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# A Non-quasiconvex Subgroup of a Hyperbolic Group with an Exotic Limit Set

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Abstract. We construct an example of a torsion free freely indecomposable finitely presented non-quasiconvex subgroup $H$ of a word hyperbolic group $G$ such that the limit set of $H$ is not the limit set of a quasiconvex subgroup of $G$. In particular, this gives a counterexample to the conjecture of G. Swarup that a finitely presented one-ended subgroup of a word hyperbolic group is quasiconvex if and only if it has finite index in its virtual normalizer.

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## 1. Introduction

A subgroup $H$ of a word hyperbolic group $G$ is *quasiconvex* (*or rational*) in $G$ if for any finite generating set $A$ of $G$ there is $\epsilon > 0$ such that every geodesic in the Cayley graph $\Gamma(G, A)$ of $G$ with both endpoints in $H$ is contained in $\epsilon$-neighborhood of $H$. The notion of a quasiconvex subgroup corresponds, roughly speaking, to that of geometric finiteness in the theory of classical hyperbolic groups (see [Swa], [KS], [Pi]). Quasiconvex subgroups of word hyperbolic groups are finitely presentable and word hyperbolic and their finite intersections are again quasiconvex. Non-quasiconvex finitely generated subgroups of word hyperbolic groups are quite rare and there are very few examples of them. We know only three basic examples of this sort. The first is based on a remarkable construction of E. Rips [R], which allows one, given an arbitrary finitely presented group $Q$, to construct a word hyperbolic group $G$ and a two-generator subgroup $H$ of $G$ such that $H$ is normal in $G$ and the quotient is isomorphic to $Q$. The second example is based on the existence of a closed hyperbolic 3-manifold fibering over a circle, provided by results of W. Thurston and T. Jorgensen. The third example is obtained using the result

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of M. Bestvina and M. Feign [BF] who proved that if $F$ is a non-abelian free group of finite rank and $\phi$ is an automorphism of $F$ without periodic conjugacy classes then the HNN-extension of $F$ along $\phi$ is word hyperbolic.

If $G$ is a word hyperbolic group then we denote the *boundary* of $G$ (see [Gr], [GH] and [CDP]) by $\partial G$. For a subgroup $H$ of $G$ the *limit set* $\partial_G(H)$ of $H$ is the set of all limits in $\partial G$ of sequences of elements of $H$.

In this note we construct an example of a non-quasiconvex finitely presented one-ended subgroup $H$ of a word hyperbolic group $G$ such that the limit set of $H$ is *exotic*. By exotic we mean that the limit set of $H$ is not the limit set of a quasiconvex subgroup of $G$. This result is of some interest since in the previously known examples non-quasiconvex subgroups were normal in the ambient hyperbolic groups and thus (see [KS]) had the same limit sets. Our subgroup $H$ also provides a counter-example to the conjecture of G. A. Swarup [Swa] which stated that a finitely presented freely indecomposable subgroup of a torsion-free word hyperbolic group is quasiconvex if and only if it has finite index in its virtual normalizer (this statement was known to be true for 3-dimensional Kleinian groups). The subgroup $H$, constructed here, coincides with its virtual normalizer. Here, by the virtual normalizer of a subgroup $H$ of a group $G$ we mean the subgroup

$$VN_G(H) = \{g \in G \mid |H : H \cap gHg^{-1}| < \infty,\ |gHg^{-1} : H \cap gHg^{-1}| < \infty\}.$$

## 2. Some Definitions and Notations

A *geodesic* in a metric space $(X, d)$ is an isometric embedding $\alpha : [0, l] \to X$ where $l \geq 0$ and $[0, l]$ is a segment of the real line. We say that a metric space $(X, d)$ is *geodesic* if any two points of $X$ can be joined by a geodesic path in $X$. A path $\beta : [0, l] \to X$ is called $\lambda$-*quasigeodesic* if it is parametrized by its arclength and for any $t_1, t_2 \in [0, l]$

$$|t_1 - t_2| \leq \lambda \cdot d(\beta(t_1), \beta(t_2)) + \lambda.$$

If $x, y$ and $z$ are points in a metric space $(X, d)$ we set

$$(x, y)_z = \frac{1}{2}(d(z, x) + d(z, y) - d(x, y)).$$

The quantity $(x, y)_z$ is called the *Gromov inner product* of $x$ and $y$ with respect to $z$.

Let $\Delta$ be a triangle in a metric space $(X, d)$ with geodesic sides $\alpha$, $\beta$ and $\gamma$ and vertices $x, y, z$. (See Figure 1.)

We say that the points $p, q, r$ on $\alpha$, $\beta$ and $\gamma$ are the vertices of the *inscribed triangle* for $\Delta$ if $d(x, p) = d(x, r) = (y, z)_x$, $d(y, p) = d(y, q) = (x, z)_y$ and $d(z, r) = d(z, q) = (x, y)_z$. In this situation $\Delta$ is called $\delta$-thin if for each $t \in [0, d(x, p)]$

$$d(p', r') \leq \delta$$

where $p', r'$ are points on $\alpha$, $\gamma$ with $d(x, p') = d(x, r') = t$ and if the symmetric condition holds for $y$ and $z$.

A geodesic metric space $(X, d)$ is called $\delta$-*hyperbolic* if there is $\delta \geq 0$ such that all geodesic triangles are $\delta$-thin.

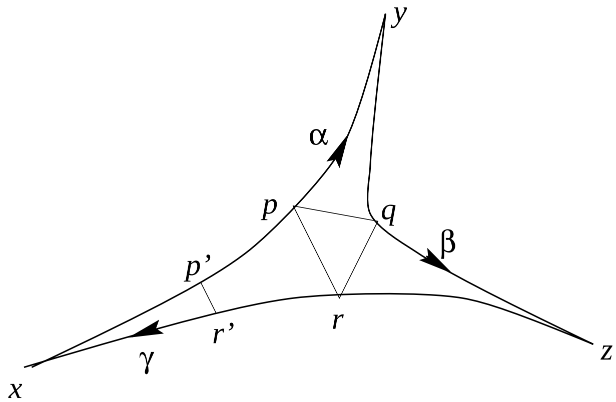


FIGURE 1

If $G$ is a finitely generated group and $\mathcal{G}$ is a finite generating set for $G$, we denote the Cayley graph of $G$ with respect to $\mathcal{G}$ by $\Gamma(G,\mathcal{G})$ and denote by $d_{\mathcal{G}}$ the word metric on $\Gamma(G,\mathcal{G})$. Also, for any $g \in G$ we define *the word length* of $g$ as $l_{\mathcal{G}}(g) = d_{\mathcal{G}}(1,g)$. It is easy to see that $(\Gamma(G,\mathcal{G}), d_{\mathcal{G}})$ is a geodesic metric space. If $w$ is a word in the generators $\mathcal{G}$, we denote by $\overline{w}$ the element of $G$ which $w$ represents.

A finitely generated group $G$ is called *word hyperbolic* if for each finite generating set $\mathcal{G}$ for $G$ there is $\delta \geq 0$ such that the Cayley graph $\Gamma(G,\mathcal{G})$ with the word metric $d_{\mathcal{G}}$ is $\delta$-hyperbolic.

A subgroup $H$ of a word hyperbolic group $G$ is called *quasiconvex* in $G$ if for some (and therefore for any) finite generating set $\mathcal{G}$ of $G$ there is $\epsilon > 0$ such that every geodesic in the Cayley graph $\Gamma(G,\mathcal{G})$ of $G$ with both endpoints in $H$ lies in the $\epsilon$-neighborhood of $H$.

If $G$ is a word hyperbolic group with a finite generating set $\mathcal{G}$, we say that a sequence of points $\{g_n \in G \,|\, n \in \mathbb{N}\}$ *defines a point at infinity* if

$$\lim_{n\to\infty} \inf_{i,j\geq n} (g_i, g_j)_1 = \infty$$

where the Gromov inner product is taken in $d_{\mathcal{G}}$-metric. Two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ defining points at infinity are called *equivalent* if

$$\lim_{n\to\infty} \inf_{i,j\geq n} (a_i, b_j)_1 = \infty.$$

*The boundary* $\partial G$ of $G$ is defined to be the set of equivalence classes of sequences defining points at infinity. If $a \in \partial G$ is the equivalence class of a sequence $(a_n)_{n\in\mathbb{N}}$, we say that $(a_n)_{n\in\mathbb{N}}$ converges to $a$ and write $\lim\limits_{n\to\infty} a_n = a$. The boundary $\partial G$ can be endowed with a natural topology which makes it a compact (and metrizable) space. It turns out that the definition of $\partial G$ and the topology on it are independent of the choice of the word metric for $G$. Moreover, $G$ acts on $\partial G$ by homeomorphisms and the action is given by $g \cdot \lim_{n\to\infty} a_n = \lim_{n\to\infty} g a_n$ where $g \in G$ and $(a_n)_{n\in\mathbb{N}}$ defines a point at infinity. If $S$ is a subset of $G$ (e.g., a subgroup of $G$), we define the *limit set* $\partial_{\mathcal{G}}(S)$ to be the set of limits in $\partial G$ of sequences of elements of $S$.

## 3. The Proofs

**Proposition A.** *Let $F$ be the fundamental group of a closed hyperbolic surface $S$. Let $\phi$ be an automorphism of $F$ induced by a pseudo-anosov homeomorphism of $S$. Take $G$ to be the mapping-torus group of $\phi$, that is*

$$G = \langle F, t \,|\, tft^{-1} = \phi(f),\, f \in F \rangle.$$

*Let $x \in F$ be an element which is not a proper power in $F$ (and so, obviously, $x$ is not a proper power in $G$). Let $G_1$ be a copy of $G$. The group $G_1$ contains a copy $F_1$ of $F$ and a copy $x_1$ of $x$. Set*

$$M = G \underset{x=x_1}{*} G_1 \tag{1}$$

*and $H = sgp(F, F_1)$.*

*Then*

1. *$M$ is torsion-free and word hyperbolic.*
2. *$H$ is finitely presented, freely indecomposable and non-quasiconvex in $M$.*

**Proof.** The group $G$ is torsion free and word hyperbolic since it is the fundamental group of a closed 3-manifold of constant negative curvature (see [Th]). Thus, $M$ is word hyperbolic by the combination theorem for negatively curved groups (see [BF], [KM]). Notice that $H = sgp(F, F_1) \cong F \underset{x=x_1}{*} F_1$ and so $H$ is torsion-free, finitely presentable and freely indecomposable. Moreover, $H$ is word hyperbolic by the same combination theorem.

Suppose $H$ is quasiconvex in $M$. It is shown in [BGSS] that $F$ is rational with respect to some automatic structure on $H$ since $H = F *_C F_1$ where $C = \langle x \rangle = \langle x_1 \rangle$ is cyclic. Therefore, $F$ is quasiconvex in $H$ (see, for example, [Swa]). Thus, since $H$ is quasiconvex in $M$ and $F$ is quasiconvex in $H$, the subgroup $F$ is quasiconvex in $M$. However, $F$ is infinite and has infinite index in its normalizer in $M$, which (see [KS]) implies that $F$ is not quasiconvex in $M$. This contradiction completes the proof of Proposition A. □

**Theorem B.** *Let $G$, $G_1$, $M$, and $H$ be as in Proposition A. Let $K$ be the limit set $\partial_M(H)$ of $H$ in the boundary $\partial M$ of $M$. Then*

$$H = Stab_M(K) = \{f \in M \,|\, fK = K\}.$$

Before proceeding with the proof, we choose a finite generating set $\mathcal{G}$ for $G$ and its copy $\mathcal{G}_1$. Then $\mathcal{G}$ defines the word length $l_{\mathcal{G}}$ and the word metric $d_{\mathcal{G}}$ for $G$. Analogously, $\mathcal{G} \cup \mathcal{G}_1$ is a finite generating set for $M = G *_C G_1$ which defines the word length $l_{\mathcal{M}}$ and the word metric $d_{\mathcal{M}}$ on $M$. Fix a $\delta > 0$ such that all $d_{\mathcal{M}}$-geodesic triangles are $\delta$-thin. We also denote by $C$ the subgroup of $M$ generated by $x = x_1$. The element $x = x_1$ will sometimes be denoted by $c$. Thus $M = G \underset{C}{*} G_1$ and $H = F \underset{C}{*} F_1$. We need to accumulate some preliminary information before proceeding with the proof of Theorem B.

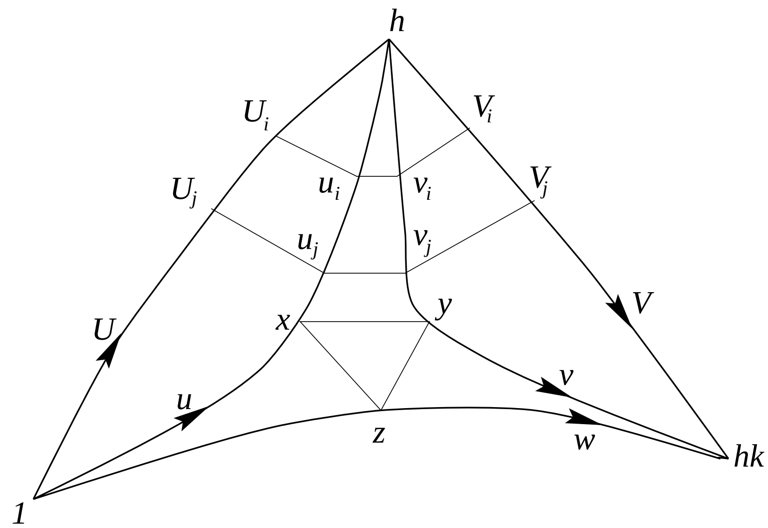


FIGURE 2

**Lemma 1.** *Let $A$ be a word hyperbolic group with a fixed finite generating set $S$. Let $H$ and $K$ be quasiconvex subgroups of $A$ such that for every $a \in A$*

$$aHa^{-1} \cap K = \{1\}$$

*Then there is a constant $r_0 > 0$ such that for every $h \in H$ and $k \in K$*

$$l_{\mathcal{A}}(hk) \geq r_0 \cdot l_{\mathcal{A}}(k)$$

**Proof.** Fix a finite generating set $\mathcal{H}$ for $H$ and a finite generating set $\mathcal{K}$ for $K$. Since $K$ and $H$ are quasiconvex in $A$, there is $\epsilon > 0$ such that every geodesic $[1, h]$, $h \in H$ in the Cayley graph of $A$ lies in the $\epsilon$-neighborhood of $H$ and every geodesic $[1, k]$, $k \in K$ lies in the $\epsilon$-neighborhood of $K$. Also, there is $\lambda > 0$ such that every $d_{\mathcal{H}}$-geodesic ($d_{\mathcal{K}}$-geodesic) word defines a $\lambda$-quasigeodesic in the Cayley graph of $A$.

Let $h \in H$ and $k \in K$. Fix $d_{\mathcal{G}}$-geodesic representatives $u, v, w$ of $h$, $k$ and $hk$ respectively. Also, fix a $d_{\mathcal{H}}$-geodesic representative $U$ of $h$ and a $d_{\mathcal{K}}$-geodesic representative $V$ of $k$. Consider the geodesic triangle $\Delta$ in the Cayley graph of $A$ with sides $u, v, w$ (see Figure 2).

Consider the inscribed triangle $xyz$ in the triangle $\Delta$ (see Figure 2). It has the following properties:

1. $d_{\mathcal{A}}(1, x) = d_{\mathcal{A}}(1, z)$, $d_{\mathcal{A}}(h, x) = d_{\mathcal{A}}(h, y)$, $d_{\mathcal{A}}(hk, y) = d_{\mathcal{A}}(hk, z)$
2. the segment $[1, u]$ of $U$ is $\delta$-uniformly close to the segment $[1, z]$ of $w$ and similar conditions hold for the other two corners of $\Delta$; in particular, $d_{\mathcal{A}}(x, y)$, $d_{\mathcal{A}}(x, z)$, $d_{\mathcal{A}}(y, z) \leq \delta$.

We claim that $d_{\mathcal{A}}(x, h) = d_{\mathcal{A}}(y, h)$ is small. More precisely, let $N$ be the number of words in the generating set of $A$ of length at most $\delta + 2\epsilon + 2$. Suppose $d_{\mathcal{A}}(h, u) > 4(\epsilon + 1)(N + 1)$. Then there is a sequence of vertices $u_1, \ldots, u_{N+1}$ on the segment of $u$ between $x$ and $h$ such that $d_{\mathcal{A}}(u_k, u_s) = 4(\epsilon + 1)|k - s|$. For each $k = 1, \dots, N+1$ there is a vertex $U_k$ on $U$ such that $d_{\mathcal{A}}(U_k, u_k) \leq \epsilon + 1$. Note that when $k \neq s$, $d_{\mathcal{A}}(U_k, U_s) \geq 4(\epsilon+1)|k-s| - 2(\epsilon+1) \geq 2(\epsilon+1) > 0$ and therefore all the vertices $U_1, \dots, U_{N+1}$ represent different elements of $A$. Also, for every $k = 1, \dots, N+1$ there is a unique vertex $v_k$ on the segment of $V$ between $h$ and $y$ such that $d_{\mathcal{A}}(h, u_k) = d_{\mathcal{A}}(h, v_k)$. Note that since the triangle $\Delta$ is $\delta$-thin, we have $d_{\mathcal{A}}(u_k, v_k) \leq \delta$, $k = 1, \dots, N+1$. Finally, for every $k = 1, \dots, N+1$ there is a vertex $V_k$ of $V$ such that $d_{\mathcal{A}}(v_k, V_k) \leq \epsilon + 1$. Thus $d_{\mathcal{A}}(U_k, V_k) \leq \delta + 2(\epsilon+1)$, $k = 1, \dots, N+1$. For each $k$ we choose a $d_{\mathcal{A}}$-geodesic path $\alpha_k$ from $U_k$ to $V_k$ in the Cayley graph of $A$. By the choice of $N$ there are $i < j$ such that $\alpha_i = \alpha_j = \alpha$. Put $a = \overline{\alpha}$. Then $a^{-1}h'a = k'$ where $h' \in H$ is the element represented by the segment of $U$ from $U_j$ to $U_i$ and $k' \in K$ is the element represented by the segment of $V$ from $V_i$ to $V_j$. Since $h' \neq 1$, this contradicts our assumption that any conjugate of $H$ intersects $K$ trivially. Thus we have established that

$$d_{\mathcal{A}}(h, u) \leq 4(\epsilon+1)(N+1) = r$$

Therefore $l(w) \geq d_{\mathcal{A}}(z, kh) = d_{\mathcal{A}}(v, kh) \geq l(v) - r$ and $l_{\mathcal{A}}(hk) = l(w) \geq \min(1/2, 1/r) \cdot l(v) = \min(1/2, 1/r) \cdot l_{\mathcal{A}}(k)$ which concludes the proof of Lemma 1. □

**Corollary 2.** *Let $A$ be a word hyperbolic group with a fixed finite generating set $\mathcal{A}$ and let $a, b$ be elements of infinite order in $A$ such that no nontrivial power of $a$ is conjugate in $A$ to a power of $b$. Then there is a constant $r_1 > 0$ such that for every $m, n \in \mathbb{Z}$*

$$l_{\mathcal{A}}(a^m b^n) \geq |n| \cdot r_1$$

**Proof.** This directly follows from Lemma 1 and the fact that cyclic subgroups of word hyperbolic groups are quasiconvex [ABC]. □

**Lemma 3.** *Assume that conditions of Theorem B are satisfied. Let $p \in C$ or $p = p_1 \dots p_s$ be a strictly alternating product of elements of $G - C$ and $G_1 - C$, where $p_s \in G_1 - C$. Let $y \in G$ be such that no power of $y$ is conjugate in $G$ to a power of $x$. Then there is a constant $D > 0$ such that for every $n \in \mathbb{Z}$*

$$l_{\mathcal{M}}(py^n) \geq D \cdot |n|$$

**Proof.** Note that if $p \in C$, then, since every conjugate of $C$ in $G$ intersects $\langle y \rangle$ trivially, Corollary 2 implies that

$$l_{\mathcal{G}}(py^n) \geq r_1 \cdot |n|$$

for some constant $r_1 > 0$ independent of $p, n$. Theorem D of [BGSS] implies that $G$ and $G_1$ are quasiconvex in $M$. Therefore there is a constant $r > 0$ such that for every $g \in G$

$$l_{\mathcal{M}}(g) \geq r \cdot l_{\mathcal{G}}(g)$$

and therefore

$$l_{\mathcal{M}}(py^n) \geq r \cdot r_1 \cdot |n|$$

From now on we assume that $p \notin C$ that is $p = p_1 \dots p_s$ is a strictly alternating product of elements of $G - C$ and $G_1 - C$ with $p_s \in G_1 - C$. To prove Lemma 3 in this case, recall that by the theorem of G. Baumslag, S. Gersten, M. Shapiro, and H. Short [BGSS], cyclic amalgamations of hyperbolic groups are automatic. In the proof of this theorem they construct an actual automatic language for a cyclic amalgam of two hyperbolic groups, which, therefore, consists of quasigeodesic words (see [ECHLPT], Theorem 3.3.4). We will explain how their procedure works in the case of the group $M = G *_C G_1$ (we use the fact that $d_{\mathcal{G}} \mid_C = d_{\mathcal{G}_1} \mid_C$). Fix a lexicographic order on the generating set $\mathcal{G}$ of $G$ and a copy of this order on the generating set $\mathcal{G}_1$ of $G_1$. We will say that a $d_{\mathcal{G}}$-geodesic word $u$ is *minimal* in the coset class $\overline{u}C$ if $l_{\mathcal{G}}(u) \le l_{\mathcal{G}}(\overline{u} \cdot c)$ for every $c \in C$ and whenever $l(u) = l(u')$ for some $d_{\mathcal{G}}$-geodesic word $u'$ with $\overline{u'} \in \overline{u}C$ then $u$ is lexicographically smaller than $u'$. It is clear that any coset class $gC$, $g \in G$ has a unique minimal representative $u$. Similarily, one defines minimal representatives for coset classes $g_1C$, $g_1 \in G_1$.

Theorem D of [BGSS] provides an explicit construction of an automatic language $L$ in the alphabet $\mathcal{M} = \mathcal{G} \cup \mathcal{G}_1$ for $M$ such that every $e \in M$ has a unique representative in $L$. Note that, in general, Theorem D of [BGSS] gives a construction of such an automatic language for $M$ in a bigger alphabet than $\mathcal{G} \cup \mathcal{G}_1$. More precisely, they need to find first a generating set $\mathcal{G}'$ containing $\mathcal{G}$ for $G$ and a generating set ${\mathcal{G}_1}'$ containing $\mathcal{G}_1$ for $G_1$ such that for some constant $\epsilon_1 > 0$

$$|l_{\mathcal{G}'}(c) - l_{{\mathcal{G}_1}'(c)}| \le \epsilon_1, \text{ for every } c \in C.$$

Then they construct the automatic language for $M$ in the alphabet $\mathcal{G}' \cup {\mathcal{G}_1}'$. However, by the choice of $M$ we already have

$$l_{\mathcal{G}}(c) = l_{\mathcal{G}_1}(c) \text{ for every } c \in C$$

and so the [BGSS] procedure gives us an automatic language $L$ with uniqueness in the alphabet $\mathcal{G} \cup \mathcal{G}_1$. (Although an automatic group has an automatic language over every finite generating set of this group, in this particular case we need not just the fact that $M$ is automatic and possesses an automatic language over $\mathcal{M}$ but, rather, the fact that $M$ has an automatic language over $\mathcal{M}$ with some very particular properties given by the [BGSS] construction). We will now describe how, given an element $e \in M$, one can find its representative in $L$.

Suppose $e \in M$. If $e \in C$, then $e = x^k$ and we take a $d_{\mathcal{G}}$-geodesic representative of $x^k$ to be the representative of $e$ in the automatic language $L$ for $M$. Suppose $e \notin C$. First, write $e$ as a strictly alternating product of elements from

$$e = e_1 \dots e_j$$

of elements from $G - C$ and $G_1 - C$. Then express $e_1$ as $e_1 = \overline{w_1}@, c^{n_1}$ where $w_1$ is the minimal representative in the coset class $e_1C$. Then express $c^{n_1}e_2$ as $c^{n_1}e_2 = \overline{w_2}@, c^{n_2}$ where $w_2$ is the minimal representative in the coset class $c^{n_1}e_2C$. And so on for $i = 1, 2, \dots, j-1$. Finally, we express $c^{n_{j-2}}e_{j-1}$ as $\overline{w_{j-1}}@, c^{n_{j-1}}$ where $w_{j-1}$ is the minimal representative in the coset class $c^{n_{j-2}}e_{j-1}C$.

We put $w_j$ to be the lexicographically minimal among all $d_{\mathcal{G}}$-geodesic ($d_{\mathcal{G}_1}$-geodesic) representatives of $c^{n_{j-1}}e_j$. As a result we obtain the word $w = w_1 \dots w_j$ such that $\overline{w} = e$. This word $w$ is the required representative of $e$ in $L$.

Note that in the case $e \notin C$ we have $\overline{w_1}@, C = e_1C$, $C\overline{w_j} = Ce_j$ and $C\overline{w_i}C = Ce_iC$ for $1 < i < j$. Note also that there is $\lambda > 0$ such that all words in $L$ define $\lambda$-quasigeodesics in the Cayley graph of $M$. This, in particular, means that for every $w \in L$

$$l(w) \leq \lambda \cdot l_{\mathcal{M}}(\overline{w}) + \lambda.$$

Suppose now that $p$ and $y$ are as in Lemma 3 and $n \in \mathbb{Z}, n \neq 0$. We will find the representative $w$ of $py^n$ in the automatic language $L$ on $M$ using the procedure described above. Note that $\langle y\rangle \cap \langle x\rangle = \{1\}$ and so $y^n \in G - C$ since $C = \langle x\rangle$. Therefore $p_1 \dots p_s y^n$ is a strictly alternating product of elements of $G - C$ and $G_1 - C$. It is clear from the construction that $w$ has the following form:

$$w = q_1 \dots q_s v$$

where

1. $q_i$ is a $d_{\mathcal{G}}$-geodesic word when $p_i \in G - C$ and $q_i$ is a $d_{\mathcal{G}_1}$-geodesic word when $p_i \in G_1 - C$;
2. $\overline{q_1}C = p_1C$ and $C\overline{q_i}C = Cp_iC$ when $i > 1$;
3. $v$ is a $d_{\mathcal{G}}$-geodesic word and $\overline{v} = cy^n$ for some $c \in C$
4. $\overline{w} = p_1 \dots p_s y^n = py^n$

Corollary 2 implies that $l(v) = l_{\mathcal{G}}(cy^n) \geq r_1 \cdot |n|$ for some $r_1 > 0$ depending only on $x, y$ and independent of $n$.

Therefore $l(w) = l(q_1 \dots q_s v) \geq l(v) \geq r_1 \cdot |n|$. Since the language $L$ consists of $\lambda$-quasigeodesics with respect to $d_{\mathcal{M}}$, we conclude that $l_{\mathcal{M}}(py^n) \geq \dfrac{l(w)}{\lambda} - \lambda \geq \dfrac{r_1}{\lambda} \cdot |n| - \lambda$ which implies the statement of Lemma 3. □

**Lemma 4.** *Suppose conditions of Theorem B are satisfied. Let $u_1 \dots u_m \notin H$ be a strictly alternating product of elements from $G - C$ and $G_1 - C$ such that $u_m \in G_1 - C$. Let $p_1 \dots p_s$ be a strictly alternating product of elements of $F - C$ and $F_1 - C$. Let $q_0$ belong to $G_1$ if $p_1 \in F - C$ and $q_0$ belong to $G$ when $p_1 \in F_1 - C$ (we allow $q_0 \in C$).*

*Then either $q_0 p_1 \dots p_s u_1 \dots u_m$ ends (when rewritten in the normal form with respect to (1)) in the element of $G_1 - C$ or $q_0 p_1 \dots p_s u_1 \dots A u_m \in C$.*

**Proof.** Indeed, $u_m \in G_1 - C$ and so $q_0 p_1 \dots p_s u_1 \dots u_m$ ends (when rewritten in the normal form with respect to (1)) in the element of $G_1 - C$ unless $u_m^{-1} \dots u_1^{-1}$ is a *terminal segment* of $q_0 p_1 \dots p_s$ that is either $m \geq s$ and

$$p_{s-m+1} \dots p_s u_1 \dots u_m \in C \tag{2}$$

or

$$q_0 p_1 \dots p_s u_1 \dots u_m \in C \tag{3}$$

It is clear that (2) is impossible since $p_{s-m+1} \dots p_s \in H = gp(F, F_1)$, $C \leq H$ and $u_1 \dots u_m \notin H$. If (3) holds, we have $q_0 p_1 \dots p_s u_1 \dots u_m \in C$ as required. Thus Lemma 4 is established. □

**Lemma 5.** *Let $z = u_1 \dots u_m \notin H$ be a strictly alternating product of elements of $G - C$ and $G_1 - C$ such that $u_m \in G_1 - C$. Let $y \in G$ be such that no nontrivial power of $y$ is conjugate in $G$ to a power of $x = c$. Then there is a constant $K_0 > 0$ with the following property.*

*Let $n \in \mathbb{Z}$ and $h \in H$. Let $v$ be a $d_{\mathcal{M}}$-geodesic from 1 to $h$ and let $u$ be a $d_{\mathcal{M}}$-geodesic from 1 to $zy^n$. Take the vertex $v(N)$ of $v$ at the distance $N$ from 1 and the vertex $u(N)$ of $u$ at the distance $N$ from 1 (see Figure 3). Then*

$$d_{\mathcal{M}}(u(N), v(N)) \geq K_0 \cdot N.$$

**Proof.** Recall that $H = sgp(F, F_1) = F *_C F_1 \leq M = G *_C G_1$. Let $\lambda > 0$ be such that any word from the automatic language $L$ on $M$ defines a $\lambda$-quasigeodesic in the Cayley graph of $M$.

Let $\hat{u}_i$ be a $d_{\mathcal{G}}$-geodesic ($d_{\mathcal{G}_1}$-geodesic) representative of $u_i$, $i \leq m$. Let $Y$ be a $d_{\mathcal{G}}$-geodesic representative of $y$. Since the element $z$ is fixed and the cyclic subgroup $\langle y \rangle$ is quasiconvex in $M$, there is a constant $\lambda_1 > 0$ such that for every $k \in \mathbb{Z}$ the word $\hat{u}_1 \dots \hat{u}_{m-1}\hat{u}_m Y^k$ is a $\lambda_1$-quasigeodesic with respect to $d_{\mathcal{M}}$. In particular, $U = \hat{u}_1 \dots \hat{u}_{m-1}\hat{u}_m Y^n$ is a $\lambda_1$-quasigeodesic representative of $zy^n$ with respect to $d_{\mathcal{M}}$. Put $\lambda_2 = max(\lambda, \lambda_1)$. Let $\epsilon > 0$ be such that any two $\lambda_2$-quasigeodesics with common endpoints in the Cayley graph of $M$ are $\epsilon$-Hausdorff-close. We will also assume that $\epsilon$ is such that for every $k \in \mathbb{Z}$ and every point $x_0$ on a $d_{\mathcal{M}}$-geodesic from 1 to $zy^k$ there is $k' \in \mathbb{Z}$, $k' \in [0, k]$, such that $d_{\mathcal{M}}(x_0, zy^{k'}) \leq \epsilon$.

If $h \in C$ and $h^{-1} = x^m$, put $p_1 = h^{-1}$ and let $w_1$ be a $d_{\mathcal{G}}$-geodesic representative of $p_1$.

If $h \notin C$ let $h^{-1} = p_1 p_2 \dots p_s$ be the strictly alternating product of elements of $F - C$ and $F_1 - C$. Note that $p_1 p_2 \dots p_s$ is also a strictly alternating product of elements of $G - C$ and $G_1 - C$. We then can find the representative $w$ of $h^{-1}$ in the automatic language $L$ on $M$ which was described in Lemma 3.

Clearly, $w = w_1 w_2 \dots w_s$, where

1. $\overline{w_1}C = p_1 C$ and $w_1$ is minimal in the coset $\overline{w_1}C$
2. for $j < s$ $C\overline{w_j}C = Cp_jC$ and $w_j$ is minimal in the coset $\overline{w_j}C$
3. $C\overline{w_s} = Cp_s$ that is $\overline{w_s} = cp_s$ for some $c \in C$
4. each $w_i$ is a $d_G$ or $d_{\mathcal{G}_1}$-geodesic word.

Note that since $p_j \in F \cup F_1$ and $C \leq F$, $C \leq F_1$, the conditions above imply that $\overline{w_j} \in F \cup F_1$ for $j = 1, \dots, s$.

Let $v$ be a $d_{\mathcal{M}}$ geodesic from 1 to $h$ and let $u$ be a $d_{\mathcal{M}}$ geodesic from 1 to $zy^n$. Assume $N$ is a positive number such that $N \leq l(v)$ and $N \leq l(u)$.

Let $v(N)$ be the point on the geodesic $v$ at the distance $N$ from 1. Let $u(N)$ be the point on the geodesic $u$ at the distance $N$ from 1 (see Figure 3).

Recall that $z = u_1 \dots u_m$ and $y$ are fixed, $u_m \in G_1 - C$. Recall further that $Y$ is a $d_{\mathcal{G}}$-geodesic representative of $y$. By the choice of $\epsilon$ there is a vertex $V(N)$ of $w$ and a vertex $U(N) = zy^k$ of $U$ such that $d_{\mathcal{M}}(u(N), U(N)) \leq \epsilon$, $d_{\mathcal{M}}(v(N), V(N)) \leq \epsilon$ (see Figure 3). The segment $S_1$ of $w$ from $V(N)$ to 1 is a terminal segment of $w = w_1 \dots w_s$, and it has the form

$$S_1 = q_i w_{i+1} \dots w_s$$

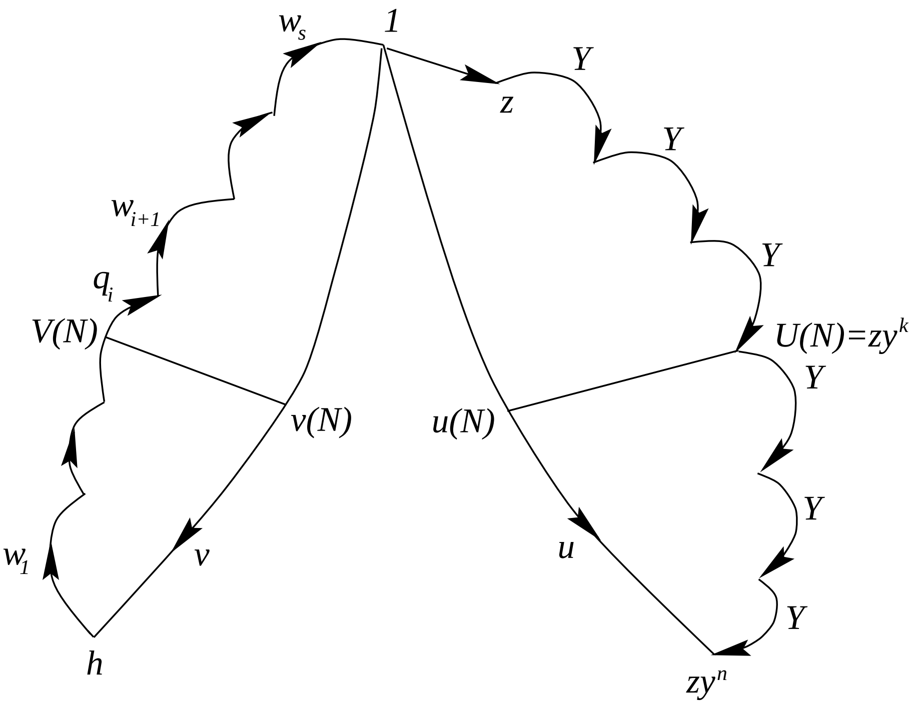


FIGURE 3

where $i \leq s$ and $q_i$ is a nonempty terminal segment of $w_i$. The segment $S_2$ of $U$ from 1 to $U(N)$ is an initial segment of $U = \hat{u}_1 \dots \hat{u}_{m-1}\hat{u}_m Y^n$ of the form

$$S_2 = \hat{u}_1 \dots \hat{u}_{m-1}\hat{u}_m Y^k$$

for some integer $k \in [0, n]$. Notice that $d_{\mathcal{M}}(1, u_1 \dots u_{m-1}\hat{u}_m y^k) \geq N - \epsilon$ and therefore

$$|k| l_{\mathcal{M}}(y) \geq l_{\mathcal{A}}(y^k) \geq N - l(\hat{u}_1 \dots \hat{u}_{m-1}\hat{u}_m) - \epsilon$$

and

$$|k| \geq (1/l_{\mathcal{A}}(y))(N - l(\hat{u}_1 \dots \hat{u}_{m-1}\hat{u}_m) - \epsilon).$$

Thus, for some constant $K_1 > 0$ independent of $h, n$, we have

$$|k| \geq K_1 \cdot N.$$

By Lemma 4, either $\overline{q}_i\overline{w}_{i+1} \dots \overline{w}_s u_1 \dots u_m \in C$ or $\overline{q}_i\overline{w}_{i+1} \dots \overline{w}_s u_1 \dots u_m$ ends in the element of $G_1 - C$, when rewritten in normal form with respect to (1). Therefore, by Lemma 3, there is a constant $D > 0$ independent of $h$, $n$ such that

$$l_{\mathcal{M}}(\overline{q}_i\overline{w}_{i+1} \dots \overline{w}_s u_1 \dots u_m y^k) = l_{\mathcal{M}}(\overline{S_1 S_2}) \geq D|k|$$

and hence $l_{\mathcal{M}}(\overline{q}_i\overline{w}_{i+1} \dots \overline{w}_s u_1 \dots u_m y^k) \geq K_1 \cdot D \cdot N$. It remains to recall that $|d_{\mathcal{M}}(u(N), v(N)) - l_{\mathcal{M}}(\overline{q}_i\overline{w}_{i+1} \dots \overline{w}_s u_1 \dots u_m y^k)| \leq 2\epsilon$ to conclude that there is a constant $K_2 > 0$ independent of $h, n$ such that $d_{\mathcal{M}}(u(N), v(N)) \geq K_2 \cdot N$. This completes the proof of Lemma 5. □

**Proof of Theorem B.** Suppose $z \in Stab_M(K)$. We will show that $z \in H$ by induction on the syllable length of $z$ with respect to presentation (1). When the syllable length of $z$ is 0, that is $z \in C$, the statement is obvious. Suppose now that $z \in Stab_M(K) - H$, the syllable length of $z$ is $m > 0$ and the statement has been proved for elements of $Stab_M(K)$ of smaller syllable length. Write $z$ as a strictly alternating product $z = u_1 \dots u_m$ of elements from $G - C$ and $G_1 - C$. If $u_m \in F \cup F_1$, then $u_m \in H \cap Stab_M(K)$, and so $u_1 \dots u_{m-1} \in Stab_M(K)$. Therefore, $u_1 \dots u_{m-1} \in H$ by the inductive hypothesis, $u_m \in H$, and so $z \in H$. Thus, $u_m \in (G - F) \cup (G_1 - F_1)$. Assume for definiteness that $u_m \in G_1 - F_1$, that is, $u_m = f_1 t_1^j$ for some $j \neq 0$, $f_1 \in F_1$.

Choose $y \in F$ so that no power of $y$ is conjugate in $G$ to a power of $x$. Fix a $d_{\mathcal{G}}$-geodesic representative $Y$ of $y$.

Let $y^+ = \lim_{n\to\infty} y^n \in \partial M$. By definition of $K$ we have $y^+ \in K$ and therefore $zy^+ \in K$. This means that for any $N > 0$ there is an element $h \in H$ and a positive power $y^n$ of $y$ such that $(h, zy^n)_1 > N$, the Gromov inner product taken in the $d_{\mathcal{M}}$-metric. This means that $l_{\mathcal{M}}(h) \geq N$, $l_{\mathcal{M}}(zy^n) \geq N$ and $d_{\mathcal{M}}(h(N), (zy^n)(N)) \leq \delta$ where $h(N)$ and $(zy^n)(N)$ are elements of $M$ represented by initial segments of length $N$ of $d_{\mathcal{M}}$-geodesic representatives of $h$ and $zy^n$.

Then $l_{\mathcal{M}}(h(N), (zy^n)(N)) \geq K_0 \cdot N$ where $K_0$ is the constant independent of $h$, $n$ which is provided by Lemma 5. Thus,

$$\delta \geq l_{\mathcal{M}}(h(N), (zy^n)(N)) \geq K_0 \cdot N$$

and therefore $N \leq (1/K_0) \cdot \delta$. This contradicts the fact that $N$ can be chosen arbitrarily big.

Therefore, $z \notin Stab_M(K)$, which completes the proof of Theorem B. □

**Corollary 6.** *Let $M$, $G$, $G_1$, $C$ and $H$ be as in Theorem B. Then*

(a) *the limit set of $H$ is not the limit set of a quasiconvex subgroup of $M$;*
(b) *the virtual normalizer $VN_M(H)$ of $H$ in $M$ is equal to $H$.*

**Proof.**

(a) Suppose there is a quasiconvex subgroup $Q_1$ of $M$ such that $\partial_{\mathcal{M}}(H) = \partial_{\mathcal{M}}(Q_1) = K$. Clearly, $Q_1$ is infinite since $K$ is nonempty. Set

$$Q = Stab_M(K) = \{y \in M | yK = K\}.$$

Since $Q_1$ is infinite and quasiconvex in $M$ and $Q = Stab_M(\partial_{\mathcal{M}}(Q_1))$, it follows from Lemma 3.9 of [KS] that $Q$ contains $Q_1$ as a subgroup of finite index and therefore $Q$ is also quasiconvex in $M$. On the other hand, Theorem B implies that $H = Stab_M(K)$, and so $H = Q$. This contradicts the fact that $H$ is not quasiconvex in $M$ by Proposition A.

(b) It is not hard to see that $A \leq VN_B(A) \leq Stab_B(\partial_B(A))$ when $A$ is an infinite subgroup of a word hyperbolic group $B$. Indeed, if $g \in VN_B(A)$, then $A_0 = A \cap gAg^{-1}$ has finite index $n$ in $A$. Let $A = A_0 \cup A_0 c_1 \cup \dots \cup A_0 c_{n-1}$, and let $D = max\{l_{\mathcal{A}}(c_i) \,|\, i = 1, \dots, n-1\}$. Suppose $p \in \partial_B(A)$. Then there is a sequence $a_m \in A$ such that $p = \lim_{m\to\infty} a_m$. For each $m$ there is $b \in B$ with

$l_B(b) \leq D + l_B(g)$ such that $ga_m b = a'_m \in A_0$. Therefore $gp \in \partial_B(A_0) = \partial_B(A)$. Since $p \in \partial_B(A)$ was chosen arbitrarily, we have $g\partial_B(A) \subseteq \partial_B(A)$. Since by the same argument $g^{-1}\partial_B(A) \subseteq \partial_B(A)$, we conclude that $g\partial_B(A) = \partial_B(A)$. Thus, $A \leq VN_B(A) \leq Stab_B(\partial_B(A))$.

For the subgroup $H$ of $M$ we have $H \leq VN_M(H) \leq Stab_M(\partial_{\mathcal{M}}(H))$. On the other hand, $Stab_M(\partial_{\mathcal{M}}(H)) = H$ by Theorem B. Therefore, $H = VN_M(H)$. □

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