notin \sigma_{BW}(T+R)$, then $\lambda \notin \sigma_{SBF_+^-}(T+R)$. As $\sigma_{SBF_+^-}(T+R) = \sigma_{SF_+^-}(T+R)$ then $\lambda \notin \sigma_{SF_+^-}(T+R)$. Thus $\lambda \notin \sigma_W(T+R) = \sigma_W(T)$. Since $\operatorname{iso}\sigma_{SF_+^-}(T) = \emptyset$ then $\sigma_W(T) = \sigma_{BW}(T)$ (see Lemma 2.2) and therefore $\lambda \notin \sigma_{BW}(T)$. Hence $\sigma_{BW}(T) \subset \sigma_{BW}(T+R)$. By symmetry, we show that $\sigma_{BW}(T) \supset \sigma_{BW}(T+R)$. Thus $\sigma_{BW}(T+R) = \sigma_{BW}(T)$.

(ii) Since R is Riesz and commutes with T, we know that $\sigma_e(T+R) = \sigma_e(T)$. As $\operatorname{iso}\sigma_e(T) = \emptyset$ then from Lemma 2.2, $\sigma_{BF}(T+R) = \sigma_e(T+R) = \sigma_e(T) = \sigma_{BF}(T)$. ∎

Lemma 2.5. *For every operator $T \in L(X)$, we have:* $\operatorname{iso}\sigma_b(T) \subset \operatorname{iso}\sigma_{ub}(T)$ *and* $\operatorname{iso}\sigma_D(T) \subset \operatorname{iso}\sigma_{LD}(T)$.

Proof. Let $\lambda \in \operatorname{iso}\sigma_b(T)$ be arbitrary; without loss of generality we can assume that $\lambda = 0$. Then there exists $\epsilon > 0$ such that $D(0,\epsilon) \setminus \{0\} \cap \sigma_b(T) = \emptyset$. To prove that $0 \in \operatorname{iso}\sigma_{ub}(T)$, it suffices to prove that $0 \in \sigma_{ub}(T)$. Assuming otherwise, then T is upper semi-Browder, so that $a(T)$ and $n(T)$ are finite. On the other hand, for every μ such that $0 < |\mu| < \epsilon$, we have $T - \mu I$ is a Fredholm operator, in particular it is an operator of topological uniform descent, see [15], and $\delta(T-\mu I)$ is finite. From [15, Corollary 4.8] we deduce that $\delta(T)$ is also finite. Thus $a(T) = \delta(T) < \infty$ and consequently $n(T) = d(T) < \infty$. Therefore $0 \notin \sigma_b(T)$, a contradiction. Hence $\operatorname{iso}\sigma_b(T) \subset \operatorname{iso}\sigma_{ub}(T)$. The proof of second assertion goes similarly. ∎

Evidently, $\sigma_{LD}(T) \subset \sigma_{ub}(T)$ and $\sigma_D(T) \subset \sigma_b(T)$ for every $T \in L(X)$, but these inclusions are proper in general. For instance, on $\ell^2(\mathbf{N})$ we consider the operator T defined by $T(x_1, x_2, x_3, \dots) = (0, 0, x_2, x_3, \dots)$. Then $\sigma_{LD}(T) = C(0,1) \subsetneq \sigma_{ub}(T) = C(0,1) \cup \{0\}$ and $\sigma_D(T) = \sigma_b(T) = D(0,1)$, where $D(0,1)$ is the closed unit disc in $\mathbf{C}$. This shows also that the first inclusion of Lemma 2.5 is proper. On the other hand, let $U \in L(\ell^2(\mathbb{N}))$ be defined by $U(x_1, x_2, x_3, \dots) = (0, x_2, x_3, \dots)$, then $\sigma_D(U) = \emptyset \subsetneq \sigma_b(U) = \{1\}$. Remark that the second inclusion of Lemma 2.5 is also proper. For this, we consider the operator $R \oplus S$, where R is the unilateral right shift operator and S defined in Example 2.1. We then have $\operatorname{iso} \sigma_D(R \oplus S) = \emptyset$ and $\operatorname{iso} \sigma_{LD}(R \oplus S) = \{0\}$.

Thus, it is natural to ask the following question: what are exactly the defect sets $\sigma_{ub}(T) \setminus \sigma_{LD}(T)$ and $\sigma_b(T) \setminus \sigma_D(T)$? The main objective of the following lemma is to give an answer to this question.

Lemma 2.6. *Let $T \in L(X)$. We have: $\sigma_{ub}(T) = \sigma_{LD}(T) \cup \operatorname{iso} \sigma_{ub}(T)$ and $\sigma_b(T) = \sigma_D(T) \cup \operatorname{iso} \sigma_b(T)$. In particular, if $\operatorname{iso} \sigma_{ub}(T) \subset \sigma_{LD}(T)$ then $\sigma_{LD}(T) = \sigma_{ub}(T)$ and $\sigma_D(T) = \sigma_b(T)$.*

Proof. Let $\lambda \in \sigma_{ub}(T) \setminus \sigma_{LD}(T)$ be arbitrary; then $a(T - \lambda I) < \infty$, $T - \lambda I$ is an upper semi-B-Fredholm operator, and in particular it is an operator of topological uniform descent, see [8]. From [8, Corollary 3.2], there exists $\epsilon > 0$ such that $T - \lambda I - \mu I$ is an upper semi-Fredholm operator for every μ such that $0 < |\mu| < \epsilon$. Let $z \in D(\lambda, \epsilon) \setminus \{\lambda\}$; then $T - zI = T - \lambda I - (z - \lambda)I$ is an upper semi-Fredholm operator. On the other hand, since $a(T - \lambda I) < \infty$, then by [15, Corollary 4.8], we deduce that $a(T - zI) < \infty$. Thus $z \notin \sigma_{ub}(T)$ and therefore $D(\lambda, \epsilon) \setminus \{\lambda\} \cap \sigma_{ub}(T) = \emptyset$. As $\lambda \in \sigma_{ub}(T)$, then $\lambda \in \operatorname{iso} \sigma_{ub}(T)$. Hence $\sigma_{ub}(T) \subset \sigma_{LD}(T) \cup \operatorname{iso} \sigma_{ub}(T)$, and since the opposite inclusion holds for every operator, then $\sigma_{ub}(T) = \sigma_{LD}(T) \cup \operatorname{iso} \sigma_{ub}(T)$. Analogously, we obtain the second equality. In particular, if $\operatorname{iso} \sigma_{ub}(T) \subset \sigma_{LD}(T)$, then $\sigma_{LD}(T) = \sigma_{ub}(T)$ and this implies that $\sigma_D(T) = \sigma_b(T)$. Observe that in this case $\operatorname{iso} \sigma_b(T) \subset \sigma_D(T)$. ■

In the next proposition we give the correct version to what has been announced in the proof of [2, Lemma 3.5] where it was affirmed that if $T \in L(X)$ and if $Q \in L(X)$ is a quasinilpotent commuting with T, then $\sigma_D(T + Q) = \sigma_D(T)$ and $\sigma_{LD}(T + Q) = \sigma_{LD}(T)$. Observe that the operator T defined in Example 2.1 satisfies $\operatorname{iso} \sigma_{ub}(T) = \{0\}$ and $\operatorname{iso} \sigma_b(T) = \{0\}$.

Proposition 2.7. *Let $T \in L(X)$ and let $R \in L(X)$ be a Riesz operator which commutes with T. The following statements hold.*

(i) *If $\operatorname{iso} \sigma_b(T) = \emptyset$ then $\sigma_D(T + R) = \sigma_D(T)$.*

(ii) *If $\operatorname{iso} \sigma_{ub}(T) = \emptyset$ then $\sigma_{LD}(T + R) = \sigma_{LD}(T)$, and in particular $\sigma_D(T + R) = \sigma_D(T)$.*

Proof. Case 1. R is of power finite rank. From [21, Theorem 2.11], $\sigma_*(T + R) = \sigma_*(T)$ for every operator T commuting with R, where $\sigma_* \in \{\sigma_{LD}, \sigma_D\}$.

Case 2. R is not power finite rank.

(i) Since R is Riesz and commutes with T, we know from [18, Corollary 8] that $\sigma_b(T + R) = \sigma_b(T)$. As $\operatorname{iso} \sigma_b(T) = \emptyset$ then by Lemma 2.6, we obtain $\sigma_D(T + R) = \sigma_b(T + R) = \sigma_b(T) = \sigma_D(T)$.

(ii) Since R is Riesz operator and commutes with T, we know from [18, Theorem 7] that $\sigma_{ub}(T + R) = \sigma_{ub}(T)$. As $\operatorname{iso} \sigma_{ub}(T) = \emptyset$ then by Lemma

2.6 we deduce that $\sigma_{LD}(T+R) = \sigma_{ub}(T+R) = \sigma_{ub}(T) = \sigma_{LD}(T)$. Since the hypothesis $\operatorname{iso}\sigma_{ub}(T) = \emptyset$ implies from Lemma 2.5 that $\operatorname{iso}\sigma_b(T) = \emptyset$, then $\sigma_D(T+R) = \sigma_D(T)$. ■

The next corollary gives the correct version of [2, Lemma 3.5].

COROLLARY 2.8. *Let $T \in L(X)$ be an operator having SVEP and let $Q \in L(X)$ be a quasinilpotent operator which commutes with T. We have:*

If $\operatorname{iso}\sigma_b(T) = \emptyset$ then $\sigma_{BW}(T+Q) = \sigma_{BW}(T) = \sigma_D(T+Q) = \sigma_D(T)$.

Moreover, if $\operatorname{iso}\sigma_{ub}(T) = \emptyset$ then $\sigma_{SBF_+^-}(T+Q) = \sigma_{SBF_+^-}(T) = \sigma_{LD}(T+Q) = \sigma_{LD}(T)$, and in particular $\sigma_{BW}(T+Q) = \sigma_{BW}(T) = \sigma_D(T+Q) = \sigma_D(T)$.

Proof. It well known in the literature on operator theory that if T has SVEP, then $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T)$ and $\sigma_{BW}(T) = \sigma_D(T)$. On the other hand, we know from [1, Corollary 2.12] that if Q is a quasinilpotent and commutes with T, then $T+Q$ has the SVEP. So $\sigma_{SBF_+^-}(T+Q) = \sigma_{LD}(T+Q)$ and $\sigma_{BW}(T+Q) = \sigma_D(T+Q)$.

Case 1. Q is nilpotent. From [6, Theorem 3.2], we have $\sigma_D(T+Q) = \sigma_D(T)$ for every operator T which commutes with Q. Hence $\sigma_{BW}(T+Q) = \sigma_D(T+Q) = \sigma_D(T) = \sigma_{BW}(T)$. (1)

Case 2. Q is not nilpotent. Since $\operatorname{iso}\sigma_b(T) = \emptyset$, then from Proposition 2.7 we have $\sigma_D(T+Q) = \sigma_D(T)$. This proves the equality (1) mentioned above.

Moreover, if $\operatorname{iso}\sigma_{ub}(T) = \emptyset$, then Proposition 2.7 entails that $\sigma_{LD}(T+Q) = \sigma_{LD}(T)$. Hence $\sigma_{SBF_+^-}(T+Q) = \sigma_{LD}(T+Q) = \sigma_{LD}(T) = \sigma_{SBF_+^-}(T)$. Since $\operatorname{iso}\sigma_{ub}(T) = \emptyset$ implies that $\operatorname{iso}\sigma_b(T) = \emptyset$, we retrieve again the equality (1). ■

Recall that an operator $T \in L(X)$ is said to be *polaroid* if $\operatorname{iso}\sigma(T) = \Pi(T)$. It was shown in [2, Lemma 3.7] that $\Pi(T+Q) = \Pi(T)$ whenever $T \in L(X)$ has SVEP and Q is a quasinilpotent operator such that $TQ = QT$. However, the operators T and Q defined in Example 2.1 show that this result is false. Indeed, T has SVEP and $TQ = QT = 0$ and $\Pi(T) = \{0, 1\}$. But $\Pi(T+Q) = \{1\}$. Note also that it was proved in [2, Theorem 3.12] that if $T \in L(X)$ has SVEP, then T is polaroid if and only if $T+Q$ is polaroid. But its proof is incorrect, since it is based on [2, Lemma 3.7] which is not true. The following example shows that in general the property "being polaroid" is not preserved under commuting quasinilpotent perturbations.

EXAMPLE 2.9. Let V denote the Volterra operator on the Banach space $C[0,1]$ defined by $V(f)(x) = \int_0^x f(t)\,dt$ for all $f \in C[0,1]$. V is injective and quasinilpotent. Let $T = 0 \in L(C[0,1])$, then T has SVEP and $TV = VT = 0$. Moreover, T is polaroid, since $\operatorname{iso}\sigma(T) = \Pi(T) = \{0\}$. But $T+V$ is not. To see this, $\operatorname{iso}\sigma(T+V) = \operatorname{iso}\sigma(V) = \{0\}$ and since $\mathcal{R}(V^n)$ is not closed for every $n \in \mathbb{N}$, then $\sigma_D(T+V) = \{0\}$. Hence $\operatorname{iso}\sigma(T+V) \neq \Pi(T+V) = \emptyset$. So $T+V = V$ is not polaroid.

The first statement of the next corollary gives the correct version of [2, Lemma 3.7 (ii)]. Its second statement gives the correct version of [2, Theorem 3.12] and [2, Corollary 3.13].

Corollary 2.10. *Let $T \in L(X)$ and let $Q \in L(X)$ be a quasinilpotent operator which commutes with T. If* $\operatorname{iso}\sigma_b(T) = \emptyset$ *then the following statements hold.*

(i) $\Pi(T+Q) = \Pi(T)$.

(ii) *T is polaroid $\Longleftrightarrow$ $T+Q$ is polaroid. In particular $T \in \mathcal{PS}(X) \Longleftrightarrow T+Q \in \mathcal{PS}(X)$, where $\mathcal{PS}(X)$ stands for the class of polaroid operators having SVEP.*

Proof. Case 1. If Q is not nilpotent. Since Q is quasinilpotent and commutes with T, we know that $\sigma(T+Q) = \sigma(T)$. As $\operatorname{iso}\sigma_b(T) = \emptyset$, then from Proposition 2.7 we have $\Pi(T+Q) = \sigma(T+Q) \setminus \sigma_D(T+Q) = \sigma(T) \setminus \sigma_D(T) = \Pi(T)$. Hence T is polaroid $\Longleftrightarrow$ $T+Q$ is polaroid. As it was already mentioned, we have T has SVEP if and only if $T+Q$ has SVEP. Thus $T \in \mathcal{PS}(X) \Longleftrightarrow T+Q \in \mathcal{PS}(X)$.

Case 2. If Q is nilpotent. Then $\Pi(T+Q) = \Pi(T)$ for every operator T commuting with Q. Thus in this case the two statements of this corollary hold without the condition $\operatorname{iso}\sigma_b(T) = \emptyset$. ■

Recall that an operator $T \in L(X)$ is called *a-polaroid* if $\operatorname{iso}\sigma_a(T) = \Pi_a(T)$. Generally, this property "being a-polaroid" is not preserved under commuting quasinilpotent perturbations. To see this, if we consider T and Q defined in Example 2.1, then $\operatorname{iso}\sigma_a(T) = \Pi_a(T) = \{0, 1\}$, i.e., T is a-polaroid. But $T+Q$ is not, since $\operatorname{iso}\sigma_a(T+Q) = \{0, 1\} \neq \Pi_a(T+Q) = \{1\}$. Nonetheless, we give in the following corollary a sufficient condition which ensure the stability of "being a-polaroid" property under commuting quasinilpotent perturbations.

Corollary 2.11. *Let $T \in L(X)$ and let $Q \in L(X)$ be a quasinilpotent operator which commutes with T. If* $\operatorname{iso}\sigma_{ub}(T) = \emptyset$ *then the following statements hold.*

(i) $\Pi_a(T+Q) = \Pi_a(T)$.

(ii) *T is a-polaroid $\Longleftrightarrow$ $T+Q$ a-polaroid. In particular, $T \in a\mathcal{PS}(X) \Longleftrightarrow T+Q \in a\mathcal{PS}(X)$, where $a\mathcal{PS}(X)$ stands for the class of a-polaroid operators having SVEP.*

Proof. Case 1. Q is not nilpotent. Since Q is quasinilpotent and commutes with T, we know that $\sigma_a(T+Q) = \sigma_a(T)$. The assumption $\operatorname{iso}\sigma_{ub}(T) = \emptyset$ entails by Proposition 2.7 that $\Pi_a(T+Q) = \sigma_a(T+Q) \setminus \sigma_{LD}(T+Q) = \sigma_a(T) \setminus \sigma_{LD}(T) = \Pi_a(T)$. This implies that T is a-polaroid $\Longleftrightarrow$ $T+Q$ a-polaroid. Hence $T \in a\mathcal{PS}(X) \Longleftrightarrow T+Q \in a\mathcal{PS}(X)$.

Case 2. Q is nilpotent. In this case it is well known from [21, Theorem 2.11] that $\Pi_a(T+Q) = \Pi_a(T)$ for any operator T commuting with Q. Hence in this case the two statements of this corollary hold without hypothesis $\operatorname{iso}\sigma_{ub}(T) = \emptyset$. ■

We finish this paper by two remarks including crucial comments about some results announced in [2].

Remark 2.12. For $T \in L(X)$, let $E(T) = \operatorname{iso}\sigma(T) \cap \sigma_p(T)$ and let $E_a(T) = \operatorname{iso}\sigma_a(T) \cap \sigma_p(T)$, where $\sigma_p(T)$ is the point spectrum of T. Generally, the set $E(T)$ is not stable under commuting quasinilpotent perturbations even if T has

SVEP, contrary to what has been announced in [2, Lemma 3.7 (i)]. Indeed, if we consider the operators T and V defined in Example 2.9, then T has SVEP, $TV = VT = 0$. But $E_a(T) = E(T) = \{0\}$ and $E_a(T+V) = E(T+V) = \emptyset$. Moreover, if we denote by $E^0(T) = \{\lambda \in E(T) : n(T-\lambda I) < \infty\}$ and $E_a^0(T) = \{\lambda \in E_a(T) : n(T-\lambda I) < \infty\}$, then we cannot in general say that $E^0(T)$ and $E_a^0(T)$ are preserved under commuting perturbations by quasinilpotent operators. To see this, consider $T = 0$ and Q defined on $\ell^2(\mathbb{N})$ by $Q(x_1, x_2, x_3, \dots) = (x_2/2, x_3/3, \dots)$, then $E^0(T) = E_a^0(T) = \emptyset$. But $E^0(T+Q) = E_a^0(T+Q) = \{0\}$.

But there are situations which ensure the preservation of these various sets of isolated eigenvalues under commuting quasinilpotent perturbations. Let $T \in L(X)$ and let $Q \in L(X)$ be a quasinilpotent operator commuting with T. For example, if $\operatorname{iso}\sigma_a(T) = \emptyset$, then $\operatorname{iso}\sigma(T) = \emptyset$ and hence $E(T) = E^0(T) = E_a(T) = E_a^0(T) = E(T+Q) = E^0(T+Q) = E_a(T+Q) = E_a^0(T+Q) = \emptyset$. As another situation, if we restrict to a finite dimensional Banach space X, then $\sigma_p(T+Q) = \sigma_p(T) = \sigma(T) = \sigma_a(T)$. So we have obviously that $E(T) = E^0(T) = E_a(T) = E_a^0(T) = E(T+Q) = E^0(T+Q) = E_a(T+Q) = E_a^0(T+Q) = \operatorname{iso}\sigma(T)$.

Remark 2.13. 1) According to [3], an operator $T \in L(X)$ is said to satisfy property (gw) if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$ or equivalently $\sigma_a(T) = \sigma_{SBF_+^-}(T) \sqcup E(T)$ where the symbol $\sqcup$ stands for the disjoint union. It was shown in [2, Theorem 3.9] that if $T \in \mathcal{PS}(X)$ and if $Q \in L(X)$ is a quasinilpotent operator commuting with T, then $(T+Q)^*$ satisfies property (gw). However, this result remains incorrect. Indeed, let $T = 0$ and let $Q = S^*$ be defined in Example 2.1, then $T \in \mathcal{PS}(X)$, Q is quasinilpotent satisfying $TQ = 0 = QT$. But $(T+Q)^*$ does not satisfy property (gw), since $\sigma_a((T+Q)^*) = \sigma_a(T+S) = \{0\}$, $\sigma_{SBF_+^-}((T+Q)^*) = \sigma_{SBF_+^-}(T+S) = \{0\}$ and $E((T+Q)^*) = E(T+S) = \{0\}$. The mistakes in the proof of [2, Theorem 3.9] originated in [2, Lemma 3.5] and in [2, Lemma 3.7] where it is affirmed that if $T \in L(X)$ has SVEP and if $Q \in L(X)$ is a quasinilpotent operator which commutes with T, then $\sigma_{SBF_+^-}((T+Q)^*) = \sigma_{SBF_+^-}(T^*)$ and $E(T^*) = E((T+Q)^*)$. But it is easily seen that this is not true, see for example, Example 2.1 and example given in the point (1) of this remark.

2) It was also found in [2, Corollary 3.14] that if $T \in \mathcal{PS}(X)$ and if $Q \in L(X)$ is a quasinilpotent operator commuting with T, then $f((T+Q)^*)$ satisfies property (gw) for every $f \in \mathcal{H}(\sigma(T))$, where $\mathcal{H}(\sigma(T))$ denotes the set of all analytic functions on a neighborhood of $\sigma(T)$. But its proof is incorrect. Indeed, take $T = 0$ and let $Q = S^*$ defined in Example 2.1; then $T \in \mathcal{PS}(X)$, Q is quasinilpotent and commutes with T. Let $f(z) = z^p$ be the polynomial on $\mathbf{C}$, then $f((T+Q)^*) = S^p$ does not satisfy property (gw), since $\sigma_a(S^p) = \sigma_{SBF_+^-}(S^p) = \{0\}$ and $E(S^p) = \{0\}$. The mistake in the proof of [2, Corollary 3.14] originated in [2, Corollary 3.13] where it is affirmed that $T \in \mathcal{PS}(X)$ if and only if $T+Q \in \mathcal{PS}(X)$ for every quasinilpotent operator Q commuting with T. But this is not true as already mentioned in Example 2.9.

3) Recall [7] that an operator T is said to satisfy generalized Weyl's theorem

if $\sigma(T) = \sigma_{BW}(T) \sqcup E(T)$ and is said to satisfy generalized a-Weyl's theorem if $\sigma_a(T) = \sigma_{SBF_+^-}(T) \sqcup E_a(T)$. It is claimed in [2, Corollary 3.15] that if $T \in \mathcal{PS}(X)$ and if $Q \in L(X)$ is a quasinilpotent operator commuting with T, then $f((T+Q)^*)$ satisfies generalized Weyl's theorem and generalized a-Weyl's theorem for every $f \in \mathcal{H}(\sigma(T))$. But its proof is based on [2, Corollary 3.14] which is false. The example defined in the point (2) shows that the result announced in [2, Corollary 3.15] does not hold in general. Indeed, $T \in \mathcal{PS}(X)$ and $f((T+Q)^*) = S^p$ does not satisfy either generalized Weyl's theorem or generalized a-Weyl's theorem, since $\sigma_a(S^p) = \sigma(S^p) = \{0\}$, $\sigma_{BW}(S^p) = \sigma_{SBF_+^-}(S^p) = \{0\}$ and $E(S^p) = E_a(S^p) = \{0\}$.

Acknowledgement. The author would like to thank the referees for their valuable comments and suggestions on this paper.

REFERENCES

[**1**] P. Aiena, *Fredholm and Local Spectral Theory, with Application to Multipliers*, Kluwer Academic Publishers, 2004.

[**2**] M. Amouch, *Polaroid operators with SVEP and perturbations of property (gw)*, Mediterr. J. Math. **6** (2009), 461–470.

[**3**] M. Amouch, M. Berkani, *On the property* (*gw*), Mediterr. J. Math. **5**, 3 (2008), 371–378.

[**4**] M. Berkani, *On a class of quasi-Fredholm operators*, Integr. Equ. Oper. Theory **34**, 2 (1999), 244–249.

[**5**] M. Berkani, *B-Weyl spectrum and poles of the resolvent*, J. Math. Anal. Appl. **272**(2) (2002), 596–603.

[**6**] M. Berkani, M. Amouch, *Preservation of property* (*gw*) *under perturbations*, Acta Sci. Math. (Szeged) **74** (2008), 769–781.

[**7**] M. Berkani, J. J. Koliha, *Weyl type theorems for bounded linear operators*, Acta Sci. Math. (Szeged) **69** (2003), 359–376.

[**8**] M. Berkani, M. Sarih, *On semi B-Fredholm operators*, Glasgow Math. J. **43** (2001), 457–465.

[**9**] M. Berkani, H. Zariouh, *Extended Weyl type theorems*, Math. Bohemica, **134**, 4 (2009), 369–378.

[**10**] M. Berkani, H. Zariouh, *Extended Weyl type theorems and perturbations*, Math. Proc. R. Ir. Acad., **110**A (1) (2010), 73–82.

[**11**] M. Berkani, H. Zariouh, *New extended Weyl type theorems*, Mat. Vesnik, **62**, 2 (2010), 145–154.

[**12**] M. Berkani, H. Zariouh, *Generalized a-Weyl's theorem and perturbations*, Func. Anal. Approx. Comput. **2**, 1 (2010), 7–18.

[**13**] M. Berkani, H. Zariouh, *Perturbation results for Weyl type theorems*, Acta Math. Univ. Comenianae, **80**, 1 (2011), 119–132.

[**14**] S. R. Caradus, W. E. Peaffenberger, Y. Bertram, *Calkin Algebras and Algebras of Operators on Banach spaces*, Marcel Deker, New York, 1974.

[**15**] S. Grabiner, *Uniform ascent and descent of bounded operators*, J. Math. Soc. Japan, **34**, 2 (1982), 317–337.

[**16**] H. Heuser, *Functional Analysis*, John Wiley & Sons Inc, New York, 1982.

[**17**] K. B. Laursen, M. M. Neumann, *An Introduction to Local Spectral Theory*, Clarendon, Oxford, 2000.

[**18**] V. Rakočević, *Semi-Browder operators and perturbations*, Studia Math., **122** (1997), 131–137.

[**19**] M. H. M. Rashid, *Property* (*gb*) *and perturbations*, J. Math. Anal. Appl. **383** (2011), 82–94.

[**20**] H. O. Tylli, *On the asymtotic behaviour of some quantities related to semi-Fredholm operators*, J. London Math. Soc., **31** (1985), 340–348.

[**21**] Q. P. Zeng, Q. F. Jiang and H. J. Zhong, *Spectra originating from semi-B-Fredholm theory and commuting perturbations*, Studia Math. **219**, 1 (2013), 1–18.

(received 20.01.2014; in revised form 31.05.2014; available online 01.07.2014)

M.B., Department of Mathematics, Science Faculty of Oujda, University Mohammed I, Operator Theory Team, Morocco

E-mail: `berkanimo@aim.com`

H.Z., Centre régional des métiers de l'éducation et de la formation, B.P 458, Oujda, Morocco et Equipe de la Théorie des Opérateurs, Université Mohammed I, Faculté des Sciences d'Oujda, Dépt. de Mathématiques, Morocco

E-mail: `h.zariouh@yahoo.fr`