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Image partition regularity over the reals

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Abstract. We show that many of the natural analogues of known characterizations of image partition regularity and weak image partition regularity of matrices with rational entries over the integers are valid for matrices with real entries over the reals.

Contents

1. Introduction

In 1933 R. Rado published [8] his famous theorem characterizing those finite matrices A with rational entries that have the property that whenever $\mathbb{N}$ is finitely colored, there must be some $\vec{x}$ in the kernel of A all of whose entries are the same color (or *monochrome*). This characterization was in terms of the *columns condition* which we shall describe below.

In 1943 Rado published a paper [9], among whose results was the fact that the same condition characterized those finite matrices with real entries that have the property that whenever $\mathbb{R}$ is finitely colored, there is some $\vec{x}$ in the kernel of A whose entries are monochrome.

Definition 1.1. Let $u, v \in \mathbb{N}$, let $S \in \{\mathbb{N}, \mathbb{Z}, \mathbb{R}^+, \mathbb{R}\}$. Let $F = \mathbb{Q}$ if $S = \mathbb{N}$ or $S = \mathbb{Z}$, and let $F = \mathbb{R}$ if $S = \mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ or $S = \mathbb{R}$. Let A be a $u \times v$ matrix with entries from F.

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(a) The matrix A is *kernel partition regular over* S provided that, whenever $S\setminus\{0\}$ is finitely colored, there exists $\vec{x} \in F^v$ such that $A\vec{x} = \vec{0}$ and all entries of $\vec{x}$ are the same color.

(b) The matrix A satisfies the *columns condition over* F if and only if there is a partition $\{I_1, I_2, \ldots, I_m\}$ of $\{1, 2, \ldots, v\}$ such that, $\sum_{i\in I_1} \vec{c_i} = \vec{0}$ and for each $t \in \{2, 3, \ldots, m\}$, if any, $\sum_{i\in I_t} \vec{c_i}$ is a linear combination over F of $\left\{\vec{c_i} : i \in \bigcup_{k=1}^{t-1} I_k\right\}$.

The results of Rado referred to above are that A is kernel partition regular over S if and only if A satisfies the columns condition over F. (Proofs of Rado's Theorem for the case $S = \mathbb{N}$ can be found in [4] and [7].)

Rado's Theorem, in any of its forms, is quite powerful. For example, it has as a corollary van der Waerden's Theorem [11], which says that whenever $\mathbb{N}$ is finitely colored, there must be arbitrarily long monochromatic arithmetic progressions. But one must be careful. For example, with a length four arithmetic progression $\{a, a+d, a+2d, a+3d\}$ one might let $x_1 = a$, $x_2 = a+d$, $x_3 = a+2d$, and $x_4 = a+3d$ and say that one was precisely asking that $x_2 - x_1 = x_3 - x_2$, and $x_3 - x_2 = x_4 - x_3$, that is that the matrix

$$\begin{pmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{pmatrix}$$

is kernel partition regular. Unfortunately, $x_1 = x_2 = x_3 = x_4$ is a solution to these equations, and one must strengthen the result, for example by demanding that the increment have the same color.

By way of contrast, if one asks that entries of the *image* of A be monochrome, van der Waerden's theorem is very simply represented. The length four version mentioned above is asking that the entries of

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} a \\ d \end{pmatrix}$$

be monochrome. Many other theorems such as Schur's Theorem [10] are naturally represented in terms of images of matrices.

Definition 1.2. Let $u, v \in \mathbb{N}$, let $S \in \{\mathbb{N}, \mathbb{Z}, \mathbb{R}^+, \mathbb{R}\}$. Let $F = \mathbb{Q}$ if $S = \mathbb{N}$ or $S = \mathbb{Z}$, and let $F = \mathbb{R}$ if $S = \mathbb{R}^+$ or $S = \mathbb{R}$. Let A be a $u \times v$ matrix with entries from F.

(a) The matrix A is *image partition regular over* S provided that, whenever $S\setminus\{0\}$ is finitely colored, there exists $\vec{x} \in (S \setminus \{0\})^v$ such that all entries of $A\vec{x}$ are the same color.

(b) The matrix A is *weakly image partition regular over* S provided that, whenever $S \setminus \{0\}$ is finitely colored, there exists $\vec{x} \in F^v$ such that all entries of $A\vec{x}$ are the same color.

Notice that the notions of weakly image partition regular over $\mathbb{N}$ and weakly image partition regular over $\mathbb{Z}$ are equivalent as are the notions of weakly image partition regular over $\mathbb{R}^+$ and weakly image partition regular over $\mathbb{R}$. (For example, given a coloring of $\mathbb{R}^+$ with r colors, one colors $\mathbb{R} \setminus \{0\}$ with $2r$ colors, using r new colors for negative values and giving $x < 0$ and $y < 0$ the same color if and only if $-x$ and $-y$ have the same color. Given $\vec{x} \in \mathbb{R}^v$ with the entries of $A\vec{x}$ the same color, one also has that the entries of $A(-\vec{x})$ are the same color.)

Certain image partition regular matrices were used by W. Deuber [1] in 1973 to prove a famous conjecture of Rado, namely that if a set $C \subseteq \mathbb{N}$ has the property (called *partition regular* in [1]) that for any $u \times v$ matrix A which is kernel partition regular over $\mathbb{N}$ there exists $\vec{x} \in C^v$ with $A\vec{x} = \vec{0}$, and C is divided into finitely many pieces, then one of those pieces also has that property.

Deuber's image partition regular matrices were examples of *first entries* matrices.

Definition 1.3. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with real entries. Then A is a *first entries matrix* if and only if:

(a) no row of A is $\vec{0}$,
(b) the first nonzero entry of each row is positive, and
(c) the first nonzero entries of any two rows are equal if they occur in the same column.

If A is a first entries matrix and d is the first nonzero entry of some row of A, then d is called a *first entry* of A.

Given the early established utility of image partition regular matrices and the fact that they naturally represent many problems of Ramsey Theory, it is surprising that characterizations of matrices with rational entries that are image partition regular over $\mathbb{N}$ or weakly image partition regular over $\mathbb{N}$ were only obtained in 1993 [5]. Additional characterizations of matrices that are image partition regular over $\mathbb{N}$ were obtained in [7] and [6]. (More attention was paid to image partition regularity than to weak image partition regularity because the former is the more natural notion — consider again van der Waerden's Theorem when the increment is allowed to be 0.)

In this paper we obtain characterizations of matrices with real entries that are image partition regular or weakly image partition regular over $\mathbb{R}$ or $\mathbb{R}^+$. These characterizations include natural analogues of several of the known characterizations of image partition regularity or weak image partition regularity over $\mathbb{N}$. They also include some characterizations of weak image partition regularity over $\mathbb{R}$ whose analogues over $\mathbb{N}$, while true, have not been previously published.

Many, but not all, of the proofs given here are simpler than the corresponding proofs for $\mathbb{N}$. (When working with a matrix A with rational entries and a vector $\vec{x}$ with integer entries, one needs to worry about when the entries of $A\vec{x}$ are integers.) The proofs dealing with weak image partition regularity over $\mathbb{R}$ are significantly simpler than the published versions of the corresponding results for $\mathbb{N}$, based on ideas of I. Leader and D. Strauss in [6].

We include proofs of all of the nonelementary facts that we use except for some results from linear algebra, as well as many of the basic facts about the algebra of the Stone-Čech compactification βS of a discrete semigroup S, for which the reader is referred to [7]. (We feel more than somewhat guilty about assuming so much, but do not want to make this paper huge.)

In Section 2 we present several preliminary lemmas. In Section 3 we give our characterizations of weak image partition regularity over $\mathbb{R}$ and $\mathbb{R}^+$. Section 4 has our characterizations of image partition regularity over $\mathbb{R}^+$.

Several of the characterizations involve the notion of *central* sets. This notion was introduced (for subsets of $\mathbb{N}$) by Furstenberg [3]. We take the following equivalent algebraic notion as our definition.

Definition 1.4. Let S be a discrete semigroup. A set $C \subseteq S$ is *central* if and only if there is an idempotent p in the smallest ideal $K(\beta S)$ of βS with $C \in p$.

It is important to note that the notion of central sets is defined in terms of a *discrete* semigroup. We shall denote by $\mathbb{R}_d$ and $\mathbb{R}_d^+$ the set of reals with the discrete topology and the set of positive reals with the discrete topology respectively.

The basic fact that we need about central sets is given by the Central Sets Theorem, which is due to Furstenberg [3] for the case $S = \mathbb{N}$. (Given a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a semigroup $(S, +)$, $FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{n \in F} x_n : F$ is a finite nonempty subset of $\mathbb{N}\}$.)

Theorem 1.5 (Central Sets Theorem). *Let $(S, +)$ be a commutative semigroup, let C be a central subset of S, let $v \in \mathbb{N}$, and for each $\ell \in \{1, 2, \ldots, v\}$, let $\langle y_{\ell,n} \rangle_{n=1}^{\infty}$ be a sequence in S. There exist a sequence $\langle a_n \rangle_{n=1}^{\infty}$ in S and a sequence $\langle H_n \rangle_{n=1}^{\infty}$ of finite nonempty subsets of $\mathbb{N}$ such that $\max H_n < \min H_{n+1}$ for each $n \in \mathbb{N}$ and such that for each $f : \mathbb{N} \to \{1, 2, \ldots, v\}$, $FS(\langle a_n + \sum_{t \in H_n} y_{f(n),t} \rangle_{n=1}^{\infty}) \subseteq C$.*

Proof. [7, Theorem 14.11]. □

We shall follow throughout the custom of denoting the entries of a matrix by the lower case version of the capital letter used to denote the matrix itself. (So $a_{2,3}$ is the entry in row 2 and column 3 of the matrix A.)

2. Preliminary results

The following lemma is standard.

Lemma 2.1. *Let $\epsilon > 0$. There is a finite coloring of $\mathbb{R}^+$ such that, if $y, z \in \mathbb{R}^+$, $y > z$, and y and z have the same color, then either $\frac{y}{z} < 1 + \epsilon$ or $\frac{y}{z} > \frac{1}{\epsilon}$.*

Proof. Let $\alpha = 1 + \epsilon$ and choose $r \in \mathbb{N}$ satisfying $r > 1 + \log_\alpha \frac{1}{\epsilon}$. For each $i \in \{1, 2, \ldots, r\}$, let $P_i = \{n \in \mathbb{N} : \lfloor \log_\alpha n \rfloor \equiv i \mod r\}$. Let $i \in \{1, 2, \ldots, r\}$ and let $y, z \in P_i$ with $y > z$. Then $\lfloor \log_\alpha y \rfloor \geq \lfloor \log_\alpha z \rfloor$.

If $\lfloor \log_\alpha y \rfloor > \lfloor \log_\alpha z \rfloor$, then $\lfloor \log_\alpha y \rfloor \geq \lfloor \log_\alpha z \rfloor + r$ and thus $y > z \cdot \alpha^{r-1} > z \cdot \frac{1}{\epsilon}$. If $\lfloor \log_\alpha y \rfloor = \lfloor \log_\alpha z \rfloor$, then $y < \alpha \cdot z = (1 + \epsilon) \cdot z$. □

Lemma 2.2. *Let A be a $u \times v$ matrix with entries from $\mathbb{R}$ and assume that A is weakly image partition regular over $\mathbb{R}^+$. There exist $m \in \mathbb{N}$ and a partition $\{I_1, I_2, \ldots, I_m\}$ of $\{1, 2, \ldots, u\}$ with the following property: for every $\epsilon > 0$, there exists $\vec{x} \in \mathbb{R}^v$ such that $\vec{y} = A\vec{x} \in (\mathbb{R}^+)^u$ and, if $i \in I_r$ and $j \in I_s$, then $1 - \epsilon < \frac{y_j}{y_i} < 1 + \epsilon$ if $r = s$ and $\frac{y_j}{y_i} < \epsilon$ if $r < s$. If A is image partition regular over $\mathbb{R}^+$, such a vector $\vec{x}$ may be chosen in $(\mathbb{R}^+)^v$.*

Proof. Suppose that $0 < \epsilon < \frac{1}{4}$. Choose a coloring of $\mathbb{R}^+$ as guaranteed by Lemma 2.1. and a vector $\vec{x} \in \mathbb{R}^v$ for which the entries of $\vec{y} = A\vec{x}$ are monochrome positive reals. If A is image partition regular over $\mathbb{R}^+$, choose such $\vec{x} \in (\mathbb{R}^+)^v$. We define a relation $\approx$ on $\{1, 2, \ldots, u\}$ by putting $i \approx j$ if and only if $1 - \epsilon < \dfrac{y_j}{y_i} < 1 + \epsilon$. Since $\epsilon < (1-\epsilon)^2 < (1+\epsilon)^2 < \dfrac{1}{\epsilon}$, it is easy to verify that this is an equivalence relation. It therefore defines a partition $\mathcal{P}(\epsilon) = \{I_1(\epsilon), I_2(\epsilon), \ldots, I_{m(\epsilon)}(\epsilon)\}$ of $\{1, 2, \ldots, u\}$. We can arrange the sets in this partition so that, if $i \in I_r(\epsilon)$, $j \in I_s(\epsilon)$, and $r < s$, then $y_j < y_i$ and so $\dfrac{y_j}{y_i} < \epsilon$. Since there are only finitely many ordered partitions of $\{1, 2, \ldots, u\}$, by the pigeon hole principle there is an infinite sequence of values of ϵ converging to 0 for which the partitions $\mathcal{P}(\epsilon)$ are all the same. □

Definition 2.3. Let $u, v \in \mathbb{N}$, let $\vec{c_1}, \vec{c_2}, \ldots, \vec{c_v}$ be in $\mathbb{R}^u$, and let $I \subseteq \{1, 2, \ldots, v\}$. The *$I$-restricted span* of $(\vec{c_1}, \vec{c_2}, \ldots, \vec{c_v})$ is

$$\{\Sigma_{i=1}^{v} \alpha_i \cdot \vec{c_i} : \text{ each } \alpha_i \in \mathbb{R} \text{ and if } i \in I, \text{ then } \alpha_i \geq 0\}\,.$$

Lemma 2.4. *Let $u, v \in \mathbb{N}$, let $\vec{c_1}, \vec{c_2}, \ldots, \vec{c_v}$ be in $\mathbb{R}^u$, and let $I \subseteq \{1, 2, \ldots, v\}$. The I-restricted span of $(\vec{c_1}, \vec{c_2}, \ldots, \vec{c_v})$ is closed in $\mathbb{R}^u$.*

Proof. This is proved in [5, Lemma 3.8] and in [7, Lemma 15.23]. In both places it is assumed that $\vec{c_1}, \vec{c_2}, \ldots, \vec{c_v} \in \mathbb{Q}^u$, but no use is made of this assumption. □

Lemma 2.5. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{R}$. If A is weakly image partition regular over $\mathbb{R}$, then there exist $m \in \{1, 2, \ldots, u\}$ and a $v \times m$ matrix G with no row equal to $\vec{0}$ such that, if $B = AG$, then B is a first entries matrix with nonnegative entries and has all of its first entries equal to 1. If A is image partition regular over $\mathbb{R}^+$, then the entries of G may be chosen to be nonnegative.*

Proof. Let $\vec{c_1}, \vec{c_2}, \ldots, \vec{c_v}$ denote the columns of A and let $\vec{e_i}$ denote the i^{th} unit vector in $\mathbb{R}^u$. Let $\{I_1, I_2, \ldots, I_m\}$ be the partition of $\{1, 2, \ldots, u\}$ guaranteed by Lemma 2.2. We claim that for each $k \in \{1, 2, \ldots, m\}$,

$$\textstyle\sum_{n \in I_k} \vec{e}_n \in c\ell\{\sum_{j=1}^{v} \alpha_j \vec{c}_j - \sum_{i=1}^{k-1} \sum_{n \in I_i} \delta_n \vec{e}_n : \text{ each } \delta_n \geq 0\}$$

and, if A is image partition regular over $\mathbb{R}^+$, then

$$\textstyle\sum_{n \in I_k} \vec{e}_n \in c\ell\{\sum_{j=1}^{v} \alpha_j \vec{c}_j - \sum_{i=1}^{k-1} \sum_{n \in I_i} \delta_n \vec{e}_n : \text{ each } \alpha_j \geq 0 \text{ and each } \delta_n \geq 0\}\,.$$

To see this, let $k \in \{1, 2, \ldots, m\}$ and let $\epsilon > 0$. Choose $\vec{x} \in \mathbb{R}^v$ such that $\vec{y} = A\vec{x} \in (\mathbb{R}^+)^u$ and, if $i \in I_r$ and $j \in I_s$, then $1 - \epsilon < \dfrac{y_j}{y_i} < 1 + \epsilon$ if $r = s$ and $\dfrac{y_j}{y_i} < \epsilon$ if $r < s$. Pick $l \in I_k$. For $j \in \{1, 2, \ldots, v\}$, let $\alpha_j = \dfrac{x_j}{y_l}$, noting that, if $x_j > 0$, then $\alpha_j > 0$. For $n \in \bigcup_{i=1}^{k-1} I_i$, let $\delta_n = \dfrac{y_n}{y_l}$. Then $\sum_{j=1}^{v} \alpha_j \vec{c}_j - \sum_{i=1}^{k-1} \sum_{n \in I_i} \delta_n \vec{e}_n - \sum_{n \in I_k} \vec{e}_n = \vec{z}$ where

$$z_n = \begin{cases} \dfrac{y_n}{y_l} & \text{if } n \in \bigcup_{i=k+1}^{m} I_i \\ \dfrac{y_n}{y_l} - 1 & \text{if } n \in I_k \\ 0 & \text{if } n \in \bigcup_{i=1}^{k-1} I_i. \end{cases}$$

In particular, $|z_n| < \epsilon$ for each $n \in \{1, 2, \ldots, u\}$.

Thus, by Lemma 2.4, we may pick $g_{j,k} \in \mathbb{R}$ for $j \in \{1, 2, \dots, v\}$ and nonnegative $b_{n,k}$ for $n \in \bigcup_{i=1}^{k-1} I_i$ such that $\sum_{n \in I_k} \vec{e}_n = \sum_{j=1}^{v} g_{j,k}\vec{c}_j - \sum_{i=1}^{k-1} \sum_{n \in I_i} b_{n,k}\vec{e}_n$. If A is image partition regular over $\mathbb{R}^+$, again by Lemma 2.4, we may assume that each $g_{j,k} \geq 0$. For $n \in I_k$, let $b_{n,k} = 1$ and for $n \in \bigcup_{i=k+1}^{m} I_i$, let $b_{n,k} = 0$.

We have thus defined a $v \times m$ matrix G and a $u \times m$ first entries matrix B with nonnegative entries and all first entries equal to 1 such that $AG = B$. If A is image partition regular over $\mathbb{R}^+$, then all entries of G are nonnegative. We may suppose that G has no row equal to $\vec{0}$, because we can add any vector in $(\mathbb{R}^+)^v$ to G as a new final column. □

We now turn to some results about central subsets of $\mathbb{R}^+$. Recall that $K(\beta S)$ is the smallest ideal of βS.

Lemma 2.6. *$K(\beta\mathbb{R}_d^+) = K(\beta(\mathbb{R}^+ \cup \{0\})_d) \cap \beta\mathbb{R}_d^+ = K(\beta\mathbb{R}_d) \cap \beta\mathbb{R}_d^+$. In particular any central subset of $\mathbb{R}^+$ is also central in $\mathbb{R}$ and in $\mathbb{R}^+ \cup \{0\}$.*

Proof. Let $I = \bigcap_{x \in \mathbb{R}^+} c\ell_{\beta\mathbb{R}}(x, \infty)$. We show that I is a left ideal of $\beta\mathbb{R}_d$. To see this, let $p \in I$. It suffices to show that for each $y \in \mathbb{R}$, $y + p \in I$ because the map $q \mapsto q + p$ is continuous. So let $y \in \mathbb{R}$, let $x \in \mathbb{R}^+$, and note that $(x + |y|, \infty) \in p$ and $(x + |y|, \infty) \subseteq -y + (x, \infty)$.

Since I is a left ideal of $\beta\mathbb{R}_d$, and thus also of $\beta(\mathbb{R}^+ \cup \{0\})_d$, it meets $K(\beta\mathbb{R}_d)$ and $K(\beta(\mathbb{R}^+ \cup \{0\})_d)$ and therefore $\beta\mathbb{R}_d^+ \cap K(\beta\mathbb{R}_d) \neq \emptyset$ and $\beta\mathbb{R}_d^+ \cap K(\beta(\mathbb{R}^+ \cup \{0\})_d) \neq \emptyset$. The conclusion then follows from [7, Theorem 1.65]. □

Lemma 2.7. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ first entries matrix with entries from $\mathbb{R}$, and let C be central in $\mathbb{R}^+$. There exist sequences $\langle x_{1,n}\rangle_{n=1}^{\infty}$, $\langle x_{2,n}\rangle_{n=1}^{\infty}$, $\dots$, $\langle x_{v,n}\rangle_{n=1}^{\infty}$ in $\mathbb{R}^+$ such that for every finite nonempty subset F of $\mathbb{N}$, $A\vec{x}_F \in C^u$, where*

$$\vec{x}_F = \begin{pmatrix} \Sigma_{n\in F} x_{1,n} \\ \Sigma_{n\in F} x_{2,n} \\ \vdots \\ \Sigma_{n\in F} x_{v,n} \end{pmatrix}.$$

Proof. Let $S = \mathbb{R}^+ \cup \{0\}$ and note that C is central in S by Lemma 2.6. We proceed by induction on v. Assume first that $v = 1$. We can assume A has no repeated rows, so in this case we have $A = (c)$ for some $c \in \mathbb{R}^+$. Pick by Theorem 1.5 a sequence $\langle k_n\rangle_{n=1}^{\infty}$ with $FS(\langle k_n\rangle_{n=1}^{\infty}) \subseteq C$ and for each $n \in \mathbb{N}$ let $x_{1,n} = \dfrac{k_n}{c}$. The sequence $\langle x_{1,n}\rangle_{n=1}^{\infty}$ is as required.

Now let $v \in \mathbb{N}$ and assume the theorem is true for v. Let A be a $u \times (v + 1)$ first entries matrix with entries from $\mathbb{R}$. By rearranging the rows of A and adding additional rows to A if need be, we may assume that we have some $r \in \{1, 2, \dots, u - 1\}$ and some $d \in \mathbb{R}^+$ such that

$$a_{i,1} = \begin{cases} 0 & \text{if } i \in \{1, 2, \dots, r\} \\ d & \text{if } i \in \{r + 1, r + 2, \dots, u\}. \end{cases}$$

Let B be the $r \times v$ matrix with entries $b_{i,j} = a_{i,j+1}$. Pick sequences $\langle z_{1,n}\rangle_{n=1}^{\infty}$, $\langle z_{2,n}\rangle_{n=1}^{\infty}$, $\dots$, $\langle z_{v,n}\rangle_{n=1}^{\infty}$ in $\mathbb{R}^+$ as guaranteed by the induction hypothesis for the matrix B. For each $i \in \{r+1, r+2, \ldots, u\}$ and each $n \in \mathbb{N}$, let

$$y_{i,n} = \sum_{j=2}^{v+1} a_{i,j} \cdot z_{j-1,n}$$

and let $y_{r,n} = 0$ for all $n \in \mathbb{N}$.

Now C is central in S, so pick by Theorem 1.5 a sequence $\langle k_n \rangle_{n=1}^{\infty}$ in S and a sequence $\langle H_n \rangle_{n=1}^{\infty}$ of finite nonempty subsets of $\mathbb{N}$ such that $\max H_n < \min H_{n+1}$ for each n and for each $i \in \{r, r+1, \ldots, u\}$, $FS(\langle k_n + \sum_{t \in H_n} y_{i,t} \rangle_{n=1}^{\infty}) \subseteq C$.

For each $n \in \mathbb{N}$, let $x_{1,n} = \dfrac{k_n}{d}$ and note that $k_n = k_n + \sum_{t \in H_n} y_{r,t} \in C \subseteq \mathbb{R}^+$. For $j \in \{2, 3, \ldots, v+1\}$, let $x_{j,n} = \sum_{t \in H_n} z_{j-1,t}$. We claim that the sequences $\langle x_{j,n} \rangle_{n=1}^{\infty}$ are as required. To see this, let F be a finite nonempty subset of $\mathbb{N}$. We need to show that for each $i \in \{1, 2, \ldots, u\}$, $\sum_{j=1}^{v+1} a_{i,j} \cdot \sum_{n \in F} x_{j,n} \in C$. So let $i \in \{1, 2, \ldots, u\}$ be given.

Case 1. $i \leq r$. Then

$$\begin{aligned} \sum_{j=1}^{v+1} a_{i,j} \cdot \sum_{n \in F} x_{j,n} &= \sum_{j=2}^{v+1} a_{i,j} \cdot \sum_{n \in F} \sum_{t \in H_n} z_{j-1,t} \\ &= \sum_{j=1}^{v} b_{i,j} \cdot \sum_{t \in G} z_{j,t} \in C \end{aligned}$$

where $G = \bigcup_{n \in F} H_n$.

Case 2. $i > r$. Then

$$\begin{aligned} \sum_{j=1}^{v+1} a_{i,j} \cdot \sum_{n \in F} x_{j,n} &= d \cdot \sum_{n \in F} x_{1,n} + \sum_{j=2}^{v+1} a_{i,j} \cdot \sum_{n \in F} x_{j,n} \\ &= \sum_{n \in F} d x_{1,n} + \sum_{n \in F} \sum_{t \in H_n} \sum_{j=2}^{v+1} a_{i,j} z_{j-1,t} \\ &= \sum_{n \in F} (k_n + \sum_{t \in H_n} y_{i,t}) \in C. \end{aligned}$$

□

Lemma 2.8. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{R}$. Define $\varphi : (\mathbb{R}^+)^v \to \mathbb{R}^u$ by $\varphi(\vec{x}) = A\vec{x}$ and let $\widetilde{\varphi} : \beta\big((\mathbb{R}_d^+)^v\big) \to (\beta\mathbb{R}_d)^u$ be its continuous extension. Let p be an idempotent in $K(\beta\mathbb{R}_d^+)$ with the property that for all $C \in p$ there exists $\vec{x} \in (\mathbb{R}^+)^v$ with $A\vec{x} \in C^u$ and let*

$$\overline{p} = \begin{pmatrix} p \\ p \\ \vdots \\ p \end{pmatrix}.$$

There exists an idempotent $q \in K\Big(\beta\big((\mathbb{R}_d^+)^v\big)\Big)$ such that $\widetilde{\varphi}(q) = \overline{p}$.

Proof. By Lemma 2.6 we have that $p \in K(\beta\mathbb{R}_d)$. Therefore by [7, Theorem 2.23] $\overline{p} \in K(\beta(\mathbb{R}_d)^u)$. By [7, Corollary 4.22] $\widetilde{\varphi}$ is a homomorphism. We claim that

$\overline{p} \in \widetilde{\varphi}\big[\beta\big((\mathbb{R}_d^+)^v\big)\big]$. This is true because $\widetilde{\varphi}\big[\big(\beta(\mathbb{R}_d^+)^v\big)\big]$ is closed and, whenever $C \in p$, $C^u \cap \varphi\big[(\mathbb{R}^+)^v\big] \neq \emptyset$.

Let $M = \{q \in \beta\big((\mathbb{R}_d^+)^v\big) : \widetilde{\varphi}(q) = \overline{p}\}$. Then M is a compact right topological semigroup so by [2, Corollary 2.10] pick an idempotent $w \in M$. By [7, Theorem 1.60] pick an idempotent $q \in K\Big(\beta\big((\mathbb{R}_d^+)^v\big)\Big)$ such that $q \leq w$. Since $\widetilde{\varphi}$ is a homomorphism, $\widetilde{\varphi}(q) \leq \overline{p}$ and thus, since p is minimal, $\widetilde{\varphi}(q) = \overline{p}$. □

3. Weak image partiton regularity over $\mathbb{R}$

In this section we obtain several characterizations of matrices that are weakly image partition regular over $\mathbb{R}$, including the nontrivial fact that weak image partition regularity over $\mathbb{R}$ implies image partition regularity over $\mathbb{R}$. Notice that conditions (d) and (i) are, by virtue of the 1943 version of Rado's Theorem, effectively computable.

Theorem 3.1. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{R}$. The following statements are equivalent:*

(a) *A is weakly image partition regular over $\mathbb{R}$.*
(b) *A is weakly image partition regular over $\mathbb{R}^+$.*
(c) *A is image partition regular over $\mathbb{R}$.*
(d) *Let $l = \mathrm{rank}(A)$. Rearrange the rows of A so that the first l rows are linearly independent over $\mathbb{R}$. Let $\vec{r_1}, \vec{r_2}, \ldots, \vec{r_u}$ be the rows of A. For each $t \in \{l+1, l+2, \ldots, u\}$, if any, let $\gamma_{t,1}, \gamma_{t,2}, \ldots, \gamma_{t,l} \in \mathbb{R}$ be determined by $\vec{r_t} = \sum_{i=1}^{l} \gamma_{t,i} \cdot \vec{r_i}$. If $u > l$, let D be the $(u-l) \times v$ matrix such that, for $t \in \{1, 2, \ldots, u-l\}$ and $i \in \{1, 2, \ldots, u\}$,*

$$d_{t,i} = \begin{cases} \gamma_{l+t,i} & \text{if } i \leq l \\ -1 & \text{if } i = l+t \\ 0 & \text{otherwise.} \end{cases}$$

Then $l = u$ or D is kernel partition regular over $\mathbb{R}$.

(e) *There exist $m \in \{1, 2, \ldots, u\}$ and a $v \times m$ matrix G with no row equal to $\vec{0}$ such that, if $B = AG$, then B is a first entries matrix with nonnegative entries which has all of its first entries equal to 1.*
(f) *There exist $m \in \{1, 2, \ldots, u\}$ and a $v \times m$ matrix G with no row equal to $\vec{0}$ such that, if $B = AG$, then B is a first entries matrix which has all of its first entries equal to 1.*
(g) *There exist $m \in \{1, 2, \ldots, u\}$ and a $u \times m$ first entries matrix B such that for all $\vec{y} \in \mathbb{R}^m$ there exists $\vec{x} \in \mathbb{R}^v$ such that $A\vec{x} = B\vec{y}$.*
(h) *For every central subset C of $\mathbb{R}^+$, there exists $\vec{x} \in \mathbb{R}^v$ such that $A\vec{x} \in C^u$.*
(i) *There exist $t_1, t_2, \ldots, t_v \in \mathbb{R} \setminus \{0\}$ such that, if*

$$T = \begin{pmatrix} t_1 & 0 & \ldots & 0 \\ 0 & t_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & t_v \end{pmatrix},$$

then $(AT - I)$ is kernel partition regular over $\mathbb{R}$, where I is the $u \times u$ identity matrix.

(j) *There exist* $b_1, b_2, \ldots, b_v \in \mathbb{R} \setminus \{0\}$ *such that, if*

$$B = \begin{pmatrix} b_1 & 0 & \ldots & 0 \\ 0 & b_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_v \end{pmatrix}, \quad \textit{then} \begin{pmatrix} B \\ A \end{pmatrix}$$

is weakly image partition regular over $\mathbb{R}$.

(k) *For each* $\vec{p} \in \mathbb{R}^v \setminus \{\vec{0}\}$ *there exists* $b \in \mathbb{R} \setminus \{0\}$ *such that* $\begin{pmatrix} b\vec{p} \\ A \end{pmatrix}$ *is weakly image partition regular over* $\mathbb{R}$.

Proof. We have already remarked that (a) $\Leftrightarrow$ (b).

We show next that (a) $\Leftrightarrow$ (d). Assume first that A is image partition regular over $\mathbb{R}$ and that $l < u$. Let $\mathbb{R} \setminus \{0\}$ be finitely colored and pick $\vec{x} \in \mathbb{R}^v$ such that the entries of $\vec{w} = A\vec{x}$ are monochrome. Then $D\vec{w} = DA\vec{x} = \mathbf{O}\vec{x} = \vec{0}$, where $\mathbf{O}$ is the $(u-l) \times v$ matrix with all zero entries.

Now assume that A satisfies (d) and assume first that $l = u$. By rearranging columns we may presume that the first l columns of A are linearly independent. Let A^* consist of those first l columns, and find $\vec{x} \in \mathbb{R}^l$ such that $(A^*)\vec{x} = \vec{1}$, the vector with all entries equal to 1. Let $y_i = x_i$ if $i \in \{1, 2, \ldots, l\}$ and let $y_i = 0$ if $i \in \{l+1, l+2, \ldots, v\}$. Then $A\vec{y} = \vec{1}$.

Now assume that $l < u$. We may again assume that the first l columns of A are linearly independent and let A^* consist of the upper left $l \times l$ corner of A. Let $\mathbb{R} \setminus \{0\}$ be finitely colored and pick a monochrome $\vec{x} \in \mathbb{R}^u$ such that $D\vec{x} = \vec{0}$. Choose $\vec{w} \in \mathbb{R}^l$ such that

$$A^*\vec{w} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_l \end{pmatrix}.$$

Let $y_j = w_j$ if $j \in \{1, 2, \ldots, l\}$ and let $y_j = 0$ if $j \in \{l+1, l+2, \ldots, v\}$. We claim that $A\vec{y} = \vec{x}$. If $t \in \{1, 2, \ldots, l\}$, then $\sum_{j=1}^{v} a_{t,j} \cdot y_j = \sum_{j=1}^{l} a_{t,j} \cdot w_j = x_t$. If $t \in \{l+1, l+2, \ldots, u\}$, then $\sum_{i=1}^{l} \gamma_{t,i} \cdot x_i = x_t$ because $D\vec{x} = 0$ and therefore

$$\begin{aligned} \sum_{j=1}^{v} a_{t,j} \cdot y_j = \sum_{j=1}^{l} a_{t,j} \cdot w_j &= \sum_{j=1}^{l} w_j \cdot \sum_{i=1}^{l} \gamma_{t,i} a_{i,j} \\ &= \sum_{i=1}^{l} \gamma_{t,j} \cdot \sum_{j=1}^{l} a_{i,j} \cdot w_j = \sum_{i=1}^{l} \gamma_{t,i} \cdot x_i = x_t. \end{aligned}$$

That (a) implies (e) follows from Lemma 2.5, and trivially (e) implies (f) and (f) implies (g).

To see that (g) implies (h), let C be a central subset of $\mathbb{R}^+$ and pick by Lemma 2.8 some $\vec{y} \in (\mathbb{R}^+)^m$ such that $B\vec{y} \in C^u$.

Since any finite partition of $\mathbb{R}^+$ must have at least one cell which is central, it is trivial that (h) implies (a).

We now show that (e) implies (i). For each $i \in \{1, 2, \ldots, v\}$ let t_i be the first nonzero entry in row i of G. Then

$$(AT - I)\begin{pmatrix} T^{-1}G \\ B \end{pmatrix} = B - B = \mathbf{O}\,.$$

The first nonzero entry in each row of $T^{-1}G$ is 1 so $\begin{pmatrix} T^{-1}G \\ B \end{pmatrix}$ is a first entries matrix.

Let $\mathbb{R}^+$ be finitely colored. Then some color class is central. Pick by Lemma 2.7 some $\vec{x} \in (\mathbb{R}^+)^m$ such that the entries of $\begin{pmatrix} T^{-1}G \\ B \end{pmatrix}\vec{x}$ all lie in this color class. Let $\vec{y} = \begin{pmatrix} T^{-1}G \\ B \end{pmatrix}\vec{x}$. Then the entries of $\vec{y}$ are monochrome and $(AT - I)\vec{y} = \mathbf{O}\vec{y} = \vec{0}$.

To see that (i) implies (j), for each $i \in \{1, 2, \ldots, v\}$ let $b_i = \dfrac{1}{t_i}$. Let $\mathbb{R} \setminus \{0\}$ be finitely colored and pick a monochrome $\vec{y} \in \mathbb{R}^{u+v}$ such that $(AT - I)\vec{y} = \vec{0}$. For $i \in \{1, 2, \ldots, v\}$ let $w_i = y_i$ and for $i \in \{1, 2, \ldots, u\}$, let $z_i = y_{v+i}$. Then $AT\vec{w} = \vec{z}$. Let $\vec{x} = T\vec{w}$. Then $\begin{pmatrix} B \\ A \end{pmatrix}\vec{x} = \vec{y}$.

To see that (j) implies (c), let $\mathbb{R} \setminus \{0\}$ be finitely colored and pick $\vec{x} \in \mathbb{R}^v$ such that the entries of $\begin{pmatrix} B \\ A \end{pmatrix}\vec{x}$ are monochrome. Then for each $i \in \{1, 2, \ldots, v\}$, $t_i \cdot x_i \neq 0$ so $\vec{x} \in (\mathbb{R} \setminus \{0\})^v$.

Trivially (c) implies (a) so we have now established that conditions (a) through (j) are equivalent.

To see that (f) implies (k), let $\vec{p} \in \mathbb{R}^v \setminus \{\vec{0}\}$. If $\vec{p}G \neq \vec{0}$, we can choose b so that the first entry of $b\vec{p}G$ is 1. If $\vec{p}G = \vec{0}$, we can choose $\vec{c} \in \mathbb{N}^v$ such that $\vec{r} \cdot \vec{c} \neq \vec{0}$ and add $\vec{c}$ to G as a new final column. In this case, we choose b so that $b\vec{r} \cdot \vec{c} = 1$. In either case, $\begin{pmatrix} b\vec{p} \\ A \end{pmatrix} G$ is a first entries matrix with all first entries equal to 1 and so statement (f) holds for $\begin{pmatrix} b\vec{p} \\ A \end{pmatrix}$ and therefore statement (a) holds for $\begin{pmatrix} b\vec{p} \\ A \end{pmatrix}$.

Trivially (k) implies (a). $\square$

4. Image partition regularity over $\mathbb{R}^+$

We now present several characterizations of the strictly stronger property of image partition regularity over $\mathbb{R}^+$. (The matrix

$$\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 4 & 6 \end{pmatrix}$$

satisfies the computable condition (i) of Theorem 3.1 but not the corresponding condition (g) of Theorem 4.1.) Notice that condition (i) of Theorem 4.1 allows one to also use the computable condition (d) of Theorem 3.1 to determine whether a matrix is image partition regular over $\mathbb{R}^+$.

Theorem 4.1. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{R}$. The following statements are equivalent:*

(a) *A is image partition regular over $\mathbb{R}^+$.*

(b) *There exist $m \in \{1, 2, \ldots, u\}$ and a $v \times m$ matrix G with nonnegative entries and no row equal to $\vec{0}$ such that, if $B = AG$, then B is a first entries matrix with nonnegative entries and has all of its first entries equal to 1.*

(c) *There exist $m \in \{1, 2, \ldots, u\}$ and a $v \times m$ matrix G with nonnegative entries and no row equal to $\vec{0}$ such that, if $B = AG$, then B is a first entries matrix with all of its first entries equal to 1.*

(d) *There exist $m \in \{1, 2, \ldots, u\}$ and a $u \times m$ first entries matrix B such that for all $\vec{y} \in (\mathbb{R}^+)^m$ there exists $\vec{x} \in (\mathbb{R}^+)^v$ such that $A\vec{x} = B\vec{y}$.*

(e) *For every central subset C of $\mathbb{R}^+$, there exists $\vec{x} \in (\mathbb{R}^+)^v$ such that $A\vec{x} \in C^u$.*

(f) *For every central subset C of $\mathbb{R}^+$, $\{\vec{x} \in (\mathbb{R}^+)^v : A\vec{x} \in C^u\}$ is central in $(\mathbb{R}^+)^v$.*

(g) *There exist $t_1, t_2, \ldots, t_v \in \mathbb{R}^+$ such that, if*

$$T = \begin{pmatrix} t_1 & 0 & \ldots & 0 \\ 0 & t_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & t_v \end{pmatrix},$$

then $(AT - I)$ is kernel partition regular over $\mathbb{R}$, where I is the $u \times u$ identity matrix.

(h) *There exist $t_1, t_2, \ldots, t_v \in \mathbb{R}^+$ such that, if*

$$T = \begin{pmatrix} t_1 & 0 & \ldots & 0 \\ 0 & t_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & t_v \end{pmatrix}, \quad \text{then} \begin{pmatrix} I \\ AT \end{pmatrix}$$

is image partition regular over $\mathbb{R}^+$, where I is the $v \times v$ identity matrix.

(i) *There exist $b_1, b_2, \ldots, b_v \in \mathbb{R}^+$ such that, if*

$$B = \begin{pmatrix} b_1 & 0 & \ldots & 0 \\ 0 & b_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_v \end{pmatrix}, \quad \text{then} \begin{pmatrix} B \\ A \end{pmatrix}$$

is weakly image partition regular over $\mathbb{R}$.

(j) *There exist $b_1, b_2, \ldots, b_v \in \mathbb{R}^+$ such that, if*

$$B = \begin{pmatrix} b_1 & 0 & \ldots & 0 \\ 0 & b_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_v \end{pmatrix}, \quad \text{then} \begin{pmatrix} B \\ A \end{pmatrix}$$

is image partition regular over $\mathbb{R}^+$.

(k) *For each $\vec{p} \in \mathbb{R}^v \setminus \{\vec{0}\}$ there exists $b \in \mathbb{R} \setminus \{0\}$ such that $\begin{pmatrix} b\vec{p} \\ A \end{pmatrix}$ is image partition regular over $\mathbb{R}^+$.*

(l) *Whenever* $m \in \mathbb{N}$ *and* $\phi_1, \phi_2, \ldots, \phi_m$ *are nonzero linear mappings from* $\mathbb{R}^v$ *to* $\mathbb{R}$*, there exists* $b \in \mathbb{R}^m$ *such that whenever* C *is central in* $\mathbb{R}^+$*, there exists* $\vec{x} \in (\mathbb{R}^+)^v$ *such that* $A\vec{x} \in C^u$ *and for each* $i \in \{1, 2, \ldots, m\}$*,* $b_i \cdot \phi_i(\vec{x}) \in C$*.*

(m) *For every central subset* C *of* $\mathbb{R}^+$*, there exists* $\vec{x} \in (\mathbb{R}^+)^v$ *such that* $\vec{y} = A\vec{x} \in C^u$*, all entries of* $\vec{x}$ *are distinct, and for all* $i \in \{1, 2, \ldots, u\}$*, if rows* i *and* j *of* A *are unequal, then* $y_i \neq y_j$*.*

Proof. That (a) implies (b) is an immediate consequence of Lemma 2.5. And trivially (b) implies (c).

To see that (c) implies (d), note that if $\vec{y} \in (\mathbb{R}^+)^m$, then since the entries of G are nonnegative and no row is $\vec{0}$, one has that $G\vec{y} \in (\mathbb{R}^+)^v$.

To see that (d) implies (e), let C be a central subset of $\mathbb{R}^+$ and pick by Lemma 2.7 some $\vec{y} \in (\mathbb{R}^+)^m$ such that $B\vec{y} \in C^u$.

We have that (e) implies (a) because given any finite partition of $\mathbb{R}^+$, one cell must be central in $\mathbb{R}^+$.

We have now established that conditions (a), (b), (c), (d), and (e) are equivalent. Notice in particular that we have established that if A is image partition regular over $\mathbb{R}^+$, then for any idempotent $p \in K(\beta\mathbb{R}_d^+)$ and any member C of p, since C is therefore central, there exists $\vec{x} \in (\mathbb{R}^+)^v$ such that $A\vec{x} \in C^u$.

To see that (a) implies (f), let C be a central subset of $\mathbb{R}^+$ and pick an idempotent $p \in K(\beta\mathbb{R}_d^+)$ such that $C \in p$. Let φ, $\widetilde{\varphi}$, and $\overline{p}$ be as in Lemma 2.8. Pick $q \in K\Big(\beta\big((\mathbb{R}_d^+)^v\big)\Big)$ such that $\widetilde{\varphi}(q) = \overline{p}$. Now $(c\ell C)^u$ is a neighborhood of $\overline{p}$ so there is a member D of q such that $\varphi[D] \subseteq C^u$. Then $D \subseteq \{\vec{x} \in (\mathbb{R}^+)^v : A\vec{x} \in C^u\}$ so $\{\vec{x} \in (\mathbb{R}^+)^v : A\vec{x} \in C^u\} \in q$ and is therefore central in $(\mathbb{R}^+)^v$.

Trivially (f) implies (e).

The proof that (b) implies (g) can be taken verbatim from the proof that (e) implies (i) in Theorem 3.1. We simply note that since the entries of G are nonnegative we have that each $t_i > 0$.

To see that (g) implies (h), let $\mathbb{R}^+$ be finitely colored. Let I_u and I_v denote the $u \times u$ and the $v \times v$ identity matrices respectively. Pick a monochrome $\vec{y} \in (\mathbb{R}^+)^{u+v}$ such that $(AT - I_u)\vec{y} = \vec{0}$. For $i \in \{1, 2, \ldots, v\}$ let $x_i = y_i$ and for $i \in \{1, 2, \ldots, u\}$ let $z_i = y_{v+i}$. Then $AT\vec{x} = \vec{z}$ and so $\begin{pmatrix} I_v \\ AT \end{pmatrix} \vec{x} = \vec{y}$.

To see that (h) implies (i), for each $i \in \{1, 2, \ldots, v\}$ let $b_i = \dfrac{1}{t_i}$. Let $\mathbb{R} \setminus \{0\}$ be finitely colored. Then $\mathbb{R}^+$ is also finitely colored, so pick $\vec{y} \in (\mathbb{R}^+)^v$ such that the entries of $\begin{pmatrix} I \\ AT \end{pmatrix} \vec{y}$ are monochrome. Let $\vec{x} = T\vec{y}$. Then $\begin{pmatrix} B \\ A \end{pmatrix} \vec{x} = \begin{pmatrix} I \\ AT \end{pmatrix} \vec{y}$.

To see that (i) implies (j), let $\mathbb{R}^+$ be finitely colored. Since $\begin{pmatrix} B \\ A \end{pmatrix}$ is weakly image partition regular over $\mathbb{R}^+$, pick some $\vec{x} \in \mathbb{R}^v$ such that the entries of $\begin{pmatrix} B \\ A \end{pmatrix} \vec{x}$ are monochrome. In particular, these entries are all positive. Since for each $i \in \{1, 2, \ldots, v\}$, we have that $b_i \cdot x_i > 0$ we in fact have each $x_i > 0$.

Trivially (j) implies (a).

We have now established that statements (a) through (j) are equivalent. The proof that (c) implies (k) can be taken verbatim from the proof that (f) implies (k) in Theorem 3.1.

Now we show that (k) implies (l). For each $i \in \{1,2,\ldots,m\}$, there exists $\vec{p}_i \in \mathbb{R}^v \setminus \{\vec{0}\}$ such that $\phi_i(\vec{x}) = \vec{p}_i \cdot \vec{x}$ for all $\vec{x} \in \mathbb{R}^v$. By applying statement (k) m times in succession (using the fact that at each stage the new matrix satisfies (k) because (a) implies (k)), we can choose $b_1, b_2, \ldots, b_m \in \mathbb{R}$ for which the matrix

$$\begin{pmatrix} b_1\vec{p}_1 \\ b_2\vec{p}_2 \\ \vdots \\ b_m\vec{p}_m \\ A \end{pmatrix}$$

is image partition regular. The conclusion then follows from the fact that every image partition regular matrix satisfies statement (e) by Lemma 2.7.

To see that (l) implies (m), we may presume that A has no repeated rows so that the conclusion regarding $\vec{y}$ becomes the statement that all entries of $\vec{y}$ are distinct. For $i \neq j$ in $\{1,2,\ldots,v\}$, let $\phi_{i,j}$ be the linear mapping from $\mathbb{R}^v$ to $\mathbb{R}$ taking $\vec{x}$ to $x_i - x_j$. For $i \neq j$ in $\{1,2,\ldots,u\}$, let $\psi_{i,j}$ be the linear mapping from $\mathbb{R}^v$ to $\mathbb{R}$ taking $\vec{x}$ to $\sum_{t=1}^{v}(a_{i,t} - a_{j,t}) \cdot x_t$. Applying statement (l) to the set $\big\{\phi_{i,j} : i \neq j \text{ in } \{1,2,\ldots,v\}\big\} \cup \big\{\psi_{i,j} : i \neq j \text{ in } \{1,2,\ldots,u\}\big\}$, we reach the desired conclusion.

It is trivial that (m) implies (e). □

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