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ON RELATIVE GORENSTEIN HOMOLOGICAL DIMENSIONS WITH RESPECT TO A DUALIZING MODULE

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Abstract. Let R be a commutative Noetherian ring. The aim of this paper is studying the properties of relative Gorenstein modules with respect to a dualizing module. It is shown that every quotient of an injective module is G_C-injective, where C is a dualizing R-module with $\mathrm{id}_R(C) \leq 1$. We also prove that if C is a dualizing module for a local integral domain, then every G_C-injective R-module is divisible. In addition, we give a characterization of dualizing modules via relative Gorenstein homological dimensions with respect to a semidualizing module.

1. Introduction

Throughout this paper R is a commutative ring and all modules are unital. The notion of a "semidualizing module" is one of the most central notion in the relative homological algebra. This notion was first introduced by Foxby [6]. Then Vasconcelos [16] and Golod [7] rediscovered these modules using different terminology for different purposes. This notion has been investigated by many authors from different points of view; see for example [1, 4, 8, 14].

Among various research areas on semidualizing modules, one basically focuses on extending the "absolute" classical notion of homological algebra to the "relative" setting with respect to a semidualizing module. For instance, this has been done for the classical and Gorenstein homological dimensions mainly through the works of Golod [7], Holm and Jørgensen [8] and White [17], and (co)homological theories have been extended to the relative setting with respect to a semidualizing module mainly through the works of Takahashi and White [14], Salimi, Tavasoli, Yassemi [11] and Salimi et al. [10].

Following this idea, the present paper aims at studying the properties of relative Gorenstein modules with respect to a dualizing module which actually strengthens the classical results. In particular, in Proposition 3.6, it is shown that every quotient of an injective module is G_C-injective, where C is a dualizing R-module

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with $\mathrm{id}_R(C) \leq 1$. We also prove that if C is a dualizing module for an integral domain, then every G_C-injective R-module is divisible, see Proposition 3.7. In addition, Theorem 3.10 is investigated whether the relative Gorenstein homological dimensions with respect to a semidualizing module have the ability to detect when a semidualizing module is dualizing. Finally, we prove that the G_C-projective dimension of a finitely generated R-module is closely related to its depth, see Theorem 3.12.

2. Preliminaries

Throughout this paper R is a commutative Noetherian ring and $\mathcal{M}(R)$ denotes the category of R-modules. We use the term "subcategory" to mean a "full, additive subcategory $\mathcal{X} \subseteq \mathcal{M}(R)$ such that, for all R-modules M and N, if $M \cong N$ and $M \in \mathcal{X}$, then $N \in \mathcal{X}$". Write $\mathcal{P}(R)$, $\mathcal{I}(R)$ and $\mathcal{F}(R)$ for the subcategories of all projective, injective and flat R-modules, respectively.

An R-complex is a sequence

$$X = \cdots \xrightarrow{\partial^X_{n+1}} X_n \xrightarrow{\partial^X_n} X_{n-1} \xrightarrow{\partial^X_{n-1}} \cdots$$

of R-modules and R-homomorphisms such that $\partial^X_{n-1}\partial^X_n = 0$ for each integer n.

Definition 2.1. Let $\mathcal{X}$ be a class of R-modules and let M be an R-module. An $\mathcal{X}$-*resolution* of M is a complex of R-modules in $\mathcal{X}$ of the form

$$X = \cdots \xrightarrow{\partial^X_2} X_1 \xrightarrow{\partial^X_1} X_0 \longrightarrow 0$$

such that $\mathrm{H}_0(X) \cong M$ and $\mathrm{H}_n(X) = 0$ for $n \geq 1$. The $\mathcal{X}$-*projective dimension* of M is the quantity

$$\mathcal{X}\text{-}\mathrm{pd}_R(M) = \inf\{\sup\{n \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

In particular, $\mathcal{X}\text{-}\mathrm{pd}_R(0) = -\infty$. The modules of $\mathcal{X}$-projective dimension zero are the non-zero modules in $\mathcal{X}$.

Dually, an $\mathcal{X}$-*coresolution* of M is a complex of R-modules in $\mathcal{X}$ of the form

$$X = 0 \longrightarrow X_0 \xrightarrow{\partial^X_0} X_{-1} \xrightarrow{\partial^X_{-1}} \cdots$$

such that $\mathrm{H}_0(X) \cong M$ and $\mathrm{H}_n(X) = 0$ for $n \leq -1$. The $\mathcal{X}$-*injective dimension* of M is the quantity

$$\mathcal{X}\text{-}\mathrm{id}_R(M) = \inf\{\sup\{n \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-coresolution of } M\}.$$

In particular, $\mathcal{X}\text{-}\mathrm{id}_R(0) = -\infty$. The modules of $\mathcal{X}$-injective dimension zero are the non-zero modules in $\mathcal{X}$.

When $\mathcal{X}$ is the class of projective R-modules we write $\mathrm{pd}_R(M)$ for the associated homological dimension and call it the projective dimension of M. Similarly, the injective dimension and flat dimension of M are denoted $\mathrm{id}_R(M)$ and $\mathrm{fd}_R(M)$ respectively.

The notion of semidualizing modules, defined next, goes back at least to Vasconcelos [16], but was rediscovered by others.

Definition 2.2. A finitely generated R-module C is called *semidualizing* if the natural homothety homomorphism $\chi_C^R : R \to \mathrm{Hom}_R(C,C)$ is an isomorphism and $\mathrm{Ext}_R^{\geq 1}(C,C)=0$. An R-module D is called *dualizing* if it is semidualizing and has finite injective dimension.

Fact 2.3 A free R-module of rank 1 is semidualizing, and indeed this is the only semidualizing module over a Gorenstein local ring.

For a semidualizing R-module C, we set

$$\mathcal{P}_C(R) = \{P \otimes_R C \mid P \text{ is a projective } R\text{-module}\},$$
$$\mathcal{F}_C(R) = \{F \otimes_R C \mid F \text{ is a flat } R\text{-module}\},$$
$$\mathcal{I}_C(R) = \{\mathrm{Hom}_R(C,I) \mid I \text{ is an injective } R\text{-module}\}.$$

The R-modules in $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$ and $\mathcal{I}_C(R)$ are called *C-projective*, *C-flat* and *C-injective*, respectively.

The next definition is due to Holm and Jørgensen [8].

Definition 2.4. Let C be a semidualizing R-module. A *complete $\mathcal{I}_C\mathcal{I}$-resolution* is a complex Y of R-modules satisfying the following:

(i) Y is exact and $\mathrm{Hom}_R(I,Y)$ is exact for each $I \in \mathcal{I}_C(R)$, and

(ii) $Y_i \in \mathcal{I}_C(R)$ for all $i \geq 0$ and Y_i is injective for all $i < 0$.

An R-module M is *G_C-injective* if there exists a complete $\mathcal{I}_C\mathcal{I}$-resolution Y such that $M \cong \mathrm{coker}(\partial_1^Y)$; in this case Y is a *complete $\mathcal{I}_C\mathcal{I}$-resolution* of M. The class of all G_C-injective R-modules is denoted by $\mathcal{GI}_C(R)$. In the case $C=R$, we use the more common terminology "complete injective resolution" and "Gorenstein injective module" and the notation $\mathcal{GI}(R)$.

A *complete $\mathcal{PP}_C$-resolution* is a complex X of R-modules such that:

(i) X is exact and $\mathrm{Hom}_R(X,P)$ is exact for each $P \in \mathcal{P}_C(R)$, and

(ii) X_i is projective for all $i \geq 0$ and $X_i \in \mathcal{P}_C(R)$ for all $i < 0$.

An R-module M is *G_C-projective* if there exists a complete $\mathcal{PP}_C$-resolution X such that $M \cong \mathrm{coker}(\partial_1^X)$; in this case X is a *complete $\mathcal{PP}_C$-resolution* of M. The class of all G_C-projective R-modules is denoted by $\mathcal{GP}_C(R)$. In the case $C=R$, we use the more common terminology "complete projective resolution" and "Gorenstein projective module" and the notation $\mathcal{GP}(R)$.

A *complete $\mathcal{FF}_C$-resolution* is a complex Z of R-modules such that:

(i) Z is exact and $Z \otimes_R I$ is exact for each $I \in \mathcal{I}_C(R)$, and

(ii) Z_i is flat for all $i \geq 0$ and $Z_i \in \mathcal{F}_C(R)$ for all $i < 0$.

An R-module M is *G_C-flat* if there exists a complete $\mathcal{FF}_C$-resolution Z such that $M \cong \mathrm{coker}(\partial_1^Z)$; in this case Z is a *complete $\mathcal{FF}_C$-resolution* of M. The class of all

G_C-flat R-modules is denoted by $\mathcal{GF}_C(R)$. In the case $C = R$, we use the more common terminology "complete flat resolution" and "Gorenstein flat module" and the notation $\mathcal{GF}(R)$.

3. Main results

In [10, Proposition 5.2] and [14, Theorem 2.11], the authors demonstrated a strong connection between the classical homological dimensions and relative homological dimensions with respect to a semidualizing R-module which are collected in the following.

FACT 3.1. Let C be a semidualizing R-module, and let M be an R-module. Then the following statements hold.

(i) $\mathcal{P}_C\text{-}\operatorname{pd}_R(M) = \operatorname{pd}_R(\operatorname{Hom}_R(C, M))$.
(ii) $\mathcal{P}_C\text{-}\operatorname{pd}_R(C \otimes_R M) = \operatorname{pd}_R(M)$.
(iii) $\mathcal{I}_C\text{-}\operatorname{id}_R(M) = \operatorname{id}_R(C \otimes_R M)$.
(iv) $\mathcal{I}_C\text{-}\operatorname{id}_R(\operatorname{Hom}_R(C, M)) = \operatorname{id}_R(M)$.
(v) $\mathcal{F}_C\text{-}\operatorname{pd}_R(M) = \operatorname{fd}_R(\operatorname{Hom}_R(C, M))$.
(vi) $\mathcal{F}_C\text{-}\operatorname{pd}_R(C \otimes_R M) = \operatorname{fd}_R(M)$.
(vii) $\mathcal{F}_C\text{-}\operatorname{pd}_R(M) \leq \mathcal{P}_C\text{-}\operatorname{pd}_R(M)$.

In [15, Proposition 2.4 and Corollary 2.5], Tang showed that in the case C is a dualizing R-module, the connection between the classical homological dimensions and relative homological dimensions with respect to C is more closed as follows.

FACT 3.2. Let C be a dualizing R-module with $\operatorname{id}_R(C) \leq n$, and let M be an R-module. Then the following statements hold.

(i) $\mathcal{F}_C\text{-}\operatorname{pd}_R(M) < \infty \Rightarrow \mathcal{P}_C\text{-}\operatorname{pd}_R(M) \leq n$.
(ii) $\mathcal{I}_C\text{-}\operatorname{id}_R(M) \leq n \Leftrightarrow \mathcal{I}_C\text{-}\operatorname{id}_R(M) < \infty \Leftrightarrow \operatorname{fd}_R(M) < \infty \Leftrightarrow \operatorname{fd}_R(M) \leq n$.
(iii) $\mathcal{F}_C\text{-}\operatorname{pd}_R(M) \leq n \Leftrightarrow \mathcal{F}_C\text{-}\operatorname{pd}_R(M) < \infty \Leftrightarrow \operatorname{id}_R(M) < \infty \Leftrightarrow \operatorname{id}_R(M) \leq n$.

Using Facts 3.1 and 3.2 we have the following result.

PROPOSITION 3.3. *Let C be a dualizing R-module with* $\operatorname{id}_R(C) \leq n$, *and let M be an R-module. Then*

(i) $\mathcal{I}_C\text{-}\operatorname{id}_R(M) < \infty \Rightarrow \operatorname{pd}_R(M) \leq n$.
(ii) $\operatorname{pd}_R(M) < \infty \Rightarrow \mathcal{I}_C\text{-}\operatorname{id}_R(M) \leq n$.
(iii) $\mathcal{P}_C\text{-}\operatorname{pd}_R(M) < \infty \Rightarrow \operatorname{id}_R(M) \leq n$.
(iv) $\operatorname{id}_R(M) < \infty \Rightarrow \mathcal{P}_C\text{-}\operatorname{pd}_R(M) \leq n$.

Proof. We just prove (i) and (ii).

(i) Let $\mathcal{I}_C\text{-}\operatorname{id}_R(M) < \infty$. Then Fact 3.2 implies that $\operatorname{fd}_R(M) \leq n$. By Fact 3.1, $\mathcal{F}_C\text{-}\operatorname{pd}_R(C \otimes_R M) \leq n$, and another use of Fact 3.2 implies that $\mathcal{P}_C\text{-}\operatorname{pd}_R(C \otimes_R M) \leq n$. Now the assertion follows from Fact 3.1.

(ii) Since $\operatorname{pd}_R(M) < \infty$, we have $\operatorname{fd}_R(M) < \infty$ and the assertion follows from Fact 3.2 ∎

In the sequel, we show that if C is a dualizing R-module, then the class of G_C-injective R-modules has nice properties as well as the class of Gorenstein modules over Gorenstein rings.

THEOREM 3.4. *Let C be a dualizing R-module with $\mathrm{id}_R(C) = n \geq 1$ and let G be an R-module. Then G is G_C-injective if and only if there exists an exact sequence*

$$G_{n-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G \longrightarrow 0,$$

where $G_{n-1}, \ldots, G_0$ are G_C-injective R-modules.

Proof. The forward implication holds by definition. For the reverse implication, we just prove the case $n = 1$. By assumption there exists a short exact sequence $(*) : 0 \to K \to G_0 \to G \to 0$ where G_0 is an G_C-injective R-module and K is an R-module. Let L be an R-module with $\mathcal{I}_C\text{-}\,\mathrm{id}_R(L) < \infty$. Then $\mathrm{pd}_R(L) \leq 1$, by Proposition 3.3. By applying the functor $\mathrm{Hom}_R(L, -)$ on the exact sequence $(*)$, we get that $\mathrm{Ext}^i_R(L, G) \cong \mathrm{Ext}^{i+1}_R(L, K)$ for all $i \geq 1$. Note that $\mathrm{Ext}^{i+1}_R(L, K) = 0$ for all $i \geq 1$, since $\mathrm{pd}_R(L) \leq 1$. So, the assertion follows from the dual of [17, Proposition 2.12]. ∎

It is known that $\mathcal{I}_C(R) \subseteq \mathcal{GI}_C(R)$ and $\mathcal{I}(R) \subseteq \mathcal{GI}_C(R)$. So we have the following result.

COROLLARY 3.5. *Let C be a dualizing R-module with $\mathrm{id}_R(C) = n \geq 1$ and let G be an R-module. Then the following statements hold.*

(i) *G is G_C-injective if and only if there exists an exact sequence*

$$\mathrm{Hom}_R(C, E_{n-1}) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(C, E_1) \longrightarrow \mathrm{Hom}_R(C, E_0) \longrightarrow G \longrightarrow 0,$$

where $E_{n-1}, \ldots, E_0$ are injective R-modules.

(ii) *If there exists an exact sequence*

$$E_{n-1} \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow G \longrightarrow 0,$$

where $E_{n-1}, \ldots, E_0$ are injective R-modules, then G is G_C-injective.

Note that the dual of Theorem 3.4 and Corollary 3.5 hold too.

PROPOSITION 3.6. *Let C be a dualizing R-module with $\mathrm{id}_R(C) \leq 1$. Then every quotient of an injective module is G_C-injective.*

Proof. Let $(*) : 0 \longrightarrow M \longrightarrow E \longrightarrow E/M \longrightarrow 0$ be a short exact sequence of R-modules such that E is injective. Let L be an R-module such that $\mathrm{pd}_R(L) < \infty$. Using Proposition 3.3, we conclude that $\mathrm{pd}_R(L) \leq 1$. By applying the functor $\mathrm{Hom}_R(L, -)$ on the sequence $(*)$, we have the following long exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(L, M) \longrightarrow \mathrm{Hom}_R(L, E) \longrightarrow \mathrm{Hom}_R(L, E/M) \longrightarrow \cdots.$$

Therefore we get $\mathrm{Ext}^i_R(L, E/M) \cong \mathrm{Ext}^{i+1}_R(L, M) = 0$ for all $i \geq 1$. By dual of [17, Proposition 2.12] and Proposition 3.3, we get the assertion. ∎

It is known that over an integral domain R, every injective R-module is divisible. In [2, Lemma 5], it is shown that over local Gorenstein integral domain R of krull dimension at most one, an R-module is Gorenstein injective if and only if it is divisible. In the following proposition we prove the relative counterpart of this result.

Proposition 3.7. *Let R be an integral domain and let C be a dualizing R-module. Then every G_C-injective R-module is divisible.*

Proof. Let M be a G_C-injective R-module and let $0 \neq r \in R$. Then $\mathrm{pd}_R(R/rR) \leq 1$. By dual of [17, Proposition 2.12] and Proposition 3.3, we have $\mathrm{Ext}^1_R(R/rR, M) = 0$. Hence $M \xrightarrow{r} M \longrightarrow 0$ is exact and therefore M is divisible. ■

It is known that in local regular rings, every module has finite homological dimensions. In [12, Corollary 3.2], it is shown that the $\mathcal{I}_C$-injective dimension and $\mathcal{P}_C$-projective dimension have the ability to detect the regularity of R, where C is a semidualizing R-module. In addition, finiteness of Gorenstein homological dimensions characterizes Gorenstein local rings as follows.

Theorem 3.8. [5, Theorem 2.19 and Corollary 3.23] *Let $(R, \mathfrak{m}, k)$ be a local ring. Then the following statements are equivalent:*

(i) *R is Gorenstein.*
(ii) $\mathrm{Gpd}_R(M) < \infty$ *for all R-modules M.*
(iii) $\mathrm{Gpd}_R(k) < \infty$.
(iv) $\mathrm{Gid}_R(M) < \infty$ *for all R-modules M.*
(v) $\mathrm{Gid}_R(k) < \infty$.

In the following theorem, we show that the relative Gorenstein homological dimensions with respect to a semidualizing module have also the ability to detect when a semidualizing module is dualizing. First, we recall the notion of trivial extension of the ring R by an R-module. If M is an R-module, then the direct sum $R \oplus M$ can be equipped with the product:

$$(a, m)(a', m') = (aa', am' + a'm),$$

where $a, a' \in R$ and $m, m' \in M$. This turns $R \oplus M$ into a ring which is called the trivial extension of R by M and denoted $R \ltimes M$. There are canonical ring homomorphisms $R \rightleftarrows R \ltimes M$, which enable us to view R-modules as $(R \ltimes M)$-modules and vice versa.

Let C be a semidualizing module. In [8], it is shown that the three G_C-dimensions always agree with the changed ring dimensions as follows.

Fact 3.9. [8, Theorem 2.16] Let C be a semidualizing R-module. The following statements hold for every R-module M.

(i) $\mathcal{GI}_C\text{-}\,\mathrm{id}_R(M) = \mathrm{Gid}_{R\ltimes C}(M)$.
(ii) $\mathcal{GP}_C\text{-}\,\mathrm{pd}_R(M) = \mathrm{Gpd}_{R\ltimes C}(M)$.
(iii) $\mathcal{GF}_C\text{-}\,\mathrm{pd}_R(M) = \mathrm{Gfd}_{R\ltimes C}(M)$.

For an R-module M, Reiten and Foxby in [6] and [9] proved that $R \ltimes M$ is Gorenstein if and only if R is Cohen-Macaulay and M is a dualizing module. Now Theorem 3.8 and Fact 3.9 imply the following result.

PROPOSITION 3.10. *Let $(R, \mathfrak{m}, k)$ be a local ring and let C be a semidualizing R-module. Then the following statements are equivalent:*

(i) *C is dualizing.*

(ii) *$\mathcal{GP}_C\text{-}\operatorname{pd}_R(M) < \infty$ for all R-modules M.*

(iii) *$\mathcal{GP}_C\text{-}\operatorname{pd}_R(k) < \infty$.*

(iv) *$\mathcal{GI}_C\text{-}\operatorname{id}_R(M) < \infty$ for all R-modules M.*

(v) *$\mathcal{GI}_C\text{-}\operatorname{id}_R(k) < \infty$.*

The projective dimension of a finitely generated R-module is closely related to its depth. This is captured by the Auslander-Buchsbaum Formula [3, Theorem 1.3.3], which states that for every finitely generated R-module M of finite projective dimension there is an equality $\operatorname{pd}_R(M) = \operatorname{depth} R - \operatorname{depth}_R M$. The Gorenstein counterpart actually strengthens the classical result; this is a recurring theme in Gorenstein homological algebra as follows.

THEOREM 3.11. [5, Theorem 1.25 and Proposition 2.16] *Let R be a local ring and let M be a finitely generated R-module with finite Gorenstein projective dimension. Then*

$$\operatorname{Gpd}_R(M) = \operatorname{depth} R - \operatorname{depth}_R M.$$

In the following theorem, we show that the G_C-projective dimension of a finitely generated R-module is also closely related to its depth.

THEOREM 3.12. *Let C be a semidualizing module for local ring R and let M be a finitely generated R-module with finite G_C-projective dimension. Then*

$$\mathcal{GP}_C\text{-}\operatorname{pd}_R(M) = \operatorname{depth} R - \operatorname{depth}_R M.$$

Proof. By Fact 3.9, we have $\mathcal{GP}_C\text{-}\operatorname{pd}_R(M) = \operatorname{Gpd}_{R\ltimes C}(M)$ and Theorem 3.11 implies that $\mathcal{GP}_C\text{-}\operatorname{pd}_R(M) = \operatorname{depth}(R \ltimes C) - \operatorname{depth}_{R\ltimes C}(M)$. Note that by [3, Exercise 1.2.26], $\operatorname{depth}_{R\ltimes C}(M) = \operatorname{depth}_R M$ and by [13, Theorem 2.2.6], $\operatorname{depth}(R \ltimes C) = \min\{\operatorname{depth} R, \operatorname{depth}_R C\} = \operatorname{depth} R$, which implies the assertion. ■

PROPOSITION 3.13. *Let R be a local ring and let C be a dualizing R-module. If M is a finitely generated R-module, then M is G_C-projective if and only if M is maximal Cohen-Macaulay.*

Proof. Note that R is Cohen-Macaulay, since R has a finitely generated module of finite injective dimension. For the forward implication, $0 = \mathcal{GP}_C\text{-}\operatorname{pd}_R(M) = \operatorname{depth} R - \operatorname{depth}_R M$. So, $\operatorname{depth}_R M = \operatorname{depth} R = \dim R$ which implies that M is maximal Cohen-Macaulay. For the reverse implication, we have $\mathcal{GP}_C\text{-}\operatorname{pd}_R(M) < \infty$ by Proposition 3.10. Now the assertion follows from Theorem 3.12. ■

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