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An up-spectral space need not be A-spectral

Othman Echi and **Riyadh Gargouri**

Abstract. An A-spectral space is a space such that its one-point compactification is a spectral space. An up-spectral space is defined to be a topological space X satisfying the axioms of a spectral space with the exception that X is not necessarily compact. This paper deals with the interactions between up-spectral spaces and A-spectral spaces. An example of up-spectral space which is not A-spectral is constructed.

Contents

Introduction

A topology $\mathcal{T}$ on a set X is said to be *spectral* [11] if the following axioms hold:

(i) $\mathcal{T}$ is sober (i.e., every nonempty irreducible closed subset of X is the closure of a unique point).

(ii) $(X, \mathcal{T})$ is compact.

(iii) The compact open subsets of X form a basis of $\mathcal{T}$.

(iv) The family of compact open subsets of X is closed under finite intersections.

Let $(X, \mathcal{T})$ be a T_0-space. Then X has a partial ordering, $\leq$, induced by $\mathcal{T}$ by letting $x \leq y$ if and only if $y \in \overline{\{x\}}$.

By an *Alexandroff-spectral space* (*A-spectral space*, for short), we mean a topological space such that its one-point compactification is a spectral space [3].

Before recalling the main result [3], let us rewrite [3, Definition 1.5]; but with a slight change.

Definitions. Let X be a topological space and U a subset of X.

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(1) U is said to be *intersection compact open*, or ICO, if for each compact open subset O of X, $U \cap O$ is compact.
(2) U is said to be *intersection compact closed*, or ICC, if for each compact closed subset O of X, $U \cap O$ is compact.
(3) U is said to be *intersection compact open closed*, or ICOC, if it is ICO and ICC.
(4) Let $\mathcal{P}$ be a property. U is said to be *co-$\mathcal{P}$* if $X \setminus U$ satisfies $\mathcal{P}$.

A complete characterization of A-spectral spaces has been given by K. Belaid, O. Echi and R. Gargouri in [3, Theorem 2.2]:

A space X is *A-spectral* if and only if the following axioms hold:

(i) $\mathcal{T}$ is sober.
(ii) X has a basis of compact open sets which is closed under, finite intersections.
(iii) For each compact closed subset C of X, there exists a cocompact ICOC open subset O of X such that $O \subseteq X \setminus C$.

An *up-spectral space* is defined to be a topological space satisfying the axioms of a spectral space with the exception that X is not necessarily compact [4]. Up-spectral spaces have been introduced and studied by Belaid and Echi [4], in order to give some substantial information on a conjecture about spectral sets raised by Lewis and Ohm in 1976 [12]. Using [3, Theorem 2.2], it is clear that an A-spectral space is up-spectral. A natural question is whether an up-spectral space is necessarily A-spectral?

This paper deals with some interactions between A-spectral spaces and up-spectral spaces. After a theoretical preliminary study (Section 1), we will be able to give an example of an up-spectral space which is not A-spectral (Section 2: The example).

The construction of our example is based on Alexandroff topologies. Recently, these topologies proved to be useful for some authors in providing examples and counterexamples in several papers dealing with topology or foliation theory (see for instance [5], [6], [7], [8] and [10]).

Thus it is of interest to recall the concept of Alexadroff topology and to give a historical background of that concept. An *Alexandroff space* is a topological space in which any intersection of open sets is open. Alexandroff spaces were first introduced by P. Alexandroff in 1937 [1] with the name of *Diskrete Räume*. In [13], A.K. Steiner has called Alexdroff spaces *principal spaces*. Mathematically, Alexandroff spaces have important role in the study of the structure of the lattice of topologies [13]. Note also that in [5], Bouacida et al. have investigated Alexandroff spaces (calling them *good spaces*); the authors have linked these spaces with spectral spaces.

It is worth noting that the interest in Alexandroff spaces was a consequence of the very important role of finite spaces in digital topology and the fact that these spaces have all the properties of finite spaces relevant for such theory.

A systematic account of many topological properties of Alexandroff spaces, independent of questions about digital applications, has been given by F. G. Arenas in [2].

In [13] and [5], the authors gave a complete description of how one may generate Alexandroff topologies on a set X. We restrict ourselves to recalling the construction of Alexandroff spaces in the T_0 context (i.e., discrete Alexandroff topologies).

Let $(X, \leq)$ be a partially ordered set and $x \in X$. The specialization of x is $S(x) = \{y \in X : y \geq x\}$, the generization of x is $G(x) = \{y \in X : y \leq x\}$. It is well-known that the collection $\mathcal{B} = \{G(x) : x \in X\}$ is a basis of a topology on X called the *discrete Alexandroff topology* (or the *left topology*) [9, Chapitre I p. 89, Exercice 2]; this topology will be denoted by $\mathcal{A}(\leq)$.

1. Preliminary results

We begin by introducing a new concept which will play a fundamental role in finding links between A-spectral spaces and up-spectral spaces. Recall from the definition (4) in the introduction that U is co-ICO in X if $X \setminus U$ is ICO.

Definition 1.1. We say $U \subset X$ is a **T**-subset if it is closed, compact and co-ICO in X.

Remarks 1.2. (1) Each closed subset of a Noetherian space is a **T**-subset.

(2) A closed compact subset need not be a **T**-subset: Consider $X = [0, \omega]$ the set of all ordinal numbers less than or equal to the first limit ordinal ω equipped with the natural order $\leq$. The discrete Alexandroff topology on X associated to the order is $\mathcal{O}(X) = \{\emptyset,\ X,\ [0, \omega[\} \cup \{(\downarrow x) : x \in X\}$, where $(\downarrow x) = \{y \in X : y \leq x\}$. Then $\{\omega\}$ is a closed compact subset of X which is not a **T**-subset.

(3) A *semispectral space* is a space in which the intersection of two compact open sets is compact [11]. If X is a semispectral space, then the following properties hold:
- (i) Any finite union of closed co-ICO subsets of X is co-ICO.
- (ii) Any finite union of **T**-subsets of X is a **T**-subset.
- (iii) The complement of a compact open set of X is co-ICO.
- (iv) The union of a co-ICO set with the complement of a compact open set of X is co-ICO.

(4) For an arbitrary space, the following properties hold:
- (i) Any finite intersection of co-ICO subsets of X is co-ICO.
- (ii) Any finite intersection of **T**-subsets of X is a **T**-subset.
- (iii) Any open set of X is co-ICO.
- (iv) The union of a co-ICO set with an open set of X is co-ICO.

Proposition 1.3. *Let X be a semispectral space with a basis of compact open sets and C a nonempty subset of X. Then the following statements are equivalent:*
- (i) *C is a closed compact co-ICOC subset of X.*
- (ii) *C is a **T**-subset of X.*
- (iii) *C is closed in X and there exist two compact open sets U and V of X such that $C = U \cap (X \setminus V)$.*

Proof. (i) $\Rightarrow$ (ii). Straightforward.

(ii) $\Rightarrow$ (iii). By the hypotheses on X, there exists a compact open subset U of X such that $C \subseteq U$, since C is compact. On the other hand, $X \setminus C$ is ICO, thus $U \cap (X \setminus C) = V$ is a compact open subset of X. It is clear that $U \cap (X \setminus V) = C$.

(iii) $\Rightarrow$ (i). Since $C = U \cap (X \setminus V)$ is a closed subset of the compact open subset U of X, it is a compact subset of X. Now, since $X \setminus U$ is closed in X and X is semispectral , we conclude that $X \setminus C = V \cup (X \setminus U)$ is ICOC. Therefore, C is a closed compact co-ICOC subset of X. □

Corollary 1.4. *Under the assumptions of Proposition 1.3, a* $\mathbf{T}$*-subset of* X *is always* ICC*.*

Now, let us turn our attention to up-spectral spaces. First, we fix some notations. Let X be a set equipped with a topology $\mathcal{T}$ and $\omega \notin X$. Set $X^\omega = X \cup \{\omega\}$. We equip X^ω with the topology $\mathcal{T}^\omega = \mathcal{T} \cup \{X^\omega\}$. Then, according to [4], we have the following proposition:

Proposition 1.5. *Let* $(X, \mathcal{T})$ *be a topological space. Then the following statements are equivalent:*

(i) $(X, \mathcal{T})$ *is up-spectral.*
(ii) $(X^\omega, \mathcal{T}^\omega)$ *is spectral.*

We need to recall the patch topology [11]. Let X be a topological space. By the *patch topology* on X, we mean the topology which has as a subbasis for its closed sets the closed sets and compact open sets of the original space. By a *patch* we mean a set closed in the patch topology. The patch topology associated to a spectral space is compact and Hausdorff [11].

Proposition 1.6. *Let* $(X, \mathcal{T})$ *be an up-spectral space and* C *a nonempty closed subset of* X*. Then the following statements are equivalent:*

(i) C *is a* $\mathbf{T}$*-subset of* X*.*
(ii) C *is compact and open in the patch topology on the spectral space* $(X^\omega, \mathcal{T}^\omega)$*.*

Proof. Let X^ω_{patch} denote the space X^ω endowed with the patch topology.

(i) $\Rightarrow$ (ii). By Proposition 1.3, there exists two compact open sets U and V of X such that $C = U \cap (X \setminus V)$. Hence C is open (in fact clopen) in X^ω_{patch}. Now, since X^ω_{patch} is compact and C is closed in X^ω_{patch}, we conclude that C is compact in X^ω_{patch}.

(ii) $\Rightarrow$ (i). Let $\Omega(X)$ be the set of all compact open sets of X. Hence

$$\{U : U \in \Omega(X)\} \cup \{X^\omega \setminus U : U \in \Omega(X)\}$$

is an open subbasis of X^ω_{patch}. Any compact open subset Q of X^ω_{patch} has the form

$$Q = (\cup\{U_i : i \in I\}) \cup (\cup\{X^\omega \setminus U_j : j \in J) \cup (\cup\{U_t \cap (X^\omega \setminus V_t) : t \in T\}),$$

where $U_i,\ U_j,\ U_t,\ V_t \in \Omega(X)$ and $I,\ J,\ T$ are finite sets. Thus Q can be written as

$$Q = U \cup (X^\omega \setminus V) \cup (\cup\{U_t \cap (X^\omega \setminus V_t) : t \in T\}), \text{ if } \omega \in Q;$$

and as

$$Q = U \cup (\cup\{U_t \cap (X^\omega \setminus V_t) : t \in T\}), \text{ if } \omega \notin Q.$$

Thus C can be written as

$$C = U \cup (\cup\{U_t \cap (X^\omega \setminus V_t) : t \in T\}), \text{ where } U, U_t, V_t \in \Omega(X).$$

Clearly, C is compact.

Now, one can prove that the set C is co-ICO by induction (on the cardinality n of T), using Remark 1.2 and the equality $W \cup (U_{n+1} \cap (X \setminus V_{n+1})) = (W \cup U_{n+1}) \cap (W \cup (X \setminus V_{n+1}))$. □

Our main result in [3] provides an intrinsic topological characterization of A-spectral spaces:

Theorem 1.7 ([3]). *Let X be a topological space. Then the following statements are equivalent:*

(1) *X is A-spectral.*
(2) *X satisfies the following properties:*
   (i) *X has a basis of compact open sets closed under finite intersections.*
   (ii) *X is sober.*
   (iii) *For each compact closed subset C of X, there exists a cocompact and ICOC open subset O of X such that $O \subseteq X \setminus C$.*

The following result gives some links between up-spectral spaces and A-spectral spaces.

Theorem 1.8. *Let X be a topological space. Then the following statements are equivalent:*

(1) *X is A-spectral.*
(2) *X satisfies the following properties:*
   (i) *X is up-spectral.*
   (ii) *For each compact closed subset C of X, there exists a $\mathbf{T}$-subset D of X such that $C \subseteq D$.*

Proof. (1) ⇒ (2). Obviously, a space X is up-spectral if and only if it satisfies conditions (i) and (ii) of Theorem 1.7. Hence the A-spectral property implies the up-spectral one, by Theorem 1.7. Let C be a compact closed subset of X. Then there exists a cocompact ICOC open subset O of X such that $O \subseteq X \setminus C$, by Theorem 1.7. Clearly, $D = X \setminus O$ is a $\mathbf{T}$-subset of X and $C \subseteq D$.

(2) ⇒ (1). Let C be a compact closed subset of X. Then there exists a $\mathbf{T}$-subset D of X such that $C \subseteq D$. According to Proposition 1.3, $O = X \setminus D$ is ICOC. Thus $X \setminus D$ is an open cocompact ICOC subset of X satisfying $O \subseteq X \setminus C$. Therefore, X is A-spectral, by Theorem 1.7. □

2. The example

This section is devoted to the construction of an up-spectral space which is not A-spectral.

The following properties of discrete Alexandroff topology, extracted from [5] and [9], can be easily checked.

Lemma 2.1. *Let $(X, \leq)$ be a partially ordered set. Then the following properties hold:*

(1) *A subset U of X is open (resp. closed) in $(X, \mathcal{A}(\leq))$ if and only if U is closed under generization (resp. specialization) i.e., $G(x) \subseteq U$, for each $x \in U$ (resp. i.e., $S(x) \subseteq U$, for each $x \in U$).*
(2) *For each $x \in X$, the closure of $\{x\}$ in $(X, \mathcal{A}(\leq))$ is $\overline{\{x\}} = S(x)$.*
(3) *$\{G(x) : x \in X\}$ is a basis of compact open subsets of $(X, \mathcal{A}(\leq))$.*
(4) *An open subset C of X is compact in $(X, \mathcal{A}(\leq))$ if and only if there exist finitely many elements x_1, x_1, … , x_n of X such that*

$$C = G(x_1) \cup G(x_2) \cup \cdots \cup G(x_n).$$

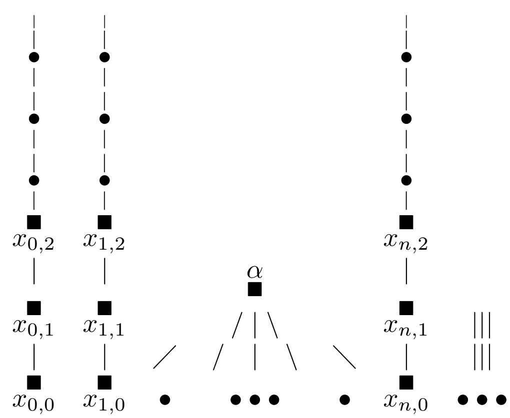
(5) $(X, \mathcal{A}(\leq))$ *is sober if and only if each decreasing sequence of the ordered set* $(X, \leq)$ *is stationary.*

Now, we are in a position to give our example.

The Example. *For each* $n \in \mathbb{N}$*, consider the subset* $X_n = \{x_{n,i} = (n, i) : i \in \mathbb{N}\}$ *of* $\mathbb{R}^2$*; and the point* $\alpha = (1/2, 1/2)$*. We equip the set* $X = (\cup\{X_n : n \in \mathbb{N}\}) \cup \{\alpha\}$ *with the order* $\leq$ *defined by:*

– *For each* $n \in \mathbb{N}$, $x_{n,i} \leq x_{n,j}$ *if and only if* $i \leq j$.
– *An element of* X_n *and an element of* X_m *are incomparable, for* $n \neq m$.
– $\alpha \leq \alpha$ *and* $x_{n,0} \leq \alpha$*, for each* $n \in \mathbb{N}$.

The ordered set $(X, \leq)$ *looks like:*



$(X, \leq)$

Equip X *with the discrete Alexandroff topology* $\mathcal{A}(\leq)$ *associated with the order* $\leq$.

We claim that $(X, \mathcal{A}(\leq))$ *is up-spectral and is not A-spectral.*

Proof. That $(X, \mathcal{A}(\leq))$ is up-spectral is straightforward:

- Clearly, each decreasing sequence of $(X, \leq)$ is stationary. Hence $(X, \mathcal{A}(\leq))$ is a sober space (see Lemma 2.1(5))
- The family $\{G(x) : x \in X\}$ is a basis of compact open sets which is closed under finite intersections.

To see that $(X, \mathcal{A}(\leq))$ is not A-spectral, note that the only compact nonempty closed subset of X is $\{\alpha\}$. If we suppose that X is A-spectral, then, according to Theorem 1.8, $\{\alpha\}$ must be a **T**-subset. But $\{\alpha\}$ is not co-ICO; since $G(\alpha)$ is a compact open subset of X and $G(\alpha) \cap (X \setminus \{\alpha\}) = \{x_{n,0} : n \in \mathbb{N}\}$ is an infinite discrete subspace of X and hence it is not compact. Thus $\{\alpha\}$ is not a **T**-subset of X. This shows that $(X, \mathcal{A}(\leq))$ is not A-spectral, by Theorem 1.8. □

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES OF TUNIS. UNIVERSITY TUNIS-EL MANAR. "CAMPUS UNIVERSITAIRE" 2092 TUNIS, TUNISIA
othechi@yahoo.com othechi@math.com

DEPARTMENT OF MATHEMATICS, INSTITUTE OF MULTIMEDIA, ROUTE MANZEL CHAKER KM 1,5, SFAX TUNISIA
tn_riadh_30@yahoo.fr

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