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ARITHMETIC PROPERTIES OF 3-REGULAR BI-PARTITIONS WITH DESIGNATED SUMMANDS

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Abstract. Recently Andrews, Lewis and Lovejoy introduced the partition functions $PD(n)$ defined by the number of partitions of n with designated summands and they found several modulo 3 and 4. In this paper, we find several congruences modulo 3 and 4 for $PBD_3(n)$, which represent the number of 3-regular bi-partitions of n with designated summands. For example, for each $n \geq 1$ and $\alpha \geq 0$ $PBD_3(4 \cdot 3^{\alpha+2}n + 10 \cdot 3^{\alpha+1}) \equiv 0 \pmod 3$.

1. Introduction

In 2002 Andrews, Lewis and Lovejoy [1] introduced a new class of partitions, partitions with designated summands which are constructed by taking ordinary partitions and tagging exactly one part among parts with equal size. With a convention that $PD(n) = 0$, for example there are 15 partitions of 5 with designated summands:

$$5', \quad 4'+1', \quad 3'+2', \quad 3'+1'+1, \quad 3'+1+1', \quad 2'+2+1', \quad 2+2'+1',$$
$$2'+1'+1+1, \quad 2'+1+1'+1, \quad 2'+1+1+1', \quad 1'+1+1+1+1,$$
$$1+1'+1+1+1, \quad 1+1+1'+1+1, \quad 1+1+1+1'+1, \quad 1+1+1+1+1'.$$

The authors [1] derived the following generating function of $PD(n)$.

$$\sum_{n=0}^{\infty} PD(n)q^n = \frac{f_6}{f_1 f_2 f_3}.$$

Throughout the paper, we use the standard q-series notation, and f_k is defined as

$$f_k := (q^k; q^k)_\infty = \lim_{n\to\infty} \prod_{l=1}^{n} (1-q^{lk}).$$

For $|ab| < 1$, Ramanujan's general theta function $f(a,b)$ is defined as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{2}$$

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Using Jacobi's triple product identity [4, Entry 19, p. 35], (2) becomes

$$f(a,b) = (-a,ab)_\infty (-b,ab)_\infty (ab,ab)_\infty .$$

The most important special cases of $f(a,b)$ are

$$\psi(q) := f\left(q,q^3\right) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = f_1.$$

The concept of partitions with designated summands goes back to MacMahon [9]. He considered partitions with designated summands and with exactly ℓ different sizes (see also Andrews and Rose [2]).

Andrews et al. [1] and N. D. Baruah and K. K. Ojah [3] have also studied $PDO(n)$, the number of partitions of n with designated summands in which all parts are odd and the generating function is given by

$$\sum_{n=0}^{\infty} PDO(n)q^n = \frac{f_4 f_6^2}{f_1 f_3 f_{12}}.$$

Thus $PDO(5) = 8$ are

$$5', \quad 3'+1'+1, \quad 3'+1+1', \quad 1'+1+1+1+1, \quad 1+1'+1+1+1,$$
$$1+1+1'+1+1, \quad 1+1+1+1'+1, \quad 1+1+1+1+1'.$$

Chen, Ji, Jin and Shen [5] have established Ramanujan type identity for the partition function $PD(3n+2)$ which implies the congruence of Andrews et al. [1] and they also gave a combinatorial interpretation of the congruence for $PD(3n+2)$ by introducing a rank for partitions with designated summands. Recently Xia [14] extended the work of deriving congruence properties of $PD(n)$ by employing the generating functions of $PD(3n)$ and $PD(3n+2)$ due to Chen et al. [5].

Mahadeva Naika et al. [10] have studied $PD_3(n)$, the number of partitions of n with designated summands whose parts are not divisible by 3 and the generating function is given by

$$\sum_{n=0}^{\infty} PD_3(n)q^n = \frac{f_6^2 f_9}{f_1 f_2 f_{18}}.$$

In [11] Mahadeva Naika et al. have established many congruences for $PD_2(n)$, the number of bipartitions of n with designated summands and the generating function is given by

$$\sum_{n=0}^{\infty} PD_2(n)q^n = \frac{f_6^2}{f_1^2 f_2^2 f_3^2}.$$

Mahadeva Naika et al. [12] have derived $PD_{2,3}(n)$, the number of partitions of n with designated summands in which parts are not multiples of 2 or 3 and generating function is given by

$$\sum_{n=0}^{\infty} PD_{2,3}(n)q^n = \frac{f_4 f_6^2 f_9 f_{36}}{f_1 f_{12}^2 f_{18}^2}.$$

Motivated by the above work, in this paper, we study $PBD_3(n)$, the number of 3-regular bi-partitions of n with designated summands and the generating function is given by

$$\sum_{n=0}^{\infty} PBD_3(n)q^n = \frac{f_6^4 f_9^2}{f_1^2 f_2^2 f_{18}^2}. \tag{3}$$

To be precise by a bipartition with designated summands we mean a pair of partitions (μ, κ) where in partitions μ and κ are partitions with designated summands. Thus $PBD_3(4) = 35$ are

$(4', \emptyset)$, $(2'+2, \emptyset)$, $(2+2', \emptyset)$, $(2'+1'+1, \emptyset)$, $(2'+1+1', \emptyset)$, $(1'+1+1+1, \emptyset)$, $(1+1'+1+1, \emptyset)$, $(1+1+1'+1, \emptyset)$, $(1+1+1+1', \emptyset)$, $(2', 2')$, $(2', 1'+1)$, $(2', 1+1')$, $(1', 1'+1+1)$, $(1', 1+1'+1)$, $(1', 1+1+1')$, $(1'+1, 1'+1)$, $(1'+1, 1+1')$, $(1+1', 1'+1)$, $(1+1', 1+1')$, $(2'+1', 1')$, $(1', 2'+1')$, $(1'+1, 2')$, $(1+1', 2')$, $(1'+1+1, 1')$, $(1+1'+1, 1')$, $(1+1+1', 1')$, $(\emptyset, 4')$, $(\emptyset, 2'+2)$, $(\emptyset, 2+2')$, $(\emptyset, 2'+1'+1)$, $(\emptyset, 2'+1+1')$, $(\emptyset, 1'+1+1+1)$, $(\emptyset, 1+1'+1+1)$, $(\emptyset, 1+1+1'+1)$, $(\emptyset, 1+1+1+1')$.

In Section 3, we prove the following theorems.

Theorem 1.1. *For $n \geq 0$ we have*

$$\sum_{n=0}^{\infty} PBD_3(2n)q^n = \frac{f_3^2 f_6^6}{f_1^6 f_{18}^2} + q\frac{f_2^4 f_3^6 f_{18}^2}{f_1^8 f_6^2 f_9^2}, \tag{4}$$

$$\sum_{n=0}^{\infty} PBD_3(2n+1)q^n = 2\frac{f_2^2 f_3^4 f_6^2}{f_1^7 f_9}. \tag{5}$$

Theorem 1.2. *For each nonnegative integer n and $\alpha \geq 0$, we have*

$$PBD_3\left(4 \times 3^{\alpha+2}n + 10 \times 3^{\alpha+1}\right) \equiv 0 \pmod{3}, \tag{6}$$

$$PBD_3\left(8 \times 3^{\alpha+2}n + 8 \times 3^{\alpha+2}\right) \equiv 0 \pmod{3}, \tag{7}$$

$$PBD_3\left(2^{\alpha+3}n\right) \equiv 2^{\alpha} PBD_3(4n) \pmod{3}, \tag{8}$$

$$\sum_{n=1}^{\infty} PBD_3(4n+2)q^n \equiv \psi(q)\psi(q^3) \pmod{3}, \tag{9}$$

$$\sum_{n=1}^{\infty} PBD_3(8n+4)q^n \equiv 2\psi(q)\psi(q^3) \pmod{3}. \tag{10}$$

Theorem 1.3. *Let p be a prime with $\left(\dfrac{-3}{p}\right) = -1$. Then for any nonnegative integer α,*

$$\sum_{n=1}^{\infty} PBD_3\left(4p^{2\alpha}n + 2p^{2\alpha}\right)q^n \equiv \psi(q)\psi(q^3) \pmod{3}, \tag{11}$$

and for $n \geq 0$, $1 \leq j \leq p-1$,

$$PBD_3\left(4p^{2\alpha+1}(pn+j)+2p^{2\alpha+2}\right) \equiv 0 \pmod 3. \tag{12}$$

THEOREM 1.4. *Let p be a prime with* $\left(\frac{-3}{p}\right) = -1$. *Then for any nonnegative integer* α,

$$\sum_{n=1}^{\infty} PBD_3\left(8p^{2\alpha}n+4p^{2\alpha}\right) q^n \equiv 2\psi(q)\psi(q^3) \pmod 3, \tag{13}$$

and for $n \geq 0$, $1 \leq j \leq p-1$,

$$PBD_3\left(8p^{2\alpha+1}(pn+j)+4p^{2\alpha+2}\right) \equiv 0 \pmod 3. \tag{14}$$

THEOREM 1.5. *For each* $n \geq 0$

$$PBD_3(12n+7) \equiv 0 \pmod 4, \tag{15}$$
$$PBD_3(12n+11) \equiv 0 \pmod 4, \tag{16}$$
$$PBD_3(24n+17) \equiv 0 \pmod 4, \tag{17}$$
$$PBD_3(36n+27) \equiv 0 \pmod 4, \tag{18}$$
$$PBD_3(72n+39) \equiv 0 \pmod 4, \tag{19}$$
$$PBD_3(72n+57) \equiv 0 \pmod 4, \tag{20}$$
$$PBD_3(216n+153) \equiv 0 \pmod 4, \tag{21}$$
$$\sum\nolimits_{n=0}^{\infty} PBD_3(72n+3) \equiv 2f_1 \pmod 4, \tag{22}$$
$$\sum\nolimits_{n=0}^{\infty} PBD_3(72n+15) \equiv 2f_1f_4 \pmod 4. \tag{23}$$

THEOREM 1.6. *For any prime* $p \geq 5$, $\alpha \geq 0$ *and* $n \geq 0$, *we have*

$$\sum_{n=0}^{\infty} PBD_3\left(72p^{2\alpha}n+3p^{3\alpha}\right) q^n \equiv 2f_1 \pmod 4. \tag{24}$$

THEOREM 1.7. *For any prime* $p \geq 5$, $\alpha \geq 0$, $n \geq 0$ *and* $l = 1, 2, ...p-1$, *we have*

$$\sum_{n=0}^{\infty} PBD_3\left(72p^{2\alpha}(pn+l)+3p^{3\alpha}\right) \equiv 0 \pmod 4. \tag{25}$$

THEOREM 1.8. *If* $p \geq 5$ *is a prime such that* $\left(\frac{-4}{p}\right) = -1$. *Then for all integers* $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} PBD_3\left(72p^{2\alpha}n+15p^{2\alpha}\right) q^n \equiv 2f_1f_4 \pmod 4. \tag{26}$$

THEOREM 1.9. *Let* $p \geq 5$ *be prime and* $\left(\frac{-4}{p}\right) = -1$. *Then for all integers* $n \geq 0$ *and* $\alpha \geq 1$,

$$PBD_3\left(72p^{2\alpha}n+p^{2\alpha-1}(15p+72j)\right) \equiv 0 \pmod 4, \tag{27}$$

where $j = 1, 2, \ldots, p-1$.

THEOREM 1.10. *For each $n \geq 0$*

$$PBD_3(18n+15) \equiv 0 \pmod{6}, \tag{28}$$
$$PBD_3(18n+3) \equiv 4f_1f_3 \pmod{6}. \tag{29}$$

THEOREM 1.11. *If $p \geq 5$ is a prime such that $\left(\dfrac{-3}{p}\right) = -1$. Then for all integers $\alpha \geq 0$,*

$$\sum_{n=0}^{\infty} PBD_3\left(18p^{2\alpha}n + 3p^{2\alpha}\right) q^n \equiv 4f_1f_3 \pmod{6}. \tag{30}$$

THEOREM 1.12. *Let $p \geq 5$ be prime and $\left(\dfrac{-3}{p}\right) = -1$. Then for all integers $n \geq 0$ and $\alpha \geq 1$,*

$$PBD_3\left(18p^{2\alpha}n + p^{2\alpha-1}(3p+18j)\right) \equiv 0 \pmod{6}, \tag{31}$$

where $j = 1, 2, \ldots, p-1$.

2. Preliminaries

We list a few dissection formulas to prove our main results.

LEMMA 2.1. *[4, Corollory, p. 49] We have*

$$\psi(q) = f(q^3, q^6) + q\psi(q^9) \tag{32}$$

LEMMA 2.2. *The following 2-dissections hold:*

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{33}$$
$$\frac{f_1}{f_3^3} = \frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q\frac{f_2^3 f_{12}^6}{f_4^2 f_6^9}. \tag{34}$$

Hirschhorn, Garvan and Borwein [7] proved equation (33). Replacing q by $-q$ in (33), we obtain (34).

LEMMA 2.3. *The following 2-dissections hold:*

$$\frac{1}{f_1f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}, \tag{35}$$
$$f_1f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}. \tag{36}$$

Equation (35) was proved by Baruah and Ojah [3]. Replacing q by $-q$ in (35) and using the fact that $(-q;-q)_\infty = \dfrac{f_2^3}{f_1f_4}$, we get (36).

Lemma 2.4. *The following 3-dissection holds:*

$$f_1f_2 = \frac{f_6f_9^4}{f_3f_{18}^2} - qf_9f_{18} - 2q^2\frac{f_3f_{18}^4}{f_6f_9^2}. \tag{37}$$

One can see this identity in [8].

Lemma 2.5. *The following 2-dissections hold:*

$$\frac{f_9}{f_1} = \frac{f_{12}^3f_{18}}{f_2^2f_6f_{36}} + q\frac{f_4^2f_6f_{36}}{f_2^3f_{12}} \tag{38}$$

and

$$\frac{f_1}{f_9} = \frac{f_2f_{12}^3}{f_4f_6f_{18}^2} - q\frac{f_4f_6f_{36}^2}{f_{12}f_{18}^3}. \tag{39}$$

Lemma 2.5 was proved by Xia and Yao [13]. Replacing q by $-q$ in (38) and using the relation $(-q;-q)_\infty = \dfrac{f_2^3}{f_1f_4}$, we obtain (39).

Lemma 2.6. *[6, Theorem 2.1] For any odd prime p,*

$$\psi(q) = \sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2}). \tag{40}$$

Furthermore, $\frac{m^2+m}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}$ *for* $0 \le m \le \frac{p-3}{2}$.

Lemma 2.7. *[6, Theorem 2.2] For any prime $p \ge 5$,*

$$f_1 = \sum_{\substack{k=-\frac{p-1}{2}\\ k\neq(\pm p-1)/6}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}.$$

Furthermore, for $-(p-1)/2 \le k \le (p-1)/2$ *and* $k \neq (\pm p-1)/6$, $\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}$.

3. Proofs of main results

3.1 Proof of Theorems 1.1 and 1.2

Substituting (38) into (3), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} PBD_3(n)q^n &= \frac{f_6^4}{f_2^2f_{18}^2}\left(\frac{f_{12}^6f_{18}^2}{f_2^4f_6^2f_{36}^2} + 2q\frac{f_4^2f_{12}^2f_{18}}{f_2^5} + q^2\frac{f_4^4f_6^2f_{36}^2}{f_2^6f_{12}^2}\right)\\ &= \frac{f_6^2f_{12}^6}{f_2^6f_{36}^2} + 2q\frac{f_4^2f_6^4f_{12}^2}{f_2^7f_{18}} + q^2\frac{f_4^4f_6^6f_{36}^2}{f_2^8f_{12}^2f_{18}^2}.\end{aligned}$$

Extracting the terms involving q^{2n} and q^{2n+1} from the above equation, we obtain (4) and (5).

By the binomial theorem, it is easy to see that for positive integers k and m,

$$f_{2k}^m \equiv f_k^{2m} \pmod 2, \tag{41}$$

$$f_{3k}^m \equiv f_k^{3m} \pmod 3 \tag{42}$$

and

$$f_{2k}^{2m} \equiv f_k^{4m} \pmod 4. \tag{43}$$

Invoking (42) in (4), we find

$$\sum_{n=0}^{\infty} PBD_3(2n)q^n \equiv 1 + q\frac{f_1 f_6^6}{f_2^2 f_3^3} \pmod 3,$$

which implies that

$$\sum_{n=1}^{\infty} PBD_3(2n)q^n \equiv q\frac{f_1 f_6^6}{f_2^2 f_3^3} \pmod 3. \tag{44}$$

Employing (34) into (44), we have

$$\sum_{n=1}^{\infty} PBD_3(2n)q^n \equiv q\frac{f_4^2 f_{12}^2}{f_2 f_6} - q^2\frac{f_2 f_{12}^6}{f_4^2 f_6^3} \pmod 3. \tag{45}$$

Extracting the terms containing q^{2n+1}, dividing throughout by q and then replacing q^2 by q from (45) and using the fact that $\psi(q) = \dfrac{f_2^2}{f_1}$, we get (9).

Substituting (32) into (9), we obtain

$$\sum_{n=1}^{\infty} PBD_3(4n+2)q^n \equiv f(q^3,q^6)\psi(q^3) + q\psi(q^3)\psi(q^9) \pmod 3, \tag{46}$$

implying

$$\sum_{n=1}^{\infty} PBD_3(12n+6)q^n \equiv \psi(q)\psi(q^3) \pmod 3. \tag{47}$$

From equations (9) and (47), we get

$$PBD_3(12n+6) \equiv PBD_3(4n+2) \pmod 3. \tag{48}$$

By using mathematical induction on α in (48), we have

$$PBD_3\left(4 \times 3^{\alpha+1}n + 2 \times 3^{\alpha+1}\right) \equiv PBD_3(4n+2) \pmod 3. \tag{49}$$

Extracting the terms containing q^{3n+2} from (46) we obtain

$$PBD_3(12n+10) \equiv 0 \pmod 3. \tag{50}$$

Using (50) in (49), we find (6).

Extracting the terms containing q^{2n} and replacing q^2 by q from (45), we get

$$\sum_{n=1}^{\infty} PBD_3(4n)q^n \equiv 2q\frac{f_1 f_6^6}{f_2^2 f_3^3} \pmod 3. \tag{51}$$

Employing (34) into (51), we obtain

$$\sum_{n=1}^{\infty} PBD_3(4n)q^n \equiv 2q\frac{f_4^2 f_{12}^2}{f_2 f_6} - 2q^2\frac{f_2 f_{12}^6}{f_4^4 f_6^3} \pmod 3. \tag{52}$$

Congruence (10) is obtained by extracting the terms containing q^{2n+1} from (52) and using the fact that $\psi(q) = \dfrac{f_2^2}{f_1}$.

Substituting (32) into (10), we have

$$\sum_{n=1}^{\infty} PBD_3(8n+4)q^n \equiv 2f(q^3,q^6)\psi(q^3) + 2q\psi(q^3)\psi(q^9) \pmod 3.$$

Extracting the terms containing q^{3n+1} and q^{3n+2} from the above equation, we obtain

$$\sum_{n=1}^{\infty} PBD_3(24n+12)q^n \equiv 2\psi(q)\psi(q^3) \pmod 3 \tag{53}$$

and

$$PBD_3(24n+20) \equiv 0 \pmod 3. \tag{54}$$

In view of the congruences (10) and (53), we get

$$PBD_3(24n+12) \equiv PBD_3(8n+4) \pmod 3. \tag{55}$$

Utilizing (55) and by mathematical induction on α, we arrive at

$$PBD_3\left(8\times 3^{\alpha+1}n + 8\times 3^{\alpha+1}\right) \equiv PBD_3(8n+4) \pmod 3. \tag{56}$$

Using (54) in (56), we obtain (7).

Extracting the terms containing q^{2n} and replacing q^2 by q from (52), we have

$$\sum_{n=1}^{\infty} PBD_3(8n)q^n \equiv q\frac{f_1f_6^6}{f_2^4f_3^3} \pmod 3. \tag{57}$$

In view of the congruences (57) and (51), we obtain

$$PBD_3(8n) \equiv 2\cdot PBD_3(4n) \pmod 3. \tag{58}$$

Utilizing (58) and by mathematical induction on α, we arrive at (8). □

3.2 Proof of Theorem 1.3

Equation (9) is the $\alpha = 0$ case of (11). If we assume that (11) holds for some $\alpha \geq 0$, then, substituting (40) in (11),

$$\begin{aligned}
&\sum_{n=1}^{\infty} PBD_3\left(4p^{2\alpha}n + 2p^{2\alpha}\right)q^n \\
&\equiv \left(\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2})\right) \\
&\times \left(\sum_{m=0}^{\frac{p-3}{2}} q^{3\frac{m^2+m}{2}} f\left(q^{3\frac{p^2+(2m+1)p}{2}}, q^{3\frac{p^2-(2m+1)p}{2}}\right) + q^{3\frac{p^2-1}{8}}\psi(q^{3p^2})\right) \pmod 3.
\end{aligned} \tag{59}$$

For any odd prime p, and $0 \leq m_1, m_2 \leq (p-3)/2$, consider the congruence

$$\frac{m_1^2+m_1}{2} + 3\frac{m_2^2+m_2}{2} \equiv \frac{4p^2-4}{8} \pmod p,$$

which implies that

$$(2m_1+1)^2 + 3(2m_2+1)^2 \equiv 0 \pmod p. \tag{60}$$

Since $\left(\frac{-3}{p}\right) = -1$, the only solution of the congruence (60) is $m_1 = m_2 = \dfrac{p-1}{2}$. Therefore, equating the coefficients of $q^{pn+\frac{4p^2-4}{8}}$ from both sides of (59), dividing throughout by $q^{\frac{4p^2-4}{8}}$ and then replacing q^p by q, we obtain

$$\sum_{n=1}^{\infty} PBD_3\left(4p^{2\alpha}\left(pn+\frac{4p^2-4}{8}\right)+2p^{2\alpha}\right)q^n \equiv \psi(q^p)\psi(q^{3p}) \pmod 3. \tag{61}$$

Equating the coefficients of q^{pn} on both sides of (61) and then replacing q^p by q, we obtain

$$\sum_{n=1}^{\infty} PBD_3\left(4p^{2\alpha+2}n+2p^{2\alpha+2}\right)q^n \equiv \psi(q)\psi(q^3) \pmod 3,$$

which is the $\alpha+1$ case of (11). Extracting the terms involving q^{pn+j} for $1 \leq j \leq p-1$ in (61), we get (12). □

3.3 Proof of Theorem 1.4

Equation (10) is the $\alpha = 0$ case of (13). If we assume that (13) holds for some $\alpha \geq 0$, then, substituting (40) in (13),

$$\begin{aligned}
&\sum_{n=1}^{\infty} PBD_3\left(8p^{2\alpha}n+4p^{2\alpha}\right)q^n \\
&\equiv 2\left(\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2})\right) \qquad (62)\\
&\times\left(\sum_{m=0}^{\frac{p-3}{2}} q^{3\frac{m^2+m}{2}} f\left(q^{3\frac{p^2+(2m+1)p}{2}}, q^{3\frac{p^2-(2m+1)p}{2}}\right) + q^{3\frac{p^2-1}{8}}\psi(q^{3p^2})\right) \pmod 3.
\end{aligned}$$

For any odd prime p, and $0 \leq m_1, m_2 \leq (p-3)/2$, consider the congruence

$$\frac{m_1^2+m_1}{2} + 3\frac{m_2^2+m_2}{2} \equiv \frac{4p^2-4}{8} \pmod p,$$

which implies that

$$(2m_1+1)^2 + 3(2m_2+1)^2 \equiv 0 \pmod p. \tag{63}$$

Since $\left(\frac{-3}{p}\right) = -1$, the only solution of the congruence (63) is $m_1 = m_2 = \dfrac{p-1}{2}$. Therefore, equating the coefficients of $q^{pn+\frac{4p^2-4}{8}}$ from both sides of (62), dividing throughout by $q^{\frac{4p^2-4}{8}}$ and then replacing q^p by q, we obtain

$$\sum_{n=1}^{\infty} PBD_3\left(8p^{2\alpha}\left(pn+\frac{4p^2-4}{8}\right)+4p^{2\alpha}\right)q^n \equiv 2\psi(q^p)\psi(q^{3p}) \pmod 3. \tag{64}$$

Equating the coefficients of q^{pn} on both sides of (64) and then replacing q^p by q, we obtain

$$\sum_{n=1}^{\infty} PBD_3\left(8p^{2\alpha+2}n+4p^{2\alpha+2}\right)q^n \equiv 2\psi(q)\psi(q^3) \pmod 3,$$

which is the $\alpha+1$ case of (13). Extracting the terms involving q^{pn+j} for $1 \leq j \leq p-1$ in (64), we arrive at (14). $\square$

3.4 Proof of Theorem 1.5

Invoking (43) in (5), we find

$$\sum_{n=0}^{\infty} PBD_3(2n+1)q^n \equiv 2\frac{f_1 f_6^4}{f_2^2 f_9} \pmod{8}. \tag{65}$$

Employing (39) into (65), we obtain

$$\sum_{n=0}^{\infty} PBD_3(2n+1)q^n \equiv 2\frac{f_6^3 f_{12}^3}{f_2 f_4 f_{18}^2} - 2q\frac{f_4 f_6^5 f_{36}^2}{f_2^2 f_{12} f_{18}^3} \pmod{8}. \tag{66}$$

Extracting the terms containing q^{2n+1}, dividing throughout by q and then replacing q^2 by q from the above equation, we get

$$\sum_{n=0}^{\infty} PBD_3(4n+3)q^n \equiv 6\frac{f_2 f_3^5 f_{18}^2}{f_1^2 f_6 f_9^3} \pmod{8}, \tag{67}$$

but

$$6\frac{f_2 f_3^5 f_{18}^2}{f_1^2 f_6 f_9^3} \equiv 6\frac{f_2 f_3^5 f_9}{f_1^2 f_6} \pmod{8}. \tag{68}$$

Invoking (41) in (68), we get

$$\sum_{n=0}^{\infty} PBD_3(4n+3)q^n \equiv 2 f_3 f_6 f_9 \pmod{4}. \tag{69}$$

Congruences (15) and (16) follow by extracting the terms containing q^{3n+1} and q^{3n+2} from (69).

Extracting the terms containing q^{3n} and replacing q^3 by q from (69). we obtain

$$\sum_{n=0}^{\infty} PBD_3(12n+3)q^n \equiv 2 f_1 f_2 f_3 \pmod{4}. \tag{70}$$

Substituting (37) into (70), we find

$$\sum_{n=0}^{\infty} PBD_3(12n+3)q^n \equiv 2\frac{f_6 f_9^4}{f_{18}^2} - 2q f_3 f_9 f_{18} \pmod{4}. \tag{71}$$

Congruence (18) is obtained by extracting the terms containing q^{3n+2} from (71).

Extracting the terms containing q^{3n} and replacing q^3 by q from the above equation we arrive at

$$\sum_{n=0}^{\infty} PBD_3(36n+3)q^n \equiv 2\frac{f_2 f_3^4}{f_6^2} \pmod{4}. \tag{72}$$

Using (41) in (72), we obtain

$$\sum_{n=0}^{\infty} PBD_3(36n+3)q^n \equiv 2 f_2 \pmod{4}. \tag{73}$$

Congruences (19) and (22) follow by extracting the terms containing q^{2n} and q^{2n+1}

from (73).

Extracting the terms containing q^{3n+1}, dividing throughout by q and then replacing q^3 by q from (71), we obtain

$$\sum_{n=0}^{\infty} PBD_3(36n+15)q^n \equiv 2f_1f_3f_6 \pmod 4. \tag{74}$$

Employing (36) into (74), we find

$$\sum_{n=0}^{\infty} PBD_3(36n+15)q^n \equiv 2\frac{f_2f_8^2f_{12}^4}{f_4^2f_{24}^2} - 2q\frac{f_4^4f_6^2f_{24}^2}{f_2f_8^2f_{12}^2} \pmod 4. \tag{75}$$

Extracting the terms containing q^{2n} and then replacing q^2 by q from (75), we obtain

$$\sum_{n=0}^{\infty} PBD_3(72n+15)q^n \equiv 2\frac{f_1f_4^2f_6^4}{f_2^2f_{12}^2} \pmod 4. \tag{76}$$

Using (41) in (76) we arrive at (23).

Extracting the terms containing q^{2n} and replacing q^2 by q from (66), we get

$$\sum_{n=0}^{\infty} PBD_3(4n+1)q^n \equiv 2\frac{f_3^3f_6^3}{f_1f_2f_9^2} \pmod 8. \tag{77}$$

Using (41) in (77), we have

$$\sum_{n=0}^{\infty} PBD_3(4n+1)q^n \equiv 2\frac{f_3^3f_6^3}{f_1f_2f_{18}} \pmod 4. \tag{78}$$

Substituting (33) into (78), we arrive at

$$\sum_{n=0}^{\infty} PBD_3(4n+1)q^n \equiv 2\frac{f_4^3f_6^5}{f_2^3f_{12}f_{18}} + 2q\frac{f_6^3f_{12}^3}{f_2f_4f_{18}} \pmod 4. \tag{79}$$

Extracting the terms containing q^{2n} and replacing q^2 by q from (79), we obtain

$$\sum_{n=0}^{\infty} PBD_3(8n+1)q^n \equiv 2\frac{f_2^3f_3^5}{f_1^3f_6f_9} \pmod 4,$$

but

$$\frac{f_2^3f_3^5}{f_1^3f_6f_9} \equiv \frac{f_2^2f_3f_6}{f_1f_9} \pmod 2.$$

This yields

$$\sum_{n=0}^{\infty} PBD_3(8n+1)q^n \equiv 2\frac{f_2^2f_3f_6}{f_1f_9} \pmod 4. \tag{80}$$

Using Jacobi's triple product identity and $\psi(q) = \frac{f_2^2}{f_1}$ in (32), we arrive at

$$\frac{f_2^2}{f_1} = \frac{f_6f_9^2}{f_3f_{18}} + q\frac{f_{18}^2}{f_9}. \tag{81}$$

Employing (81) into (80), we get

$$\sum_{n=0}^{\infty} PBD_3(8n+1)q^n \equiv 2\frac{f_6^2f_9}{f_{18}} + 2q\frac{f_3f_6f_{18}^2}{f_9^2} \pmod 4. \tag{82}$$

Congruence (17) is obtained by extracting the terms containing q^{3n+2} from the above equation.

Extracting the terms containing q^{3n+1}, dividing throughout by q and then replacing q^3 by q from (82), we obtain

$$\sum_{n=0}^{\infty} PBD_3(24n+9)q^n \equiv 2\frac{f_1 f_2 f_6^2}{f_3^2} \pmod{4}. \tag{83}$$

Using (41) in (83), we have

$$\sum_{n=0}^{\infty} PBD_3(24n+9)q^n \equiv 2f_1 f_2 f_6 \pmod{4}. \tag{84}$$

Substituting (37) into (84), we obtain

$$\sum_{n=0}^{\infty} PBD_3(24n+9)q^n \equiv 2\frac{f_6^2 f_9^4}{f_3 f_{18}^2} - 2qf_6 f_9 f_{18} \pmod{4}. \tag{85}$$

Congruence (20) follows from (85) and extracting the terms containing q^{3n} and replacing q^3 by q from the above equation. we find

$$\sum_{n=0}^{\infty} PBD_3(72n+9)q^n \equiv 2\frac{f_2^2 f_3^4}{f_1 f_6^2} \pmod{4}. \tag{86}$$

Using (41) in (86), we get

$$\sum_{n=0}^{\infty} PBD_3(72n+9)q^n \equiv 2\frac{f_2^2}{f_1} \equiv 2\psi(q) \pmod{4}. \tag{87}$$

Substituting (32) into (87) and extracting the terms containing q^{3n+2}, we arrive at (21). $\square$

3.5 Proof of Theorem 1.6

Employing Lemma (2.7) into (22), it can be see that

$$\sum_{n=0}^{\infty} PBD_3\left(72\left(pn+\frac{p^2-1}{24}\right)+3\right)q^n \equiv 2f_p \pmod{4}, \tag{88}$$

which implies that

$$\sum_{n=0}^{\infty} PBD_3\left(72p^2 n+3p^3\right)q^n \equiv 2f_1 \pmod{4}.$$

Therefore, $PBD_3\left(72p^2n+3p^3\right) \equiv PBD_3(72n+3) \pmod{4}$.

Using the above relation and by induction on α, we arrive at (24). $\square$

3.6 Proof of Theorem 1.7

Combining (88) with Theorem (1.6), we derive that for $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} PBD_3\left(72p^{2\alpha+1}n+3p^{3\alpha}\right) \equiv 2f_p \pmod{4}.$$

Therefore, it follows that

$$\sum_{n=0}^{\infty} PBD_3\left(72p^{2\alpha+1}(pn+l)+3p^{3\alpha}\right) \equiv 0 \pmod 4.$$

where $l = 1, 2, ..., p-1$, and we obtain (25). $\square$

3.7 Proof of Theorem 1.8

For a prime $p \geq 5$ and $-(p-1)/2 \leq k, m \leq (p-1)/2$, consider

$$\frac{3k^2+k}{2} + 4 \times \frac{3m^2+m}{2} \equiv \frac{5p^2-5}{24} \pmod p.$$

This is equivalent to $(6k+1)^2 + 4(6m+1)^2 \equiv 0 \pmod p$. Since $\left(\frac{-4}{p}\right) = -1$, the only solution of the above congruence is $k = m = (\pm p - 1)/6$. Therefore, from Lemma 2.7,

$$\sum_{n=0}^{\infty} PBD_3\left(72\left(p^2 n + 5 \times \frac{p^2-1}{24}\right) + 15\right) q^n \equiv 2f_1 f_4 \pmod 4. \tag{89}$$

Using (23), (89), and induction on α, we get (26). $\square$

3.8 Proof of Theorem 1.9

From Lemma 2.7 and Theorem 1.8, for each $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} PBD_3\left(72\left(p^2 n + 5 \times \frac{p^2-1}{24}\right) + 15\right) q^n \equiv 2f_1 f_4 \pmod 4.$$

That is,

$$\sum_{n=0}^{\infty} PBD_3\left(72p^{2\alpha+1}n + 15p^{2\alpha+2}\right) q^n \equiv 2f_p f_{4p} \pmod 4. \tag{90}$$

Since there are no terms on the right of (90) where the powers of q are congruent to $1, 2, \ldots, p-1$ modulo p,

$$PBD_3\left(72p^{2\alpha+1}(pn+j) + 15p^{2\alpha+2}\right) \equiv 0 \pmod 4,$$

for $j = 1, 2, \ldots, p-1$. Therefore, for $j = 1, 2, \ldots, p-1$ and $\alpha \geq 1$, we arrive at (27). $\square$

3.9 Proof of Theorem 1.10

By the binomial theorem, it is easy to see that for positive integers k and m,

$$f_{3k}^{3m} \equiv f_k^{9m} \pmod 9, \tag{91}$$

Invoking (91) in (5), we have

$$\sum_{n=0}^{\infty} PBD_3(2n+1) q^n \equiv 2\frac{f_1^2 f_2^2 f_3 f_6^2}{f_9} \pmod{18}. \tag{92}$$

Employing (37) into (92) and extracting the terms containing q^{3n+1}, dividing throughout by q and then replacing q^3 by q from (92), we obtain

$$\sum_{n=0}^{\infty} PBD_3(6n+3)q^n \equiv 14\frac{f_2^3 f_3^4}{f_6} + 8q\frac{f_1^3 f_6^8}{f_3^5} \pmod{18}. \tag{93}$$

Invoking (42) in (93), we see that

$$\sum_{n=0}^{\infty} PBD_3(6n+3)q^n \equiv 4f_3^4 + 4q\frac{f_6^8}{f_3^4} \pmod{6}. \tag{94}$$

Congruence (28) follows by extracting the terms containing q^{3n+2} from the above equation.

Extracting the terms containing q^{3n} and replacing q^3 by q from (94), we arrive at

$$\sum_{n=0}^{\infty} PBD_3(18n+3)q^n \equiv 4f_1^4 \pmod{6},$$

which implies

$$\sum_{n=0}^{\infty} PBD_3(18n+3)q^n \equiv 4f_1 f_1^3 \pmod{6}. \tag{95}$$

Invoking (42) in (95) we get (29). □

3.10 Proof of Theorem 1.11

For a prime $p \geq 5$ and $-(p-1)/2 \leq k, m \leq (p-1)/2$, consider

$$\frac{3k^2+k}{2} + 3 \times \frac{3m^2+m}{2} \equiv \frac{4p^2-4}{24} \pmod{p}.$$

This is equivalent to $(6k+1)^2 + 3(6m+1)^2 \equiv 0 \pmod{p}$.

Since $\left(\frac{-3}{p}\right) = -1$, the only solution of the above congruence is $k = m = (\pm p - 1)/6$. Therefore, from Lemma 2.7,

$$\sum_{n=0}^{\infty} PBD_3\left(18\left(p^2 n + 4 \times \frac{p^2-1}{24}\right) + 3\right) q^n \equiv 4f_1 f_3 \pmod{6}. \tag{96}$$

Using (29), (96), and induction on α, we arrive at (30). □

3.11 Proof of Theorem 1.12

From Lemma 2.7 and Theorem 1.11, for each $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} PBD_3\left(18\left(p^2 n + 4 \times \frac{p^2-1}{24}\right) + 3\right) q^n \equiv 4f_1 f_3 \pmod{6}.$$

That is,

$$\sum_{n=0}^{\infty} PBD_3\left(18p^{2\alpha+1}n + 3p^{2\alpha+2}\right) q^n \equiv 4f_p f_{3p} \pmod{6}. \tag{97}$$

Since there are no terms on the right of (97) where the powers of q are congruent to $1, 2, \ldots, p-1$ modulo p,

$$PBD_3\left(18p^{2\alpha+1}(pn+j) + 3p^{2\alpha+2}\right) \equiv 0 \pmod{6},$$

for $j = 1, 2, \ldots, p-1$. Therefore, for $j = 1, 2, \ldots, p-1$ and $\alpha \geq 1$, we obtain (31). $\square$

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