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Lifting Möbius Groups: Addendum

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Abstract. We show that if Γ is a group with $H^2(\Gamma;\mathbb{Z}_2) = 0$ then every representation of Γ into $\mathrm{PSL}(2,\mathbb{C})$ lifts to $\mathrm{SL}(2,\mathbb{C})$, but the converse does not hold.

In [2] we consider the natural surjection $\pi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$ and we examine the question of when a subgroup of $\mathrm{PSL}(2,\mathbb{C})$ can be lifted to $\mathrm{SL}(2,\mathbb{C})$. In Section 3 of that paper we let Γ be an abstract group and ask when a representation $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ can be lifted to a representation $\bar{\rho} : \Gamma \to \mathrm{SL}(2,\mathbb{C})$; that is $\rho = \pi\bar{\rho}$. It is pointed out there that if ρ is a faithful representation then the question is the same as whether the subgroup $\rho(\Gamma)$ has a lift, but it is not the same in general. A question that has been raised on this topic is which groups Γ have the property that every representation $\rho \in \mathrm{Hom}\,(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$ can be lifted; certainly any free group possesses this property.

We first show the following proposition which is known and is outlined in [3, p. 118], [4, p. 174] and [1, p. 755].

Proposition 1. *If* $H^2(\Gamma;\mathbb{Z}_2) = 0$ *then every representation* ρ *of* Γ *into* $\mathrm{PSL}(2,\mathbb{C})$ *can be lifted to* $\mathrm{SL}(2,\mathbb{C})$.

Proof. Given such a ρ, we seek a group $\hat{\Gamma}$ with $\hat{\Gamma}/\mathbb{Z}_2 = \Gamma$ and, if $\phi : \hat{\Gamma} \to \Gamma$ is the natural surjection, a representation $\hat{\rho} : \hat{\Gamma} \to \mathrm{SL}(2,\mathbb{C})$ with $\pi\hat{\rho} = \rho\phi$. If so then $\hat{\Gamma}$ is a central extension of $\mathbb{Z}_2$ by Γ, although note that $(\hat{\Gamma}, \phi, \hat{\rho})$ is not uniquely determined by these conditions: if ρ does lift to $\bar{\rho}$ and $\phi : \hat{\Gamma} \to \Gamma$ is any central extension of $\mathbb{Z}_2$ by Γ then setting $\hat{\rho} = \bar{\rho}\phi$ will do. In our case we define $\hat{\Gamma}$ using the pullback; this means that

$$\hat{\Gamma} = \{(\gamma, x) : \gamma \in \Gamma, x \in \pi^{-1}(\rho(\gamma))\},$$

$\phi(\gamma, x) = \gamma$ and $\hat{\rho}$ is the homomorphism from $\hat{\Gamma}$ to $\mathrm{SL}(2,\mathbb{C})$ defined by $\hat{\rho}(\gamma, x) = x$. Then if $\phi : \hat{\Gamma} \to \Gamma$ is the trivial central extension, so that $\hat{\Gamma} \cong \Gamma \times \mathbb{Z}_2$ and we can define the homomorphism $\phi^{-1} : \Gamma \to \Gamma \times \{I\}$, we get a lift of ρ by setting $\bar{\rho} = \hat{\rho}\phi^{-1}$. Moreover if ρ can be lifted to $\bar{\rho}$ then the function $(\gamma, x) \mapsto x(\bar{\rho}(\gamma))^{-1}$ is a surjective homomorphism from $\hat{\Gamma}$ to $\pm I$ with its kernel projecting down under ϕ to Γ, thus $\hat{\Gamma}$ must be the trivial central extension of Γ.

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However the central extensions of $\mathbb{Z}_2$ by Γ are described by the group $H^2(\Gamma;\mathbb{Z}_2)$ and so if $H^2(\Gamma;\mathbb{Z}_2)=0$ then there is only the trivial central extension which means that from above every representation of Γ into $\mathrm{PSL}(2,\mathbb{C})$ must lift. $\square$

A good source to look for examples of groups Γ with $H^2(\Gamma;\mathbb{Z}_2)=0$ is the fundamental group $\pi_1 X$ of an aspherical 3-manifold X, in which case the (co)homology groups of X are the same as those of $\pi_1 X$. We know for M a compact aspherical 3-manifold with boundary that

$$H^2(\pi_1 M;\mathbb{Z}_2)=H^2(M;\mathbb{Z}_2)=H_1(M,\partial M;\mathbb{Z}_2)$$

by Poincaré duality. If such a 3-manifold has one boundary component that is a torus and $H_1(M;\mathbb{Z})=\mathbb{Z}+G$ for G odd and finite then $H_1(M,\partial M;\mathbb{Z}_2)=0$ and so all representations can be lifted; included in these are all knot complements in S^3. If M is a closed aspherical 3-manifold then $H^2(\pi_1 M;\mathbb{Z}_2)=H_1(M;\mathbb{Z}_2)$ which is 0 if and only if $H_1(M;\mathbb{Z})$ is finite and has odd order. We know of one example: the Siefert-Wieber dodecahedral space is hyperbolic, thus aspherical, and has $H_1(M;\mathbb{Z})=(\mathbb{Z}_5)^3$ (see [5]) so every representation of its fundamental group lifts.

This begs the question: if all representations of Γ lift then is $H^2(\Gamma;\mathbb{Z}_2)=0$? The obvious candidates for counterexamples are simple groups that do not embed in $\mathrm{PSL}(2,\mathbb{C})$ so that the only representation is the identity.

Proposition 2. *The alternating groups A_n for $n\geq 8$ have every* $\mathrm{PSL}(2,\mathbb{C})$*-representation liftable but $H^2(A_n;\mathbb{Z}_2)\neq 0$.*

Proof. For $n\geq 5$ we know that A_n is simple, and although A_5 is a subgroup of $\mathrm{PSL}(2,\mathbb{C})$ so that its faithful representation cannot be lifted, A_n is certainly not a subgroup of $\mathrm{PSL}(2,\mathbb{C})$ for $n\geq 6$ so that here only the identity representation exists which certainly can be lifted. The Schur multiplier of A_n is $\mathbb{Z}_2$ for $n\geq 8$, which means that there exists a non-trivial central extension of $\mathbb{Z}_2$ by A_n. Thus $H^2(A_n;\mathbb{Z}_2)\neq 0$ but every representation lifts. $\square$

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