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The Geometry of Badly Approximable Vectors

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Abstract. A vector $\mathbf{v} = (v_1, v_2, \ldots, v_k)$ in $\mathbb{R}^k$ is ϵ-*badly approximable* if for all m, and for $1 \le j \le k$, the distance $\|mv_j\|$ from mv_j to the nearest integer satisfies $\|mv_j\| > \epsilon m^{-1/k}$. A badly approximable vector is a vector that is ϵ-badly approximable for some $\epsilon > 0$. For the case of $k = 1$, these are just the badly approximable numbers, that is, the ones with a continued fraction expansion for which the partial quotients are bounded. One main result is that if $\mathbf{v}$ is a badly approximable vector in R^k then as $x \to \infty$ there is a lattice $\Lambda(\mathbf{v}, x)$, said lattice not too terribly far from cubic, so that most of the multiples $k\mathbf{v}$ mod 1, $1 \le k \le x$, of $\mathbf{v}$ fall into one of $O(x^{1/(k+1)})$ translates of $\Lambda(\mathbf{v}, x)$. Each translate of this lattice has on the order of $x^{k/(k+1)}$ of these elements. The lattice has a basis in which the basis vectors each have length comparable to $x^{-1/(k+1)}$, and can be listed in order so that the angle between each, and the subspace spanned by those prior to it in the list, is bounded below by a constant, so that the determinant of $\Lambda(\mathbf{v}, x)$ is comparable to $x^{-k/(k+1)}$.

A second main result is that given a badly approximable vector $\mathbf{v} = (v_1, v_2, \ldots, v_k)$, for all sufficiently large x there exist integer vectors $\mathbf{n}_j, 1 \le j \le k+1 \in \mathbb{Z}^{k+1}$ with euclidean norms comparable to x, so that the angle, between each $\mathbf{n}_j$ and the span of the $\mathbf{n}_i$ with $i < j$, is comparable to $x^{-1-1/k}$, and the angle between $(v_1, v_2, \ldots, v_k, 1)$ and each $\mathbf{n}_j$ is likewise comparable to $x^{-1-1/k}$. The determinant of of the matrix with rows $\mathbf{n}_j, 1 \le j \le k+1$ is bounded. This is analogous to what is known for badly approximable numbers α but for the case $k = 1$ we can arrange that the determinant be always 1.

Contents

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1. Introduction and notation

Given a list of k real numbers, we may ask how the successive multiples of the list are distributed with respect to the integer lattice $\mathbb{Z}^k$. Equivalently, we may require that the original list consist of numbers $0 \le v_j < 1$ and observe the successive locations of a point $\mathbf{p}_n$ in the unit k-cube I_k, when the first point $\mathbf{p}_0$ is the origin and where the entry at position j in $\mathbf{p}_n$ is the fractional part of nv_j, which we will also denote $nv_j \mod 1$. If any of the entries v_j is zero, the points are confined to a single face of I_k. If any is rational, they are confined to a finite number of hyperplanes parallel to one of those faces. If all entries are irrational, the distribution is asymptotically uniform, but even in this case the points need not spread themselves out as evenly as geometry will permit. The vectors for which the resulting distribution is always reasonably even are precisely those for which no integer multiple of the original $\mathbf{v}$ ever falls much nearer zero, mod 1, than need be. These are the badly approximable vectors, and even in this case, the point set

$$P(\mathbf{v}, x) := \{n\mathbf{v} \mod 1 : 1 \le n \le x\}$$

has large scale structures and regularities. It is these which we study here.

The badly approximable vectors are a generalization of badly approximable real numbers. These are much better understood, for there is the characterization by way of their continued fraction partial quotients. A real number α is badly approximable if there exists $\epsilon > 0$ so that for all positive integers m, the distance from $m\alpha$ to the nearest integer is at least ϵ/m. Badly approximable real numbers are precisely the real numbers for which the continued fraction expansion has bounded partial quotients [13]. We now turn to establishing the notation.

For a real number α, we define $\|\alpha\|$ to be the distance from α to the nearest integer. This distance function is subadditive but its scaling properties are not those of a true norm:

$$\|c\alpha\| = c\|\alpha\| \text{ if and only if } 0 \le c \le \frac{1}{2\|\alpha\|}.$$

Given a real vector $\mathbf{v} = (v_1, v_2, \dots, v_k)$, we define $\|\mathbf{v}\|$ to be the maximum value of $\|v_j\|$ for $1 \le j \le k$.

Definition 1. An ϵ-*badly approximable* vector in $\mathbb{R}^k$ is a vector $\mathbf{v} = (v_1, v_2, \dots, v_k)$ so that for all $m \ge 1$ and for all $1 \le j \le k$, the distance $\|mv_j\|$ from mv_j to the nearest integer satisfies $\|mv_j\| > \epsilon m^{-1/k}$. Equivalently, $\|m\mathbf{v} \mod 1\| > \epsilon m^{-1/k}$. A badly approximable vector is a vector that is ϵ-badly approximable for some $\epsilon > 0$.

The angle between two vectors $\mathbf{u}, \mathbf{v}$ is

$$\arg[\mathbf{u}, \mathbf{v}] = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\cdot|\mathbf{v}|}\right).$$

The span of $\{\mathbf{u}_1, \dots, \mathbf{u}_k\} \subset \mathbb{R}^n$ is denoted by $\langle \mathbf{u}_1, \dots, \mathbf{u}_k\rangle$. The angle between a vector $\mathbf{v}$ and a subspace B of $\mathbb{R}^n$ is the minimum angle between $\mathbf{v}$ and a non-zero vector $\mathbf{b} \in B$. The minimizing $\mathbf{b}$ is the orthogonal projection of $\mathbf{v}$ onto B. A vector $\mathbf{v} \in \mathbb{R}^k$ is badly approximable if and only if for some $\epsilon' > 0$, the angle between the vector $(v_1, v_2, \dots, v_k, 1)$ and every nonzero lattice vector $(a_1, a_2, \dots, a_{k+1}) \in \mathbb{Z}^{k+1}$ satisfies $\arg[\mathbf{v}, \mathbf{a}] \ge \epsilon'|\mathbf{a}|^{-1-1/k}$. This observation prompts us to define a badly approximable direction.

Definition 2. For any nonzero vector $\mathbf{v} \in \mathbb{R}^{k+1}$, the *direction* or one-dimensional subspace of $\mathbb{R}^{k+1}$ determined by $\mathbf{v}$, $\{c(v_1, v_2, \ldots, v_{k+1}) : c \in \mathbb{R}\}$, is badly approximable if and only if for some $\epsilon' > 0$, the angle between $(v_1, v_2, \ldots, v_k, v_{k+1})$ and every nonzero lattice vector $(a_1, a_2, \ldots, a_{k+1})$ satisfies $\arg[\mathbf{v}, \mathbf{a}] \geq \epsilon' |\mathbf{a}|^{-1-1/k}$.

No entry of a badly approximable direction is zero. Permuting the entries, and scaling the vector, has no effect on the geometry we study, so without loss of generality we restrict attention to those badly approximable directions in $\mathbb{R}^{k+1}$ determined by vectors $\mathbf{v}$ with $|v_1|, \ldots, |v_k| \leq 1$ and $v_{k+1} = 1$. The vector $(v_1, v_2, \ldots, v_k)$ is badly approximable if and only if the direction determined by $(v_1, v_2, \ldots, v_k, 1)$ is a badly approximable direction, and the corresponding ϵ and ϵ' are comparable. That is, given $k \geq 1$, there exists $C_k > 0$ so that if $\mathbf{v} \in \mathbb{R}^k$ is ϵ-badly approximable then $\{c(v_1, v_2, \ldots, v_k, 1) : c \in \mathbb{R}\}$ is an ϵ/C_k-badly approximable direction in $\mathbb{R}^{k+1}$ and vice versa.

There is an explicit class of examples of badly approximable vectors of the form (α_1, α_2) where the degree of the field extension $[Q(\alpha_1, \alpha_2) : Q] = 3$; a similar construction provides badly approximable vectors of all dimensions [4]. Davenport [2] shows that there are uncountably many such vectors; the proof is given in full detail for dimension 2 but generalizes to arbitrary dimension. W. Schmidt [11, 12] showed that the set of badly approximable vectors has full Hausdorff dimension. Lagarias [6] gives an algorithm based on the Lenstra-Lenstra-Lovasz lattice reduction algorithm; his algorithm yields a sequence of tolerably close to best-possible integer approximations to a given direction $\mathbf{v} = [v_1, v_2, v_3]$; equivalently it gives simultaneous rational approximations to $[v_1/v_3, v_2/v_3]$ by rational pairs $[s/q, t/q]$. See also [7]. Diamond and Pomerance [3] study the issue of very-nearly-parallel integer vectors. Moschevitin [9] shows that for some vectors, all best integer approximations lie in a three-dimensional sublattice. These last two results treat as it were the opposite extreme to the vectors we are concerned with here.

In this work, the notation $|\cdot|$ will denote the Euclidean norm in real n-space, $|\cdot|_1$ will be the L_1 norm, and as already mentioned $\|\cdot\|$ will apply to real numbers or vectors and denotes the distance or maximal coordinate distance to the nearest integer or lattice point respectively. To say that $A(\cdot) \asymp B(\cdot)$ will mean that there exist positive constants C_1 and C_2 so that in all cases, $C_1 A < B < C_2 A$. That is, the two quantities are comparable.

One main result is that given an ϵ-badly approximable vector $\mathbf{v} \in \mathbb{R}^k$, and a sufficiently large x, there exist geometric arrays, that is, translates of a subset of a full-dimensional lattice, into which most of the elements of $P(\mathbf{v}, x)$ (the multiples $n\mathbf{v}$ mod 1, $1 \leq n \leq x$) fall. Each array has on the order of $x^{k/(k+1)}$ elements. The differences between elements of any one of these arrays are themselves elements of a full-dimensional lattice $\Lambda(\mathbf{v}, x)$ in $\mathbb{R}^k$. This lattice has a lattice basis in which the vectors each have length comparable to $x^{-1/(k+1)}$, and they can be listed in order so that the dihedral angle between each, and the span of those listed previously, is bounded below by a constant depending only on k and ϵ. These basis vectors are congruent mod 1 to multiples of $\mathbf{v}$.

Remark. This result can be illustrated in striking fashion for $k = 3$. Pick some badly approximable vector $\mathbf{v} = (v_1, v_2, v_3)$, and some x on the order of 1000, and use a computer algebra system to display the list of multiples of $\mathbf{v}$ mod 1 as dots in a virtual cube. From most perspectives, the points will look rather well distributed, but with the right twist, they can be seen to lie in a series of parallel arrays. The Mathematica code snippets

```
v = N[{Sqrt[2], Sqrt[3], Sqrt[6]}, 20];
pts = Table[Map[# - Floor[#] &, n v], {n, 1, 1000}];

pts2 =Map[Point, pts];

lookat[u_] :=Show[Graphics3D[{PointSize[0.005], pts2}, ViewPoint
-> u]];

Do[lookat[10 Cos[n/60]{2, 2, 1} + 10 Sin[n/60]{2, -1, -2}], {n, 1,
300}]
```

serve nicely. The chosen $\mathbf{v}$ is a member of the class of explicit examples mentioned above.

The result is reminiscent of G. Marsaglia's famous observation [8] that (linear congruential) random numbers fall mainly in the planes.

A second main result is analogous to what is known for badly approximable numbers α. It is not hard to see that if α is a badly approximable number, then the vector $(\alpha, 1)$ determines a badly approximable direction. Furthermore, for all x there exists a unimodular integer basis $\{\mathbf{n}_1, \mathbf{n}_2\}$ of $\mathbb{R}^2$ so that $|\mathbf{n}_i| \asymp x$. But in our dimension k result, we only have that the determinant is bounded and non-zero.

2. Lemmas

No vector can escape approximation in direction entirely. The points on the unit ball in $\mathbb{R}^{k+1}$ representing the intersection of lines through the origin and another lattice point are, after all, dense. Likewise, for any vector $\mathbf{v} \in \mathbb{R}^k$ there exist positive integers n for which $\|n\mathbf{v}\|$ is particularly small.

Lemma 1. *Given a vector $\mathbf{v} = (v_1, v_2, \ldots, v_k) \in \mathbb{R}^k$, and $x > 2^k$, there exists an integer m, $1 \le m \le x$, so that for all $1 \le j \le k$, $|mv_j \mod 1| < 2x^{-1/k}$.*

Proof. take $y := [\sqrt[k]{x}]$, and partition the unit cube in $\mathbb{R}^k$ into y^k cubes of side $1/y$. Since there are more integers in $\{0, 1, \ldots, x\}$ than cubes, some two multiples m_1 and m_2 of v must belong to the same cube mod 1. Thus, $|(m_1 - m_2)v_j| \le 1/y$ for all j. Since $y \ge x^{1/k}/2$, $\|m\mathbf{v}\| \le 2x^{-1/k}$. □

Although our main results have to do with the geometry of the set of multiples $m\mathbf{v}$, $1 \le m \le x$, taken mod 1 in the unit cube $[0, 1)^k$, and so are independent of the concept of badly approximable directions, our proofs are based on a discussion of approximating integer vectors $\mathbf{n} \in \mathbb{Z}^{k+1}$ to the direction of the corresponding vector $(\mathbf{v}, 1) := (v_1, v_2, \ldots, v_k, 1)$, and their geometry.

Given a nonzero vector $\mathbf{w} \in \mathbb{R}^{k+1}$, recall that $\{c\mathbf{w} : c > 0\}$ is a badly approximable direction if there exists $\epsilon' > 0$ so that for all nonzero vectors $\mathbf{n} \in \mathbb{Z}^{k+1}$, $\arg[\mathbf{v}, \mathbf{n}] > \epsilon' |\mathbf{n}|^{-1-1/k}$. As in the case of badly approximable vectors, no entry of a vector $\mathbf{w}$ determining a badly approximable direction can be zero. Since scaling and permutation of the entries of a badly approximable direction do not affect the relevant geometric properties, without loss of generality we may and do require that vectors $\mathbf{w}$ determining a badly approximable direction satisfy the condition that for $1 \le j \le k$, $0 < |w_j| \le 1$, while $w_{k+1} = 1$. An elementary calculation shows that if $\mathbf{w}$ determines a badly approximable direction, then the truncated vector $\mathbf{w}' = (w_1, w_2, \ldots, w_k)$ is a badly approximable vector, and conversely. The corresponding values of ϵ need not be equal, but they are comparable to within a factor of $O(k)$. For $\mathbf{v} \in \mathbb{R}^k$, and $a \in \mathbb{R}$, let $(\mathbf{v}, a)$ denote the vector $(v_1, v_2, \ldots, v_k, a) \in \mathbb{R}^{k+1}$.

Lemma 2. *If* $\mathbf{v} = (v_1, v_2, \ldots, v_k)$ *with* $|v_i| < 1$ *for* $1 \le i \le k$, *and if* $|m\mathbf{v} - \mathbf{n}| = \delta > 0$, *then*

$$(1) \qquad \delta/(2\sqrt{k+1}|(\mathbf{n},m)|) \le \delta/(2\sqrt{k}m) \le \arg[(\mathbf{v},1),(\mathbf{n},m)] \le 2\delta/m \le 2\sqrt{k+1}\delta/|(\mathbf{n},m)|.$$

Proof. Use the series expansion of $\arg[(\mathbf{v},1),(\mathbf{n},m)] = \cos^{-1}(y)$ where $y = (m(\mathbf{v},1)\cdot(\mathbf{n},m))/|\mathbf{n} + (\mathbf{u},m)| \cdot |(\mathbf{n},m)|$, and $\mathbf{u} = m\mathbf{v} - \mathbf{n}$ so that $|u| \le \delta$. □

The Dirichlet box-principle observation of Lemma 1 above has a formulation involving angles.

Lemma 3. *If* $(\mathbf{v},1) \in \mathbb{R}^{k+1}$ *with* $|v_j| < 1$ *for* $1 \le j \le k$ *determines an* ϵ*-badly approximable direction, then for all* $x > 0$, *there exists an integer vector* $(\mathbf{n},m) \in \mathbb{Z}^{k+1}$ *so that*

$$(2) \qquad \arg[(\mathbf{n},m),(\mathbf{v},1)] \le 6k|(\mathbf{n},m)|^{-1-1/k} \text{ and } x \le |(\mathbf{n},m)| \le (5k)^k\epsilon^{-k}x.$$

Proof. For any direction vector $(\mathbf{v},1) = (v_1, v_2, \ldots v_k, 1)$ with $|v_i| < 1$ for $1 \le i \le k$, and for any $x > k$, there exists an integer $m \le (k+1)^{-1/2}(5k)^k\epsilon^k x$ so that for the integer vector $\mathbf{n}$ nearest $m\mathbf{v}$,

$$\|m\mathbf{v} - \mathbf{n}\| = \|m(\mathbf{v},1) - (\mathbf{n},m)\| \le 2(k+1)^{-1/(2k)}(5k)^{-1}\epsilon x^{-1/k}.$$

Since each $|n_j| \le m$, $|(\mathbf{n},m)| \le \sqrt{k+1}m \le (5k)^k\epsilon^{-k}x$. Now $|m(\mathbf{v},1) - (\mathbf{n},m)| \le 2\sqrt{k}(k+1)^{-1/(2k)}(5k)^{-1}\epsilon x^{-1/k}$, so

$$\arg[(\mathbf{v},1),(\mathbf{n},m)] \le \frac{1}{2}(k+1)^{-1/(2k)}k^{-1/2}\epsilon x^{-1/k}|(\mathbf{n},m)|^{-1}.$$

On the other hand, by hypothesis, $(\mathbf{v},1)$ is ϵ-badly approximable, meaning that $\arg[(\mathbf{v},1),(\mathbf{n},m)] \ge \epsilon|(\mathbf{n},m)|^{-1-1/k}$. Thus $|(\mathbf{n},m)| \ge 2^k\sqrt{k+1}k^{k/2}\epsilon^{-k}x$. Now $|(\mathbf{n},m)| \le (5k)^k\epsilon^{-k}x$, so $x^{-1/k} \le 5k|(\mathbf{n},m)|^{-1/k}\epsilon^{-1}$. Thus $\arg[(\mathbf{v},1),(\mathbf{n},m)] \le (5/2)(k+1)^{-1/(2k)}k^{1/2}|(\mathbf{n},m)|^{-1-1/k}$.

Hence $\arg[(\mathbf{v},1),(\mathbf{n},m)] \le 3k^{1/2}|(\mathbf{n},m)|^{-1-1/k}$ and so $x < 2^k\sqrt{k+1}k^{k/2}\epsilon^{-k}x \le |\mathbf{n},m)| \le (5k)^k\epsilon^{-k}x$. From this it follows that $\arg[(\mathbf{n},m),(\mathbf{v},1)] \le 6k|(\mathbf{n},m)|^{-1-1/k}$ as claimed in (2). □

Lemma 4. *The first* k *entries of a vector determining a badly approximable direction in* $\mathbb{R}^{k+1}$ *are irrational if the last entry is* 1.

Proof. If one of the other entries were rational, say $v_1 = a/b$, then the set of multiples $\{mb\mathbf{v} : 1 \le m \le x/b\}$ would include approximations to zero mod 1 with error $O((x/b)^{-1/(k-1)})$. For x sufficiently large, this will be less than $\epsilon x^{-1/k}$. □

Lemma 5. *Given linearly independent vectors* $\mathbf{u}_1, \ldots, \mathbf{u}_k \in \mathbb{R}^n$, *where* $k \le n$, *the image of the unit sphere in* $\mathbb{R}^k$, *under the mapping which takes* $(c_1, c_2, \ldots, c_k)$ *to* $\sum_1^k c_i\mathbf{u}_i$, *is an ellipsoid. The maximum of the absolute value of the change in* *direction of the latter, divided by the change in direction of the former, is the ratio of the longest to the shortest vector in this ellipsoid, which is the square root of the ratio of the largest to least eigenvalue of the matrix U^tU where the columns of U are the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$. The eigenvectors of U^tU corresponding to these eigenvalues are orthogonal.*

Proof. An ellipsoid in $\mathbb{R}^n$ may be defined as any translate of the set of all points $\sum_1^m x_j\mathbf{e}_j$ so that $\sum_1^k x_j^2 \le 1$ where the vectors e_j are mutually orthogonal. Any ellipsoid is the limit of, and a cross section of, a full-rank ellipsoid where $m = n$, for we may supply additional short vectors $\mathbf{e}_j, m+1 \le j \le n$. Any invertible linear transformation $\mathcal{L}$ on $\mathbb{R}^n$ carries the unit ball in $\mathbb{R}^n$ to a full-rank ellipsoid: Let L be the matrix of $\mathcal{L}$ with respect to the standard basis. Let $M := (L^tL)^{1/2}$. (That is, M is the symmetric positive definite matrix so that $M^2 = L^tL$.) Then for any $L\mathbf{x}$ so that $|\mathbf{x}| = 1$, there exists $\mathbf{z}$ so that $M\mathbf{z} = L\mathbf{x}$ and $|\mathbf{z}| = 1$: Take $\mathbf{z} = M^{-1}L\mathbf{x}$ and note that $|\mathbf{z}|^2 = \mathbf{x}^tL^tM^{-2}L\mathbf{x} = \mathbf{x}^t\mathbf{x} = 1$. Any non-invertible linear transformation $\mathcal{L}$ of rank $m < n$ is nevertheless invertible when restricted to its own span, and the image under $\mathcal{L}$ of the unit ball in $\mathbb{R}^n$ is equal to the image under $\mathcal{L}$ of the intersection of that ball with the orthogonal complement of the null space of $\mathcal{L}$. This is an isometric embedding into $\mathbb{R}^n$ of the image under L' of the unit ball in $\mathbb{R}^m$ where L' is the matrix of the restriction of $\mathcal{L}$ to this orthogonal complement, with respect to an orthonormal basis $\{\hat{\mathbf{b}}_j, 1 \le j \le m\}$ of that complement of the null space of $\mathcal{L}$. But the image in $\mathbb{R}^k$ of the unit ball in $\mathbb{R}^k$ under L' is, as we have already seen, an ellipsoid, which makes its isometric embedding in $\mathbb{R}^n$ under the mapping $(c_1, c_2, \ldots, c_m) \to \sum_1^m c_j\mathbf{b}_j$ also an ellipsoid. The claim about angle ratios is a consequence of the fact that if $\mathcal{L}$ is a linear transformation on $\mathbb{R}^n$, if $\mathbf{x}$ is a unit vector, and $\mathbf{h}$ is a vector orthogonal to $\mathbf{x}$, if L is the matrix of $\mathcal{L}$ with respect to the standard basis, and λ and Λ are the least and greatest eigenvalues of L^tL respectively, then

$$|L\mathbf{h}|^2 = \mathbf{h}^tL^tLh \le |\mathbf{h}|^2\Lambda$$

while

$$|L\mathbf{x}|^2 = \mathbf{x}^tL^tLx \ge |x|^2\lambda$$

The derivative of $\arg[\mathcal{L}(\mathbf{x}+t\mathbf{h}), \mathcal{L}\mathbf{x}]$ with respect to t at $t = 0$ is at most $|\mathcal{L}\mathbf{h}|/|\mathcal{L}\mathbf{x}|$ while, for $\mathbf{h}$ normal to $\mathbf{x}$ and at $t = 0$, the derivative of $\arg[(\mathbf{x}+t\mathbf{h}), \mathbf{x}]$ is $|\mathbf{h}|/|\mathbf{x}|$ which completes the proof. □

Lemma 6. *If $\mathbf{v} = (v_1, v_2, \ldots, v_k, 1)$ with $|v_j| < 1$ for $1 \le j \le k$ determines a badly approximable direction, if $\mathbf{n}_1, \mathbf{n}_2 \ldots, \mathbf{n}_k \in \mathbb{Z}^{k+1}$ are linearly independent over $\mathbb{R}$, and if $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in \mathbb{R}^k$ are given by $\mathbf{u}_j = m_j\mathbf{v} - \mathbf{n}_j$ where m_j is the $(k+1)^{st}$ (and last) entry of $\mathbf{n}_j$, then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ are also linearly independent.*

Proof. Suppose $\sum_1^k c_j\mathbf{u}_j = \mathbf{0}$. Then $\sum_1^k c_j(m_j\mathbf{v} - \mathbf{n}_j) = \mathbf{0}$ so that $(\sum_1^k c_jm_j)\mathbf{v} = \sum_1^k c_j\mathbf{n}_j$. But $\mathbf{v}$ cannot be a linear combination of k integer vectors in $\mathbb{R}^{k+1}$, for it would then belong to a k dimensional subspace with an integer vector basis. The coordinate vector $\mathbf{v}^*$, with respect to this basis, of $\mathbf{v}$ would be a vector in $\mathbb{R}^k$ and it would have integer approximations $\mathbf{p}$ in direction to within an angle of $O(|\mathbf{p}|^{-1-1/(k-1)})$.

Thus there would exist arbitrarily large integer vectors $\mathbf{p} \in \mathbb{Z}^k$ for which

$$\arg[\mathbf{v}^*, \mathbf{p}] \ll |\mathbf{p}|^{-k/(k-1)}.$$

Consider the angle between $\mathbf{v}$ and $\sum_1^k p_j\mathbf{n}_j$. Consider also two objects: the k-dimensional sphere S_n in $\mathbb{R}^{k+1}$ which is the intersection of the unit sphere in $\mathbb{R}^{k+1}$ with the span of $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_k$, and the unit sphere S^* in $\mathbb{R}^k$. The mapping which takes a vector $\sum_1^k c_j\mathbf{n}_j \in S_n$ to the vector $(c_1, c_2, \ldots, c_k)/\sqrt{\sum_1^k c_j^2} \in S^*$ is continuous and invertible. It also satisfies a Lipschitz condition: There are positive, finite upper and lower bounds, depending on the vectors $\mathbf{n}_i$, to the ratio of the lengths of the change in output to the change in input vectors.

Hence, for x sufficiently large, the good approximations in direction to $\mathbf{v}^*$ by integer vectors in $\mathbb{R}^k$ correspond to approximations to $\mathbf{v}$ by integer combinations of the $\mathbf{n}_j$ which are too good to be compatible with the definition of 'badly approximable', which was to have been a property of $\mathbf{v}$. □

3. The main results

Theorem 1. *If* $\mathbf{e}_1, \mathbf{e}_2 \ldots, \mathbf{e}_k \in \mathbb{R}^n$ *are linearly independent vectors, with* $1 \le |\mathbf{e}_i| \le C$ *for* $1 \le i \le k$, *and if* $\arg[\mathbf{e}_i, \langle \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{i-1}\rangle] \ge \theta > 0$ *for* $2 \le i \le k$, *then there exists* $K_1 = K_1(C, k, \theta) > 0$ *and* $K_2(C, k, \theta) > 0$ *so that for arbitrary* $(c_1, \ldots, c_k) \in \mathbb{R}^k$,

$$K_1^2\left|\sum_1^k c_i\mathbf{e}_i\right|^2 \le \sum_1^k c_i^2|\mathbf{e}_i|^2 \le K_2^2\left|\sum_1^k c_i\mathbf{e}_i\right|^2.$$

Proof. For the lower bound, we have

$$\sum_1^k c_i^2|\mathbf{e}_i|^2 \ge \max_{1\le i\le k} c_i^2|\mathbf{e}_i|^2 \ge \left(k^{-1}\sum_1^k |c_i\mathbf{e}_i|\right)^2 \ge k^{-2}\left|\sum_1^k c_i\mathbf{e}_i\right|^2.$$

Thus we may take $K_1 = k^{-1}$.

For the upper bound, we first put $\mathbf{e}_i' := \mathbf{e}_i/|\mathbf{e}_i|$, and $b_i := c_i|\mathbf{e}_i|$. Without loss of generality we take $\sum_1^k b_i^2 = 1$, and now we prove that $|\sum b_i\mathbf{e}_i'|^2 \ge 1/K_2$. The lattice with basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\}$ has determinant $\prod_1^k |\mathbf{e}_i| \prod_2^k \sin\theta_i \ge (\sin\theta)^{k-1}\prod_1^k |\mathbf{e}_i|$ (where θ_i is the angle between e_i and $\langle \mathbf{e}_1 \ldots \mathbf{e}_{i-1}\rangle$, so that $\theta_i \ge \theta$). Now if $b_k \ge (1/(K_2\sin\theta))$, then $|\sum_1^k b_i\mathbf{e}_i'| \ge 1/K_2$. If not, then $b_k\mathbf{e}_k' + b_{k-1}\mathbf{e}_{k-1}' = b_{k-1}\mathbf{e}_{k-1}' + \Theta(1/(K_2\sin\theta))$, where $\Theta(u)$ denotes a number of absolute value no greater than $|u|$. So in this case, if $|b_{k-1}| \ge (1/(K_2\sin^2\theta) + 1/(K_2\sin\theta))$, then the component of $b_{k-1}\mathbf{e}_{k-1}' + b_k\mathbf{e}_k'$ normal to the span of the first $k-2$ $\mathbf{e}$'s is $\ge 1/(K_2\sin\theta)$ and again the whole sum is large enough. Continuing in this fashion, a necessary condition for failure of the required inequality is that $|b_{k-j}| \le (1/K_2)(z + z^2 + \cdots + z^{j+1})$ where $z = 1/\sin\theta$. But with $K_2 = k\sin^{-k}\theta$, this is impossible and the conclusion must hold. □

Theorem 2. *Suppose* $\mathbf{v} = (v_1, v_2, \ldots, v_k, 1)$ *with* $k \ge 2$ *is a vector determining a badly approximable direction, with* $|v_j| < 1$ *for* $1 \le j \le k$, *and suppose*

$$\arg[\mathbf{v}, \mathbf{n}] > \epsilon|\mathbf{n}|^{-1-1/k}$$

for all non-zero $\mathbf{n} \in \mathbb{Z}^{k+1}$. *Then there exist positive constants* C, X, *and* λ, depending only on k and ϵ, *such that for each* $x > X$ *one can find linearly independent integer vectors* $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{k+1}$ *so that*:

(i) $x \le |\mathbf{n}_j| \le Cx$ *for* $1 \le j \le k+1$.

(ii) $\arg[\mathbf{n}_j, \langle \mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{j-1}\rangle] \geq \lambda x^{-1-1/k}$ *for* $2 \leq j \leq k+1$.
(iii) $\arg[\mathbf{n}_j, \mathbf{v}] \leq 6k|\mathbf{n}_j|^{-1-1/k}$.

Proof. We have already seen the case $j = 1$ of this, in Lemma 1. There one had $C = (5k)^{5k}\epsilon^{-k}$. We give a proof by induction, but starting from $j = 2$ which we prove from scratch. The inductive stage asserts that there exist constants $C_1, C_2, \ldots, C_j$ and $\lambda_2, \ldots, \lambda_j$, depending on k and ϵ so that for x sufficiently large,and for any ϵ-badly approximable $\mathbf{v}$, there exist integer vectors $\mathbf{n}_1, \ldots, \mathbf{n}_j$ so that $x \leq |\mathbf{n}_i| \leq C_i x$ for $1 \leq i \leq j$, while for $2 \leq i \leq j$, $\arg[\mathbf{n}_i, \langle \mathbf{n}_1, \ldots, \mathbf{n}_{i-1}\rangle] \geq \lambda_i x^{-(k+1)/k}$ yet $\arg[\mathbf{n}_i, \mathbf{v}] \leq R|\mathbf{n}_i|^{-(k+1)/k}$.

Let $R := 6k$ and $Q := (5k)^k\epsilon^{-k}$. Now for the case $j = 2$, take

$$K = (2R)^{k/(k+1)}\epsilon^{-k/(k+1)}Q.$$

By Lemma 3, there exists $\mathbf{n}_2 \in [Kx, QKx]$ so that $\arg[\mathbf{n}_2, \mathbf{v}] \leq R(Kx)^{-(k+1)/k}$. But then

$$\begin{aligned}\arg[\mathbf{n}_2, \mathbf{n}_1] &\geq \arg[\mathbf{n}_1, \mathbf{v}] - \arg[\mathbf{n}_2, \mathbf{v}] \\ &\geq \left(\epsilon Q^{-(k+1)/k} - RK^{-(k+1)/k}\right) x^{-(k+1)/k} \\ &\geq (\epsilon/2)Q^{-(k+1)/k}\end{aligned}$$

which gets the induction started with $\lambda_2 = \epsilon/2$, $C_1 = Q$, and $C_2 = KQx$.

Now consider a general j, $2 \leq j \leq k$, and assume that the inductive hypothesis holds up to j. We will show that there are constants C_{j+1} and λ_{j+1} so that the inductive assertion holds. Let w be the orthogonal projection of $\mathbf{v}$ onto $\langle \mathbf{n}_1, \ldots, \mathbf{n}_j\rangle$. Let λ denote some constant less than $\lambda_j/2$. We note that $\lambda_2 < \epsilon$, and that the λ's will form a decreasing sequence. We consider two possibilities: Either $\arg[\mathbf{v}, \mathbf{w}] > \lambda x^{-(k+1)/k}$, or not. If so, then we take M so that $RM^{-(k+1)/k} = \lambda/6$, and by Lemma 1, we take $\mathbf{n}$ to be a vector with $Mx \leq |\mathbf{n}| \leq QMx$ so that $\arg[\mathbf{n}, \mathbf{v}] < R|\mathbf{n}|^{-(k+1)/k} \leq R(Mx)^{-(k+1)/k}$. The desired inequalities then follow from $\arg[\mathbf{n}, \langle \mathbf{n}_1, \ldots, \mathbf{n}_j\rangle] \geq \arg[\mathbf{v}, \mathbf{w}] - \arg[\mathbf{n}, \mathbf{w}]$. We can take $\mathbf{n}_{j+1} = \mathbf{n}$, $\lambda_{j+1} = \lambda$, and $C_{j+1} = QM$.

The alternative is that for all $\lambda > 0$ there exist ϵ-badly approximable vectors $\mathbf{v}$, arbitrarily large values of x, and corresponding choices of $\mathbf{n}_1, \ldots, \mathbf{n}_j$, so that $\arg[\mathbf{v}, \mathbf{w}] \leq \lambda x^{-(k+1)/k}$. This, however, cannot happen, because $\mathbf{w}$ sits in a j-dimensional lattice and thus has good lattice vector approximations. These are so nearly parallel to $\mathbf{w}$ that, if $\mathbf{w}$ were in turn nearly parallel to $\mathbf{v}$, this would violate the defining condition for ϵ-badly approximable vectors. There is a catch, though. The lattice generated by the $\mathbf{n}$'s is not a cubic lattice, and is in fact rather far from it. The Dirichlet principle, and Lemma 1, only guarantee good approximations within a cubic lattice. We will have to have information specific to the lattice Λ generated by $\mathbf{n}_1, \ldots, \mathbf{n}_j$, and some way to relate it to cubic lattices. Not surprisingly, Lenstra-Lenstra-Lovasz reduced bases play a key rôle here.

Let Λ be the j-dimensional lattice in $\mathbb{R}^{k+1}$ generated by the integer vectors $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_j$. The assumptions in force are that $x \leq |\mathbf{n}_i| \leq C_i x$ for $1 \leq i \leq j$, and that $\arg[\mathbf{n}_i, \langle \mathbf{n}_1, \ldots, \mathbf{n}_{i-1}\rangle] \geq \lambda_i x^{-1-1/k}$ for $1 \leq i \leq j$, while

$$\begin{aligned}\arg[\mathbf{n}_i, \langle \mathbf{n}_1, \ldots, \mathbf{n}_{i-1}\rangle] &\leq \arg[\mathbf{n}_i, \mathbf{n}_{i-1}] \leq \arg[\mathbf{n}_i, \mathbf{v}] + \arg[\mathbf{v}, \mathbf{n}_{i-1}] \\ &\leq R|\mathbf{n}_i|^{-1-1/k} + R|\mathbf{n}_{i-1}|^{-1-1/k} < 2Rx^{-1-1/k}.\end{aligned}$$

Thus,

$$\det\Lambda = \prod_1^j |\mathbf{n}_i| \prod_2^j \sin\arg[\mathbf{n}_i, \langle \mathbf{n}_1, \ldots, \mathbf{n}_{i-1}\rangle] \le (2R)^{j-1} Q x^{1-(j-1)/k}.$$

On the other hand, if $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_j\}$ is a LLL-reduced lattice basis for Λ, then $2^{-j(j-1)/4}\prod_1^j |e_i| \le \det\Lambda \le \prod_1^j |\mathbf{e}_i|$. Furthermore, for such a basis,

$$\sin\arg[\mathbf{e}_i, \langle \mathbf{e}_1, \ldots, \mathbf{e}_{i-1}\rangle] \ge 2^{-(i-1)/2}$$

[1, p. 84].

Thus,

$$\prod_1^j |\mathbf{e}_i| \le 2^{j(j-1)/4}\det\Lambda \le 2^{j(j-1)/4}(2R)^{j-1} Q x^{1-(j-1)/k}.$$

Keeping $\mathbf{w} = \text{proj}\ [\mathbf{v}, \langle \mathbf{n}_1, \ldots, \mathbf{n}_{i-1}\rangle]$, write $\mathbf{w} = \sum_1^j \beta_i \mathbf{e}_i$. Let $\alpha_i = \beta_i / \sum_1^j |\beta_n||\mathbf{e}_n|$, so that $\sum_1^j |\alpha_i|\,|\mathbf{e}_i| = 1$.

We now introduce y and $z := (\epsilon/2)^{-(j-1)}(2y)^{(j-1)/k}(4(j-1))^{j-1}$, taking y large enough that $16^{j-1}z < y$. Let $b_i := 4z^{-1/(j-1)}|\mathbf{e}_i|^{-1}$, and let $u_{i,n} := [(n-1)b_i, nb_i)$. Partition the unit cube $[0,1)^{j-1}$ by partitioning the i^{th} axis, $2 \le i \le j$, into intervals $u_{i,n}$, $1 \le n \le 1 + \lfloor 1/b_i \rfloor$. Since each $\mathbf{e}_i$ is a nonzero integer vector, $|\mathbf{e}_i| \ge 1$ for all i, so that $b_i \le 1/4$. Thus, there are between $4^{-(j-1)} z \prod_2^j |e_i|$ and $(5/16)^{j-1} z \prod_2^j |e_i|$ boxes. Our choice of z thus means that there are no more than

$$(5/4)^{j-1}(\epsilon/2)^{-(j-1)}(j-1)^{j-1}(2y)^{(j-1)/k}\prod_2^j |e_i|$$

boxes in the partition. If $|\alpha_1|y$ is greater than this, then there must be two non-negative integers l_1 and l_2, both less than $|\alpha_1|y$ so that $(l_i/|\alpha_1|)(\sum_2^j \alpha_i e_i) \mod 1$ belong to the same box. Taking $l := |l_2 - l_1|$, this gives

$$(l/|\alpha_1|)\sum_1^j \alpha_i \mathbf{e}_i = \sum_1^j m_i \mathbf{e}_i + \sum_2^j \delta_i \mathbf{e}_i$$

where $m_i \in \mathbb{Z}$, $\delta_1 = 0$, and $|\delta_i| \le 4z^{-1/(j-1)}|\mathbf{e}_i|^{-1}$ for $2 \le i \le j$ so that with $\delta := \sum_1^j \delta_i \mathbf{e}_i$,

$$|\delta| \le 4(j-1)z^{-1/(j-1)}.$$

The same would hold if, instead of using $|\alpha_1|y$ as the upper bound for l, and partitioning the product of the last $j-1$ unit intervals, we used $|\alpha_r|$ in place of $|\alpha_1|$. Thus, the only way to avoid the existence of an $l \le |\alpha_r|y$ and δ satisfying the bound above, would be that for $1 \le r \le j$,

$$|\alpha_r|y < (5/16)^{j-1} z \prod_{i \ne r}^j |\mathbf{e}_i|.$$

But then,

$$y = \sum_1^j |\alpha_i||\mathbf{e}_i|y < j(5/16)^{j-1} z \prod_1^j |\mathbf{e}_i|.$$

Yet we took $z = (\epsilon/2)^{-j-1}(2y)^{(j-1)/k}(4(j-1))^{j-1}$, so this would mean that

$$y < j(5/16)^{j-1}(\epsilon/2)^{-j-1}(2y)^{(j-1)/k}(4(j-1))^{j-1}\prod_1^j |e_i|.$$

Now

$$\prod_1^j |\mathbf{e}_i| \le 2^{j)j-1)/4}\det\Lambda \le 2^{j(j-1)/4}(2R)^{j-1}Qx^{1-(j-1)/k}$$

so this gives $y < Mx$ where M is a constant that depends on j, ϵ, and k only. That is, the only way to avoid the existence of such a δ and l is to choose $y < Mx$. Thus, we take $y = 2Mx$ and the existence of the δ and l is assured. That is, there exists a positive integer l, an integer $1 \le r \le k$, and a vector $\delta \in \mathbb{R}^{k+1}$ so that $l \le |\alpha_r|y$ and $l\mathbf{v} = \sum_1^j m_i\mathbf{e}_i + \delta$, and

$$|\delta| \le 4(j-1)z^{-1/(j-1)}.$$

Now by Theorem 1, $|\sum_1^j m_i\mathbf{e}_i|$ is comparable to $\sum_1^j |m_i||\mathbf{e}_i|$. From the definition of z and the now-chosen value of $y = 2Mx$, we have $|\delta| \le (\epsilon/2)(4Mx)^{-1/k}$. Now with $\mathbf{m} = \sum_1^j m_i\mathbf{e}_i$, we have a multiple of $\mathbf{w}$, namely $l\sum_1^j(\alpha_i/\alpha_r)\mathbf{e}_i$, which is equal to $\mathbf{m}$ to within an error δ with $|\delta| \le 4(j-1)z^{-1/(j-1)} = (\epsilon/2)(4Mx)^{-1/k}$. But

$$|\mathbf{m}| \le \sum_1^k |m_i||\mathbf{e}_i| \le (1+o(1))\sum_1^j |l\alpha_i/\alpha_r||\mathbf{e}_i|.$$

Since $l \le |\alpha_r|y$, $|m_i| \le |\alpha_i|y$ so that $|\mathbf{m}| \le y$. Thus,

$$\arg[\mathbf{w},\mathbf{m}] \le (\epsilon/2)|\mathbf{m}|^{-1-1/k}.$$

Together with $\arg[\mathbf{w},\mathbf{v}] \le \lambda x^{-1-1/k}$, this gives

$$\arg[\mathbf{v},\mathbf{m}] \le (\epsilon/2)|\mathbf{m}|^{-1-1/k} + \lambda x^{-1-1/k}.$$

But now since $|m| \le 4Mx$, we have

$$\arg[\mathbf{v},\mathbf{m}] \le (\epsilon/2 + (4M)^{1+1/k}\lambda)|\mathbf{m}|^{-1-1/k}$$

and for λ sufficiently small, (in terms of M and ϵ, and thus ultimately in terms of j, k, and ϵ), we have

$$\arg[\mathbf{v},\mathbf{m}] < \epsilon|\mathbf{m}|^{-1-1/k}.$$

This contradicts the definition of a badly-approximable vector, and shows that the angle between $\mathbf{v}$ and the span of the j nearly-parallel vectors $\mathbf{n}_1,\ldots,\mathbf{n}_j$ must exceed some $\lambda_{j+1}x^{-1-1/k}$. This completes the induction. □

4. Lattice-like structures in the set of multiples mod 1 of a badly approximable vector

We now come to the topic of why it is that there is a lattice structure to the set of points $n\mathbf{v} \mod 1$ in the cube $[0,1)^k$. We maintain our convention that the standard form for vectors $\mathbf{v}$ determining our ϵ-badly approximable directions have entries $(v_1, v_2, \ldots, v_k, 1)$, with all entries but the last of absolute value less than 1. This imposes no real constraint.

Let $P(\mathbf{v}, x) := \{n\mathbf{v} \mod 1 : 1 \le n \le x\} \subseteq [0,1)^k$. According to Theorem 2, there exist constants $C_1(k,\epsilon) > 0$, $C_2(k,\epsilon) > 0$ so that for all y sufficiently large, there exist integer vectors $\mathbf{n}_1, \ldots, \mathbf{n}_k \in \mathbb{R}^{k+1}$ so that $y \leq |\mathbf{n}_j| \leq C_1(k,\epsilon)y$ and so that $\|m_j\mathbf{v}\| \leq C_2(k,\epsilon)y^{-1/k}$, where $\|\mathbf{w}\|$ denotes the maximum distance of any entry to the nearest integer, and where m_j denotes the $(k+1)$ entry of $\mathbf{n}_j$. Note also that $\|m_j\mathbf{v}\| \gg \epsilon y^{-1/k}$ from the definition of ϵ-badly approximable vectors and from the comparability of the lengths of the $\mathbf{n}_j$ to y. Without loss of generality we may require that $C_1(k,\epsilon) \geq 2C_2(k,\epsilon)$, and it will be convenient to impose this condition for later use.

Consider the remainder vectors $\mathbf{r}_j := m_j\mathbf{v} - \mathbf{n}_j$. We first observe that $\{\mathbf{r}_1, \ldots, \mathbf{r}_k\}$ is linearly independent over $\mathbb{R}$, because if $\sum_1^k c_j\mathbf{r}_j = 0$ we would have $\sum_1^k c_j m_j \mathbf{v} = \sum_1^k c_j \mathbf{n}_j$. But this puts $\mathbf{v}$ in the span of a set of k integer vectors $\{\mathbf{n}_1, \ldots, \mathbf{n}_k\}$. The coefficient vector $\mathbf{u} = (u_1, \ldots, u_k)$ of $\mathbf{v}$ with respect to this set has integer approximations in direction that are accurate to within an angle of $O(|\mathbf{u}|)^{-1-1/(k-1)})$. From Lemma 3 though, angles between underlying vectors are no greater than a bounded multiple of angles between the corresponding coefficient vectors. Thus, because $1/(k-1) > 1/k$, $\mathbf{v}$ itself would eventually have integer approximations in direction more accurate than allowed an ϵ-badly approximable direction.

A more exact and quantitative version of this line of thought shows that in fact the vectors $\mathbf{r}_j$ are not only linearly independent over $\mathbb{R}$, but that there exists a positive constants $C_4(k,\epsilon)$ so that for x sufficiently large, and for any ϵ-badly approximable direction determined by a v in standard form, there exist vectors $\mathbf{r}_j = m_j\mathbf{v} - \mathbf{n}_j$ so that

$$\arg[\mathbf{r}_j, \langle \mathbf{r}_1, \ldots, \mathbf{r}_{j-1}\rangle] \geq C_4(k,\epsilon)$$

for $2 \leq j \leq k$ That is, the vectors $\mathbf{r}_1, \ldots, \mathbf{r}_k$ form a basis for $\mathbb{R}^k$ in which the lengths are all comparable and the angles between the next vector, and the span of the previous ones, are all bounded below by a constant that depends on ϵ and k but on nothing else. But before proving this, we show that most points $n\mathbf{v} \mod 1$, $1 \leq n \leq x$ fall into a relatively few lattice equivalence classes. Given a lattice $\Lambda \subset \mathbb{R}^k$, a fixed vector $\mathbf{v} = (v_1, v_2, \ldots, v_k, 1)$, and integers m_1 and m_2, we say $m_1 \equiv m_2 \mod (\Lambda, \mathbf{v})$ if $((m_1(v_1, \ldots, v_k)) \mod 1) - ((m_2(v_1, \ldots, v_k)) \mod 1) \in \Lambda$. The equivalence class $[m, \Lambda, \mathbf{v}]$ denotes $\{m' : m' \equiv m \mod (\Lambda, \mathbf{v})\}$.

Theorem 3. *Given a vector $\mathbf{v} = (v_1, v_2, \ldots, v_k, 1) \in \mathbb{R}^{k+1}$ determining an ϵ-badly approximable direction, for sufficiently large x there exists a lattice $\Lambda = \Lambda(\mathbf{v}, x) \subset \mathbb{R}^k$ of determinant comparable to $x^{-k/(k+1)}$ with the following property: For almost all $m \leq x$,*

$$[m, \Lambda, \mathbf{v}] \cap \{1, 2, \ldots, x\}$$

has on the order of $x^{1-1/(k+1)}$ elements. In more detail, the result reads: For $\epsilon > 0$ and $k \geq 1$, there exist positive constants X, K_1, and K_2, and a function $z_{k,\epsilon}(x)$ tending to zero as $x \to \infty$ so that for all $x \geq X$ there are at least $x(1 - z_{k,\epsilon}(x))$ integers between 1 and x for which $[m, \Lambda, v] \cap \{1, 2, \ldots, x\}$ has between $K_1 x^{1-1/(k+1)}$ and $K_2 x^{1-1/(k+1)}$ elements.

Proof. We continue to use the notation about m_i, $\mathbf{r}_i$, and $\mathbf{n}_i$ of the discussion above. Our lattice is the lattice with basis $\mathbf{r}_1, \ldots, \mathbf{r}_k$ from the discussion above, taking $y = x^{k/(k+1)}$. One half of the assertion about its determinant is evident; the basis vectors each have length comparable to $x^{1/(k+1)}$ so $\det \Lambda \ll x^{-k/(k+1)}$. That it is comparable to this, follows from Theorem 4 below.

For the proof of the other assertions, consider

$$T(n,\mathbf{v},y) := \left\{(l_1,l_2,\dots,l_k)\in\mathbb{Z}^k : 1\le n+\sum_1^k l_i m_i \le y \text{ and } \sum_1^k l_i\|\mathbf{r}_i\|\le 1/2\right\}.$$

Since $m_i \le C_1(k,\epsilon)y^{k/(k+1)}$, if two conditions on l_i are satisfied, that

$$l_i \le \frac{1}{2kC_2(k,\epsilon)}y^{1/(k+1)} \text{ and that } \frac{1-n}{k} \le c_1(k,\epsilon)y^{k/(k+1)}l_i \le \frac{y-n}{k}$$

then $\mathbf{l}\in T(n,\mathbf{v},y)$. There are on the order of $y^{k/(k+1)}$ choices of $\mathbf{l}$ which satisfy the second condition, and since $C_1 \ge 2C_2$, these also satisfy the first condition. Thus, there are at least on the order of $y^{k/(k+1)}$ points in $\mathbb{R}^k$, arranged in lattice array, equivalent modulo the lattice to $n\mathbf{v}$, and all within the cube $n\mathbf{v} \mod 1 + [-1/2,1/2]^k$.

For most $n\le x$, a sizeable proportion of these points will be within the unit cube $[0,1)$. Thus, most points $n\mathbf{v} \mod 1, 1\le n\le x$ belong to a class of points, equivalent modulo this lattice, all of the form $n'\mathbf{v} \mod 1$ with $1\le n'\le x$, with on the order of $x^{k/(k+1)}$ elements. □

Remark, In one dimension already this makes sense. Most of the fractional parts of $n\sqrt{2}$ for which $1\le n\le x$ belong to one of $O(x^{1/2})$ arithmetic progressions of points $(n'+l_1m_1)\sqrt{2} \mod 1$ in which m_1 is comparable to $x^{1/2}$, $\|m_1\sqrt{2}\|$ is comparable to $x^{-1/2}$, and l_1 ranges through an interval of length comparable to $x^{1/2}$. If we allow one 'wraparound', then all the points belong to such an arithmetic progression.

Theorem 4. *If $\mathbf{v}\in\mathbb{R}^{k+1}$ determines an ϵ-badly approximable direction in $\mathbb{R}^{k+1}$, and if $\mathbf{n}_1,\mathbf{n}_2,\dots,\mathbf{n}_{k+1}$ are integer approximations to $\mathbf{v}$ in direction, with $x\le|\mathbf{n}_i|\le Cx$, linearly independent, and satisfying $\arg[\mathbf{n}_i,\mathbf{v}]\le 6k|\mathbf{n}_i|^{-1-1/k}$, then the angles $\arg[\mathbf{r}_i,\langle\mathbf{r}_1,\dots,\mathbf{r}_{i-1}\rangle]$ are bounded below, independently of $\mathbf{v}$ and of x. Here, C is the constant in the statement of Theorem 2. Both C, and the bounds of this result, do depend on k and ϵ.*

Remark. Thus, the lattice in Theorem 3 has generating vectors which are not far from being a reduced basis, and which correspond to values of m comparable to x. There is a related result, due to Lagarias [7, Theorem 5.1], that bounds the determinant of the matrix whose columns are consecutive best approximations to a given vector. The determinant is $\ll \exp(O(k^2\log(1/\epsilon)))$, with finitely many exceptions.

Proof. We first recall that all the $\mathbf{r}_i$ have lengths comparable to $x^{-1/(k+1)}$, and that the implicit constant in "comparable" depends only on k and ϵ. We note further that for any remainder vector $n\mathbf{v} \mod 1$ we have $\|n\mathbf{v}\|\gg \epsilon n^{-1/k}$.

Now let $\theta_2 := \arg[\mathbf{r}_1,\mathbf{r}_2]$. Let $\mathbf{r}_2'$ be the orthogonal projection of $\mathbf{r}_2$ onto $\mathbf{r}_1$. Then for all positive integers N, there exists $n, 1\le n\le N$ and n' so that $|n\mathbf{r}_2'-n'\mathbf{r}_1|\le|\mathbf{r}_1|/N$. Let $n\mathbf{r}_2' \mod \mathbf{r}_1 := n\mathbf{r}_2'-n'\mathbf{r}_1$. Then

$$|n\mathbf{r}_2-n'\mathbf{r}_1|^2 = |n\mathbf{r}_2' \mod \mathbf{r}_1|^2 + (n|\mathbf{r}_2|\sin\theta_2)^2.$$

On the other hand, $n\mathbf{r}_2-n'\mathbf{r}_1 = (nm_2\pm n'm_1)\mathbf{v} \mod 1 \ll Nx^{k/(k+1)}$. Hence

$$|n\mathbf{r}_2-n'\mathbf{r}_1| \gg (Nx^{k/(k+1)})^{-1/k}.$$

Thus

$$n^2|\mathbf{r}_2|^2\sin^2\theta_2 + O(N^{-2}x^{-2/(k+1)}) \gg N^{-2/k}x^{-2/(k+1)}$$

from which it follows that

$$\sin^2\theta_2 + O(N^{-2}) \gg N^{-2/k}.$$

Whatever the implicit constant in the O and in the $\gg$, there exists a choice of N on the order of 1 which maximizes this lower bound for $\sin^2\theta_2$ and the resulting bound is on the order of 1. That is, there exists $\epsilon_2(k,\epsilon)$ so that for all sufficiently large x, for all ϵ-badly approximable v, and for all linearly independent choices of $\mathbf{n}_1, \mathbf{n}_2$ with $y = x^{k/(k+1)}$ and $y \le m_1, m_2 \le C(k,\epsilon)y$ and $\mathbf{r}_i = m_i\mathbf{v} - \mathbf{n}_i$ with $|\mathbf{r}_i K(k,\epsilon)x^{-1/(k+1)}$, $\arg[\mathbf{r}_1, \mathbf{r}_2] \ge \theta_2(k,\epsilon)$.

This begins an induction. We will now proceed with freer use of O and $\gg$, it being understood that the implicit constants claimed are uniform over choices of x and $\mathbf{v}$ and the vectors $\mathbf{n}_i$ of length comparable to y that approximate $\mathbf{v}$ in direction, but do depend, (quite strongly, in fact), on k and ϵ.

Fix ϵ and k. Assume we have $\epsilon_2, \dots, \epsilon_{j-1}$ so that for sufficiently large x, for arbitrary ϵ-badly approximable $\mathbf{v}$, with $y = x^{k/(k+1)}$, approximating integer vectors $\mathbf{n}_i$, $1 \le i \le j$ with corresponding last entries $m_1, \dots, m_j$, so that $y \le m_i \le Cy$ for $1 \le y \le j$. Assume $\mathbf{r}_i = m_i\mathbf{v} - \mathbf{n}_i$ for $1 \le i \le j$. Then $\arg[\mathbf{r}_i, \langle \mathbf{r}_1, \dots, \mathbf{r}_{i-1}\rangle] \ge \epsilon_i$ for $1 \le i \le j-1$. We must show that there exists also ϵ_j with the same properties.

Consider $\mathbf{r}'_j$, the orthogonal projection of $\mathbf{r}_j$ onto $\langle \mathbf{r}_1, \dots, \mathbf{r}_{j-1}\rangle$. For arbitrary large N, but now taking N also to be the k^{th} power of an integer, there exists $1 \le n \le N$ so that $n\mathbf{r}'_j = \sum_1^{j-1} a_i\mathbf{r}_i + \sum_1^{j-1} u_i\mathbf{r}_i$ where the a_i are integers and $|u_i| \le N^{-1/(j-1)}$. We first claim that $\sum_1^{j-1}|a_i| \ll N$.

Otherwise, there would be instances in which some $|a_i|/N$ was arbitrarily large. But $|n\mathbf{r}'_j| \le |n\mathbf{r}_j| \ll Nx^{-1/(k+1)}$, while $|\sum_1^{j-1} a_i\mathbf{r}_i|^2$ satisfies the following inequalities:

$$\left|\sum_1^{j-1} a_i\mathbf{r}_i\right|^2 \ge a_{j-1}^2|\mathbf{r}_{j-1}|^2\sin^2\epsilon_{j-1}$$

$$\left|\sum_1^{j-1} a_i\mathbf{r}_i\right|^2 \ge (|a_{j-2}||\mathbf{r}_{j-2}\sin\epsilon_{j-2} - |a_{j-1}||\mathbf{r}_{j-1}|\mathbf{r}_{j-1}|)^2$$

$$\left|\sum_1^{j-1} a_i\mathbf{r}_i\right|^2 \ge (|a_{j-3}|\mathbf{r}_{j-3}|\sin\epsilon_{j-3} - |a_{j-1}|\mathbf{r}_{j-1}| - |a_{j-2}||\mathbf{r}_{j-2})^2,$$

etc.

But $|\sum_1^{j-1} a_i\mathbf{r}_i| \ll |n\mathbf{r}'_j| \ll Nx^{-1/(k+1)}$. Thus

$$N^2x^{-2/(k+1)} \gg a_{j-1}^2|\mathbf{r}_{j-1}|^2\sin^2\epsilon_{j-1}$$

$$N^2x^{-2/(k+1)} \gg (|a_{j-2}||\mathbf{r}_{j-2}\sin\epsilon_{j-2} - |a_{j-1}||\mathbf{r}_{j-1}|\mathbf{r}_{j-1}|)^2$$

and so on. From the first of these inequalities we conclude that $|a_{j-1}| \ll N$ since $|r_{j-1}| \asymp x^{-1/(k+1)}$. The next inequality gives

$$x^{-2/(k+1)} \gg \Big(\frac{a_{j-2}}{N}|\mathbf{r}_{j-2}|\sin\epsilon_{j-2} - O(x^{-1/(k+1)})\Big)^2$$

Now if $|a_{j-2}|/N$ cannot be arbitrarily large because this would contradict the bound $|\mathbf{r}_{j-2}| \gg x^{-1/(k+1)}$. Continuing in this way we see that for each $i, 1 \le i \le j-1$, $|a_i|/N$ is bounded. Thus, $\sum_1^{j-1} |a_i| \ll N$ as claimed. But now we have

$$\left| n\mathbf{r}_j - \sum_1^{j-1} a_i \mathbf{r}_i \right|^2 = \left| n\mathbf{r}'_j - \sum_1^{j-1} a_i \mathbf{r}_i \right|^2 + |n\mathbf{r}_j|^2 \sin^2\theta_j$$

The first term on the right is $\ll N^{-2/(j-1)} x^{-2/(k+1)}$, while on the left the expression is $\gg (nm_j + \sum_1^{j-1} |a_i| m_i)^{-2/k}$ since

$$\left(n\mathbf{r}_j - \sum_1^{j-1} a_i \mathbf{r}_i \right) = \left(nm_i - \sum_1^{j-1} a_i m_i \right) \mathbf{v} - \mathbf{n}$$

where n is an integer vector. Thus

$$\left| n\mathbf{r}_j - \sum_1^{j-1} a_i \mathbf{r}_i \right|^2 \gg (N x^{k/(k+1)})^{-2/k} = N^{-2/k} x^{-2/(k+1)}.$$

On the other hand, the right side of the equation above is

$$O(N^{-2/(j-1)} x^{-2/(k+1)}) + O(x^{-2/(k+1)} \sin^2\theta_j).$$

For N sufficiently large, the term involving $N^{-2/(j-1)}$ becomes negligible in comparison to the other term, so the right side is $O(x^{-2/(k+1)} \sin^2\theta_j)$. But we can take N large enough to bring to prominence this term, while keeping N bounded, and then $x^{-2/(k+1)} \sin^2\theta_j \gg N^{-2/k} x^{-2/(k+1)} \Rightarrow \sin^2\theta_j \gg 1$. Thus, there exists a constant $\epsilon_j(k, \epsilon) > 0$, independent of x, v, or the various $\mathbf{n}_i$. This completes the induction and proves Theorem 4. □

There is a converse to all of this:

Theorem 5. *If a vector* $\mathbf{v} \in \mathbb{R}^{k+1}$ *has the property that for all* x *there exist* $k+1$ *linearly independent integer vectors that approximate* $\mathbf{v}$ *in direction, to within* $Rx^{-1-1/k}$ *in angle, and that are comparable in length to* x*, then it is* ϵ*-badly approximable for some positive* ϵ*.*

To prove this theorem we first establish Lemma 7 below, which contains the bulk of the work. Fix $\mathbf{v}$, and for arbitrary large x let $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{k+1}$ be the linearly independent integer approximations to $\mathbf{v}$ in direction. The absolute value of the determinant of the matrix N with columns $\mathbf{n}_i$, $1 \le i \le k+1$, is a positive integer, and so is at least one. On the other hand, it is $(k+1)!$ times the volume of the simplex with vertices $\mathbf{n}_i, 1 \le i \le k+1$ together with the origin.

Scaling these vectors each to length exactly x (multiplying by c_i say) will not affect the volume by more than a bounded factor, but the volume of the resulting simplex is bounded by $(k+1)^{-1}$ times the k-dimensional volume of the simplex with vertices $c_i\mathbf{n}_i$, $1 \le i \le k+1$. For each of these, the distance to any of the others is $O(x^{-1/k})$. Thus the k-dimensional volume of this simplex is $O(x^{-k/k})$ so that the volume of the $k+1$-dimensional simplex is $O(1)$. Thus $\det N \asymp 1$.

Lemma 7. *If* $\mathbf{n}$ *has length comparable to* x*, and if* $\arg[\mathbf{n}, \mathbf{v}] \ll x^{-(k+1)/k}$*, then the coefficients* c_i *in* $\mathbf{n} = \sum_1^{k+1} c_i \mathbf{n}_i$ *are bounded. More precisely, suppose* $C_1 > 1$ *and* C_2 *are positive constants. Then there exists a positive constant* C_3 *so that if* $x \ge 2+$

$2C_2$, if $\mathbf{n} \in \mathbb{Z}^{k+1}$ with $x \le |\mathbf{n}| \le C_1 x$, if $\mathbf{v} \ne \mathbf{0} \in \mathbb{R}^{k+1}$ with $\arg[\mathbf{n}, \mathbf{v}] \le C_2 x^{-1-1/k}$, and if there exist $k+1$ linearly independent integer vectors $\mathbf{n}_j, 1 \le j \le k+1$ so that $x \le |n_j| \le C_1 x$ and $\arg[\mathbf{n}_j, \mathbf{v}] \le C_2 x^{-1-1/k}$ for $1 \le j \le k+1$, then the coefficients c_i in the representation of $\mathbf{n} = \sum_1^{k+1} c_i \mathbf{n}_i$ all satisfy $|c_i| \le C_3$.

Proof. Without loss of generality, $|\mathbf{v}| = 1$. Let $y = \mathbf{n} \cdot \mathbf{v}$. Since $\arg[\mathbf{n}, \mathbf{v}] \le 1/2$ and $|\mathbf{n}| \ge x$, $x/2 \le y \le C_1 x$. Let $\tilde{\mathbf{n}}_j = ((\mathbf{n} \cdot \mathbf{v})/(\mathbf{n}_j \cdot \mathbf{v}))\mathbf{n}_j$, so that $\tilde{\mathbf{n}}_j \cdot \mathbf{v} = \mathbf{n} \cdot \mathbf{v}$. It will suffice to show that the coefficients $\tilde{c}_j$ in $\mathbf{n} = \sum_{j=1}^{k+1} \tilde{c}_j \tilde{\mathbf{n}}_j$ are bounded (independent of x, $\mathbf{n}$ and the $\mathbf{n}_j$; the bound will depend on C_1 and C_2 and k.) We note that $\sum_1^{k+1} \tilde{c}_j = 1$. Now $|\det[\mathbf{n}_1 \ldots \mathbf{n}_{k+1}]| \ge 1$ so $|\det[\tilde{\mathbf{n}}_1 \ldots \tilde{\mathbf{n}}_{k+1}]| \gg 1$. The $\tilde{\mathbf{n}}_j$ all lie in a certain hyperplane orthogonal to $\mathbf{v}$. Let S denote the convex hull of these points. Then the volume of the $k+1$-dimensional simplex with base S and extra vertex $\mathbf{0}$ is $\gg 1$ so that the k-dimensional volume of S is $\gg x^{-1}$ and that of $x^{1/k}S$ is $\gg 1$. This prompts us to consider the vectors $\mathbf{p}_j$ defined to be the component of $x^{1/k}\tilde{\mathbf{n}}$ orthogonal to $\mathbf{v}$. Because $\arg[\mathbf{n_j}, \mathbf{v}] \ll x^{-1-1/k}$, $|\mathbf{p}_j| \ll 1$.

Now let $e_j = \tilde{c}_j - 1/(k+1)$ for $1 \le j \le k+1$, so that $\sum_1^{k+1} e_j = 0$. It will suffice now to prove that $|e_{k+1}| \ll 1$. First we note that the component of $\mathbf{n}$ orthogonal to $\mathbf{v}$ has norm $\ll x^{-1-1/k}$ by hypothesis, so that $\left|\sum_1^{k+1} \tilde{c}_j \mathbf{p}_j\right| \ll 1$. Now let $\mathbf{u} = \sum_1^{k+1} e_j \mathbf{p}_j = \sum_1^{k+1} \tilde{c}_j \mathbf{p}_j - (k+1)^{-1} \sum_1^{k+1} \mathbf{p}_j$. Since $|\mathbf{p}_j| \ll 1$, $|\mathbf{u}| \ll 1$. But $\sum_1^k e_j \mathbf{p}_j = \mathbf{u} - e_{k+1}\mathbf{p}_{k+1}$ so that

$$\mathbf{p}_{k+1} = \frac{-1}{e_{k+1}}\mathbf{u} + \sum_{j=1}^{k+1} \left(\frac{e_j}{\sum_{i=1}^k e_i} \right) \mathbf{p}_j.$$

The last term in this identity is an affine combination of the vertices of one face of S so that the distance from that face, to the opposite vertex $\mathbf{p}_{k+1}$, is $\ll 1/|e_{k+1}|$. But the edges of the face in question have bounded length so the $k-1$ dimensional volume of that face is bounded. Thus the distance to the opposite face cannot be arbitrarily small, from which it follows that the $|e_j|$ cannot be arbitrarily large, nor can the $|c_j|$. This is what was claimed. (A more detailed calculation along these lines shows that $|e_j| \le 2^{2k+1} C_1^{2k+1} C_2^k$ so that $|c_j| \le 2^{2k+2} C_1^{2k+2} C_2^k + C_1$.) □

To recapitulate: the result of the lemma is that if $\mathbf{v}$ has the property that for all x there exist integer vectors $\mathbf{n}_i, 1 \le i \le k+1$, linearly independent, of length comparable to x, and all within an angle of $Rx^{-1-1/k}$ of $\mathbf{v}$, then if $\mathbf{n}$ is another vector (integer or not) similarly near to parallel to $\mathbf{v}$ and also of length comparable to x , then in the representation $\mathbf{n} = \sum_1^{k+1} c_i \mathbf{n}_i$, the $|c_i|$ are bounded. This bound will depend upon the dimension, and upon the implicit constant in the premise about lengths 'comparable to x', but not upon $\mathbf{v}$ or x or the $\mathbf{n}_i$'s themselves.

Proof of Theorem 5. Assume to the contrary that for all $\epsilon > 0$, there exists an integer vector $\mathbf{n}$ so that $\arg[\mathbf{n}, \mathbf{v}] \le \epsilon |\mathbf{n}|^{-1-1/k}$. Let $x := \epsilon^{-k/(k+1)}|\mathbf{n}|$, and consider the integer vectors $\mathbf{n}_i$, $1 \le i \le k+1$ of length comparable to x and with $\arg[\mathbf{n}_i, \mathbf{v}] \ll x^{-1-1/k} \asymp \epsilon |\mathbf{n}|^{-1-1/k}$ given by our premise on $\mathbf{v}$. Let $\mathbf{n}' := \epsilon^{-k/(k+1)}\mathbf{n}$. According to the lemma, in the representation $\mathbf{n}' = \sum_1^{k+1} c_i \mathbf{n}_i$, the $|c_i|$ are bounded. But that would mean that for the integer vector $\mathbf{n}$, the coefficients $c_i' = c_i \epsilon^{k/(k+1)}$ in $\mathbf{n} = \sum_1^{k+1} c_i' \mathbf{n}_i$ were all small (arbitrarily close to zero). Yet the determinant D of the matrix with columns $\mathbf{n}_i$ is bounded, so that any integer vector's representation in terms of the $\mathbf{n}_i$ must have coefficients which are integer multiples of $1/D$. This contradiction shows that there cannot be integer vectors $\mathbf{n}$ for which $|n|^{1+1/k} \arg[\mathbf{n}, \mathbf{v}]$ is arbitrarily small, and that is Theorem 5. □

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