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Asymptotic Density in Combined Number Systems

Karen Yeats

Abstract. Necessary and sufficient conditions are found for a combination of additive number systems and a combination of multiplicative number systems to preserve the property that all partition sets have asymptotic density. These results cover and extend several special cases mentioned in the literature and give partial solutions to two problems in [6].

Contents

1. Introduction

The recent ancestry of the problems examined in this paper began in 1937 when abstract number systems with real valued norms were introduced by Arne Beurling [5] for the purpose of finding minimal sufficient conditions to prove prime number theorems. This is also the subject of Paul Bateman and Harold Diamond's article [2]. John Knopfmacher focused on density questions for classes of well known algebraic and topological structures in his books [12] and [13].

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Interest in number systems with the property that is central to the discussion below, namely that all partition sets have asymptotic density, started with Kevin Compton's work relating combinatorics to logic [9], [10], [11]. He proves logical limit laws on certain classes of structures using enumeration methods. Stanley Burris's book *Number Theoretic Density and Logical Limit Laws* [6] pulls the abstract number systems and the logical asymptotic combinatorics together.

In [6] Burris poses the following problem (Problem 5.20): Find necessary and sufficient conditions on two additive number systems $\mathcal{A}_1$ and $\mathcal{A}_2$ for all partition sets of $\mathcal{A}_1 * \mathcal{A}_2$ to have asymptotic density. He also poses the corresponding problem for multiplicative number systems (Problem 11.25): Find necessary and sufficient conditions on two multiplicative number systems $\mathcal{A}_1$ and $\mathcal{A}_2$ for all partition sets of $\mathcal{A}_1 * \mathcal{A}_2$ to have global asymptotic density.[1]

This paper considers these problems when $\mathcal{A}_1$ and $\mathcal{A}_2$ themselves have the property that all partition sets have asymptotic density. Elegant and popular special cases for additive and multiplicative systems will be considered first, followed by the general cases.

2. Preliminaries

Definition 1. $\mathbb{N} = \{0, 1, 2, \dots\}$.

Definition 2. A *number system*

$$\mathcal{A} = (\mathsf{A}, \mathsf{P}, *, \mathsf{e}, \|\ \|)$$

consists of a countable free commutative monoid $(\mathsf{A}, *, \mathsf{e})$, where P is the nonempty set of indecomposable elements, and $\|\ \|$ a norm.

Definition 3 ([6], 2.5 and 2.7)**.** An *additive number system* is a number system for which $\|\ \|$ is an *additive norm*, that is, a mapping from A to the nonnegative integers such that $\|\mathsf{a}\| = 0$ iff $\mathsf{a} = \mathsf{e}$, $\|\mathsf{a} * \mathsf{b}\| = \|\mathsf{a}\| + \|\mathsf{b}\|$, and for every $n \in \mathbb{N}$ the set $\{\mathsf{a} \in \mathsf{A} : \|\mathsf{a}\| = n\}$ is finite.

Definition 4 ([6], 8.1 and 8.2)**.** A *multiplicative number system* is a number system for which $\|\ \|$ is a *multiplicative norm*, that is, a mapping from A to the positive integers such that $\|\mathsf{a}\| = 1$ iff $\mathsf{a} = \mathsf{e}$, $\|\mathsf{a} * \mathsf{b}\| = \|\mathsf{a}\| \cdot \|\mathsf{b}\|$, and for every positive integer n the set $\{\mathsf{a} \in \mathsf{A} : \|\mathsf{a}\| = n\}$ is finite.

Definition 5 ([6], 2.12 and 2.28)**.** Given a number system $\mathcal{A}$, for each set $\mathsf{B} \subseteq \mathsf{A}$ the (*local*) *counting function* of B is

$$b(n) \;=\; \big|\{\mathsf{b} \in \mathsf{B} : \|\mathsf{b}\| = n\}\big|,$$

and the *global counting function* of B is

$$B(x) \;=\; \sum_{n \le x} b(n).$$

Notice that $B(x)$ is nondecreasing. As special cases we have $a(n)$ and $p(n)$, the (local) counting functions of A and P, respectively, and $A(x)$, the global counting function of A. We will refer to $a(n)$ and $p(n)$ as the (local) counting functions of $\mathcal{A}$ and to $A(x)$ as the global counting function of $\mathcal{A}$.

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[1] Burris in [6] uses $+$ for the combination of additive number systems and $\times$ for the combination of multiplicative number systems.

Definition 6 ([6], 3.21, 9.20)**.** For $\mathcal{A}$ a number system and $\mathsf{B}_1, \ldots, \mathsf{B}_k \subseteq \mathsf{A}$, let

$$\mathsf{B}_1 * \cdots * \mathsf{B}_k \;=\; \{\mathsf{b}_1 * \cdots * \mathsf{b}_k : \mathsf{b}_i \in \mathsf{B}_i\}.$$

Definition 7 ([6], 3.23, 9.22)**.** For $\mathcal{A}$ a number system and $\mathsf{B} \subseteq \mathsf{A}$,

$$\begin{aligned} \mathsf{B}^0 &= \{\mathsf{e}\} \\ \mathsf{B}^m &= \underbrace{\mathsf{B} * \cdots * \mathsf{B}}_{m \text{ times}} \quad \text{for } m > 0 \\ \mathsf{B}^{\leq m} &= \mathsf{B}^0 \cup \cdots \cup \mathsf{B}^m \quad \text{for } m \geq 0 \\ \mathsf{B}^{\geq m} &= \bigcup_{n \geq m} \mathsf{B}^n \quad \text{for } m \geq 0. \end{aligned}$$

Definition 8 ([6], 3.25, 9.24)**.** Given a number system $\mathcal{A}$ a subset B of A is a *partition set* of $\mathcal{A}$ if B can be written in the form

$$\mathsf{B} \;=\; \mathsf{P}_1^{\gamma_1} * \cdots * \mathsf{P}_k^{\gamma_k}$$

where $\mathsf{P}_1, \ldots, \mathsf{P}_k$ is a finite partition of the set P of indecomposables of $\mathcal{A}$, and each γ_i is of the form m, $\leq m$, or $\geq m$.

Definition 9 ([6], 4.14, 10.4)**.** Given number systems $\mathcal{A}_1$ and $\mathcal{A}_2$, both additive or both multiplicative, define $\mathcal{A}_1 * \mathcal{A}_2$ to be the number system whose underlying monoid is the direct product of the two monoids, and if $\mathcal{A}_1$ and $\mathcal{A}_2$ are additive then the norm is the sum of the coordinate norms, that is, $\|(\mathsf{a}_1, \mathsf{a}_2)\| = \|\mathsf{a}_1\|_1 + \|\mathsf{a}_2\|_2$, while if $\mathcal{A}_1$ and $\mathcal{A}_2$ are multiplicative then the norm is the product of the coordinate norms, that is, $\|(\mathsf{a}_1, \mathsf{a}_2)\| = \|\mathsf{a}_1\|_1 \cdot \|\mathsf{a}_2\|_2$.

Note that $\mathcal{A}_1 * \mathcal{A}_2$ is additive if $\mathcal{A}_1$ and $\mathcal{A}_2$ are additive and is multiplicative if $\mathcal{A}_1$ and $\mathcal{A}_2$ are multiplicative. The set of indecomposables of $\mathcal{A}_1 * \mathcal{A}_2$ is

$$\{(\mathsf{p}_1, \mathsf{e}) : \mathsf{p}_1 \in \mathsf{P}_1\} \cup \{(\mathsf{e}, \mathsf{p}_2) : \mathsf{p}_2 \in \mathsf{P}_2\}.$$

Let $a_i(n)$ and $p_i(n)$ be the local counting functions of $\mathcal{A}_i$ for $i = 1, 2$, and let $a(n)$ and $p(n)$ be the local counting functions of $\mathcal{A}_1 * \mathcal{A}_2$. Then

$$\begin{aligned} p(n) &= p_1(n) + p_2(n) \\ a(n) &= \begin{cases} \displaystyle\sum_{i+j=n} a_1(i)a_2(j) & \text{for } \mathcal{A} \text{ an additive number system} \\ \displaystyle\sum_{i \cdot j=n} a_1(i)a_2(j) & \text{for } \mathcal{A} \text{ a multiplicative number system.} \end{cases} \end{aligned}$$

Lemma 10. *Let* $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$ *be a multiplicative number system and let* $\mathsf{B} = \mathsf{B}_1 \times \mathsf{B}_2$ *where* $\mathsf{B}_i \subseteq \mathsf{A}_i$. *Then*

$$B(x) = \sum_{1 \leq k \leq x} b_1(k) B_2(x/k).$$

Proof. Theorem 3.10 from [1]. □

Lemma 11. *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *be number systems both additive or both multiplicative. Then any partition set* B *of* $\mathcal{A}_1 * \mathcal{A}_2$ *can be written in the form*

$$\mathsf{B} = \bigcup_{j=1}^{k} \mathsf{B}_{1j} \times \mathsf{B}_{2j}$$

for some $k \geq 1$ *where the union is disjoint and each* B_{ij} *is a partition set of* $\mathcal{A}_i$.

Proof. The proof is routine and thus is left to the reader to complete by using the definitions of m, $\geq m$, and $\leq m$, and the fact that if B, C, and D are sets of numbers from a number system, then $(\mathsf{B} \cup \mathsf{C}) * \mathsf{D} = (\mathsf{B} * \mathsf{D}) \cup (\mathsf{C} * \mathsf{D})$. $\square$

3. The nice cases

In this section we will deal with the most popular additive and multiplicative systems, namely the reduced additive systems and the strictly multiplicative systems. In Section 4 we look at the general situation.

3.1. Additive. *Throughout the additive subsections of this paper $\mathcal{A}$, $\mathcal{A}_1$, and $\mathcal{A}_2$ will denote additive number systems.* The most desirable of the additive systems are those which are reduced.

Definition 12 ([6], 2.41)**.** For $f : \mathbb{N} \to \mathbb{N}$ the *support* of f is

$$\mathsf{supp}\ f(n) \ = \ \{n \in \mathbb{N} : f(n) > 0\}.$$

Definition 13 ([6], 2.43)**.** $\mathcal{A}$ is *reduced* if $\gcd(\mathsf{supp}\ p(n)) = 1$.

Notice that $\mathcal{A}_1 * \mathcal{A}_2$ may be reduced even when $\mathcal{A}_1$ and $\mathcal{A}_2$ are not. However in this subsection we will only consider reduced systems $\mathcal{A}_i$.

Definition 14 ([6], 2.12)**.** Given $\mathcal{A}$, for each $\mathsf{B} \subseteq \mathsf{A}$ the *generating series* of B is

$$\mathbf{B}(x) \ = \ \sum_{n \geq 0} b(n) x^n.$$

Definition 15 ([6], 1.24 and 3.18)**.** Given $\rho \geq 0$, a real-valued function $f(n)$ that is eventually defined on $\mathbb{N}$ and eventually positive is in RT_ρ if

$$\lim_{n \to \infty} \frac{f(n-1)}{f(n)} \ = \ \rho,$$

that is, the radius of convergence of $\sum f(n) x^n$ can be found by the ratio test.

A power series is in RT_ρ if the sequence of coefficients is in RT_ρ. $\mathcal{A}$ is in RT_ρ if $a(n) \in \mathsf{RT}_\rho$, which can only occur if $a(n)$ is eventually nonzero.

Definition 16 ([6], 3.1)**.** For $\mathsf{B} \subseteq \mathsf{A}$, the *asymptotic density* $\delta(\mathsf{B})$ of B is as follows, provided the limit exists:

$$\delta(\mathsf{B}) \ = \ \lim_{\substack{n \to \infty \\ a(n) \neq 0}} \frac{b(n)}{a(n)}.$$

We will use the following notational conventions.

The radius of convergence of the generating function $\mathbf{A}_i(x)$ of $\mathcal{A}_i$ is ρ_i, the counting functions are $a_i(n)$ and $p_i(n)$, and when defined $\delta_i(\mathsf{B}_i)$ is the asymptotic density of B_i with respect to $\mathcal{A}_i$ for $\mathsf{B}_i \subseteq \mathsf{A}_i$. For $\mathcal{A}$ the preceding apply with the removal of the subscript i.

Since P, P_1, and P_2 are nonempty by definition, we know that ρ, ρ_1, and ρ_2 are in $[0, \infty)$. From Lemma 2.23 of [6] we know that ρ, ρ_1, and ρ_2 are in $[0, 1]$.

Lemma 17. *Let $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$ and $\mathsf{B} = \mathsf{B}_1 \times \mathsf{B}_2$ where $\mathsf{B}_i \subseteq \mathsf{A}_i$. Then:*

1. $\mathbf{B}(x) = \mathbf{B}_1(x) \cdot \mathbf{B}_2(x)$.
2. $\displaystyle\sum_{n \geq 0} B(n) x^n = \sum_{n \geq 0} B_1(n) x^n \cdot \mathbf{B}_2(x)$.

Proof. Both items follow from Proposition 3.33 of [6]. For the second item notice that $B(n) = \sum_{k\le n} B_1(k)b_2(n-k)$ is the same relation as that which holds between $b(n)$, $b_1(n)$, and $b_2(n)$. □

Lemma 18. *If* $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$ *then* $\rho = \min\{\rho_1, \rho_2\}$.

Proof. $\rho = \min\{\rho_1, \rho_2\}$ by Lemma 17 since $\mathbf{A}_1(x)$ and $\mathbf{A}_2(x)$ are power series with nonnegative coefficients. □

Definition 19. $\mathcal{A}$ has Property I if all partition sets of $\mathcal{A}$ have asymptotic density. $\mathcal{A}$ has Property II if $\rho > 0$, $\mathbf{A}(\rho) = \infty$, and $a(n) \in \mathsf{RT}_\rho$.

Our goal is to analyse when Property I holds. Some results of Bell, Bell et al., Burris and Sárközy, and Warlimont follow; then we will proceed towards the main result of the additive subsection, Theorem 28.

Proposition 20 ([6], 3.28)**.** *If* $\mathcal{A}$ *has Property* I *then* $\delta(\mathsf{P}) = 0$.

Proposition 21 ([3], [15], [4])**.** *If* $\delta(\mathsf{P}) = 0$ *then* $\rho > 0$ *and* $\mathbf{A}(\rho) = \infty$.

Proposition 22 ([6], 3.30)**.** $\mathcal{A}$ *has Property* I *and is reduced implies* $a(n) \in \mathsf{RT}_\rho$.

Corollary 23. $\mathcal{A}$ *has Property* I *and is reduced implies* $\mathcal{A}$ *has Property* II.

Proposition 24. *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *have Property* II *and be reduced, and let* $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$. *If* $\rho_1 \le \rho_2$, *then:*

- $\displaystyle\lim_{n\to\infty} \frac{a_1(n)}{a(n)} = \begin{cases} 0 & \text{if } \rho_1 = \rho_2 \\ \mathbf{A}_2(\rho_1)^{-1} & \text{if } \rho_1 < \rho_2. \end{cases}$
- $\displaystyle\lim_{n\to\infty} \frac{a_2(n)}{a(n)} = 0.$

Proof. For the second item take n large enough that $a_2(n) > 0$. Then for $m < n$

$$\frac{a(n)}{a_2(n)} \;=\; \sum_{k=0}^{n} a_1(k)\frac{a_2(n-k)}{a_2(n)} \;\ge\; \sum_{k=0}^{m} a_1(k)\frac{a_2(n-k)}{a_2(n)}.$$

So

$$\liminf_{n\to\infty} \frac{a(n)}{a_2(n)} \;\ge\; \sum_{k=0}^{m} a_1(k)\rho_2^k.$$

Now $\mathbf{A}_1(\rho_2) = \infty$; so by taking the limit as $m \to \infty$ we get the desired result.

For the first item if $\rho_1 = \rho_2$ apply the above proof with 1 and 2 interchanged; if $\rho_1 < \rho_2$ then apply Schur's Theorem.[2] □

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[2]By this is meant Schur's Tauberian Theorem which can be found in [6], 3.42. It states: Let $\mathbf{S}(x)$ and $\mathbf{T}(x)$ be two power series such that, for some $\rho \ge 0$, $\mathbf{T}(x) \in \mathsf{RT}_\rho$, and $\mathbf{S}(x)$ has radius of convergence greater than ρ. Then

$$\lim_{n\to\infty} \frac{[x^n]\big(\mathbf{S}(x)\cdot\mathbf{T}(x)\big)}{[x^n]\,\mathbf{T}(x)} = \mathbf{S}(\rho).$$

If $\mathbf{S}(\rho) > 0$ then this can be expressed as $[x^n]\big(\mathbf{S}(x)\cdot\mathbf{T}(x)\big) \sim \mathbf{S}(\rho)\cdot[x^n]\,\mathbf{T}(x)$.

Proposition 25. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ have Property* I *and be reduced with $\rho_1 = \rho_2$. Then $\mathcal{A}_1 * \mathcal{A}_2$ has Property* I *and for* B *a partition set of $\mathcal{A}_1 * \mathcal{A}_2$ and B_{ij} as in Lemma* 11 *we get*

$$\delta(\mathsf{B}) = \sum_{j=1}^{k} \delta_1(\mathsf{B}_{1j})\delta_2(\mathsf{B}_{2j}).$$

Proof. Let $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$.

First, consider partition sets of $\mathcal{A}$ of the form $\mathsf{B} = \mathsf{B}_1 \times \mathsf{B}_2$ where each B_i is a partition set of $\mathcal{A}_i$.

From $b_i(n)/a_i(n) \to \delta_i(\mathsf{B}_i)$ it follows that for all $\epsilon > 0$ there is an N such that for $m, n \geq N$,

$$\big|b_1(m)b_2(n) - \delta_1(\mathsf{B}_1)\delta_2(\mathsf{B}_2)a_1(m)a_2(n)\big| \;\leq\; \epsilon a_1(m)a_2(n).$$

Then, using Proposition 24, and the fact that a linear combination $\sum r_i\, o\big(a(n)\big)$ is $o\big(a(n)\big)$, we have

$$\begin{aligned}
&\big|b(n) - \delta_1(\mathsf{B}_1)\delta_2(\mathsf{B}_2)a(n)\big| \\
&\quad = \left|\sum_{j=0}^{n} b_1(j)b_2(n-j) - \delta_1(\mathsf{B}_1)\delta_2(\mathsf{B}_2)a_1(j)a_2(n-j)\right| \\
&\quad \leq \left|\sum_{j=N}^{n-N} b_1(j)b_2(n-j) - \delta_1(\mathsf{B}_1)\delta_2(\mathsf{B}_2)a_1(j)a_2(n-j)\right| \\
&\qquad + \left|\sum_{j=0}^{N-1} b_1(j)b_2(n-j) - \delta_1(\mathsf{B}_1)\delta_2(\mathsf{B}_2)a_1(j)a_2(n-j)\right| \\
&\qquad\quad + \left|\sum_{j=0}^{N-1} b_1(n-j)b_2(j) - \delta_1(\mathsf{B}_1)\delta_2(\mathsf{B}_2)a_1(n-j)a_2(j)\right| \\
&\quad \leq \; \epsilon a(n) + \left|\sum_{j=0}^{N-1} b_1(j)o\big(a(n)\big) - \delta_1(\mathsf{B}_1)\delta_2(\mathsf{B}_2)a_1(j)o\big(a(n)\big)\right| \\
&\qquad + \left|\sum_{j=0}^{N-1} o\big(a(n)\big)b_2(j) - \delta_1(\mathsf{B}_1)\delta_2(\mathsf{B}_2)o\big(a(n)\big)a_2(j)\right| \\
&\quad = \; \epsilon a(n) + o\big(a(n)\big).
\end{aligned}$$

Therefore $\delta(\mathsf{B}) = \delta_1(\mathsf{B}_1)\delta_2(\mathsf{B}_2)$.

Thus for a general partition set B of $\mathcal{A}$

$$\delta(\mathsf{B}) = \sum_{j=1}^{k} \delta_1(\mathsf{B}_{1j})\delta_2(\mathsf{B}_{2j})$$

by Lemma 11 and Lemma 3.2 from [6]. Therefore $\mathcal{A}$ has Property I. ☐

Lemma 26. *If $\lim_{n\to\infty} c(n)/d(n) = \ell > 0$ and $d(n) \in \mathsf{RT}_\rho$ then $c(n) \in \mathsf{RT}_\rho$.*

Proof. The hypotheses imply $c(n)$ is eventually nonzero, and then

$$\lim_{n\to\infty}\frac{c(n-1)}{c(n)} \;=\; \lim_{n\to\infty}\frac{c(n-1)}{d(n-1)}\frac{d(n)}{c(n)}\frac{d(n-1)}{d(n)} \;=\; \rho.$$

□

Proposition 27. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ have Property* I *and be reduced with $\rho_1 < \rho_2$. Then $\mathcal{A}_1 * \mathcal{A}_2$ has Property* I*. Let B be a partition set of $\mathcal{A}_1 * \mathcal{A}_2$ and let B_{ij} be as in Lemma* 11*. Then*

$$\delta(\mathsf{B}) = \sum_{j=1}^{k} \delta_1(\mathsf{B}_{1j})\frac{\mathbf{B}_{2j}(\rho_1)}{\mathbf{A}_2(\rho_1)}.$$

Proof. Let $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$. We know $\mathcal{A}_1 \in \mathsf{RT}_\rho$ by Proposition 22. Thus $\mathcal{A} \in \mathsf{RT}_\rho$ by Proposition 3.44 of [6].

Consider partition sets of $\mathcal{A}$ of the form $\mathsf{B} = \mathsf{B}_1 \times \mathsf{B}_2$ where B_i is a partition set of $\mathcal{A}_i$, $i = 1, 2$.

First, suppose $\delta_1(\mathsf{B}_1) \neq 0$. Then $\mathbf{B}_1(x) \in \mathsf{RT}_\rho$ by Lemma 26. Thus by Lemma 17 we can apply Schur's Theorem to get $b(n) \sim b_1(n)\mathbf{B}_2(\rho_1)$ and $a(n) \sim a_1(n)\mathbf{A}_2(\rho_1)$. Hence

$$\delta(\mathsf{B}) \;=\; \delta_1(\mathsf{B}_1)\frac{\mathbf{B}_2(\rho_1)}{\mathbf{A}_2(\rho_1)}.$$

Second, suppose $\delta_1(\mathsf{B}_1) = 0$; then for all $\epsilon > 0$ there exists an N such that $b_1(n)/a_1(n) < \epsilon$ for $n \geq N$. Thus, since $b_2(n) \leq a_2(n)$ for all n,

$$\begin{aligned}
b(n) \;&=\; \sum_{j=0}^{n} b_1(n-j)b_2(j)\\
&\leq\; \sum_{j=0}^{n-N} b_1(n-j)a_2(j) \;+\; \sum_{j=0}^{N-1} b_1(j)o\big(a(n)\big) \quad \text{by Proposition 24}\\
&\leq\; \epsilon\sum_{j=0}^{n-N} a_1(n-j)a_2(j) \;+\; o\big(a(n)\big)\\
&\leq\; \epsilon a(n) + o\big(a(n)\big)
\end{aligned}$$

which gives $\delta(\mathsf{B}) = 0$. Therefore in both cases $\delta(\mathsf{B}) = \delta_1(\mathsf{B}_1)\dfrac{\mathbf{B}_2(\rho_1)}{\mathbf{A}_2(\rho_1)}$.

Hence for a general partition set B of $\mathcal{A}$

$$\delta(\mathsf{B}) = \sum_{j=1}^{k} \delta_1(\mathsf{B}_{1j})\frac{\mathbf{B}_{2j}(\rho_1)}{\mathbf{A}_2(\rho_1)}$$

by Lemma 11 and Lemma 3.2 from [6]. Therefore $\mathcal{A}$ has Property I. □

Theorem 28. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ have Property* I *and be reduced with $\rho_1 \leq \rho_2$. Then $\mathcal{A}_1 * \mathcal{A}_2$ has Property* I *and for B a partition set of $\mathcal{A}_1 * \mathcal{A}_2$ and B_{ij} as in Lemma* 11 *we get*

$$\delta(\mathsf{B}) = \begin{cases} \displaystyle\sum_{j=1}^{k} \delta_1(\mathsf{B}_{1j}) \frac{\mathbf{B}_{2j}(\rho_1)}{\mathbf{A}_2(\rho_1)} & \text{if } \rho_1 < \rho_2, \\ \displaystyle\sum_{j=1}^{k} \delta_1(\mathsf{B}_{1j})\delta_2(\mathsf{B}_{2j}) & \text{if } \rho_1 = \rho_2. \end{cases}$$

Proof. If $\rho_1 = \rho_2$ apply Proposition 25; otherwise apply Proposition 27. □

Remark 29. If we assume only that $\mathcal{A}_1 * \mathcal{A}_2$ has Property I then we can not conclude that $\mathcal{A}_1$ and $\mathcal{A}_2$ have Property I. For instance if $1 < q_1 < q_2$, where q_2 is an integer, and we take $\mathcal{A}_1$ to be the additive number system with $p_1(n) = \lfloor q_1^n/n^2 \rfloor$ and $\mathcal{A}_2$ to be the additive number system with $p_2(n) = q_2^n$, then $p(n) = q_2^n + \lfloor q_1^n/n^2 \rfloor = q_2^n + O(q_1^n)$; so by Theorem 5.17 of [6], based on the asymptotics of Knopfmacher, Knopfmacher, and Warlimont, $\mathcal{A}_1 * \mathcal{A}_2$ has Property I. However $\rho_1 = 1/q_1$ and $\mathbf{P}_1(1/q_1) < \infty$ which gives $\mathbf{A}_1(1/q_1) < \infty$; so $\mathcal{A}_1$ does not have Property I.[3]

3.2. Multiplicative. *Throughout the multiplicative subsections of this paper, unless otherwise specified, $\mathcal{A}$, $\mathcal{A}_1$, and $\mathcal{A}_2$ will denote multiplicative number systems.*

Similarly to the additive situation, the most desirable multiplicative systems are those which are strictly multiplicative.

Definition 30 ([6], 9.39 and 9.48)**.** $\mathcal{A}$ is *discrete* if there is a positive integer λ such that $\|\mathsf{a}\|$ is an integer power of λ for any $\mathsf{a} \in \mathsf{A}$. $\mathcal{A}$ is *strictly multiplicative* if it is not discrete.

Strictly multiplicative systems will be the focus of this subsection.

Definition 31 ([6], 8.6)**.** Given $\mathcal{A}$, for each set $\mathsf{B} \subseteq \mathsf{A}$ the *generating series* of B is the Dirichlet series:

$$\mathbf{B}(x) \;=\; \sum_{n \geq 1} b(n) n^{-x}.$$

Definition 32 ([6], 7.19 and 9.16)**.** Given $\alpha \in \mathbb{R}$, a function $f(x)$ that is eventually defined on $\mathbb{R}$, and eventually positive, is in RV_α if for all $y > 0$

$$\lim_{x \to \infty} \frac{f(xy)}{f(x)} \;=\; y^\alpha,$$

that is, $f(x)$ has *regular variation at infinity of index α.* $\mathcal{A}$ is in RV_α if $A(x) \in \mathsf{RV}_\alpha$.

Definition 33 ([6], 9.3)**.** For $\mathsf{B} \subseteq \mathsf{A}$, the *global asymptotic density* $\Delta(\mathsf{B})$ of B is as follows, provided the limit exists:

$$\Delta(\mathsf{B}) \;=\; \lim_{n \to \infty} \frac{B(n)}{A(n)}.$$

We will use the following notational conventions.

The abscissa of convergence of the generating function $\mathbf{A}_i(x)$ of $\mathcal{A}_i$ is α_i, the functions $a_i(n)$, $p_i(n)$, and $A_i(x)$ are the local and global counting functions, and when defined $\Delta_i(\mathsf{B}_i)$ is the global asymptotic density of B_i with respect to $\mathcal{A}_i$ for $\mathsf{B}_i \subseteq \mathsf{A}_i$. For $\mathcal{A}$ the preceding apply with the removal of the subscript i.

[3] This counterexample is inspired by one suggested by the referee.

Since P, P_1, and P_2 are nonempty by definition, we know that α, α_1, and α_2 are nonnegative.

Lemma 34. *Let $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$ and $\mathsf{B} = \mathsf{B}_1 \times \mathsf{B}_2$ where $\mathsf{B}_i \subseteq \mathsf{A}_i$. Then*

$$\mathbf{B}(x) = \mathbf{B}_1(x) \cdot \mathbf{B}_2(x).$$

Proof. Apply Proposition 9.30 from [6]. □

Lemma 35. *If $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$ then $\alpha = \max\{\alpha_1, \alpha_2\}$.*

Proof. This follows from Lemma 34 since $\mathbf{A}_1(x)$ and $\mathbf{A}_2(x)$ both have nonnegative coefficients. □

Definition 36. $\mathcal{A}$ has Property I means that all partition sets of $\mathcal{A}$ have global asymptotic density. $\mathcal{A}$ has Property II if $\alpha < \infty$, $\mathbf{A}(\alpha) = \infty$, and $\mathcal{A} \in \mathsf{RV}_\alpha$.

As in the additive case Property I is the central property of interest. Some important results of Burris and Sárközy, and Warlimont follow; then we will proceed towards the main result of the multiplicative subsection, Theorem 47. Proposition 39 is a recently announced result of Warlimont proving conjecture 9.70 from [6].

Proposition 37 ([6], 9.28)**.** *If $\mathcal{A}$ has Property* I *then $\Delta(\mathsf{P}) = 0$.*

Proposition 38 ([14])**.** *If $\Delta(\mathsf{P}) = 0$ then $\alpha < \infty$.*

Proposition 39 ([16])**.** *If $\Delta(\mathsf{P}) = 0$ then $\mathbf{A}(\alpha) = \infty$.*

Proposition 40 ([6], 9.50)**.** *If $\mathcal{A}$ is strictly multiplicative and has Property* I *then $\mathcal{A} \in \mathsf{RV}_\alpha$.*

Corollary 41. *$\mathcal{A}$ has Property* I *and is strictly multiplicative implies $\mathcal{A}$ has Property* II.

We also need a multiplicative analogue of Schur's Theorem:

Proposition 42 ([6, 9.53])**.** *Given $\alpha \in \mathbb{R}$, suppose $\mathbf{S}(x)$ and $\mathbf{T}(x)$ are two Dirichlet series such that $\mathbf{T}(x)$ has nonnegative coefficients, $T(x) \in \mathsf{RV}_\alpha$, and the abscissa of absolute convergence of $\mathbf{S}(x)$ is less than α. Let $\mathbf{R}(x) = \mathbf{S}(x) \cdot \mathbf{T}(x)$. Then*

$$\lim_{x\to\infty} \frac{R(x)}{T(x)} = \mathbf{S}(\alpha).$$

If $\mathbf{S}(\alpha) > 0$ then this can be expressed as $R(x) \sim \mathbf{S}(\alpha) \cdot T(x)$.

Proposition 43. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ have Property* II *and be strictly multiplicative, and let $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$. If $\alpha_1 \geq \alpha_2$, then:*

- $\displaystyle \lim_{n\to\infty} \frac{A_1(n)}{A(n)} = \begin{cases} 0 & \text{if } \alpha_1 = \alpha_2 \\ \mathbf{A}_2(\alpha_1)^{-1} & \text{if } \alpha_1 > \alpha_2. \end{cases}$
- $\displaystyle \lim_{n\to\infty} \frac{A_2(n)}{A(n)} = 0.$

Proof.

$$\frac{A(n)}{A_2(n)} \;=\; \sum_{i\leq n} a_1(i) \frac{A_2(n/i)}{A_2(n)} \;\geq\; \sum_{i\leq m} a_1(i) \frac{A_2(n/i)}{A_2(n)}$$

for fixed $m < n$. So

$$\liminf_{n\to\infty} \frac{A(n)}{A_2(n)} \;\geq\; \sum_{i\leq m} a_1(i) i^{-\alpha_2}.$$

Now $\mathbf{A}_1(\alpha_2) = \infty$; so by taking the limit as $m \to \infty$ we get the second item.

For the first item, if $\alpha_1 = \alpha_2$ apply the above proof with 1 and 2 interchanged; if $\alpha_1 > \alpha_2$ then apply Proposition 42. □

Proposition 44. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ have Property* I *and be strictly multiplicative with $\alpha_1 = \alpha_2$. Then $\mathcal{A}_1 * \mathcal{A}_2$ has Property* I. *Let B be a partition set of $\mathcal{A}_1 * \mathcal{A}_2$ and let B_{ij} be as in Lemma* 11. *Then*

$$\Delta(\mathsf{B}) = \sum_{j=1}^{k} \Delta_1(\mathsf{B}_{1j})\Delta_2(\mathsf{B}_{2j}).$$

Proof. Let $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$.

First, consider partition sets of $\mathcal{A}$ of the form $\mathsf{B} = \mathsf{B}_1 \times \mathsf{B}_2$ where each B_i is a partition set of $\mathcal{A}_i$.

Take an $\epsilon > 0$. Let N be such that $\left|B_i(n) - \Delta_i(\mathsf{B}_i)A_i(n)\right| < \epsilon A_i(n)$ for $n \geq N$ and $i = 1, 2$. Notice that

$$\sum_{j=1}^{n} b_1(j)A_2(n/j) \;=\; \sum_{j=1}^{n} B_1(n/j)a_2(j), \tag{1}$$

since both sides are equal to the global counting function of $\mathsf{B}_1 \times \mathsf{A}_2$. Then

$$\begin{aligned}
&\left|B(n) - \Delta_1(\mathsf{B}_1)\Delta_2(\mathsf{B}_2)A(n)\right| \\
&\quad= \left|\sum_{j=1}^{n} b_1(j)B_2(n/j) - \Delta_1(\mathsf{B}_1)\Delta_2(\mathsf{B}_2)a_1(j)A_2(n/j)\right| \\
&\quad\leq \left|\sum_{j=1}^{n} b_1(j)B_2(n/j) - \Delta_2(\mathsf{B}_2)b_1(j)A_2(n/j)\right| \\
&\qquad + \left|\sum_{j=1}^{n} \Delta_2(\mathsf{B}_2)b_1(j)A_2(n/j) - \Delta_1(\mathsf{B}_1)\Delta_2(\mathsf{B}_2)a_1(j)A_2(n/j)\right| \\
&\quad= \underbrace{\left|\sum_{j=1}^{n} b_1(j)B_2(n/j) - \Delta_2(\mathsf{B}_2)b_1(j)A_2(n/j)\right|}_{\mathrm{I}} \\
&\qquad + \Delta_2(\mathsf{B}_2)\underbrace{\left|\sum_{j=1}^{n} B_1(n/j)a_2(j) - \Delta_1(\mathsf{B}_1)A_1(n/j)a_2(j)\right|}_{\mathrm{II}} \quad \text{by (1).}
\end{aligned}$$

Applying the triangle equality gives

$$\begin{aligned}
\text{Term I} \leq & \sum_{1\leq j\leq n/N} b_1(j)\big|B_2(n/j) - \Delta_2(\mathsf{B}_2)A_2(n/j)\big| \\
& + \sum_{n/N<j\leq n} b_1(j)\big|B_2(n/j) - \Delta_2(\mathsf{B}_2)A_2(n/j)\big| \\
\leq & \ \epsilon \sum_{1\leq j\leq n/N} b_1(j)A_2(n/j) \\
& + \sum_{1\leq k<N} \ \sum_{\frac{n}{k+1}<j\leq \frac{n}{k}} b_1(j)\big|B_2(k) - \Delta_2(\mathsf{B}_2)A_2(k)\big| \\
\leq & \ \epsilon A(n) + \sum_{k=1}^{N-1} \left(B_1\left(\tfrac{n}{k}\right) - B_1\left(\tfrac{n}{k+1}\right)\right)\big|B_2(k) - \Delta_2(\mathsf{B}_2)A_2(k)\big| \\
= & \ \epsilon A(n) + o\big(A(n)\big) \quad \text{by Proposition 43.}
\end{aligned}$$

Term II can be treated in the same way. Therefore $\Delta(\mathsf{B}) = \Delta_1(\mathsf{B}_1)\Delta_2(\mathsf{B}_2)$. Thus for a general partition set B of $\mathcal{A}$

$$\Delta(\mathsf{B}) = \sum_{j=1}^{k} \Delta_1(\mathsf{B}_{1j})\Delta_2(\mathsf{B}_{2j})$$

by Lemma 11 and Lemma 9.4 from [6]. Therefore $\mathcal{A}$ has Property I. □

Lemma 45. *Let B be a partition set of $\mathcal{A}$. If $\Delta(\mathsf{B}) \neq 0$ and $A(x) \in \mathsf{RV}_\alpha$ then $B(x) \in \mathsf{RV}_\alpha$.*

Proof. $B(x)$ is eventually nonzero, since $\Delta(\mathsf{B}) \neq 0$. Then, for all $y > 0$,

$$\begin{aligned}
\lim_{x\to\infty} \frac{B(xy)}{B(x)} &= \lim_{x\to\infty} \frac{B(xy)}{A(xy)}\frac{A(x)}{B(x)}\frac{A(xy)}{A(x)} \\
&= \lim_{x\to\infty} \frac{B(\lfloor xy\rfloor)}{A(\lfloor xy\rfloor)}\frac{A(\lfloor x\rfloor)}{B(\lfloor x\rfloor)}\frac{A(xy)}{A(x)} \ = \ y^\alpha.
\end{aligned}$$

□

Proposition 46. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ have Property* I *and be strictly multiplicative with $\alpha_1 > \alpha_2$. Then $\mathcal{A}_1 * \mathcal{A}_2$ has Property* I. *Also for B a partition set of $\mathcal{A}_1 * \mathcal{A}_2$ and for B_{ij} as in Lemma* 11 *we get*

$$\Delta(\mathsf{B}) = \sum_{j=1}^{k} \Delta_1(\mathsf{B}_{1j})\frac{\mathbf{B}_{2j}(\alpha_1)}{\mathbf{A}_2(\alpha_1)}.$$

Proof. Let $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$.

Consider partition sets of $\mathcal{A}$ of the form $\mathsf{B} = \mathsf{B}_1 \times \mathsf{B}_2$ where B_i is a partition set of $\mathcal{A}_i$, $i = 1, 2$.

First, suppose $\Delta_1(\mathsf{B}_1) \neq 0$. Then $B_1(x) \in \mathsf{RV}_{\alpha_1}$ by Lemma 45. Thus by Lemma 34 we can apply Proposition 42 to get $B(x) \sim B_1(x)\mathbf{B}_2(\alpha_1)$ and $A(x) \sim A_1(x)\mathbf{A}_2(\alpha_1)$. So, as $n \to \infty$,

$$\frac{B(n)}{A(n)} \ \to \ \Delta_1(\mathsf{B}_1)\frac{\mathbf{B}_2(\alpha_1)}{\mathbf{A}_2(\alpha_1)}.$$

Second, suppose $\Delta_1(\mathsf{B}_1) = 0$. Then for all $\epsilon > 0$ there is an N such that $B_1(n)/A_1(n) < \epsilon$ for $n \geq N$. Thus

$$\begin{aligned} B(n) &= \sum_{1\leq j\leq n/N} B_1(n/j)b_2(j) + \sum_{n/N<j\leq n} B_1(n/j)b_2(j) \\ &\leq \sum_{1\leq j\leq n/N} B_1(n/j)b_2(j) + \sum_{1\leq k<N} \sum_{\frac{n}{k+1}<j\leq \frac{n}{k}} B_1(k)b_2(j) \\ &\leq \epsilon \sum_{1\leq j\leq n/N} A_1(n/j)a_2(j) + \sum_{1\leq k<N} B_1(k)\big(B_2\big(\tfrac{n}{k}\big) - B_2\big(\tfrac{n}{k+1}\big)\big) \\ &\leq \epsilon A(n) + \sum_{1\leq k<N} B_1(k)o\big(A(n)\big) \qquad \text{by Proposition 43} \\ &\leq \epsilon A(n) + o\big(A(n)\big) \end{aligned}$$

which gives $\Delta(\mathsf{B}) = 0$. Therefore in both cases $\Delta(\mathsf{B}) = \Delta_1(\mathsf{B}_1)\dfrac{\mathbf{B}_2(\alpha_1)}{\mathbf{A}_2(\alpha_1)}$.

Hence for a general partition set B of $\mathcal{A}$

$$\Delta(\mathsf{B}) = \sum_{j=1}^{k} \Delta_1(\mathsf{B}_{1j})\frac{\mathbf{B}_{2j}(\alpha_1)}{\mathbf{A}_2(\alpha_1)}$$

by Lemma 11 and Lemma 9.4 from [6]. Therefore $\mathcal{A}$ has Property I. □

Theorem 47. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ have Property* I *and be strictly multiplicative with $\alpha_1 \geq \alpha_2$. Then $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$ has Property* I. *Also for B a partition set of $\mathcal{A}_1 * \mathcal{A}_2$ and for B_{ij} as in Lemma* 11 *we get*

$$\Delta(\mathsf{B}) = \begin{cases} \displaystyle\sum_{j=1}^{k} \Delta_1(\mathsf{B}_{1j})\frac{\mathbf{B}_{2j}(\alpha_1)}{\mathbf{A}_2(\alpha_1)} & \text{if } \alpha_1 > \alpha_2, \\ \displaystyle\sum_{j=1}^{k} \Delta_1(\mathsf{B}_{1j})\Delta_2(\mathsf{B}_{2j}) & \text{if } \alpha_1 = \alpha_2. \end{cases}$$

Proof. If $\alpha_1 = \alpha_2$ apply Proposition 44; otherwise apply Proposition 46. □

4. The general situation

4.1. Additive. Now, returning to the additive notational conventions, we extend the results to the case when $\mathcal{A}_1$ and $\mathcal{A}_2$ are not necessarily reduced.

Definition 48. Let $d = \gcd(\mathsf{supp}\ p(n))$ and let $j|d$. Define $\mathcal{A}^{*j}$ to be the additive number system obtained from $\mathcal{A}$ by altering the norm so that $\|a\|^{*j} = \|a\|/j$. Let $a^{*j}(n)$ and $p^{*j}(n)$ be the counting functions of $\mathcal{A}^{*j}$, and let ρ^{*j} be the radius of convergence of $\mathbf{A}^{*j}(x)$, the generating function of $\mathcal{A}^{*j}$.

Notice that $a^{*j}(n) = a(nj)$, $p^{*j}(n) = p(nj)$, $\gcd(\mathsf{supp}\ p^{*j}(n)) = 1$, and $\rho^{*j} = \rho^j$. If $d = j$ then we drop the j in the exponent and write $\mathcal{A}^*$, which is a reduced additive system called the *reduced form of* $\mathcal{A}$. Notice also that if $\mathcal{A}$ is not reduced then $\mathcal{A} \notin \mathsf{RT}_\rho$ since $a(n)$ is infinitely often zero.

As an additional notational convention we will let $d_i = \gcd(\mathsf{supp}\ p_i(n))$ and $d = \gcd(\mathsf{supp}\ p(n))$.

Here is an example of the value of working with nonreduced systems. For q a power of a prime and d a positive integer, define the additive number system $\mathcal{A}_{q,d}$ as the system formed from the monic elements of $\mathbb{F}_q[x^d]$ with polynomial multiplication as the operation and $\|g(x)\| = \deg g(x)$.

Let $a_{q,d}(n)$ be the local counting function for $\mathcal{A}_{q,d}$. Then

$$a_{q,d}(n) = \begin{cases} q^m & \text{if } dm = n \\ 0 & \text{if } d \nmid n. \end{cases}$$

So the radius of convergence of the generating function of $\mathcal{A}_{q,d}$ is

$$\rho = \frac{1}{q^{1/d}}.$$

Now, for all d, $\mathcal{A}^*_{q,d}$ looks like $\mathcal{A}_{q,1}$ under the mapping $g(x^d) \mapsto g(x)$. From $a_{q,1}(n) = q^n$ we see by the Knopfmacher, Knopfmacher, and Warlimont asymptotics (see Theorem 5.17, [6]) that $\mathcal{A}_{q,1}$ has Property I and so, as we will see in Lemma 51, $\mathcal{A}_{q,d}$ has Property I for all d.

Determining when $\mathcal{A}_{q_1,d_1} * \mathcal{A}_{q_2,d_2}$ has Property I is rather more involved.

Theorem 57, the main theorem of this subsection, shows that $\mathcal{A}_{q_1,d_1} * \mathcal{A}_{q_2,d_2}$ has Property I iff

- $1/q_1^{1/d_1} < 1/q_2^{1/d_2}$ and $d_1|d_2$, or
- $1/q_1^{1/d_1} > 1/q_2^{1/d_2}$ and $d_2|d_1$, or
- $1/q_1^{1/d_1} = 1/q_2^{1/d_2}$.

For instance, $\mathcal{A}_{2,2} * \mathcal{A}_{2,4}$ and $\mathcal{A}_{2,1} * \mathcal{A}_{2,3}$ have Property I but $\mathcal{A}_{2,2} * \mathcal{A}_{2,3}$ does not. Such facts require a careful study of nonreduced systems.

Lemma 49. *For all* $\mathsf{B} \subseteq \mathsf{A}$, $\delta(\mathsf{B}) = \lim_{n\to\infty} \frac{b^*(n)}{a^*(n)}$, *provided* $\delta(\mathsf{B})$ *exists.*

Proof. From Hua's Theorem (as found for instance in [6], 2.49), $a^*(n)$ is eventually nonzero; so $\lim_{n\to\infty} \frac{b^*(n)}{a^*(n)} = \lim_{\substack{n\to\infty\\ a(n)\neq 0}} \frac{b(n)}{a(n)}$. □

The next lemma follows easily from the definitions.

Lemma 50. *Let* $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$*; then* $d = \gcd(d_1, d_2)$*. Let* $j|d$*; then* $\mathcal{A}^{*_j} = \mathcal{A}_1^{*_j} * \mathcal{A}_2^{*_j}$ *iff* $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$*.*

Lemma 51. $\mathcal{A}^{*_j}$ *has Property* I *iff* $\mathcal{A}$ *has Property* I*, whenever* $j|d$*.*

Proof. $\mathcal{A}$ and $\mathcal{A}^{*_j}$ have the same reduced form, so apply Lemma 3.3 from [6]. □

From this we get a slight variation on Proposition 22.

Proposition 52. $\mathcal{A}$ *has Property* I *implies* $a^*(n) \in \mathsf{RT}_{\rho^*}$*.*

The key to the more general case is Proposition 54, a modified version of Proposition 24. In order to prove Proposition 54 we will need the following Lemma.

Lemma 53. *If* $f(n) \in \mathsf{RT}_\rho$*,* $\rho > 0$*, and* $\sum_{n\geq 0} f(n)\rho^n = \infty$ *then for all* k *and all* $m \geq 1$ *we have* $\sum_{n\equiv k \bmod m} f(n)\rho^n = \infty$*.*

Proof. Pick an $\epsilon > 0$. Let N be such that $f(j+1) > 0$ and $f(j)/f(j+1) < \epsilon + \rho$ for $j > N$. Then, for suitable constants C_1 and C,

$$\begin{aligned}\sum_{j\equiv k \bmod m} f(j)\rho^j &= \sum_{\substack{j\equiv k \bmod m\\ j\leq N}} f(j)\rho^j + \sum_{\substack{j\equiv k \bmod m\\ j>N}} f(j)\rho^j \\ &\leq C_1 + (\epsilon+\rho) \sum_{\substack{j\equiv k \bmod m\\ j>N}} f(j+1)\rho^j \\ &= C + \left(\frac{\epsilon+\rho}{\rho}\right) \sum_{j\equiv k+1 \bmod m} f(j)\rho^j.\end{aligned}$$

Now $\sum_{n\geq 0} f(n)\rho^n = \infty$ implies that for all m we have $\sum_{j\equiv k \bmod m} f(j)\rho^j = \infty$ for some k. Then by the above the latter equation holds for all k $\square$

Proposition 54. *Let $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$, $\gcd(d_1, d_2) = 1$, $\mathbf{A}_1(\rho_1) = \infty$, and $\mathcal{A}_2^* \in \mathsf{RT}_{\rho_2^*}$. If $\rho_1 \leq \rho_2$ then $\lim_{n\to\infty} a_2(n-j)/a(n) = 0$ for all integers $j \geq 0$.*

Proof. Let us restrict out attention to n large enough that $a(n-j) > 0$. This is possible since $d = \gcd(d_1, d_2) = 1$; so $\mathcal{A}$ is reduced. For $d_2 \nmid n-j$ we have $a_2(n-j) = 0$, and so $a_2(n-j)/a(n) = 0$. Assume $d_2 | n-j$, and let $d_2 m = n-j$. Now

$$\begin{aligned}\frac{a(d_2m+j)}{a_2(d_2m)} &= \sum_{k=0}^{d_2m+j} a_1(k)\frac{a_2(d_2m+j-k)}{a_2(d_2m)} \\ &= \sum_{-j/d_2\leq i\leq m} a_1(d_2i+j)\frac{a_2(d_2m-d_2i)}{a_2(d_2m)} \\ &= \sum_{-j/d_2\leq i\leq m} a_1(d_2i+j)\frac{a_2^*(m-i)}{a_2^*(m)} \\ &\geq \sum_{i=0}^{k} a_1(d_2i+j)\frac{a_2^*(m-i)}{a_2^*(m)}\end{aligned}$$

for fixed $k < m$. Thus

$$\begin{aligned}\liminf_{m\to\infty}\frac{a(d_2m+j)}{a_2(d_2m)} &\geq \sum_{i=0}^{k} a_1(d_2i+j)\liminf_{m\to\infty}\frac{a_2^*(m-i)}{a_2^*(m)} \\ &= \sum_{i=0}^{k} a_1(d_2i+j)\rho_2^{d_2i}.\end{aligned}$$

Taking the limit as $k\to\infty$, we have

$$\begin{aligned}\liminf_{m\to\infty}\frac{a(d_2m+j)}{a_2(d_2m)} &\geq \sum_{i=0}^{\infty} a_1(d_2i+j)\rho_2^{d_2i} \\ &= \rho_2^{-j}\sum_{d_1\ell\equiv j \bmod d_2} a_1(d_1\ell)\rho_2^{d_1\ell} \\ &\geq \rho_2^{-j}\sum_{d_1\ell\equiv j \bmod d_2} a_1^*(\ell)(\rho_1^*)^{\ell}\end{aligned}$$

since $\rho_2^{d_1} \geq \rho_1^{d_1} = \rho_1^*$. Now since $\gcd(d_1, d_2) = 1$, $d_1\ell \equiv j \mod d_2$ is the same as $\ell \equiv c \mod d_2$ for some c. We know that $\sum_{\ell \geq 0} a_1^*(\ell)(\rho_1^*)^\ell = \infty$, and $\rho_1 > 0$ since $\mathbf{A}_1(\rho_1) = \infty$. Apply Lemma 53 to get $\lim_{n\to\infty} a_2(n-j)/a(n) = 0$. □

Proposition 55. *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *have Property* I *with* $\rho_1 = \rho_2$*. Then* $\mathcal{A}_1 * \mathcal{A}_2$ *has Property* I*. Let* B *be a partition set of* $\mathcal{A}_1 * \mathcal{A}_2$ *and let* B_{ij} *be as in Lemma* 11*. Then*

$$\delta(\mathsf{B}) = \sum_{j=1}^{k} \delta_1(\mathsf{B}_{1j})\delta_2(\mathsf{B}_{2j}).$$

Proof. Let $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$. By Lemmas 50 and 51 we need only consider $d = 1$. Since $\rho_1 = \rho_2$ we can switch the roles of $\mathcal{A}_1$ and $\mathcal{A}_2$ in Proposition 54 to get $a_1(n-j) = o\big(a(n)\big)$ for all integers $j \geq 0$.

From $b_i(nd_i)/a_i(nd_i) \to \delta_i(\mathsf{B}_i)$ it follows that for all $\epsilon > 0$ there is an N such that for $md_1, nd_2 \geq N$,

$$\big|b_1(md_1)b_2(nd_2) - \delta_1(\mathsf{B}_1)\delta_2(\mathsf{B}_2)a_1(md_1)a_2(nd_2)\big| \;<\; \epsilon a_1(md_1)a_2(nd_2).$$

Clearly $a_i(k) = 0$ gives $b_i(k) = 0$ for all k; so for $j, k \geq N$,

$$\big|b_1(j)b_2(k) - \delta_1(\mathsf{B}_1)\delta_2(\mathsf{B}_2)a_1(j)a_2(k)\big| \;\leq\; \epsilon a_1(j)a_2(k).$$

Now continue as in the proof of Proposition 25 using Proposition 54 in place of Proposition 24. □

Proposition 56. *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *have Property* I *with* $\rho_1 < \rho_2$*. Then* $\mathcal{A}_1 * \mathcal{A}_2$ *has Property* I *iff* $d_1|d_2$*. If* $d_1|d_2$ *then for* B *a partition set of* $\mathcal{A}_1 * \mathcal{A}_2$ *and for* B_{ij} *as in Lemma* 11 *we get*

$$\delta(\mathsf{B}) = \sum_{j=1}^{k} \delta_1(\mathsf{B}_{1j})\frac{\mathbf{B}_{2j}(\rho_1)}{\mathbf{A}_2(\rho_1)}.$$

Proof. Let $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$. $d_1|d_2$ iff $d_1 = d$ since $d = \gcd(d_1, d_2)$. We need only consider the case when $d = 1$, that is, $\mathcal{A}$ is reduced, because $d_1|d_2$ iff $\frac{d_1}{d}\big|\frac{d_2}{d}$ and Lemmas 50 and 51 give $\mathcal{A}_1 * \mathcal{A}_2$ has Property I iff $\mathcal{A}_1^{*d} * \mathcal{A}_2^{*d}$ has Property I. $\mathcal{A}_1 \in \mathsf{RT}_\rho$ iff $\mathcal{A} \in \mathsf{RT}_\rho$ by Proposition 3.44 of [6].

($\Rightarrow$) Suppose that $\mathcal{A}$ has Property I and radius of convergence ρ. Then $\mathcal{A} \in \mathsf{RT}_\rho$ since $\mathcal{A}$ is reduced, and so $\mathcal{A}_1 \in \mathsf{RT}_\rho$. Therefore $\mathcal{A}_1$ is reduced, and so $d_1 = 1 = d$.

($\Leftarrow$) Suppose that $d_1 = 1 = d$; then $\mathcal{A}_1 \in \mathsf{RT}_{\rho_1}$. Now continue as in the proof of Proposition 27 using Proposition 54 in place of Proposition 24. □

Theorem 57. *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *have Property* I *with* $\rho_1 \leq \rho_2$*. Then* $\mathcal{A}_1 * \mathcal{A}_2$ *has Property* I *iff* $d_1|d_2$ *or* $\rho_1 = \rho_2$*. If* $\mathcal{A}_1 * \mathcal{A}_2$ *has Property* I *then for* B *a partition set of* $\mathcal{A}_1 * \mathcal{A}_2$ *and* B_{ij} *as in Lemma* 11 *we get*

$$\delta(\mathsf{B}) = \begin{cases} \displaystyle\sum_{j=1}^{k} \delta_1(\mathsf{B}_{1j})\frac{\mathbf{B}_{2j}(\rho_1)}{\mathbf{A}_2(\rho_1)} & \text{if } \rho_1 < \rho_2, \\ \displaystyle\sum_{j=1}^{k} \delta_1(\mathsf{B}_{1j})\delta_2(\mathsf{B}_{2j}) & \text{if } \rho_1 = \rho_2. \end{cases}$$

Proof. If $\rho_1 = \rho_2$ apply Proposition 55; otherwise apply Proposition 56. □

For any additive number system $\mathcal{A}$ define $\mathcal{A}_{\mathsf{q}}$, the extension of $\mathcal{A}$ by an indecomposable q, to be the additive system formed by adding a new indecomposable q to P, by letting A_{q} be the set of formal expressions $\mathsf{q}^m * \mathsf{a}$, for m a nonnegative integer and $\mathsf{a} \in \mathsf{A}$, and by extending the norm with the definition $\|\mathsf{q}^m * \mathsf{a}\| = m\|\mathsf{q}\| + \|\mathsf{a}\|$.

Corollary 58. *Suppose $\mathcal{A}$ has Property* I. *An extension $\mathcal{A}_{\mathsf{q}}$ has Property* I *iff $d \big| \|\mathsf{q}\|$ or $\rho = 1$.*

Proof. We can form an additive number system $\mathcal{Q} = (\mathsf{A}_{\mathcal{Q}}, \mathsf{P}_{\mathcal{Q}}, *, \mathsf{e}, \|\ \|)$ generated by q by letting $\mathsf{P}_{\mathcal{Q}} = \{\mathsf{q}\}$, $\mathsf{A}_{\mathcal{Q}} = \{\mathsf{q}^m : m \geq 0\}$, $\mathsf{q}^{m_1} * \mathsf{q}^{m_2} = \mathsf{q}^{m_1+m_2}$, $\mathsf{e} = \mathsf{q}^0$, and $\|\mathsf{q}^m\| = m\|\mathsf{q}\|$. Then $\mathcal{A}_{\mathsf{q}} \cong \mathcal{A} * \mathcal{Q}$, $\mathcal{Q}$ has Property I, $\rho_{\mathcal{Q}} = 1$, and $d_{\mathcal{Q}} = \|\mathsf{q}\|$. Now apply Theorem 57. $\square$

If $\mathcal{A}$ is reduced in the above corollary then we have Theorem 3.59 of [6].

Corollary 59. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be additive number systems with $\mathcal{A}_1^*, \ldots, \mathcal{A}_n^*$ in RT_1. Then $(\mathcal{A}_1 * \cdots * \mathcal{A}_n)^*$ is also in RT_1.*

Proof. Let $\mathcal{A} = \mathcal{A}_1 * \cdots * \mathcal{A}_n$. It suffices to prove the theorem for $n = 2$. By Theorem 4.2 of [6] $\mathcal{A}_i^* \in \mathsf{RT}_1$ iff $\mathcal{A}_i^*$ has Property I and $\rho_i = 1$, and likewise for $\mathcal{A}^*$. Now apply Theorem 57. $\square$

This gives the additive half of Theorem 16.1 from [7]. If all the $\mathcal{A}_i$ are reduced then we have Stewart's Theorem ([6], 4.15). We can also use Theorem 57 to prove the following corollary, extending a result of Bateman and Erdős which in our notation says: if $p(n) \leq 1$, for $n \geq 1$, then $\mathcal{A}^* \in \mathsf{RT}_1$, ([6], 4.13).

Corollary 60. *If $p(n) \leq c$, for $n \geq 1$, then $\mathcal{A}^* \in \mathsf{RT}_1$*

Proof. Let $m = \max\{p(n) : n \geq 1\}$, which exists since $p(n) \leq c$ is integer valued. Then we can construct additive number systems $\mathcal{A}_i$, $1 \leq i \leq m$, such that $p_i(n) \leq 1$ for $1 \leq i \leq m$ and $\mathcal{A} = \mathcal{A}_1 * \cdots * \mathcal{A}_m$. By the Bateman and Erdős result each $\mathcal{A}_i^* \in \mathsf{RT}_1$; so by Corollary 59 $\mathcal{A}^* \in \mathsf{RT}_1$. $\square$

4.2. Multiplicative. Now, using the multiplicative notational conventions, we extend the results to the case when $\mathcal{A}_1$ and $\mathcal{A}_2$ may be discrete.

As an additional notational convention, if $\mathcal{A}$ is discrete λ will denote the largest integer such that $\|\mathsf{a}\|$ is an integer power of λ for all $\mathsf{a} \in \mathsf{A}$; and similarly for λ_i when $\mathcal{A}_i$ is discrete.

Although $\mathcal{A}$ having Property I does not imply $\mathcal{A} \in \mathsf{RV}_\alpha$ the following lemma gives a weaker result of a similar flavour.

Lemma 61. *For any multiplicative number system $\mathcal{A}$ with Property* I *and abscissa of convergence α, there exists a $C > 0$ such that $\liminf_{n\to\infty} \dfrac{A(n/i)}{A(n)} \geq Ci^{-\alpha}$, for all integers $i > 0$.*

Proof. If $\mathcal{A}$ is strictly multiplicative then, by Corollary 9.50 from [6], $\mathcal{A} \in \mathsf{RV}_\alpha$, and so the lemma holds with $C = 1$.

Suppose $\mathcal{A}$ is discrete multiplicative; then $\widetilde{\mathcal{A}} = (\mathsf{A}, \mathsf{P}, *, \mathsf{e}, \log_\lambda \|\ \|)$ is a reduced *additive* number system with all partition sets having global asymptotic density, and with the radius of convergence of $\widetilde{\mathbf{A}}(x)$ being $\lambda^{-\alpha}$. Notice also that $\widetilde{A}(x) = A(\lambda^x)$

(Section 9.8.2 of [6] discusses relationships between $\mathcal{A}$ and $\widetilde{\mathcal{A}}$). Thus, by Proposition 3.20 and Lemma 3.29 from [6], $\widetilde{A}(n) \in \mathsf{RT}_{\lambda^{-\alpha}}$; so

$$\begin{aligned} \frac{A(n/i)}{A(n)} &= \frac{\widetilde{A}(\log_\lambda(n) - \log_\lambda(i))}{\widetilde{A}(\log_\lambda(n))} \\ &\geq \frac{\widetilde{A}(\lfloor \log_\lambda(n) \rfloor - \lceil \log_\lambda(i) \rceil)}{\widetilde{A}(\lfloor \log_\lambda(n) \rfloor)} \\ &\to \lambda^{-\alpha \lceil \log_\lambda i \rceil}. \end{aligned}$$

Therefore

$$\liminf_{n\to\infty} \frac{A(n/i)}{A(n)} \geq \lambda^{-\alpha\lceil \log_\lambda i\rceil} \geq \lambda^{-\alpha(\log_\lambda i + 1)} = \lambda^{-\alpha} i^{-\alpha}.$$

So let $C = \lambda^{-\alpha}$. □

Proposition 62. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ have Property* I *and let $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$. If $\alpha_1 \geq \alpha_2$, then:*

- $\displaystyle\lim_{n\to\infty} \frac{A_1(n)}{A(n)} = \begin{cases} 0 & \text{if } \alpha_1 = \alpha_2 \\ \mathbf{A}_2(\alpha_1)^{-1} & \text{if } \alpha_1 > \alpha_2. \end{cases}$
- $\displaystyle\lim_{n\to\infty} \frac{A_2(n)}{A(n)} = 0.$

Proof.

$$\frac{A(n)}{A_2(n)} = \sum_{i\leq n} a_1(i) \frac{A_2(n/i)}{A_2(n)} \geq \sum_{i \leq m} a_1(i) \frac{A_2(n/i)}{A_2(n)}$$

for fixed $m < n$. So

$$\liminf_{n\to\infty} \frac{A(n)}{A_2(n)} \geq \liminf_{n\to\infty} \sum_{i\leq m} a_1(i) \frac{A_2(n/i)}{A_2(n)} \geq C \sum_{i\leq m} a_1(i) i^{-\alpha}$$

by Lemma 61. By taking the limit as $m \to \infty$ we get the second item.

For the first item, if $\alpha_1 = \alpha_2$ apply the above proof with 1 and 2 interchanged; if $\alpha_1 > \alpha_2$ then apply Proposition 42. □

Proposition 63. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ have Property* I *with $\alpha_1 = \alpha_2$. Then $\mathcal{A}_1 * \mathcal{A}_2$ has Property* I. *Let* B *be a partition set of $\mathcal{A}_1 * \mathcal{A}_2$ and let B_{ij} be as in Lemma* 11. *Then*

$$\Delta(\mathsf{B}) = \sum_{j=1}^{k} \Delta_1(\mathsf{B}_{1j})\Delta_2(\mathsf{B}_{2j}).$$

Proof. Follow the proof to Proposition 44 replacing references to Lemma 43 with references to Lemma 62. □

Proposition 64. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ have Property* I *with $\alpha_1 > \alpha_2$, and let $\mathcal{A}_1$ be strictly multiplicative. Then $\mathcal{A}_1 * \mathcal{A}_2$ has Property* I. *Let* B *be a partition set of $\mathcal{A}_1 * \mathcal{A}_2$ and let B_{ij} be as in Lemma* 11. *Then*

$$\Delta(\mathsf{B}) = \sum_{j=1}^{k} \Delta_1(\mathsf{B}_{1j}) \frac{\mathbf{B}_{2j}(\alpha_1)}{\mathbf{A}_2(\alpha_1)}.$$

Proof. Follow the proof to Proposition 46 replacing references to Lemma 43 with references to Lemma 62. □

For the purposes of the next lemma we will briefly escape from our multiplicative notational conventions. This lemma corresponds to Proposition 3.44 from [6], but is concerned with global counting functions rather than local counting functions.

Lemma 65. *Suppose $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$, all three of which are additive number systems. Additionally suppose $\sum_{n\geq 0} A_1(n)x^n$ has radius of convergence ρ and $\sum_{n\geq 0} A_2(n)x^n$ has radius of convergence $\rho_2 > \rho$. Then $A(n) \in \mathsf{RT}_\rho$ iff $A_1(n) \in \mathsf{RT}_\rho$.*

Proof. By Lemma 2.31 from [6] the radius of convergence of $\mathbf{A}_2(x)$ is also ρ_2. By Corollary 2.39 from [6] we have that $[\mathbf{1}/\mathbf{A}_2](x)$ has radius of convergence at least ρ_2, where $[\mathbf{1}/\mathbf{A}_2](x)$ is the power series expansion of $1/\mathbf{A}_2(x)$. Then, by Lemma 17

$$\begin{aligned} \sum_{n\geq 0} A(n)x^n &= \sum_{n\geq 0} A_1(n)x^n \cdot \mathbf{A}_2(x) \\ \sum_{n\geq 0} A_1(n)x^n &= [\mathbf{1}/\mathbf{A}_2](x) \cdot \sum_{n\geq 0} A(n)x^n. \end{aligned}$$

It follows from Schur's Theorem that $A(n) \in \mathsf{RT}_\rho$ iff $A_1(n) \in \mathsf{RT}_\rho$. □

Proposition 66. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ have Property* I *with $\alpha_1 > \alpha_2$, and let $\mathcal{A}_1$ and $\mathcal{A}_2$ be discrete with λ_2 a power of λ_1. Then $\mathcal{A}_1 * \mathcal{A}_2$ has Property* I. *Let B be a partition set of $\mathcal{A}_1 * \mathcal{A}_2$ and let B_{ij} be as in Lemma* 11. *Then*

$$\Delta(\mathsf{B}) = \sum_{j=1}^{k} \Delta_1(\mathsf{B}_{1j}) \frac{\mathbf{B}_{2j}(\alpha_1)}{\mathbf{A}_2(\alpha_1)}.$$

Proof. Let $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$. From the hypotheses $\mathcal{A}$ is discrete with $\lambda = \lambda_1$. Therefore each $\widetilde{\mathcal{A}}_i = (\mathsf{A}_i, \mathsf{P}_i, *, \mathsf{e}, \log_\lambda \| \ \|_i)$ is an additive number system and $\widetilde{\mathcal{A}} = (\mathsf{A}, \mathsf{P}, *, \mathsf{e}, \log_\lambda \| \ \|)$ and $\widetilde{\mathcal{A}}_1$ are reduced additive number systems.

Notice that $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}_1 * \widetilde{\mathcal{A}}_2$ since $\log_\lambda \|(\mathsf{a}_1, \mathsf{a}_2)\| = \log_\lambda \|\mathsf{a}_1\|_1 + \log_\lambda \|\mathsf{a}_2\|_2$ for all $\mathsf{a}_1 \in \mathsf{A}_1$ and $\mathsf{a}_2 \in \mathsf{A}_2$. Also $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ have all partition sets with global asymptotic density since $\mathcal{A}_1$ and $\mathcal{A}_2$ do.

Consider partition sets of $\mathcal{A}$ of the form $\mathsf{B} = \mathsf{B}_1 \times \mathsf{B}_2$ where B_i is a partition set of $\mathcal{A}_i$, $i = 1, 2$.

First, suppose $\Delta_1(\mathsf{B}_1) \neq 0$. Then $\Delta_1(\widetilde{\mathsf{B}}_1) \neq 0$. So $\widetilde{B}_1(n) \in \mathsf{RT}_{\lambda^{-\alpha}}$ by Lemma 26. Thus $\widetilde{B}(n) \sim \widetilde{B}_1(n)\widetilde{\mathbf{B}}_2(\lambda^{-\alpha_1})$ and $\widetilde{A}(n) \sim \widetilde{A}_1(n)\widetilde{\mathbf{A}}_2(\lambda^{-\alpha_1})$ by Lemma 17 and Schur's Theorem. So, as $n \to \infty$,

$$\frac{B(n)}{A(n)} = \frac{\widetilde{B}(\lfloor \log_\lambda n \rfloor)}{\widetilde{A}(\lfloor \log_\lambda n \rfloor)} \to \Delta_1(\widetilde{\mathsf{B}}_1) \frac{\widetilde{\mathbf{B}}_2(\lambda^{-\alpha_1})}{\widetilde{\mathbf{A}}_2(\lambda^{-\alpha_1})} = \Delta_1(\mathsf{B}_1) \frac{\mathbf{B}_2(\alpha_1)}{\mathbf{A}_2(\alpha_1)}.$$

Second, suppose $\Delta_1(\mathsf{B}_1) = 0$. The proof that $\Delta(\mathsf{B}) = 0$ is exactly as in the proof of Proposition 46. Therefore in both cases $\Delta(\mathsf{B}) = \Delta_1(\mathsf{B}_1) \dfrac{\mathbf{B}_2(\alpha_1)}{\mathbf{A}_2(\alpha_1)}$.

Hence for a general partition set B of $\mathcal{A}$

$$\Delta(\mathsf{B}) = \sum_{j=1}^{k} \Delta_1(\mathsf{B}_{1j}) \frac{\mathbf{B}_{2j}(\alpha_1)}{\mathbf{A}_2(\alpha_1)}$$

by Lemma 11 and Lemma 9.4 from [6]. Therefore $\mathcal{A}$ has Property I. □

Proposition 67. *If* $\mathcal{A}_1 * \mathcal{A}_2$ *has Property* I *and* $\alpha_1 \geq \alpha_2$*, then*

- $\alpha_1 = \alpha_2$ *or*
- $\alpha_1 > \alpha_2$ *and* $\mathcal{A}_1$ *is strictly multiplicative or*
- $\alpha_1 > \alpha_2$*,* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are discrete, and* λ_2 *is a power of* λ_1*.*

Proof. Let $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$. Assume $\alpha_1 > \alpha_2$.

Suppose $\mathcal{A}$ is strictly multiplicative; then $\mathcal{A} \in \mathsf{RV}_\alpha$ by Corollary 9.50 from [6]. Thus $\mathcal{A}_1 \in \mathsf{RV}_\alpha$ by Proposition 9.55 from [6]. Then $\mathcal{A}_1$ is strictly multiplicative; for otherwise $\alpha_1 = 0$ by Proposition 9.51 from [6] which contradicts $\alpha_1 > \alpha_2$.

Suppose $\mathcal{A}$ is discrete; then $\|(\mathsf{a}_1, \mathsf{a}_2)\|$ is an integer power of λ for $\mathsf{a}_i \in \mathsf{A}_i$. Thus if there existed an $\mathsf{a}_1 \in \mathsf{A}_1$ such that $\|\mathsf{a}_1\|_1$ was not an integer power of λ then $\|(\mathsf{a}_1, \mathsf{e})\| = \|\mathsf{a}_1\|_1$ would not be an integer power of λ, and likewise for $\mathsf{a}_2 \in \mathsf{A}_2$. Therefore for all $\mathsf{a}_i \in \mathsf{A}_i$, $i = 1, 2$, $\|\mathsf{a}_i\|_i$ is an integer power of λ; so $\mathcal{A}_1$ and $\mathcal{A}_2$ are discrete. Hence λ_1 and λ_2 are powers of λ, each $\widetilde{\mathcal{A}}_i = (\mathsf{A}_i, \mathsf{P}_i, *, \mathsf{e}, \log_\lambda \| \ \|_i)$ is an additive number system, and $\widetilde{\mathcal{A}} = (\mathsf{A}, \mathsf{P}, *, \mathsf{e}, \log_\lambda \| \ \|)$ is a reduced additive number system.

As in Proposition 66 $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}_1 * \widetilde{\mathcal{A}}_2$ and $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ have all partition sets with global asymptotic density. So by Proposition 3.20 and Lemma 3.29 from [6], $\widetilde{A}(n) \in \mathsf{RT}_{\lambda^{-\alpha}}$; so $\widetilde{A}_1(n) \in \mathsf{RT}_{\lambda^{-\alpha}}$ by Lemma 65. Now $\alpha = \alpha_1 > \alpha_2 \geq 0$; so $\lambda^{-\alpha} < 1$. If $\widetilde{\mathcal{A}}_1$ is not reduced then $\widetilde{A}_1(n-1)/\widetilde{A}_1(n)$ is infinitely often 1 giving a contradiction. So $\widetilde{\mathcal{A}}_1$ is reduced, and thus $\lambda = \lambda_1$. Therefore λ_2 is a power of λ_1. □

Theorem 68. *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *have Property* I *with* $\alpha_1 \geq \alpha_2$*. Then* $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$ *has Property* I *iff*

- $\alpha_1 = \alpha_2$ *or*
- $\alpha_1 > \alpha_2$ *and* $\mathcal{A}_1$ *is strictly multiplicative or*
- $\alpha_1 > \alpha_2$*,* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are discrete, and* λ_2 *is a power of* λ_1*.*

If $\mathcal{A}_1 * \mathcal{A}_2$ *has Property* I *then for* B *a partition set of* $\mathcal{A}_1 * \mathcal{A}_2$ *and for* B_{ij} *as in Lemma* 11 *we get*

$$\Delta(\mathsf{B}) = \begin{cases} \displaystyle\sum_{j=1}^{k} \Delta_1(\mathsf{B}_{1j}) \frac{\mathbf{B}_{2j}(\alpha_1)}{\mathbf{A}_2(\alpha_1)} & \text{if } \alpha_1 > \alpha_2, \\ \displaystyle\sum_{j=1}^{k} \Delta_1(\mathsf{B}_{1j}) \Delta_2(\mathsf{B}_{2j}) & \text{if } \alpha_1 = \alpha_2. \end{cases}$$

Proof. In the if direction apply Propositions 63, 64, and 66, and in the only if direction apply Proposition 67. □

Because of the connection between discrete multiplicative systems and additive systems used in Lemma 61 and Proposition 66, global density results for multiplicative systems give global density results for additive systems. Each additive system can be seen as a multiplicative system (as shown in Section 9.8.1 of [6]); this translation maps additive number systems in RT_ρ to multiplicative number systems in $\mathsf{RV}_{-\log_\lambda \rho}$ for integer $\lambda > 1$ and preserves partition sets and density. In the next corollary we will also need the fact that if the generating series of the additive number system diverges at its radius of convergence then the Dirichlet series of the corresponding multiplicative number system diverges at its abscissa of convergence.

For the purposes of this corollary we are returning to the notational conventions of the first half of the paper.

Corollary 69. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be additive number systems such that every partition set has global asymptotic density. Then every partition set of $\mathcal{A}_1 * \mathcal{A}_2$ has global asymptotic density iff $d_1|d_2$ or $\rho_1 = \rho_2$.*

Proof. Let $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$. Pick an integer $\lambda_0 > 1$. Then $\widetilde{\mathcal{A}}_i = (\mathsf{A}_i, \mathsf{P}_i, *, \mathsf{e}, \lambda_0^{\| \|_i})$ and $\widetilde{\mathcal{A}} = (\mathsf{A}, \mathsf{P}, *, \mathsf{e}, \lambda_0^{\| \|})$ are discrete multiplicative number systems with $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}_1 * \widetilde{\mathcal{A}}_2$. Apply Theorem 68. Notice that in the case $\rho_1 < \rho_2$ the condition $d_1|d_2$ is equivalent to the condition λ_2 is a power of λ_1 for the discrete systems. □

Corollary 70. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be multiplicative number systems in RV_0. Then $\mathcal{A}_1 * \cdots * \mathcal{A}_n \in \mathsf{RV}_0$.*

Proof. It suffices to prove the theorem for $n = 2$. $\mathcal{A}_i \in \mathsf{RV}_0$ iff $\mathcal{A}_i$ has Property I and the abscissa of convergence of $\mathcal{A}_i$ is 0 by Theorem 10.2 of [6]; the same applies to $\mathcal{A}_1 * \mathcal{A}_2$. Now apply Theorem 68. □

This gives Odlyzko's Theorem ([6], 10.5) and the multiplicative half of Theorem 16.1 from [7].

For any multiplicative number system $\mathcal{A}$ define $\mathcal{A}_{\mathsf{q}}$, the extension of $\mathcal{A}$ by an indecomposable q, to be the multiplicative number system formed by adding a new indecomposable q to P, by letting A_{q} be the set of formal expressions $\mathsf{q}^m * \mathsf{a}$, for m a nonnegative integer and $\mathsf{a} \in \mathsf{A}$, and by extending the norm by the definition $\|\mathsf{q}^m * \mathsf{a}\| = \|\mathsf{q}\|^m \|\mathsf{a}\|$.

Corollary 71. *Suppose that $\mathcal{A}$ has Property* I. *An extension $\mathcal{A}_{\mathsf{q}}$ has Property* I *iff $\mathcal{A}$ is strictly multiplicative, or $\mathcal{A}$ is discrete and $\|\mathsf{q}\|$ is an integer power of λ, or $\alpha = 0$.*

Proof. We can form a multiplicative number system $\mathcal{Q} = (\mathsf{A}_{\mathcal{Q}}, \mathsf{P}_{\mathcal{Q}}, \cdot, \mathsf{1}, \| \|)$ generated by q by letting $\mathsf{P}_{\mathcal{Q}} = \{\mathsf{q}\}$, $\mathsf{A}_{\mathcal{Q}} = \{\mathsf{q}^m : m \geq 0\}$, $\mathsf{1} = \mathsf{q}^0$, $\mathsf{q}^{m_1} * \mathsf{q}^{m_2} = \mathsf{q}^{m_1+m_2}$, and $\|\mathsf{q}^m\| = \|\mathsf{q}\|^m$. Then $\mathcal{A}_{\mathsf{q}} \cong \mathcal{A} * \mathcal{Q}$, and $\mathcal{Q}$ has Property I, $\alpha_{\mathcal{Q}} = 0$. Now apply Theorem 68. □

If $\mathcal{A}$ is strictly multiplicative then Corollary 71 gives Theorem 9.69 from [6].

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Department of Pure Mathematics, University of Waterloo, Waterloo Ontario, N2L 3G1, Canada
kayeats@uwaterloo.ca

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