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On C^*-Algebras Associated to Certain Endomorphisms of Discrete Groups

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Abstract. Let $\alpha : G \to G$ be an endomorphism of a discrete amenable group such that $[G : \alpha(G)] < \infty$. We study the structure of the C^* algebra generated by the left convolution operators acting on the left regular representation space, along with the isometry of the space induced by the endomorphism.

Contents

1. Introduction

Some interesting examples of non-invertible topological dynamical systems, such as the n-fold covering of the circle, and the one-sided full symbolic shifts on n letters, arise as surjective endomorphisms of compact abelian groups.

If H is such an abelian group, and $T : H \twoheadrightarrow H$ is the map, then T induces an isometry on $L^2(H, \text{Haar})$ via the pull-back. The algebra $C(H)$ acts on L^2 by multiplication, and we want to study the structure of the C^*-algebra generated by those multiplication operators and the isometry, which is analogous to (a representation of) the crossed product algebra in the case of an action of an automorphism.

Such surjective endomorphisms are in duality with injective endomorphisms of the (discrete) dual groups. Thus this problem can be conveniently reformulated as studying the structure of the C^*-algebra generated by the convolution operators on the l^2 space of a discrete group, along with the isometry induced on the space by the given injective endomorphism of the discrete group. Aside from the fact that this is technically more convenient, it allows an immediate generalization of the question by dropping the requirement that the group be commutative.

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The situation is analogous to that arising in the construction of the crossed product by an endomorphism, however the algebra is not a crossed product by an endomorphism in the sense alluded to in [C]. The main difference is that the endomorphism here not implemented by the associated isometry (i.e., if α is the endomorphism, S is the associated isometry and a is an element of the group C^*-algebra, then $SaS^* \neq \alpha(a)$), but rather is only intertwined by it ($Sa = \alpha(a)S$). Additional conditions relevant to the situation seemed required here, so as to limit the size of the algebra.

The construction considered in this paper is related to a situation considered by Deaconu ([De]) concerning self covering maps of compact Hausdorff spaces (the two situations overlap when our group is abelian, in which case we can dualize to get a map from the dual group onto itself — see above). Deaconu used a groupoid approach to that problem. Exel's approach to the crossed product by an endomorphism ([E]) places the examples considered in this paper in a more general framework (see Remark 1.4 below).

Definition 1.1. Let G be a discrete group, and let $\alpha : G \hookrightarrow G$ be an injective homomorphism. We call α *pure* if $\bigcap_{n=0}^{\infty} \alpha^n(G) = \{1\}$.

We will assume throughout that $[G : \alpha(G)] < \infty$**.**

We consider the universal C^*-algebra $\mathcal{E}_\alpha$ generated by $\mathbb{C}G$ and and an isometry S, satisfying the relations:

1. $S\delta_x = \delta_{\alpha(x)}S$ for $x \in G$.
2. $S^*\delta_x S = 0$ if $x \notin \alpha(G)$.
3. For any complete list of right coset representatives $x_1, \ldots, x_n \in G$ of $\alpha(G)$

$$\sum_{k=1}^{n} \delta_{x_k^{-1}} SS^* \delta_{x_k} = 1.$$

Here, δ_x denotes the image of x in $\mathbb{C}G$.

Remark 1.2. Note that we could fix in relation (3) an arbitrary list of coset representatives; relation (1) shows that

$$\sum \delta_{x_k^{-1}} SS^* \delta_{x_k} = \sum \delta_{y_k^{-1}} SS^* \delta_{y_k}$$

if $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$ are two sets of representatives.

There is an action γ of $\mathbb{T}$ on $\mathcal{E}_\alpha$ given by

$$\gamma_t(S) = tS, \ \ \gamma_t(a) = a \ \forall a \in \mathbb{C}G \ , \ t \in \mathbb{T} \subseteq \mathbb{C}.$$

Remark 1.3. Note that $\mathcal{E}_\alpha$ has a represenation λ extending the left regular representation of the group G, given by the usual left regular representation on $l^2(G)$, $\lambda(g)\xi_h = \xi_{gh}$, and with $\lambda(S)\xi_h = \xi_{\alpha(h)}$, where $\{\xi_h \,|\, h \in G\}$ is the standard basis for $l^2(G)$. It is easily verified that this representation satisfies relations (1)–(3). In particular, this implies that if G is amenable then $\mathcal{E}_\alpha$ always contains a copy of $C^*(G)$. **We shall assume from now on that** G **is amenable**.

Remark 1.4. $\mathcal{E}_\alpha$ can be described using the language of [E] as the crossed product of $C^*(G)$ by α with transfer operator corresponding to the expectation of $C^*(G)$ onto $\alpha(C^*(G))$ (see [E], Proposition 2.6). We will not make use of results from [E] in this paper, though.

2. The γ-invariant subalgebra

Let $\mathcal{F}$ denote the γ-invariant subalgebra. We denote $P_n = S^n S^{*n}$.

Claim 2.1. $\mathcal{F} = \overline{\operatorname{span}\{\delta_x P_n \delta_y \mid x, y \in G, n \in \mathbb{N}\}}$

Proof. We have an expectation map $E_{\mathcal{F}} : \mathcal{E}_\alpha \to \mathcal{F}$ defined by

$$E_{\mathcal{F}}(a) = \int_{\mathbb{T}} \gamma_t(a) dt.$$

$\mathcal{E}_\alpha$ is densely spanned by the words with letters in $\{S, S^*, \delta_x, x \in G\}$, and it is clear that if w is such a word, then $E_{\mathcal{F}}(w) = 0$ unless w has the same number of S's and S^*'s. If w is a word with n S's and n S^*'s, then it is easy to see that it can be rewritten as a word of the form $\delta_x P_n \delta_y$, by using the commutation relations defining the algebra. □

Now fix n and let $\mathcal{F}_n = \overline{\operatorname{span}\{\delta_x P_n \delta_y \mid x, y \in G\}}$. It is clear that $\mathcal{F}_n$ is a subalgebra of $\mathcal{F}$. The spanning elements of $\mathcal{F}_n$ satisfy the following multiplication rule:

$$\delta_x P_n \delta_y \delta_z P_n \delta_w = \begin{cases} \delta_{xyz} P_n \delta_w & yz \in \alpha^n(G) \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the words $\delta_x P_n \delta_y$ are not all distinct - clearly, $\delta_x P_n \delta_y = \delta_{xz} P_n \delta_{z^{-1}y}$ for any $z \in \alpha^n(G)$. We can force uniqueness by fixing representatives for the conjugacy classes of $\alpha^n(G)\backslash G$ and always choosing y to be one of them, for example. A more 'symmetric' way will be to introduce the following notation. Let R_n be fixed sets of representatives of right cosets, chosen so that $\alpha(R_n) \subseteq R_{n+1}$. This is done as follows: we fix R_1, and then define recursively $R_{n+1} = \{\alpha(s)r \mid s \in R_n\ ,\ r \in R_1\}$. We may assume that $1 \in R_1$.

Notation. $(x, k, y) = \delta_{x\alpha^n(k)} P_n \delta_y$, where $y \in R_n,\ x \in R_n^{-1},\ k \in G$.

In this notation, the multiplication rule assumes the simpler form:

$$(x, k, y)(z, l, w) = \begin{cases} (x, kl, w) & y = z^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Note that this multiplication rule is simply that of matrices with entries in $\mathbb{C}G$. More explicitly, we can define a map $M_{|R_n|}(\mathbb{C}G) \to \mathcal{F}_n$ by $ge_{xy} \mapsto (x^{-1}, g, y)$ where we index the matrix entries by elements of R_n and e_{xy} denotes the usual matrix unit. This map will extend to an isomorphism from

$$M_{|R_n|}(C^*(G)) \cong \mathcal{K}(l^2(\alpha(G)\backslash G))^{\otimes n} \otimes C^*(G)$$

onto $\mathcal{F}_n$ (by Remark 1.3).

Relation (3) implies that $\mathcal{F}_{n+1} \supseteq \mathcal{F}_n$, and $\mathcal{F} = \overline{\bigcup_{n=0}^{\infty} \mathcal{F}_n}$.

Denote by ι the inclusion map $\mathcal{F}_n \to \mathcal{F}_{n+1}$. We now describe ι in terms of the 'triples' notation, so that we can effectively identify $\mathcal{F}_n$ with the familiar algebra $\mathcal{K}(l^2(\alpha(G)\backslash G))^{\otimes n} \otimes C^*(G)$. Since we have to consider words with P_n and with P_{n+1} simultaneously, we'll put a subscript to keep track of that and denote $(x, k, y)_j = \delta_x P_j \delta_{\alpha^j(k)y},\ x^{-1}, y \in R_j$.

Now, $(x,k,y)_n(z,l,w)_{n+1} = 0$ unless $yz \in \alpha^n(G)$, in which case the product is $\delta_{x\alpha^n(k)yz\alpha^{n+1}(l)}P_{n+1}\delta_w$. We need to convert this to 'triples' notation. Let $z = y^{-1}\alpha^n(j)$. So, now, the product is

$$\delta_{x\alpha^n(kj)\alpha^{n+1}(l)}P_{n+1}\delta_w.$$

Let $kj = q\alpha(g)$ for $q \in R_1^{-1}$. Recall that $x \in R_n^{-1}$, so $x = r_0\alpha(r_1)\ldots\alpha^{n-1}(r_{n-1})$ for some $r_0,\ldots,r_{n-1} \in R_1^{-1}$. Therefore $x\alpha^n(q) \in R_{n+1}^{-1}$. So, the above expression in 'triples' notation is:

$$(x\alpha^n(q), gl, w)_{n+1}.$$

So, we obtain

$$\iota((x,k,y)_n) = \sum_{q\in R_1^{-1}} (x\alpha^n(q), p, \alpha^n(s)y)_{n+1}$$

where $p \in G$,$s \in R_1$ are given by the equation $\alpha(p)s = q^{-1}k$.

Remark 2.2. We know that $\mathcal{F}_{n+1} \cong \mathcal{F}_n \otimes \mathcal{K}(l^2(\alpha(G)\backslash G))$. Writing $\mathcal{F}_n \cong \mathcal{F}_0 \otimes \mathcal{K}(l^2(\alpha(G)\backslash G))^{\otimes n}$, $\mathcal{F}_{n+1} \cong \mathcal{F}_1 \otimes \mathcal{K}(l^2(\alpha(G)\backslash G))^{\otimes n}$, by looking at the formula for ι, we can see that $\iota = \iota_0 \otimes id$, where ι_0 is the inclusion map from $\mathcal{F}_0$ to $\mathcal{F}_1$.

Theorem 2.3. *If G is amenable and α is pure then $\mathcal{F}$ is simple.*

To prove the theorem, we need a lemma. We first fix some notation.

Denote by tr the von Neumann trace on $C^*(G)$ (which is faithful since G was taken to be amenable). α extends to a unital endomorphism of $C^*(G)$, which we denote by α as well. For $a \in C^*(G)$ and $g \in G$ we denote by $\hat{a}(g)$ the g-th Fourier coefficient of a, i.e., $\hat{a}(g) = \mathrm{tr}(a\delta_{g^{-1}})$. Denote by $\alpha^*(a)$ the element of $C^*(G)$ satisfyting $\widehat{\alpha^*(a)}(g) = \hat{a}(\alpha(g))$. Observe that $(1,1,1)_1\iota(a)(1,1,1)_1 = \alpha^*(a)(1,1,1)_1$ (when here $C^*(G)$ is identified with $\mathcal{F}_0$).

Lemma 2.4. *If $a \in C^*(G)$ is an element such that $\hat{a}(1) = 0$, then $\|\alpha^{*n}(a)\| \to 0$.*

Proof. We know that the image of $\mathbb{C}G$ is dense in $C^*(G)$. Fix $\epsilon > 0$, and let $a' \in \mathbb{C}G$ satisfy $\|a' - a\| < \epsilon$. We can assume without loss of generality that $\hat{a'}(1) = 0$. Observe that α^* is a contraction. Since α is pure, there is some m' such that for all $m \geq m'$, $\alpha^{*m}(a') = 0$ (since all its Fourier coefficients vanish). Therefore, for all $m \geq m'$, $\|\alpha^{*m}(a)\| < \epsilon$, which is what we wanted to show. □

Corollary 2.5. *For all $a \in C^*(G)$, $\alpha^{*n}(a) \to \hat{a}(1)1$ in norm.*

Proof of Theorem 2.3. Suppose $\mathcal{J} \triangleleft \mathcal{F}$. We know that

$$\mathcal{J} = \overline{\bigcup_{n=0}^{\infty} \mathcal{J} \cap \mathcal{F}_n}$$

where we think of $\mathcal{F}_n$ as being inside $\mathcal{F}$ (the connecting maps are injective). Denote $\mathcal{J}_n = \mathcal{J} \cap \mathcal{F}_n$. Notice that $\iota(\mathcal{J}_n) \subseteq \mathcal{J}_{n+1}$, and that $\mathcal{J}_n \cong \mathcal{J}_0 \otimes \mathcal{K}(l^2(\alpha(G)\backslash G))^{\otimes n}$. By the observation before the lemma, this implies that if $a \in \mathcal{J}_0$ then $\alpha^*(a) \in \mathcal{J}_0$ as well.

Suppose $\mathcal{J}_0 \neq 0$. Let $a \in \mathcal{J}_0$ be a nonzero positive element. Since tr is faithful, $\hat{a}(1) = \mathrm{tr}(a) > 0$. By Corollary 2.5, since $\mathcal{J}_0$ is closed, we see that $1 \in \mathcal{J}_0$, so $\mathcal{J}_0$ is trivial, and hence $\mathcal{J}_n$ are trivial for all n, hence $\mathcal{J}$ is trivial, hence $\mathcal{F}$ is simple. □

Examples.

1. Take $G = \mathbb{Z}$, and α to be multiplication by a positive integer $N > 1$. We identify δ_n with the function z^n (where z denotes both the complex variable and the inclusion map $\mathbb{T} \to \mathbb{C}$). We pick $R_1 = \{0, 1, 2, \ldots, N-1\}$, and we identify $\mathcal{K}(l^2(\mathbb{Z}/N\mathbb{Z}))$ with M_N, where we label the columns $0, 1, 2, \ldots, N-1$.

   By the above remark, in order to see what the map ι looks like, it is enough to look at the map from $\mathcal{F}_0 \cong C(\mathbb{T})$ to $\mathcal{F}_1 \cong M_N(C(\mathbb{T}))$.

   The only triples in $\mathcal{F}_0$ are of the form $(0, n, 0)_0$ so we simply write them as z^n, and we write $(-a, n, b)_1$, $a, b \in \{0, 1, \ldots, N-1\}$ as $z^n e_{a,b}$, where $e_{a,b}$ is the standard matrix unit (the matrix whose entry in the a-th row and b-th column is 1, and is zero everywhere else).

   Since $C(\mathbb{T})$ is generated by z, it suffices to see to where z maps. Translating the above formula to additive notation and to our special case, and replacing q by $-q$ for convenience, we see that z maps to $\sum_{q=0}^{N-1} z^p e_{q,s}$ where p,s are given by the equation $Np + s = q + 1$. It is easy to solve this equation: we have $p = 0$, $s = q + 1$ except for the last term, in which case we have $p = 1$, $s = 0$.

   Writing this in matrix form, we get

$$z \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ z & 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

   This is the same map which appears in the construction of the Bunce-Deddens algebras, so $\mathcal{F}$ is isomporphic to the Bunce-Deddens algebra corresponding to the supernatural number N^∞.

2. Fix a finite group H, let $G = H \times H \times H \times \ldots$ (the algebraic sum, so each element has only finitely many nontrivial terms), and let α be the left shift $\alpha(a_1, a_2, a_3, \ldots) = (1, a_1, a_2, a_3, \ldots)$. We pick R_1 to be $H \times 1 \times 1 \times \ldots$, which we identify with H in the obvious way. As in the previous example, it is enough to understand the map $\mathcal{F}_0 \to \mathcal{F}_1$. We denote the images of G in $\mathcal{F}_0 = C^*(G)$ by $\delta_{a,\vec{b}}$, with $a \in H$ and $\vec{b}$ denoting the tail. The formula yields in this case: $\delta_{a,\vec{b}} \mapsto \delta_{\vec{b}} \sum_{q \in H} e_{q,qa}$, i.e., the image is $\delta_{\vec{b}}$ tensored by the permutation matrix coming from the image of a under the right regular representation of H (notice that we write $\delta_z e_{x,y} = (x^{-1}, z, y)$, $x, y \in H$).

   Since $a, \vec{b}$ has only finitely many nonzero entries, after sufficiently many applications of ι to this, we will have a zero-one matrix. ι applied to a scalar matrix simply embeds it in the standard multiplicity $|H|$ embedding into the higher matrix algebra. Therefore, the image of each element of the dense subalgebra $\mathbb{C}G \otimes \mathcal{K}(l^2(\alpha(G)\backslash G))^{\otimes n} \subseteq \mathcal{F}_n$ in $\mathcal{F}$ is the same as an image of a scalar matrix in some higher $\mathcal{F}_m$. Therefore, the images of the scalar matrices are dense in $\mathcal{F}$, so $\mathcal{F} = \overline{\bigcup_{k=0}^{\infty} M_{|H|^k}}$, with the appropriate inclusion, and therefore $\mathcal{F}$ is isomorphic to the UHF algebra with supernatural number $|H|^\infty$.

3. Let $G = \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2 = b^2 = 1 \rangle$, and let $\alpha : G \to G$ be given by $\alpha(a) = aba$, $\alpha(b) = bab$. It is easily verified that $[G : \alpha(G)] = 3$, and $R_1 = R_1^{-1} = \{1, a, b\}$ is a list of both left and right coset representatives. Notice that $\alpha(G)$ is not normal in G in this case. It is known ([B] 6.10.4) that $K_0(C^*(G)) \cong \mathbb{Z}^3$, and $K_1(C^*(G)) = 0$, and that if we denote $P_a = (1+\delta_a)/2$, $P_b = (1+\delta_b)/2$, then $[1]$,$[P_a]$,$[P_b]$ generate $K_0(C^*(G))$ as a free abelian group. Identifying $\mathcal{F}_1$ with $M_3(C^*(G))$, and making the first row in the matrix correspond to $1 \in R_1$, the second to $a \in R_1$ and the third to $b \in R_1$, we see that

$$\iota(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \iota(\delta_a) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \delta_b \end{pmatrix} \quad \iota(\delta_b) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \delta_a & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

so

$$\iota(P_a) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & P_b \end{pmatrix} \quad \iota(P_b) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & P_a & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

and therefore the induced map on K_0 is given by

$$\iota_*([1]) = 3 \cdot [1] \qquad \iota_*([P_a]) = [1] + [P_b] \qquad \iota_*([P_b]) = [1] + [P_a]$$

which in matrix form is given by

$$\begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

$K_0(\mathcal{F})$ is the inductive limit of the inductive system given by $\mathbb{Z}^3 \to \mathbb{Z}^3 \to \mathbb{Z}^3 \to \cdots$, where the maps are given by this matrix. This group is isomorphic to $\mathbb{Z}\left[\frac{1}{3}\right] \oplus \mathbb{Z} \oplus \mathbb{Z}$.

2.1. An abelian subalgebra. Let $\mathcal{A}_n = \operatorname{span}\{(x^{-1}, 1, x)_n | x \in R_n\} \subseteq \mathcal{F}_n$. $\mathcal{A}_n$ is an abelian finite dimensional algebra (of dimension $|R_n|$). Notice that if we identify $\mathcal{F}_n$ with $C^*(G) \otimes M_m$ (where $m = |R_n| = [G : \alpha^n(G)]$), then A_n is identified with $\mathbb{C}1 \otimes \mathrm{Diag}$, Diag being the diagonal subalgebra of M_m. Let Δ denote the standard faithful expectation map $M_m \to \mathrm{Diag}$, and let $E_{\mathcal{A}_n} : \mathcal{F}_n \to \mathcal{A}_n$ be the map given by $E_{\mathcal{A}_n} = \mathrm{tr}(\cdot)1 \otimes \Delta$, then $E_{\mathcal{A}_n}$ is a faithful expectation. Notice that $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ and furthermore, the $E_{\mathcal{A}_n}$'s are consistent with the inclusion maps, i.e., $E_{\mathcal{A}_{n+1}}\iota = \iota\lceil_{\mathcal{A}_n} E_{\mathcal{A}_n}$. Therefore, this consistent system of faithful expectation maps passes on to the direct limit, and so we obtain a faithful expectation $E_{\mathcal{A}} : \mathcal{F} \to \mathcal{A}$, where $\mathcal{A}$ is the direct limit of the $\mathcal{A}_n$'s.

$\hat{\mathcal{A}}$, the Gelfand spectrum of $\mathcal{A}$, is homeomorphic to the Cantor set. It can be thought of as a product of a countable number of copies of R_1 with the Tychonof topology, where there Gelfand transform of $(x^{-1}, 1, x)_n$, where

$$x = \alpha^{n-1}(r_{n-1})\alpha^{n-2}(r_{n-2}) \cdots \alpha(r_1)r_0,$$

$r_0, \ldots, r_{n-1} \in R_1$, is the characteristic function of the basic cylindrical clopen set $\{r_0\} \times \{r_1\} \times \cdots \times \{r_{n-1}\} \times R_1 \times R_1 \times \cdots$.

For any $a \in \mathcal{A}$, and any $x \in G$, we have $\delta_x a \delta_{x^{-1}} \in \mathcal{A}$. Denote this left action by β: $\beta_x(a) = \delta_x a \delta_{x^{-1}}$. That gives rise to an action of G on $\hat{\mathcal{A}}$. Let us describe this action. Notice that G acts on R_1, where $r \cdot x = r'$ for the unique $r' \in R_1$ such that $rx \in Gr'$. Similarly, we have an action on R_2, and the action on R_1 is the quotient action (because of the way we constructed R_2), and so on. $\hat{\mathcal{A}} \cong \lim_{\leftarrow} R_n$, so the consistency gives us an action on $\hat{\mathcal{A}}$, which is the dual of the action on $\mathcal{A}$. If $\omega = (r_0, r_1, \dots) \in \hat{\mathcal{A}}$ (when we identify it with $R_1 \times R_1 \times \cdots$), then the stabilizer of ω is the decreasing intersection of the stabilizers of $\omega_n = (r_0, \dots, r_n) \in R_{n+1}$. Write $x_n = \alpha^n(r_n) \cdots \alpha(r_1) r_0$, then the stabilizer of ω_n is $x_n^{-1} \alpha^{n+1}(G) x_n$. Therefore the stabilizer of ω is $\bigcap_{n=0}^{\infty} x_n^{-1} \alpha^{n+1}(G) x_n$.

Lemma 2.6. *$\mathcal{F} \cong \mathcal{A} \times_\beta G$.*

Proof. Any element of the form $\delta_x P_n \delta_y$ can be written in the form $\delta_{t^{-1}} P_n \delta_t \delta_z$, for some $t \in R_n$. If we take two such elements, $\delta_{t^{-1}} P_n \delta_t \delta_z$, $\delta_{s^{-1}} P_n \delta_s \delta_w$, then their product satisfies $\delta_{t^{-1}} P_n \delta_t \delta_z \delta_{s^{-1}} P_n \delta_s \delta_w = \delta_{t^{-1}} P_n \delta_t (\delta_z \delta_{s^{-1}} P_n \delta_s \delta_{z^{-1}}) \delta_z \delta_w = \delta_{t^{-1}} P_n \delta_t \beta_z (\delta_{s^{-1}} P_n \delta_s) \delta_{zw}$ where β denotes the action of G on $\mathcal{A}$.

So, $\mathrm{span}\{a\delta_x \mid a \in \mathcal{A}\ ,\ x \in G\}$ is a dense subalgebra of $\mathcal{F}$, which is isomorphic to the twisted convolution algebra used in the definition of the crossed product. Therefore, $\mathcal{F}$ is a quotient of the crossed product $\mathcal{A} \times_\beta G$. Let $\varphi : \mathcal{A} \times_\beta G \to \mathcal{F}$ denote this quotient map. It is clear that $E_{\mathcal{A}} \circ \varphi$ coincides with the canonical expectation $\mathcal{A} \times_\beta G \to \mathcal{A}$, and therefore $\ker\varphi = 0$, which is what we wanted. □

Remark 2.7. When the endomorphism is the multiplication by n map on the integers, the action of $\mathbb{Z}$ on the Cantor set we obtain is the odometer action (with n digits). This gives us another way to see that the Bunce-Deddens algebras can be obtained as crossed products by the odometer actions (see [Da]).

3. A crossed product representation

Define $\Phi : \mathcal{F} \to \mathcal{F}$ by $\Phi(a) = SaS^*$. Φ is a non-unital injective endomorphism of $\mathcal{F}$. Following the procedure from [C], we repeat Φ and form a direct system $\mathcal{F}^{(0)} \longrightarrow \mathcal{F}^{(1)} \longrightarrow \mathcal{F}^{(2)} \longrightarrow \cdots$ where the $\mathcal{F}^{(n)}$ are just copies of $\mathcal{F}$. We denote the direct limit by $\vec{\mathcal{F}}$, and we denote by $\vec{\Phi}$ the automorphism of $\vec{\mathcal{F}}$ induced by Φ. We denote by P the image of $1_{\mathcal{F}^{(0)}}$ in $\vec{\mathcal{F}}$.

Let $\widetilde{\mathcal{E}_\alpha} = \vec{\mathcal{F}} \times_{\vec{\Phi}} \mathbb{Z}$, then $\mathcal{E}_\alpha \cong P\widetilde{\mathcal{E}_\alpha}P$, so $\mathcal{E}_\alpha$ and $\widetilde{\mathcal{E}_\alpha}$ are strongly Morita equivalent.

Notice that in 'triples' notation, $\Phi((x,k,y)_n) = (\alpha(x), k, \alpha(y))_{n+1}$, and therefore Φ is unitarily equivalent to the multiplicity one embedding of $\mathcal{F}$ into the upper corner of $M_{[G:\alpha(G)]}(\mathcal{F}) \cong \mathcal{F}$, and in particular, it induces an isomorphism on K-theory, so $K(\mathcal{F}) \cong K(\vec{\mathcal{F}})$.

The map Φ, thought of as a map on the inductive system $\cdots \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \cdots$ has the effect of shifting the sequence by 1 (and stabilizing by $M_{|R_n|}$), and so the action on $K_*(\mathcal{F})$ will be ι_*^{-1} (the action of the shift). The Pimsner-Voiculescu 6-term exact sequence for crossed products by the $\mathbb{Z}$ now gives us a procedure to compute $K_*(\mathcal{E}_\alpha)$.

Examples. We return to the examples from the previous section.

1. Let us compute $K_*(\mathcal{E}_\alpha)$. It can be seen from the inductive limit description above that in this case, $K_0(\mathcal{F}) \cong \mathbb{Z}\left[\frac{1}{N}\right]$, $K_1(\mathcal{F}) \cong \mathbb{Z}$. $\vec{\Phi}$ induces the identity map on K_1, and the multiplication by $\frac{1}{N}$ map on K_0. An application of the

Pimsner-Voiculescu sequence yields then that

$$K_0(\mathcal{E}_\alpha) \cong \mathbb{Z} \oplus Z_{N-1}, \qquad K_1(\mathcal{E}_\alpha) \cong Z.$$

2. Here the algebra $\mathcal{E}_\alpha$ we obtain is isomorphic to the Cuntz algebra O_n, with $n = |H|$.
3. The induced action on K_0 is given by

$$\begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1}$$

Applying the Pimsner-Voiculescu sequence yields

$$K_0(\mathcal{E}_\alpha) \cong \mathbb{Z} \oplus Z_2, \qquad K_1(\mathcal{E}_\alpha) \cong \mathbb{Z}.$$

Unlike the first two cases, it is not clear at this time whether $\mathcal{E}_\alpha$ is simple.

The expectation $E_{\mathcal{F}}$ is the restriction of the faithful canonical expectation map $\widetilde{\mathcal{E}_\alpha} \longrightarrow \vec{\mathcal{F}}$ from the general theory of crossed products, and therefore it is faithful as well.

Denote $E = E_{\mathcal{A}} \circ E_{\mathcal{F}} : \mathcal{E}_\alpha \longrightarrow \mathcal{A}$. This is also a faithful expectation.

Let π be a nondegenerate representation of $\mathcal{E}_\alpha$. $\pi\lceil_{\mathcal{A}}$ is a representation of $\mathcal{A}$, and as such is given by a spectral measure Λ on $\hat{\mathcal{A}}$. We henceforth view vectors of the representation space H_π as sections over $\hat{\mathcal{A}}$. Since $\mathcal{F}$ is simple, assuming $\pi \neq 0$, π must be faithful when restricted to $\mathcal{F}$, and therefore it must be faithful when restricted to $\mathcal{A}$ as well, so we must have $\overline{\text{supp}}(\Lambda) = \hat{\mathcal{A}}$, and therefore

$$\|\pi(a)\| = \sup\{|\hat{a}(\omega)| \ \mid \omega \in \hat{\mathcal{A}}\}.$$

We know that S, δ_x normalize $\mathcal{A}$ (i.e., $S\mathcal{A}S^* \subseteq \mathcal{A}$, $S^*\mathcal{A}S \subseteq \mathcal{A}$, $\delta_x\mathcal{A}\delta_x^* \subseteq \mathcal{A}$ for all $x \in G$), and therefore the semigroup generated by those elements acts (partially) on $\hat{\mathcal{A}}$. Denote by $p \cdot U$ the action of this semigroup on $\hat{\mathcal{A}}$, where $p \in \hat{\mathcal{A}}$ and U is in the semigroup.

Observation 3.1. *Let U be as above, and let $\xi \in H_\pi$, then $\overline{\text{supp}}(U\xi) \subseteq \overline{\text{supp}}\,\xi \cdot U$.*

Proof. It suffices to verify this for ξ such that $\overline{\text{supp}}\,\xi$ is a cylindrical subset of $\hat{\mathcal{A}}$ of the form $B = \{r_1\} \times \{r_2\} \times \cdots \times \{r_n\} \times R_1 \times R_1 \times \cdots$. For the purpose of this proof, we freely identify elements of $\mathcal{A}$ with functions in $C(\hat{\mathcal{A}})$. So, $\xi = \chi_B\xi$. Notice that $U^*U \in \mathcal{A}$. Write $f = U^*U \in C(\hat{\mathcal{A}})$. So, $U\xi = UU^*U\xi = Uf\xi = Uf\chi_B\xi = U\chi_B f\xi = (U\chi_B U^*)U\xi = \chi_{B\cdot U}U\xi$, and $\overline{\text{supp}}\,\chi_{B\cdot U}U\xi \subseteq \overline{\text{supp}}\,\chi_{B\cdot U} = B \cdot U = \overline{\text{supp}}\,\xi \cdot U$ which is what we wanted. □

For $x \in G$ we can associate an element $(x_0, x_1, \dots)$ of $R_1^{\mathbb{N}}$ ($\cong \hat{\mathcal{A}}$) by requiring that $x \in \alpha^{n+1}(G)\alpha^n(x_n)\cdots\alpha(x_1)x_0$. Clearly this is well defined.

Definition 3.2. We call the above sequence the *R-sequence* of x, and denote it by $R(x)$.

This gives us a map $R : G \longrightarrow \hat{\mathcal{A}}$.

Lemma 3.3. *The map R is injective.*

Proof. We must show that if $R(x) = R(y)$, $x, y \in G$, then $x = y$. $R(x) = R(y)$ is equivalent to saying that $\alpha^n(G)x = \alpha^n(G)y$ for all $n \in \mathbb{N}$, i.e., $xy^{-1} \in \alpha^n(G)$ for all n, and since α was assumed to be pure, we obtain $xy^{-1} = 1$. □

Definition 3.4. We say that the endomorphism α is *totally normal* if $\alpha^n(G)$ is normal in G for all n.

From now on, we assume that α is totally normal. Notice that this implies that the action of G on $\hat{\mathcal{A}}$ is free (from the remarks in the previous section).

We define maps $\Psi_k^n : R_1^{n+k} \to R_1^n$ (for $n > k$) by taking $\Psi_k^n(x_1, \ldots, x_{n+k})$ to be the element of R_1^n corresponding to the coset representative of

$$x_1^{-1}\alpha(x_2^{-1}) \cdots \alpha^{n-1}(x_n^{-1})\alpha^{n-1}(x_{n+k}) \cdots \alpha(x_{k+2})x_{k+1}.$$

From the normality codition, it follows that Ψ_k^n is $|R_1|^k$–to–1. Furthermore, the following diagram commutes.

$$\begin{array}{ccc} R_1^{n+k+1} & \xrightarrow{\Psi_k^{n+1}} & R_1^{n+1} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ R_1^{n+k} & \xrightarrow{\Psi_k^{n}} & R_1^{n} \end{array}$$

where the π's denote the projections onto the first $n+k$ and n coordinates, respectively.

Therefore those maps form a consistent system, and so we get maps on the projective limit $\Psi_k : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$.

Let μ be the probability product measure on $\hat{\mathcal{A}}$ (thought of as $R_1^{\mathbb{N}}$), where the measure on each factor is the one giving all points the same measure $\left(\frac{1}{|R_1|}\right)$. The canonical maps $\hat{\mathcal{A}} \longrightarrow R_1^m$ for $m \in \mathbb{N}$ induce product push-forward probability measures μ_m on R_1^m (assigning the measure $1/|R_1|^m$ to each point). The fact that the maps Ψ_k^n are $|R_1|^k$–to–1 implies that they are measure preserving with those measures, and therefore the maps Ψ_k are all measure perserving.

Definition 3.5. We say that a point $\omega \in R_1^{\mathbb{N}} \cong \hat{\mathcal{A}}$ is *tail intersecting* if there is some nontrivial $x \in G$ such that ωx is a tail of ω.

Lemma 3.6. *If ω is tail intersecting then there is some k such that $\Psi_k(\omega) \in R(G)$ (i.e., is an R-sequence for some group element).*

Proof. Suppose $\omega = (x_1, x_2, \ldots)$, and there is some $y \in G$ such that $\omega y = (x_k, x_{k+1}, \ldots)$. Then for each $n \in \mathbb{N}$, we must have

$$y \in \alpha^n(G)x_1^{-1}\alpha(x_2^{-1}) \cdots \alpha^{n-1}(x_n^{-1})\alpha^{n-1}(x_{n+k}) \cdots \alpha(x_{k+2})x_{k+1},$$

and therefore, we must have $\Psi_k(\omega) = R(y)$. □

Corollary 3.7. $\{\omega \mid \omega \text{ is not tail intersecting}\}$ *is dense in $R_1^{\mathbb{N}}$.*

Proof. Since $R(G)$ is countable, and the measure μ is nonatomic, it has μ-measure 0. Since the Ψ_k are measure preserving, we see that $\mu(\bigcup_k \Psi_k^{-1}(R(G))) = 0$, and therefore its complement, which contains the set in which we're interested, has full measure, and in particular is dense. □

Lemma 3.8. *For all $a \in \mathcal{E}_\alpha$, $\|\pi(a)\| \geq \|\pi(E(a))\|$.*

Before proving the lemma, we state and prove its main consequence.

Theorem 3.9. *If G is amenable and α is pure and totally normal then $\mathcal{E}_\alpha$ is simple.*

Proof of Theorem 3.9 (assuming Lemma 3.8)**.** Let π be a representation of $\mathcal{E}_\alpha$, and let $a \in \ker(\pi)$ be a positive element, then $E(a) \neq 0$, since E is faithful. By the lemma, $E(a) \in \ker(\pi)$, so $\ker(\pi) \cap \mathcal{F} \neq 0$, and therefore $\ker(\pi) \supseteq \mathcal{F}$, so $1 \in \ker(\pi)$, so $\pi = 0$. So any nonzero representation of $\mathcal{E}_\alpha$ must be faithful, i.e., $\mathcal{E}_\alpha$ is simple. □

Proof of Lemma 3.8. By the remarks preceeding the lemma, it suffices to show that $\|\pi(a)\| \geq |\widehat{E(a)}(\omega)|$ for a dense collection of $\omega \in \hat{\mathcal{A}}$. It also suffices to verify it for a dense collection of $a \in \mathcal{E}_\alpha$. So we assume that a is taken from the *-algebra generated by S and the δ_x, $x \in G$. Each such element is in the span of the semigroup from above. Notice that if ω is not tail intersecting then the only semigroup elements that fix it are the idempotents which have it in their domain. We pick ω to be in this dense set.

We can decompose the representation space H_π into a direct sum $H_\pi = L^2(\hat{\mathcal{A}}, \nu) \oplus H'$, where $\overline{\text{supp}}(\nu) = \hat{\mathcal{A}}$, and $\mathcal{A}$ acts on $L^2(\hat{\mathcal{A}}, \nu)$ by multiplication (by the Gelfand transform). We also assume, without loss of generality, that $\nu(\hat{\mathcal{A}}) = 1$. Suppose ω corresponds to the sequence $r_1, r_2, r_3, \ldots$. Denote $B_n = \{r_1\} \times \{r_2\} \times \cdots \times \{r_n\} \times R_1 \times R_1 \times \cdots$, let ξ'_n be the vector $\chi_{B_n} \oplus 0$, and let $\xi_n = \frac{\xi'_n}{\|\xi'_n\|}$. Denote $\xi_0 = \chi_{\hat{\mathcal{A}}} \oplus 0$ (and notice that $\|\xi_0\| = 1$), so $\xi'_n = \pi(\chi_{B_n})\xi_0$.

For any $a \in \mathcal{A}$, we have

$$\hat{a}(\omega) = \lim_{n\to\infty} \langle \pi(a)\xi_n, \xi_n \rangle = \lim_{n\to\infty} \frac{1}{\|\xi_n\|^2} \langle \pi(\chi_{B_n} a \chi_{B_n})\xi_0, \xi_0 \rangle$$

If U in the semigroup is not a projection, then because of our choice of ω, for sufficiently large n, $B_n \cap (B_n \cdot U) = \emptyset$, so $\chi_{B_n} U \chi_{B_n} = 0$. Also, notice that the intersection of the semigroup with $\mathcal{A}$ is exactly the projections. Therefore, for an element a in this dense subalgebra and for such an ω, for all sufficiently large n we have $\langle \pi(\chi_{B_n} a \chi_{B_n})\xi_0, \xi_0 \rangle = \langle \pi(\chi_{B_n} E(a) \chi_{B_n})\xi_0, \xi_0 \rangle$. So,

$$\|\pi(a)\| \geq |\langle \pi(a)\xi_n, \xi_n \rangle| = |\langle \pi(E(a))\xi_n, \xi_n \rangle| \to |\widehat{E(a)}(\omega)|$$

which is what we wanted. □

Remark 3.10. A groupoid approach could have been used in this case. We represented the γ-invariant subalgebra as a crossed product with an action of a discrete group on a compact Hausdorff set, and therefore it can be viewed as the groupoid algebra of the associated transformation groupoid. We could then present $\mathcal{E}_\alpha$ as the groupoid algebra of a groupoid which is the restriction of the semi-direct product groupoid of the above transformation groupoid by an action of the integers (cf. [R]). Equivalently, we could view this groupoid as a certain restriction of the universal groupoid associated to the inverse semigroup generated by G and S (cf. [Pa]). In this language, we showed that under our conditions, the groupoid thus obtained is essentially principal ([R]) with no invariant sets in the unit space. The proof of the last lemma is essentially that of [R], with the simplifications that arise from the fact that our situation is less general.

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