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The Topological Snake Lemma and Corona Algebras

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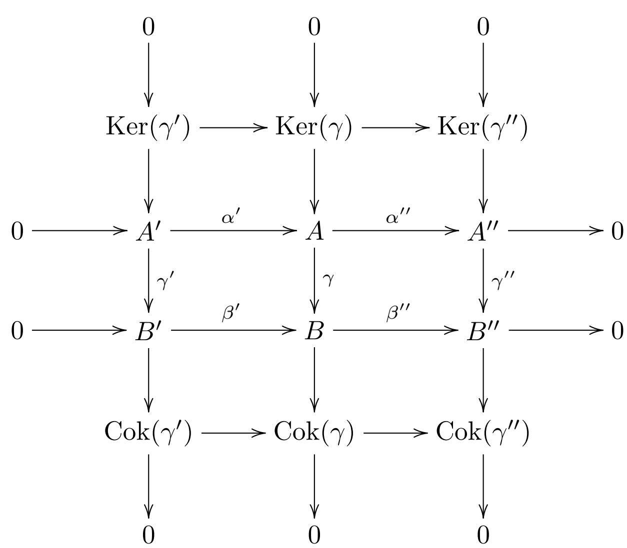
Abstract. We establish versions of the Snake Lemma from homological algebra in the context of topological groups, Banach spaces, and operator algebras. We apply this tool to demonstrate that if $f : B \to B'$ is a quasi-unital C^*-map of separable C^*-algebras, so that it induces a map of Corona algebras $\bar{f} : \mathcal{Q}B \to \mathcal{Q}B'$, and if f is mono, then the induced map $\bar{f}$ is also mono.

This paper presents a cross-cultural result: we use ideas from homological algebra, suitable topologized, in order to establish a functional analytic result.

The Snake Lemma (also known as the Kernel-Cokernel Sequence) [1] is a basic result in homological algebra. Here is what it says. Suppose that one is given a commutative diagram

(1)



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[1]and fondly recalled as the one serious mathematical theorem ever to appear in a major motion picture: "It's My Turn," starring Jill Clayburgh (1980).

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with exact rows in some abelian category.[2] The Snake Lemma (cf. [Mac, page 50]) asserts that there is a morphism

$$\delta : \mathrm{Ker}(\gamma'') \to \mathrm{Cok}(\gamma')$$

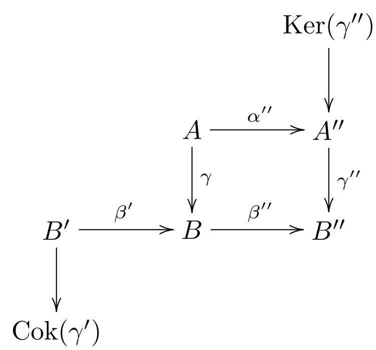
which is natural with respect to diagrams and a long exact sequence

(2)

$$0 \longrightarrow \mathrm{Ker}(\gamma') \longrightarrow \mathrm{Ker}(\gamma) \longrightarrow \mathrm{Ker}(\gamma'') \xrightarrow{\delta} \mathrm{Cok}(\gamma') \longrightarrow \mathrm{Cok}(\gamma) \longrightarrow \mathrm{Cok}(\gamma'') \longrightarrow 0.$$

The maps not explicitly labeled in (1) and (2) are induced by α', α'', β' and β'' in the obvious way.

The boundary map δ is defined as follows.[3]



Let $a'' \in A''$ be an element of $\mathrm{Ker}(\gamma'')$. Since α'' is onto, there is some $a \in A$ with $\alpha''(a) = a''$. Then

$$\beta''\gamma(a) = \gamma''\alpha''(a) = \gamma''(a'') = 0$$

and so $\gamma(a) \in \mathrm{Ker}(\beta'') = \mathrm{Im}(\beta')$. Thus there is some unique $b' \in B'$ with $\beta'(b') = \gamma(a)$. Finally, define

$$\delta(a'') = [b'] \in B'/\mathrm{Im}(\gamma') = \mathrm{Cok}(\gamma').$$

The map δ is well-defined and it is a morphism in the category.[4]

We suppose that the following proposition is well-known. The notation refers to (1).

Proposition 3. *Suppose that A is a ring, A' is an ideal, A'' is the quotient ring, and similarly for B. Further, suppose that the maps γ', γ, and γ'' are ring homomorphisms, and that $\gamma'(A')$ is an ideal in B'. Then the map δ is a ring homomorphism.*

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[2]For instance, modules over some commutative ring. Eventually the ring will be the complex numbers.

[3]Here we assume for convenience that we are working with a category of modules over a commutative ring so that our objects have elements. This is not necessary, strictly speaking, but the alternative is to be far more abstract than is needed for present purposes.

[4]Clayburgh defines δ and proves that it is well-defined in the opening credits of the movie. Her proof is correct.

Proof. We check directly using the definition of δ. Suppose that $a''_1, a''_2 \in \mathrm{Ker}(\gamma'')$. We wish to show that

$$\delta(a''_1 a''_2) = \delta(a''_1)\delta(a''_2).$$

Choose elements $a_i, a \in A$ with $\alpha''(a_i) = a''_i$ and $\alpha''(a) = a''_1 a''_2 \in \mathrm{Ker}(\gamma'')$. Then

$$\alpha''(a - a_1 a_2) = a''_1 a''_2 - a''_1 a''_2 = 0$$

and so

$$a - a_1 a_2 \in \mathrm{Ker}(\alpha'') = \mathrm{Im}(\alpha').$$

Let $a' \in A'$ be the unique element with

$$\alpha'(a') = a - a_1 a_2.$$

We have

$$\beta''\gamma(a_i) = \gamma''\alpha''(a_i) = \gamma(a''_i) = 0$$

and

$$\beta''\gamma(a) = \gamma''\alpha''(a) = \gamma(a''_1 a''_2) = 0$$

so that $\gamma(a)$ and both $\gamma(a_i)$ lie in $\mathrm{Ker}(\beta'') = \mathrm{Im}(\beta')$. Thus there exist unique elements $b'_i, b' \in B'$ with

$$\beta'(b'_i) = \gamma(a_i) \quad \text{and} \quad \beta'(b') = \gamma(a).$$

Of course

$$\delta(a''_i) = [b'_i] \in \mathrm{Cok}(\gamma')$$

and

$$\delta(a''_1 a''_2) = [b'] \in \mathrm{Cok}(\gamma')$$

so to complete this proof we must show that $[b'_1][b'_2] = [b']$. Now

$$[b'] - [b'_1][b'_2] = [b' - b'_1 b'_2]$$

so it suffices to show that $b' - b'_1 b'_2 \in \mathrm{Im}(\gamma')$. We compute:

$$\beta'(b' - b'_1 b'_2) = \gamma(a - a_1 a_2) = \gamma\alpha'(a') = \beta'\gamma'(a')$$

and since β' is mono we have

$$b' - b'_1 b'_2 = \gamma(a') \in \mathrm{Im}(\gamma')$$

as required. This implies that the map δ is a ring map. □

Now we start to impose topological conditions upon diagram (1).

Proposition 4. *Suppose that A is a topological group with subgroup A' and quotient group A'', and similarly for B, and suppose that the maps γ', γ, and γ'' are continuous. Give the various kernels the subgroup topology and the various cokernels the quotient group topology. Then all of the maps in the 6-term sequence (2) are continuous.*

Proof. It is necessary only to show that δ is continuous. Let $U \subset \mathrm{Cok}(\gamma')$ be an open set. We must show that $\delta^{-1}(U)$ is an open set in $\mathrm{Ker}(\gamma'')$.

Let $\pi : B' \to \mathrm{Cok}(\gamma')$ be the natural map. It is continuous, so the set $\pi^{-1}(U)$ is open in B'. As B' has the relative topology in B, this means that there is some open set $V \subset B$ with

$$\pi^{-1}(U) = B' \cap V.$$

Then $\gamma^{-1}(V)$ is open in A, since γ is continuous, and $\alpha''\gamma^{-1}(V)$ is open in A'', since α'' is an open map. Thus

$$\alpha''\gamma^{-1}(V) \cap \operatorname{Ker}(\gamma'')$$

is an open set in $\operatorname{Ker}(\gamma'')$. To complete the argument it will thus suffice to establish that

$$(*) \qquad \delta^{-1}(U) = \alpha''\gamma^{-1}(V) \cap \operatorname{Ker}(\gamma'').$$

This is a direct check. Suppose that $a'' \in \delta^{-1}(U)$. Then $\delta(a'') \in U$. But $\delta(a'') = [b']$ for some $b' \in B'$ given as per the definition of δ, and so $b' \in \pi^{-1}(U)$. Then

$$\beta' b' \in B' \cap V \subseteq V$$

and $\beta'(b') = \gamma(a)$ with $\alpha''(a) = x$ by the definition of δ, so $a \in \gamma^{-1}(V)$. Then

$$a'' = \alpha''(a) \in \alpha''\gamma^{-1}(V)$$

as required.

In the opposite direction, let $a'' \in \alpha''\gamma^{-1}(V) \cap \operatorname{Ker}(\gamma'')$. Then $a'' = \alpha''(a)$ with $a \in \gamma^{-1}(V)$, so $\gamma(a) \in V$. Also, $\gamma(a) \in \beta' B'$, since $a'' \in \operatorname{Ker}(\gamma'')$. Thus

$$\gamma(a) \in \beta' B' \cap V = \pi^{-1}(U)$$

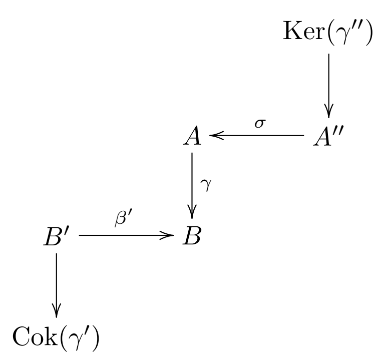
and so $\delta(x) = [\gamma(a)] \in U$. □

Recall that if $\alpha'' : A \to A''$ is a continuous surjection of Banach spaces then it has a continuous cross-section $\sigma : A'' \to A$ by the Bartle-Graves theorem ([BG, Theorem 4], [Mi, Corollary on page 364]). We may use this section to explicitly realize the map δ.

Proposition 5. *Suppose that A is a Banach space, A' is a closed Banach subspace, and A'' is the quotient Banach space, and similarly for B, and suppose that the vertical maps are continuous. Then we may realize the map*

$$\delta : \operatorname{Ker}(\gamma'') \longrightarrow \operatorname{Cok}(\gamma')$$

in terms of the Bartle-Graves section via the diagram



Proof. As any section (continuous or not) of α'' may be used in the definition of δ, we may as well use the section σ. Then the composition $\gamma\sigma : \operatorname{Ker}(\gamma'') \to B$ is obviously continuous. Its image lies in the image of β', and since B' has the relative topology in B we may conclude that $\gamma\sigma : \operatorname{Ker}(\gamma'') \to B'$ is also continuous. Composing with the continuous projection $B' \to \operatorname{Cok}(\gamma')$ yields δ. □

Note that as a consequence of the proof we see that all Bartle-Graves sections yield the same map δ.

We continue to assume that (1) is a diagram in the category of Banach spaces and closed subspaces as in the previous proposition.

Proposition 6 (K. Thomsen)**.** *If the map γ is a monomorphism, then the map δ is an isometry.*

Proof. This is a direct calculation. Let $a'' \in A''$ and choose some $a \in A$ with $\alpha''(a) = a''$. Then

$$\begin{aligned} ||\delta(a'')|| &= \inf_{a' \in A'} ||\gamma(a) - \beta'\gamma'(a')|| \\ &= \inf_{a' \in A'} ||\gamma(a) - \gamma\alpha'(a')|| \end{aligned}$$

but γ is mono, hence an isometry

$$\begin{aligned} &= \inf_{a' \in A'} ||a - \alpha'(a')|| \\ &= ||a''|| \end{aligned}$$

completing the proof. □

We turn our attention to C^*-algebras.

Theorem 7. *Suppose in Diagram* (1) *that A is a C^*-algebra, A' is a closed ideal, and A'' is the quotient algebra, and similarly for B, and suppose that the vertical maps are C^*-maps. Then*

1. *the Snake sequence*

$$0 \to \mathrm{Ker}(\gamma') \to \mathrm{Ker}(\gamma) \to \mathrm{Ker}(\gamma'') \xrightarrow{\delta} \mathrm{Cok}(\gamma') \to \mathrm{Cok}(\gamma) \to \mathrm{Cok}(\gamma'') \to 0$$

   is an exact sequence of Banach spaces.

2. *The sequence*

$$0 \longrightarrow \mathrm{Ker}(\gamma') \longrightarrow \mathrm{Ker}(\gamma) \longrightarrow \mathrm{Ker}(\gamma'')$$

   is an exact sequence of C^*-algebras and C^*-maps.

3. *If γ is a monomorphism then δ is an isometry and the sequence reduces to the sequence*

$$0 \longrightarrow \mathrm{Ker}(\gamma'') \xrightarrow{\delta} \mathrm{Cok}(\gamma') \longrightarrow \mathrm{Cok}(\gamma) \longrightarrow \mathrm{Cok}(\gamma'') \longrightarrow 0$$

4. *If $\gamma'(A')$ is a closed ideal in B' then the map*

$$\delta : \mathrm{Ker}(\gamma'') \to \mathrm{Cok}(\gamma')$$

   is also a map of C^*-algebras.

Proof. This simply applies the earlier results to the context of C^*-algebras. The only point to check is that δ preserves the $*$-operation, and this we leave as an exercise. □

If B is a C^*-algebra then the multiplier algebra of B is denoted by $\mathcal{M}B$ and the Corona algebra is denoted $\mathcal{Q}B = \mathcal{M}B/B$.

Recall [H, 1.1.6], [T, 2.6] that a $*$-homomorphism $f : B \to B'$ is *quasi-unital* when there is a projection $p \in \mathcal{M}B'$ such that the closed linear span of $f(B)B'$ has the form pB'. Thomsen shows that a $*$-homomorphism $f : B \to B'$ extends to a $*$-homomorphism $\mathcal{M}f : \mathcal{M}B \to \mathcal{M}B'$ which is strictly continuous on the unit ball if and only if f is quasi-unital. Of course if f does extend then there is an induced map $\bar{f} : \mathcal{Q}B \to \mathcal{Q}B'$. Thomsen also shows that if f is a monomorphism then so is $\mathcal{M}f$.[5]

Proposition 8. *Suppose that B and B' are C^*-algebras and $f : B \to B'$ is a quasi-unital map. Then the natural diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & \mathcal{M}B & \longrightarrow & \mathcal{Q}B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathcal{M}f} & & \downarrow{\scriptstyle \bar{f}} & & \\ 0 & \longrightarrow & B' & \longrightarrow & \mathcal{M}B' & \longrightarrow & \mathcal{Q}B' & \longrightarrow & 0 \end{array}$$

leads to the exact sequence of Banach spaces

$$0 \to \mathrm{Ker}(f) \to \mathrm{Ker}(\mathcal{M}f) \to \mathrm{Ker}(\bar{f}) \xrightarrow{\delta} \mathrm{Cok}(f) \to \mathrm{Cok}(\mathcal{M}f) \to \mathrm{Cok}(\bar{f}) \to 0.$$

The map δ is continuous. If $\mathcal{M}f$ is mono then δ is an isometry and the sequence degenerates to the exact sequence

$$0 \longrightarrow \mathrm{Ker}(\bar{f}) \xrightarrow{\delta} \mathrm{Cok}(f) \longrightarrow \mathrm{Cok}(\mathcal{M}f) \longrightarrow \mathrm{Cok}(\bar{f}) \longrightarrow 0$$

and if f is the inclusion of an ideal then δ is a C^*-map.

Proof. This follows by specializing the general results above. □

Theorem 9. *Suppose that B and B' are separable C^*-algebras and that $f : B \to B'$ is a quasi-unital monomorphism. Then the natural map*

$$\bar{f} : \mathcal{Q}B \to \mathcal{Q}B'$$

is a monomorphism.

Proof. We apply Proposition 8 to obtain the sequence

$$0 \longrightarrow \mathrm{Ker}(\bar{f}) \xrightarrow{\delta} \mathrm{Cok}(f) \longrightarrow \cdots$$

Now $\mathrm{Cok}(f)$ is a quotient of the separable C^*-algebra B' (as a metric vector space) and hence is separable. This, plus the fact that δ is an isometry, implies that $\mathrm{Ker}(\bar{f})$ is separable. On the other hand, $\mathrm{Ker}(\bar{f})$ is an ideal in $\mathcal{Q}B$ and we know from L. G. Brown [Br, Corollary 6] that $\mathcal{Q}B$ has no non-trivial separable ideals. The conclusion is that $\mathrm{Ker}(\bar{f}) = 0$ and $\bar{f}$ is mono. □

Remark 10. Klaus Thomsen has found a direct proof of the above result. It will be included in [S]. The original impetus for this work came from wanting an explicit realization of the map $KK_1(A, B) \to KK_1(A, B')$ induced from a C^*-map $B \to B'$. This is indeed possible, via the induced map $\bar{f} : \mathcal{Q}B \to \mathcal{Q}B'$. It is vital there to know that if f is mono then so is $\bar{f}$. For details see [S].

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[5] Here is the argument. Let $m \in \mathcal{M}B$ be such that $Mf(m) = 0$. Then $f(mb) = Mf(m)f(b) = 0$ for all $b \in B$. Since f is injective this means that $mb = 0$ for all $b \in B$ and hence $m = 0$.

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