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ON STARRABLE LATTICES

Hossain Khass, Ali Reza Ashrafi and Behnam Bazigaran

Abstract. A starrable lattice is one with a cancellative semigroup structure satisfying $(x \vee y)(x \wedge y) = xy$. If the cancellative semigroup is a group, then we say that the lattice is fully starrable. In this paper, it is proved that distributivity is a strict generalization of starrability. We also show that a lattice $(X, \leq)$ is distributive if and only if there is an abelian group $(G, +)$ and an injection $f : X \to G$ such that $f(x) + f(y) = f(x \vee y) + f(x \wedge y)$ for all $x, y \in X$, while it is fully starrable if and only if there is an abelian group $(G, +)$ and a bijection $f : X \to G$ such that $f(x) + f(y) = f(x \vee y) + f(x \wedge y)$, for all $x, y \in X$.

1. Introduction

We first introduce some terminologies and notations that will be used along the paper. Let $(X, \leq)$ be a poset. The set of all *down-sets* of $(X, \leq)$ is $\{A \subseteq X \mid A = \downarrow A\}$ and will be denoted by $\mathcal{O}(X)$; where $\downarrow A = \{x \in X \mid (\exists \in A)(x \leq a)\}$. $(\mathcal{O}(X), \subseteq)$ is a sublattice of $(\mathcal{P}(X), \subseteq)$; where $\mathcal{P}(X)$ is the set of all subsets of X.

Let $(X, \leq)$ be a lattice. The set of all $\vee$-irreducible elements of X will be denoted by $J(X)$ and we assume a bottom element of X is $\vee$-irreducible. The spectrum of $a \in X$ in the lattice $(X, \leq)$ is defined by $\mathrm{spec}(a) = \{x \in J(X) \mid x \leq a\}$. If $(R, \leq)$ is a Boolean lattice and $a \in R$, the complement of a will be denoted a^c. By a linear lattice we mean a chain. For any $n \in \mathbb{N}$, D_n denotes the set of all positive divisors of n. For sets A and B, B^A is the set of all functions $f : A \to B$.

Definition 1. $(S, \leq, \star)$ is said to be a *starred lattice* if $(S, \leq)$ is a lattice, $(S, \star)$ is a cancellative semigroup, and for every $x, y \in S$,

$$(x \vee y) \star (x \wedge y) = x \star y.$$

A lattice $(X, \leq)$ is said to be *starrable* if there is a binary operation $*$ on X such that $(X, \leq, *)$ is a starred lattice. A semigroup structure on a lattice $(X, \leq)$, with binary operation $*$, is said to be *compatible* with $(X, \leq)$ if $(X, \leq, *)$ is a starred lattice.

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We know that the class of all cancellative semigroups is not a variety of algebras, because it is not closed under homomorphic images. So if we think of a starred lattice $(X, \leq, *)$ as an algebraic structure $(X, \wedge, \vee, *)$, where $\wedge$ and $\vee$ represent the meet and join operations in the lattice $(X, \leq)$, then the class of all starred lattices is not a variety of algebras. Note that any semigroup is a starred lattice with any linear order.

If $(S, \leq, *)$ is a starred lattice, $(S, *)$ is clearly commutative; and if $(X, \leq)$ is a starrable lattice then it is easy to see that $(X, \leq^{-1})$ is also starrable. In fact, if $(X, \leq, *)$ is a starred lattice then $(X, \leq^{-1}, *)$ is also a starred lattice. One of the simplest starrable lattice is the lattice $(\mathbb{N}, |)$. In fact, for every $m, n \in \mathbb{N}$, $(m, n)[m, n] = mn$.

The proofs of the following two lemmas are straightforward and omitted for brevity.

LEMMA 1. *Let $(X, \leq)$ be a lattice and $1 \in X$ be the greatest element in it. If $(X \setminus \{1\}, \leq)$ is a lattice then it is a sublattice of $(X, \leq)$.*

LEMMA 2. *Let $(S, \cdot)$ be a (cancellative) semigroup, 1 be an element with $1 \notin S$ and $*$ be a binary operation on $S \cup \{1\}$ defined by*

$$a * b = ab,\ a * 1 = 1 * a = a,\ 1 * 1 = 1$$

for every $a, b \in S$. Then $(S \cup \{1\}, *)$ is a (cancellative) semigroup.

2. Main properties of starrable lattices

In this section, the main properties of starrable lattices are investigated.

THEOREM 3. *Let $(X, \leq)$ be a lattice and $1 \in X$ be a greatest element. If $(X \setminus \{1\}, \leq)$ is a starrable lattice, then $(X, \leq)$ is starrable.*

Proof. Let $(X \setminus \{1\}, \leq, \cdot)$ be a starred lattice. For every $a, b \in X \setminus \{1\}$, define

$$a * b = ab,\ a * 1 = 1 * a = a,\ 1 * 1 = 1.$$

By Lemma 2, $(X, *)$ is a cancellative semigroup. Let $a, b \in X \setminus \{1\}$. By Lemma 1, infimum and supremum of a and b are the same in $(X, \leq)$ and $(X \setminus \{1\}, \leq)$. Now it can be verified easily that $(X, \leq, *)$ is a starred lattice. ∎

Let $(X, \leq)$ be a lattice and $0 \in X$ be the smallest element. By duality, if $(X \setminus \{0\}, \leq)$ is a starrable lattice, then $(X, \leq)$ is starrable. On the other hand, by the previous theorem, the lattice $(\mathbb{N} \cup \{0\}, |)$ is starrable, because $(\mathbb{N}, |)$ is a starrable lattice.

THEOREM 4. *Every starrable lattice is distributive.*

Proof. Let $(X, \leq, \cdot)$ be a starred lattice. Let $a, b, t \in X$ and suppose $a \vee t = b \vee t$ and $a \wedge t = b \wedge t$. It suffices to show $a = b$ (see [1, Theorem 5.1]). We have $at = (a \wedge t)(a \vee t) = (b \wedge t)(b \vee t) = bt$. Thus $a = b$. Therefore for every $a, b, t \in X$,

$$a \vee t = b \vee t,\ a \wedge t = b \wedge t \ \rightarrow\ a = b.$$

Now it is clear that $(X, \leq)$ cannot have a sublattice order-isomorphic to N_5 or M_3. This shows that $(X, \leq)$ is distributive. ∎

Let $((X_i, \leq))_{i \in I}$ be a collection of posets. The product poset $(\prod_{i \in I} X_i, \leq)$ is defined so that for every $f, g \in \prod_{i \in I} X_i$,

$$f \leq g \ \leftrightarrow \ (\forall i \in I)\,(f(i) \leq g(i))\,.$$

Let the posets $(X_i, \leq)$ be lattices, then $(\prod_{i \in I} X_i, \leq)$ is lattice with

$$(\forall i \in I)\,((f \vee g)(i) = f(i) \vee g(i))$$

and

$$(\forall i \in I)\,((f \wedge g)(i) = f(i) \wedge g(i))\,.$$

Let $((X_i, .))_{i \in I}$ be a collection of semigroups. The product semigroup $(\prod_{i \in I} X_i, \cdot)$ is defined by $(fg)(i) = f(i)g(i)$. Suppose the $(X_i, \cdot)$ are cancellative. Then $(\prod_{i \in I} X_i, \cdot)$ is cancellative.

Definition 2. $(G, \leq, \star)$ is said to be a *fully starred lattice* if $(G, \leq)$ is a lattice, $(G, \star)$ is a group, and for every $x, y \in G$,

$$(x \vee y) \star (x \wedge y) = x \star y.$$

A lattice $(X, \leq)$ is said to be *fully starrable* if there is a binary operation $*$ on X such that $(X, \leq, *)$ is a fully starred lattice. A group structure on a lattice $(X, \leq)$, with binary operation $*$, is said to be *compatible* with $(X, \leq)$ if $(X, \leq, *)$ is a fully starred lattice.

Since every finite cancellative semigroup is a group, every finite starrable lattice is fully starrable.

Theorem 5. *The product of a collection of (fully) starrable lattices is (fully) starrable.*

Proof. Let $((X, \leq, \cdot))_{i \in I}$ be a collection of (fully) starred lattices. $(\prod_{i \in I} X_i, \leq, \cdot)$ is a (fully) starred lattice. Actually, if $f, g \in \prod_{i \in I} X_i$, then for every $i \in I$,

$$\begin{aligned} ((f \vee g)(f \wedge g))\,(i) &= (f \vee g)(i)(f \wedge g)(i) \\ &= (f(i) \vee g(i))(f(i) \wedge g(i)) \\ &= f(i)g(i) = (fg)(i), \end{aligned}$$

and so $(f \vee g)(f \wedge g) = fg$. ■

As for starred lattices, we can assume that a fully starred lattice is an algebraic structure. It is clear that a subalgebra of this algebra is also a starred lattice. It can be seen easily, as in the theorem above, that a direct product of fully starred lattices is again a fully starred lattice. Also a homomorphic image of a fully starred lattice, as an algebraic structure, is also a fully starred lattice. This means that, unlike the class of all starred lattices, the class of all fully starred lattices is a variety of algebras.

It is known that the axiom of choice is equivalent to this proposition: Every nonempty set allows a group structure [3]. The following simple lemma uses the axiom of choice to prove a more general result:

Lemma 6. *Every set containing an element a, allows an abelian group structure with identity element a.*

It can be proved easily. One can also see [3] for a proof.

Theorem 7. *Every nonempty linear lattice is fully starrable. More generally, if $(X, \leq)$ is a linear lattice and $a \in X$, then there is a compatible group structure on X with identity element a.*

Proof. By Lemma 6, there is an abelian group structure on X with identity element a. For every $x, y \in X$, one of the following is satisfied:

- $x \leq y$. Then $(x \vee y)(x \wedge y) = yx = xy$.
- $y \leq x$. Then $(x \vee y)(x \wedge y) = xy$.

Therefore $(X, \leq, \cdot)$ is a fully starred lattice. ∎

Theorem 8. *Let A be a set. Then $(\mathcal{P}(A), \subseteq)$ is a fully starrable lattice.*

One can prove this theorem by proving that the function $\psi : \{0,1\}^A \to \mathcal{P}(A)$ by $\psi(f) = \{a \in A \mid f(a) = 1\}$ is an order-isomorphism. More generally, every Boolean lattice is fully starrable. In fact, if $(R, \leq)$ is a Boolean lattice, for every $a, b \in R$ we can define $a + b = (a \vee b) \wedge (a \wedge b)^c$ and we can know that $(R, +, \wedge)$ is a Boolean ring. $(R, +)$ is an abelian group and always $(a \vee b) + (a \wedge b) = a + b$.

Next we try to define direct sums for lattices. Let $((X_i, \leq))_{i \in I}$ be a collection of lattices and for each $i \in I$, let $a_i \in X_i$. The sublattice $(\bigoplus_{i \in I}(X_i, a_i), \leq)$ of $(\prod_{i \in I} X_i, \leq)$ is defined by:

$$\bigoplus_{i \in I}(X_i, a_i) = \left\{ f \in \prod_{i \in I} X_i \;\middle|\; \text{there is a finite set } J \subseteq I \text{ such that } (\forall i \in I \setminus J)(f(i) = a_i) \right\}$$

and is called a *direct sum* of the $(X_i, \leq)$.

Let $((G_i, \leq, .))_{i \in I}$ be a collection of fully starred lattices and suppose for each $i \in I$, 1_i is the identity element of the group $(G_i, .)$. Then the lattice $(\bigoplus_{i \in I}(G_i, 1_i), \leq)$ is fully starrable, since $\bigoplus_{i \in I}(G_i, 1_i)$ is a subgroup and a sublattice of $\prod_{i \in I} G_i$.

Example 1. Let $p_1, p_2, p_3, \ldots$ be the sequence of all prime numbers. For each $n \in \mathbb{N}$, let $M_{p_n} = \{p_n^k \mid k \in \mathbb{N} \cup \{0\}\}$. $(M_{p_k}, |)$ is a linear lattice and so it is fully starrable. So we can assume that $(M_{p_k}, |)$ is a fully starred lattice with identity element 1 (note that this does not need axiom of choice, because M_{p_k} is finite). Define:

$$\psi : \bigoplus_{n \in \mathbb{N}}(M_n, 1) \to \mathbb{N}, \qquad \psi(f) = \prod_{n \in \mathbb{N}} f(n).$$

Clearly ψ is well-defined and onto. Suppose $\psi(f) \mid \psi(g)$. Then

$$\prod_{n \in \mathbb{N}} f(n) \mid \prod_{n \in \mathbb{N}} g(n).$$

For each $n \in \mathbb{N}$, there are $\alpha, \beta \in \mathbb{N} \cup \{0\}$ with $f(n) = p_n^\alpha$ and $g(n) = p_n^\beta$. Clearly $\alpha \leq \beta$ and so $f(n) \mid g(n)$. Therefore $f \mid g$. This shows that ψ is one-to-one and ψ^{-1} is increasing. Let $f, g \in \bigoplus_{n\in\mathbb{N}}(M_n, 1)$ and suppose $f \mid g$. For each $n \in \mathbb{N}$, $f(n) \mid g(n)$. Thus $\psi(f) \mid \psi(g)$. This shows that ψ is increasing. We have proved that $(\mathbb{N}, |)$ is order-isomorphic to a direct sum of linear lattices and so $(\mathbb{N}, |)$ is fully starrable.

We will need the theorem below in the following section.

THEOREM 9. *A lattice is distributive if and only if it can be lattice-embedded in a (fully) starrable lattice.*

Proof. Let $(X, \leq)$ be a lattice. If $(X, \leq)$ is distributive lattice, then by [2, Theorem 119], there is a set A such that $(X, \leq)$ can be lattice-embedded in $\mathcal{P}(A)$ which is a fully starrable lattice by Theorem 8. If $(X, \leq)$ can be lattice-embedded in a (fully) starrable lattice, then by Theorem 4, it is distributive. ■

3. A non-starrable distributive lattice

By Theorem 4, every starrable lattice is distributive. In this section we show that the converse is not true and there is some distributive lattice which is not starrable.

THEOREM 10. *A lattice $(X, \leq)$ is*

1) *distributive if and only if there is an abelian group $(G, +)$ and an injection $f : X \to G$ such that for any $x, y \in X$:*
$$f(x) + f(y) = f(x \vee y) + f(x \wedge y).$$
2) *fully starrable if and only if there is an abelian group $(G, +)$ and a bijection $f : X \to G$ such that for any $x, y \in X$:*
$$f(x) + f(y) = f(x \vee y) + f(x \wedge y).$$

Proof. 1) Let $(X, \leq)$ be distributive. By Theorem 9, there is a fully starred lattice $(G, \leq, +)$ and a lattice-embedding $f : X \to G$. For every $x, y \in X$ we have:
$$f(x \vee y) + f(x \wedge y) = (f(x) \vee f(y)) + (f(x) \wedge f(y)) = f(x) + f(y).$$
Conversely, suppose there is some abelian $(G, +)$ and an injection $f : X \to G$ satisfying
$$f(x) + f(y) = f(x \vee y) + f(x \wedge y),$$
for all $x, y \in X$. Suppose $a, b, t \in X$, $a \vee t = b \vee t$ and $a \wedge t = b \wedge t$. By [1, Theorem 5.1], it suffices to show that $a = b$. We have:
$$f(a) + f(t) = f(a \vee t) + f(a \wedge t) = f(b \vee t) + f(b \wedge t) = f(b) + f(t).$$
Thus $f(a) = f(b)$, which implies that $a = b$.

2) Let $(X, \leq)$ be fully starrable. There is a group operation $+$ on X such that $(X, \leq, +)$ is a starred lattice. Let $f : X \to X$ be the identity function. For every $x, y \in X$, we have:

$$f(x) + f(y) = x + y = (x \vee y) + (x \wedge y) = f(x \vee y) + f(x \wedge y).$$

Conversely, suppose there is an abelian group $(G, +)$ and a bijection $f : X \to G$ satisfying

$$f(x) + f(y) = f(x \vee y) + f(x \wedge y).$$

Define the operator $\star$ on X by

$$x \star y = f^{-1}(f(x) + f(y)).$$

Clearly, $(X, \star)$ is a group and $f : X \to G$ is a group isomorphism. For every $x, y \in X$,

$$(x \vee y) \star (x \wedge y) = f^{-1}(f(x \vee y) + f(x \wedge y)) = f^{-1}(f(x) + f(y)) = x \star y.$$

Therefore, $(X, \leq, \star)$ is a fully starred lattice. ■

Definition 3. Let $(X, \leq)$ be a finite lattice. A *distributer* on $(X, \leq)$ is a function $f : X \to (G, +)$, where $(G, +)$ is an abelian group, such that for every $x, y \in X$,

$$f(x) + f(y) = f(x \vee y) + f(x \wedge y).$$

By the previous theorem, a lattice is distributive if and only if there is an injective distributer on it and is fully starrable if and only if there is a bijective distributer on it. If $f : (X, \leq) \to (G, +)$ is a (bijective)(injective) distributer, then clearly for any $a \in G$, $f + a$ is also a (bijective)(injective) distributer.

Let $(X, \leq)$ be a finite distributive lattice. Let $0 = \min X$ and for any $A \subseteq X$ let $A^* = A \setminus \{0\}$. For any $a \in X$, unlike [gratzer], we assume $0 \in \operatorname{spec} a$. Let $x, y \in X$, then by [2, Theorem 107], we have $\operatorname{spec}(x \vee y)^* = \operatorname{spec}(x)^* \cup \operatorname{spec}(y)^*$ and so

$$\operatorname{spec}(x \vee y) = \operatorname{spec}(x) \cup \operatorname{spec}(y). \tag{1}$$

Also because $\operatorname{spec}(x \wedge y)^* = \operatorname{spec}(x)^* \cap \operatorname{spec}(y)^*$, we have

$$\operatorname{spec}(x \wedge y) = \operatorname{spec}(x) \cap \operatorname{spec}(y). \tag{2}$$

Lemma 11. *Let $(X, \leq)$ be a finite distributive lattice and $\theta : J(X) \to (G, +)$ be a function, where $(G, +)$ is an abelian group. Then the function $f : X \to G$ defined by*

$$f(x) = \sum_{t \in \operatorname{spec}(x)} \theta(t)$$

is a distributer on $(X, \leq)$.

Proof. For every $x, y \in X$,

$$\begin{aligned} *f(x \vee y) + f(x \wedge y) &= \sum_{t \in \mathrm{spec}(x \vee y)} \theta(t) + \sum_{t \in \mathrm{spec}(x \wedge y)} \theta(t) \\ &= \sum_{t \in \mathrm{spec}(x) \cup \mathrm{spec}(y)} \theta(t) + \sum_{t \in \mathrm{spec}(x) \cap \mathrm{spec}(y)} \theta(t) \\ &= \sum_{t \in \mathrm{spec}(x)} \theta(t) + \sum_{t \in \mathrm{spec}(y)} \theta(t) = f(x) + f(y), \end{aligned}$$

in which we used (1) and (2). ∎

LEMMA 12. *Let $(X, \leq)$ be a finite distributive lattice and $f : X \to (G, +)$ be a distributer on it. There is a unique function $\theta : J(X) \to G$ such that for every $x \in X$,*

$$f(x) = \sum_{t \in \mathrm{spec}(x)} \theta(t)$$

Proof. For every $x \in J(X)$, by induction we define:

$$\theta(x) = f(x) - \sum_{\substack{t \in J(X) \\ t < x}} \theta(t).$$

Note that we assume $0 \in J(X)$ and so by definition, $\theta(0) = f(0)$. Define the function $g : X \to G$ by:

$$g(x) = \sum_{t \in \mathrm{spec}(x)} \theta(t).$$

By the previous lemma, g is a distributer on $(X, \leq)$. We need to show that $f = g$. For every $x \in J(X)$,

$$f(x) = \theta(x) + \sum_{\substack{t \in J(X) \\ t < x}} \theta(t) = \sum_{\substack{t \in J(X) \\ t \leq x}} \theta(t) = \sum_{t \in \mathrm{spec}(x)} \theta(t) = g(x). \tag{3}$$

Using well-founded induction, we show that for every $x \in X$, $f(x) = g(x)$. Suppose $x \in X$ and $f(y) = g(y)$ hold for all $y \in X$ with $y < x$. If $x \in J(X)$ then by (3), $f(x) = g(x)$. Otherwise, there are $a, b \in X$ with $x = a \vee b$, $a < x$, $b < x$ and we have:

$$\begin{aligned} f(x) &= f(a \vee b) \\ &= f(a \vee b) + f(a \wedge b) - f(a \wedge b) \\ &= f(a) + f(b) - f(a \wedge b)) \\ &= g(a) + g(b) - g(a \wedge b) \\ &= g(a \vee b) + g(a \wedge b) - g(a \wedge b) \\ &= g(a \vee b) = g(x). \end{aligned}$$

Let $\delta : J(X) \to G$ be another function such that for every $x \in X$,

$$f(x) = \sum_{t \in \mathrm{spec}(x)} \delta(t).$$

For every $x \in J(X)$, we have:

$$f(x) = \delta(x) + \sum_{\substack{t \in J(X) \\ t < x}} \delta(t)$$

and so

$$\delta(x) = f(x) - \sum_{\substack{t \in J(X) \\ t < x}} \delta(t).$$

By well-founded induction, we prove that $\delta = \theta$. Let $x \in J(X)$ and suppose for every $t \in J(X)$ with $t < x$, we have $\delta(t) = \theta(t)$. Then

$$\delta(x) = f(x) - \sum_{\substack{t \in J(X) \\ t < x}} \delta(t) = f(x) - \sum_{\substack{t \in J(X) \\ t < x}} \theta(t) = \theta(x).$$

This completes the proof. ■

Definition 4. The unique function θ in the previous theorem will be called the *foundation function* of f and will be denoted by θ_f.

The following theorem is an immediate consequence of the previous lemmas:

Theorem 13. *Let $(X, \leq)$ be a lattice and $(G, +)$ be an abelian group. There is a one-to-one correspondence between the set of all distributers of the form $f : X \to G$ and the set of all functions of the form $\theta : J(X) \to G$.*

Example 2. Let $X = \{1, 2, 3, 4, 5, 6\}$ and let $<$ be the relation $\{(1, 4), (1, 5), (2, 4), (2, 6), (3, 5), (3, 6)\}$. Then $(X, \leq)$ is a poset with Hasse diagram depicted in Figure 1.

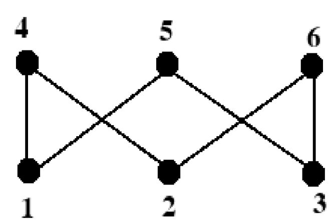

Fig. 1. The Hasse diagram of $(X, \leq)$

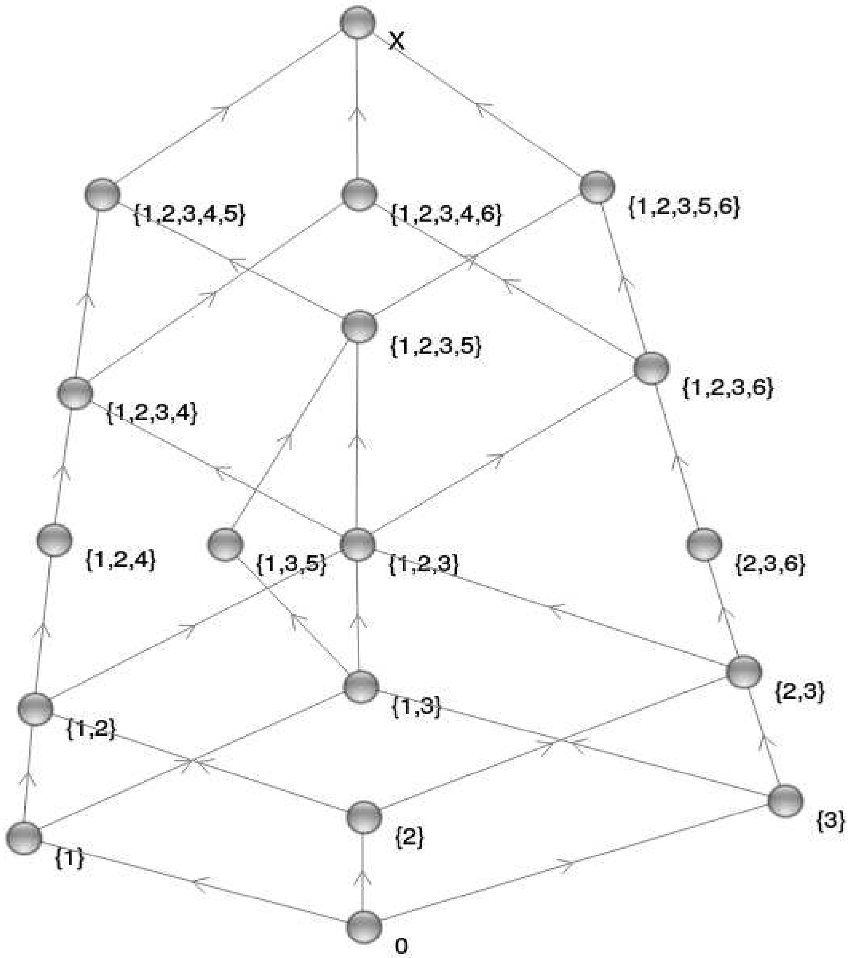

Fig. 2. The Hasse diagram of $\mathcal{O}(X)$

The set of all downsets of this poset is the lattice:

$$\begin{aligned}\mathcal{O}(X) = \{&\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\},\\ &\{1,2,3\}, \{1,2,4\}, \{1,3,5\}, \{2,3,6\}, \{1,2,3,4\}, \{1,2,3,5\}, \{1,2,3,6\},\\ &\{1,2,3,4,5\}, \{1,2,3,4,6\}, \{1,2,3,5,6\}, X\}\end{aligned}$$

which is a free lattice on three generators.

We have:

$$J(\mathcal{O}(X)) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2,4\}, \{1,3,5\}, \{2,3,6\}\}.$$

$(\mathcal{O}(X), \subseteq)$ is a distributive lattice, but it is not starrable. To prove this, we need to show that there is not any injective distributer $f : \mathcal{O}(X) \to (G, +)$, where $(G, +)$ is an abelian group with $|G| = 18$. Thus, we need to show that there are no injective distributers of the form $f : \mathcal{O}(X) \to \mathbb{Z}_{18}$ and there is no injective distributer of the form $f : \mathcal{O}(X) \to \mathbb{Z}_3 \times \mathbb{Z}_6$. To see this, we need to write a computer program which generates all possible distributers $f : \mathcal{O}(X) \to \mathbb{Z}_{18}$ and $f : \mathcal{O}(X) \to \mathbb{Z}_3 \times \mathbb{Z}_6$ by generating their foundation functions. Such a computer program can check that none of these distributers is injective. An algorithm for such a program is as follows:

- Find the set Ξ of all functions $\theta : J(\mathcal{O}(X)) \to \mathbb{Z}_{18}$.
- For each $\theta \in \Xi$, if the quantities $\sum_{\substack{T \in J(\mathcal{O}(X)) \\ T \subseteq A}} \theta(T)$ are distinct for distinct values of $A \in \mathcal{O}(X)$, then print "*An injective distributer found. The lattice* $(\mathcal{O}(X), \subseteq)$ *is starrable*" and then exit.
- Find the set Λ of all functions $\theta : J(\mathcal{O}(X)) \to \mathbb{Z}_6 \times \mathbb{Z}_3$.
- For each $\theta \in \Xi$, if the quantities $\sum_{\substack{T \in J(\mathcal{O}(X)) \\ T \subseteq A}} \theta(T)$ are distinct, for distinct values of $A \in \mathcal{O}(X)$, then print "*An injective distributer found. The lattice* $(\mathcal{O}(X), \subseteq)$ *is starrable*" and then exit.
- Print "*No injective distributer found. The lattice* $(\mathcal{O}(X), \subseteq)$ *is not starrable*".

When run, this algorithm will print "*No injective distributer found. The lattice* $(\mathcal{O}(X), \subseteq)$ *is not starrable*" which proves that $(\mathcal{O}(X), \subseteq)$ is not starrable.

We have prepared a C# console computer program based on the above algorithm for our calculations which was presented in Computational Algebra, Computational Number Theory and Applications Conference in Unversity of Kashan 2014. This program is accessible from `http://rextester.com/BQHR88905` or from the authors upon request. For the algorithm above, clearly we lose nothing if we assume always $\theta(\emptyset) = 0$. This simplification was implemented in our computer program.

Also, the algorithm can be simplified if the orbits of the action of $\mathrm{Aut}(G)$ on $G^6 = G \times G \times G \times G \times G \times G$ are known; where G is $\mathbb{Z}_{18}$ or $\mathbb{Z}_6 \times \mathbb{Z}_3$. Then the function θ in the algorithm may assume only one representative of each orbit. This can reduce the volume of calculations. However, we did not try to use this simplification in our algorithm because it is fast enough to run in a few seconds in rextester.com compiler.

4. Conclusion

We have introduced starrability and proceeded just a few steps. There are still a lot of unanswered questions and unsolved problems. For example:

- Construct an infinite class of finite non starrable distributive lattices.
- Is there an inifinite distributive lattice which is not starrable?
- Find a not starrable sublattice of a starrable lattice, if any.
- Is there a starrable lattice, which is not fully starrable?
- Is there a criterion for starrability similar to Birkhoff's distributivity criterion?

and many other similar questions. We have tried to answer some and ignored others. Also, we have tried to circumvent the computer program given above. But we have not succeeded.

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Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan 87317-51167, I. R. Iran

E-mail: ashrafi_1385@yahoo.co.in