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Subalgebras of graph C*-algebras

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Dedicated to the memory of Gert Kjærgård Pedersen

Abstract. We prove a spectral theorem for bimodules in the context of graph C*-algebras. A bimodule over a suitable abelian algebra is determined by its spectrum (i.e., its groupoid partial order) iff it is generated by the Cuntz–Krieger partial isometries which it contains iff it is invariant under the gauge automorphisms. We study 1-cocycles on the Cuntz–Krieger groupoid associated with a graph C*-algebra, obtaining results on when integer valued or bounded cocycles on the natural AF subgroupoid extend. To a finite graph with a total order, we associate a nest subalgebra of the graph C*-algebra and then determine its spectrum. This is used to investigate properties of the nest subalgebra. We give a characterization of the partial isometries in a graph C*-algebra which normalize a natural diagonal subalgebra and use this to show that gauge invariant generating triangular subalgebras are classified by their spectra.

Contents

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1. Introduction

Groupoid techniques ("coordinatization") play a major role in the study of nonselfadjoint subalgebras of C*-algebras. The primary focus of this approach has been on subalgebras of AF C*-algebras. In this paper we apply groupoid techniques to the study of subalgebras of another extremely important class of C*-algebras: the graph C*-algebras. We develop a spectral theorem for bimodules which differs somewhat from the similar theorem for AF C*-algebras. Cocycles are a vital tool in the study of analytic subalgebras of AF C*-algebras; accordingly, we investigate cocycles in the Cuntz–Krieger groupoid context. We also apply our spectral theorem for bimodules to study nest subalgebras of graph C*-algebras. Classification of triangular subalgebras by their spectra is a central result in the AF context. We extend this result to the graph C*-algebra context via a characterization of normalizing partial isometries which is similar to, and depends on, the AF analog.

Graph C*-algebras are constructed from directed graphs. We shall make one minor modification in the usual notation for this process: when concatenating edges to form paths we will read right to left. We do this because edges (and paths) correspond to partial isometries in the graph C*-algebra and composition of partial isometries is always read from right to left. This forces some changes in terminology from what appears elsewhere in the graph C*-algebra literature (relevant changes are mentioned in Section 2), but we believe that it is worth paying this small price to make some of the proofs more natural. Furthermore, our conventions are in conformity with the ones now in use in the study of higher rank graph C*-algebras and in the study of quiver algebras. Section 2 also provides some background material needed for the proof of the spectral theorem for bimodules.

Graph C*-algebras are groupoid C*-algebras, as shown in [7]. Since we make substantial use of the groupoid, and in order to establish terminology, we sketch this construction in Section 4. The bimodules which appear in the spectral theorem for bimodules are bimodules over a natural abelian subalgebra of the graph–groupoid C*-algebra. From the graph point of view, this is the C*-subalgebra generated by all the initial and final projections of the partial isometries associated with paths (the Cuntz–Krieger partial isometries). From the groupoid point of view, this abelian algebra is the algebra of continuous functions (vanishing at infinity) on the space of units. This abelian algebra need not be maximal; in Section 5 we show that it is maximal abelian if, and only if, every loop in the graph has an entrance.

In order to define the spectrum of a bimodule, we need to be able to view all elements of the groupoid C*-algebra as functions on the groupoid. This is possible for r-discrete groupoids (and all the groupoids in this paper are r-discrete) when they are amenable. It is proven in [7] and [10] that path space groupoids are always amenable, so spectral techniques are readily available to us.

The spectral theorem for bimodules was first proven by Muhly and Solel [9] for groupoids which are r-discrete and principal. The groupoids which arise from graph C*-algebras are r-discrete but, in general, not principal. In the r-discrete principal groupoid context, every bimodule is determined by its spectrum. This is false for graph C*-algebras. (It is false even for the Cuntz algebra O_n.) In Sections 3 and 6 we provide two conditions, each of which is necessasry and sufficient for a bimodule $\mathcal{B}$ to be determined by its spectrum. One condition is that $\mathcal{B}$ is determined by the Cuntz–Krieger partial isometries which it contains; the other is that $\mathcal{B}$ is invariant under the gauge automorphisms. As it happens, the fact that these two conditions are equivalent to each other can be proven without use of the groupoid model. We prove the equivalence of these two conditions in Section 3, which appears before the description of the groupoid model, and we give the full spectral theorem for bimodules in Section 6. The argument in Section 3 appeals only to the spectral bimodule theorem in the AF C*-algebra case. (See [15], for example.) In Section 8 we extend the spectral theorem for bimodules by showing that we can replace the gauge automorphisms by the one-parameter automorphism group naturally associated with any locally constant real valued cocycle satisfying a mild technical constraint.

Analytic subalgebras play a major role in the study of subalgebras of AF C*-algebras. Analytic subalgebras are most conveniently described in terms of cocycles on the AF groupoid. Two special classes of cocycles of particular importance are the integer valued cocycles and the bounded cocycles. The Cuntz–Krieger groupoids which arise from range finite graphs share some, but not all, of the properties of AF groupoids. This results in interesting differences between the cocycle theories in the two contexts. In Section 7 we introduce techniques for studying cocycles on the Cuntz–Krieger groupoid and apply these techniques to investigate both bounded and integer valued cocycles. Every Cuntz–Krieger groupoid contains a natural AF subgroupoid; Section 7 is particularly concerned with the question of when a cocycle on the AF subgroupoid extends to a cocycle on the whole groupoid.

The third author (Power) initiated the study of nest subalgebras of Cuntz C*-algebras in [12] in 1985. This topic then lay dormant until the first two authors (Hopenwasser and Peters) revisited the topic using groupoid techniques in [2]. It turns out that everything which was done for nest subalgebras of Cuntz C*-algebras can be extended to the graph C*-algebra context (for a finite graph), provided that a suitable order is imposed on the edges of the graph. Definitions of an ordered graph and of an associated nest and nest algebra are given in Section 9. We characterize the Cuntz–Krieger partial isometries in the nest algebra and, in turn, the spectrum of the nest algebra. This enables us to deduce several results about these nest subalgebras of graph C*-algebras; for example, the radical is equal to the closed commutator ideal.

In [13], it was shown that the triangular subalgebras $\mathcal{A}$ of AF C*-algebras $\mathcal{B}$ for which $\mathcal{A} \cap \mathcal{A}^*$ is a standard AF masa are classified up to isometric isomorphism by their associated topological binary relation, or spectrum. This reduction of the issue of isomorphism for TAF algebras to that of classifying their groupoid partial orders has proven to be a standard tool for the classification of many families. We shall obtain an analogous reduction for triangular subalgebras A of a wide class of graph C*-algebras where $\mathcal{A} \cap \mathcal{A}^*$ is the standard masa determined by the generators of $C^*(G)$. As in [13] the key step for the proof is the identification of the partial isometries in $C^*(G)$ which normalise $\mathcal{D}$ as the elements v for which

$$\|pvq\| = 0 \text{ or } 1, \text{ for all projections } p, q \in \mathcal{D}.$$

We obtain this characterisation in Section 10 and apply it to gauge invariant triangular subalgebras in Section 11.

Recall that the tensor (or quiver) algebras of directed graphs correspond to the norm closed nonselfadjoint subalgebras of graph C*-algebras generated by the

Cuntz–Krieger generators and that for various forms of isomorphism these algebras are known to be in bijective correspondence with their underlying graphs. (See [4, 17, 3].) We remark that the algebras studied here, being bimodules over the canonical masa, are quite distinct from these algebras and present more subtle problems of isomorphism type.

2. Preliminaries

Let $G = (V, E, r, s)$ be a directed graph. As usual, V denotes the set of vertices and E the set of edges. The range and source maps are r and s. In this paper, we shall modify slightly the usual procedure (as it appears in most of the literature) for associating a graph C*-algebra $C^*(G)$ to G. (As a consequence, the description of the groupoid underlying $C^*(G)$ will also be slightly modified). This minor change is just notational: a finite path $\alpha = \alpha_1 \dots \alpha_n$ is a finite sequence of edges, or a word, which satisfies $r(\alpha_{i+1}) = s(\alpha_i)$ for $i = 1, \dots, n-1$. Infinite paths will be infinite sequences with the same condition for all i. Edges and finite paths in G correspond to partial isometries in the graph C*-algebra; with this notational change the path $\alpha_1\alpha_2$, for example, corrresponds to $S_{\alpha_1}S_{\alpha_2}$. This notational change will result in modification of some of the usual conditions concerning graphs which appear in the literature; for example, the condition that every loop has an exit will be replaced by the condition that every loop has an entrance; no sinks will be replaced by no sources, etc. Although we are deviating from the usual terminology used in most of the literature on graph C*-algebras, we are in conformity with the conventions used for higher rank graph C*-algebras (e.g., in [5]) and also for free semigroup(oid) algebras and quiver algebras.

Throughout this paper we denote the set of finite paths from G by F and the set of infinite paths by P. Range and source maps are defined on F as follows: if $\alpha = \alpha_1 \dots \alpha_n$ then $r(\alpha) = r(\alpha_1)$ and $s(\alpha) = s(\alpha_n)$. Due to our choice of notation for paths, only the range map can be defined on P; this we do in the obvious way. Also, if $\alpha = \alpha_1 \dots \alpha_n \in F$ then the *length* of α (which is n) is denoted by $|\alpha|$.

We assume that the graph G satisfies the property that $r^{-1}(v)$ is a finite set, for each vertex v. When this property is satisfied, we say that G is *range finite.* (This corresponds to 'row finite' in the literature on graph C*-algebras.) The graph C*-algebra $C^*(G)$ associated with G is the universal C*-algebra generated by a set of partial isometries $\{S_e\}_{e\in E}$ which satisfy the Cuntz–Krieger relations:

$$S_e^* S_e = \sum_{\{f \mid r(f)=s(e)\}} S_f S_f^*.$$

(This minor variation on the usual Cuntz–Krieger relations is made to conform to our notation for paths.) Since we assume throughout this paper that the graph has no sources, we do not need to explicitly include a projection for each vertex. (If v is a vertex, there is an edge e with $s(e) = v$ and $P_v = S_e^* S_e$.)

For any finite path α, let $S_\alpha = S_{\alpha_1} \dots S_{\alpha_k}$. The Cuntz–Krieger relations imply that any product of the generators and their adjoints can be written in the form $S_\alpha S_\beta^*$. These are the *Cuntz–Krieger partial isometries* in $\mathcal{A}$

If $\{S_e\}$ are Cuntz–Krieger generators for $\mathcal{A}$ and if z is a complex number of absolute value one, then $\{zS_e\}$ is another Cuntz–Krieger family which generates $\mathcal{A}$. By the universality of $\mathcal{A}$, there is an automorphism γ_z such that $\gamma_z(S_e) = zS_e$, for all edges e. These are the gauge automorphisms of $\mathcal{A}$. Note that for any Cuntz–Krieger partial isometry $S_\alpha S_\beta^*$, we have $\gamma_z(S_\alpha S_\beta^*) = z^{|\alpha|-|\beta|} S_\alpha S_\beta^*$.

The gauge automorphisms are used in [1] to determine when the C*-algebra generated by a representation of the graph G is isomorphic to the graph C*-algebra. As part of that analysis the authors identify the fixed point algebra of the gauge automorphisms as the natural AF subalgebra of $\mathcal{A}$ and describe a faithful projection of $\mathcal{A}$ onto the fixed point algebra. It is clear that any Cuntz–Krieger partial isometry $S_\alpha S_\beta^*$ with $|\alpha| = |\beta|$ is in the fixed point algebra of the gauge automorphisms. In fact, these partial isometries generate the fixed point algebra, which we shall denote by $\mathcal{F}$. It is proven in [1] than $\mathcal{F}$ is an AF C*-algebra.

The projection from $\mathcal{A}$ onto $\mathcal{F}$ described in [1] is the usual expectation:

$$\Phi_0(f) = \int_{\mathbb{T}} \gamma_z(f)\, dz.$$

This is positive, has norm 1, and is faithful in the sense that $\Phi_0(f^*f) = 0$ implies that $f = 0$.

Let $B^*(G)$ denote the *-algebra generated by $\{S_e \mid e \in E\}$, the Cuntz–Krieger generators of $\mathcal{A}$. So, $B^*(G)$ is just the linear span of the Cuntz–Krieger partial isometries. If $a \in B^*(G)$, then a has an expansion as a finite sum

$$a = \sum_m \sum_{|\lambda|-|\mu|=m} a_{\lambda\mu} S_\lambda S_\mu^*.$$

While this expansion is not unique, each term of the form $\displaystyle\sum_{|\lambda|-|\mu|=m} a_{\lambda\mu} S_\lambda S_\mu^*$ is completely determined by a. Given a represented as above, let

$$\Phi_m(a) = \sum_{|\lambda|-|\mu|=m} a_{\lambda\mu} S_\lambda S_\mu^*.$$

Since for any α and β, $\gamma_z(S_\alpha S_\beta^*) = z^{|\alpha|-|\beta|} S_\alpha S_\beta^*$, we have

$$\int_{\mathbb{T}} z^{-m} \gamma_z(S_\alpha S_\beta^*)\, dz = \begin{cases} S_\alpha S_\beta^*, & \text{if } |\alpha| - |\beta| = m, \\ 0, & \text{if } |\alpha| - |\beta| \neq m. \end{cases}$$

It follows that

$$\Phi_m(a) = \int_{\mathbb{T}} z^{-m} \gamma_z(a)\, dz$$

for all $a \in B^*(G)$. Since Φ_m is well-defined, linear, and norm decreasing on $B^*(G)$; it extends to all of $\mathcal{A}$.

Now fix $a \in \mathcal{A}$ and consider the function $f \colon \mathbb{T} \to \mathcal{A}$ given by $f(z) = \gamma_z(a)$. The Fourier coefficients for f are just the elements $\Phi_m(a)$ of $\mathcal{A}$ and we have the Fourier series $f \sim \sum_{m\in\mathbb{Z}} \Phi_m(a) z^m$. While the infinite sum need not be convergent, the Cesaro means converge uniformly to f. Since $f(1) = a$, we obtain the fact that a is in the closed linear span of the elements $\Phi_m(a)$. Thus we have the formal series

$$a \sim \sum_{m\in\mathbb{Z}} \Phi_m(a)$$

with a Cesaro convergence of the series. We reiterate that Φ_0 maps $\mathcal{A}$ onto the core AF subalgebra $\mathcal{F}$.

3. The spectral theorem for bimodules. Part I

A portion of the spectral theorem for bimodules can be proven without reference to the groupoid model. The full theorem will appear in Section 6 and a further extension is given in Section 8.

Let $\mathcal{D}$ be the abelian subalgebra of $\mathcal{A}$ generated by all projections of the form $S_\alpha S_\alpha^*$ and $S_\alpha^* S_\alpha$. This is clearly a subalgebra of the core AF algebra $\mathcal{F}$. In general, $\mathcal{D}$ need not be maximal abelian in $\mathcal{A}$ (though it will be maximal abelian in $\mathcal{F}$). We discuss when $\mathcal{D}$ will be maximal abelian in $\mathcal{A}$ in Section 5.

Theorem 3.1. *Let G be a range finite graph with no sources. Let $\mathcal{B} \subseteq \mathcal{A}$ be a bimodule over $\mathcal{D}$. Then $\mathcal{B}$ is generated by the Cuntz–Krieger partial isometries which it contains if, and only if, it is invariant under the gauge automorphisms.*

Proof. It is trivial that a bimodule generated by its Cuntz–Krieger partial isometries is invariant under the gauge automorphisms, so we need only prove the converse.

Let $\mathcal{B}$ be a gauge invariant bimodule over $\mathcal{D}$. First note that for each m, $\Phi_m(\mathcal{B}) \subseteq \mathcal{B}$. For each path $\nu \in F$, let

$$\mathcal{B}^\nu = \{b \in \mathcal{F} \mid S_\nu b \in \mathcal{B}\}.$$

We claim that $\mathcal{B}^\nu$ is a closed bimodule over $\mathcal{D}$. It is trivial to see that $\mathcal{B}^\nu$ is closed and a right module. Since $\mathcal{D}$ is generated by projections of the form $S_\alpha S_\alpha^*$, we can show that $\mathcal{B}^\nu$ is a left module by showing that for each $b \in \mathcal{B}$ and each finite path α, the element $S_\nu(S_\alpha S_\alpha^*)b \in \mathcal{B}$. Such an element is nonzero when $S_\alpha S_\alpha^* \leq S_\nu^* S_\nu$, and in this case

$$S_\nu(S_\alpha S_\alpha^*)b = S_\nu S_\alpha S_\alpha^* S_\nu^* S_\nu b = (S_\nu S_\alpha S_\alpha^* S_\nu^*) S_\nu b.$$

This is in $\mathcal{B}$, since $S_\nu S_\alpha S_\alpha^* S_\nu^* \in \mathcal{D}$. Similarly, the spaces

$$\mathcal{B}_\nu = \{b \in \mathcal{F} \mid b S_\nu^* \in \mathcal{B}\}$$

are also closed $\mathcal{D}$-bimodules. Since $\mathcal{D}$ is a canonical masa in the AF algebra $\mathcal{F}$ and the $\mathcal{B}^\nu$ and $\mathcal{B}_\nu$ are $\mathcal{D}$-bimodules in $\mathcal{F}$, the spectral theorem for bimodules in the AF case implies that each of $\mathcal{B}^\nu$ and $\mathcal{B}_\nu$ is spanned by the matrix unit elements $S_\alpha S_\beta^*$ in $\mathcal{B}^\nu$ or $\mathcal{B}_\nu$ (as appropriate) with $|\alpha| = |\beta|$. Thus, the spaces $S_\nu \mathcal{B}^\nu$ and $\mathcal{B}_\nu S_\nu^*$ are generated by their Cuntz–Krieger partial isometries.

We claim that it follows that the spaces $\Phi_m(\mathcal{B})$ are also generated by their Cuntz–Krieger partial isometries. In view of Cesaro convergence and the fact that the $\Phi_m(\mathcal{B})$ spaces are subspaces of $\mathcal{B}$, this implies that $\mathcal{B}$ is generated by its Cuntz–Krieger partial isometries.

The claim is elementary to confirm in the case of a finite graph, since $\Phi_m(\mathcal{B})$ is then the finite linear span of the spaces $S_\nu \mathcal{B}^\nu$ or $\mathcal{B}_\nu S_\nu^*$ with $|\nu| = m$ or $|\nu| = -m$, as appropriate, and the isometries S_ν have orthogonal ranges. In general the claim will follow if we show that $\Phi_m(\mathcal{B})$ is the closed linear span of these subspaces.

The case when the graph is infinite can be reduced to the finite graph case as follows: recall that the Cuntz–Krieger partial isometries in $\Phi_m(\mathcal{A})$ are precisely the $S_\alpha S_\beta^*$ with $|\alpha| - |\beta| = m$. Let F_n be a sequence of finite subsets of the Cuntz–Krieger partial isometries in $\Phi_m(\mathcal{A})$ such that $\bigcup F_n$ is the set of all Cuntz–Krieger partial isometries in $\Phi_m(\mathcal{A})$. Also, let P_n denote the projection onto the closed linear span of the ranges of the partial isometries in F_n.

Any element b in $\mathcal{A}$ can be approximated by a linear combination of Cuntz–Krieger partial isometries. But Φ_m is contractive, acts as the identity on Cuntz–Krieger partial isometries in $\Phi_m(\mathcal{A})$ and maps all other Cuntz–Krieger partial isometries to 0; consequently, when $b \in \Phi_m(\mathcal{A})$ it can be approximated by linear combinations of Cuntz–Krieger partial isometies in $\Phi_m(\mathcal{A})$. In particular, there is a sequence $a_n \in \Phi_m(\mathcal{A})$ such that $P_n a_n = a_n$ and $a_n \to b$. Now, suppose further that $b \in \Phi_m(\mathcal{B})$. Since $P_n b - b = P_n(b - a_n) + a_n - b$, we have $P_n b \to b$. Also $P_n b \in P_n \Phi_m(\mathcal{B}) = \Phi_m(P_n \mathcal{B})$. By the result for finite graphs, each $P_n b$ can be approximated by linear combinations of Cuntz–Krieger partial isometries in $\Phi_m(P_n \mathcal{B})$. It follows that b can be approximated by linear combinations of Cuntz–Krieger partial isometries in $\Phi_m(\mathcal{B})$. $\square$

4. The groupoid model

In [7], Kumjian, Pask, Raeburn and Renault construct a locally compact r-discrete groupoid $\mathcal{G}$ such that the groupoid C*-algebra $C^*(\mathcal{G})$ is the graph C*-algebra for G. We sketch below a slightly modified version of this construction.

We shall assume that every vertex is the range of at least one edge. (The graph has no sources.) It follows that every edge is part of an infinite path (notationally, infinite to the right). Infinite path space P is topologized by taking as a basis of open sets the following cylinder sets: for each finite path α of length k,

$$\begin{aligned} Z(\alpha) &= \{x \in P \mid x_1 = \alpha_1, \ldots, x_k = \alpha_k\} \\ &= \{\alpha y \mid y \in P \text{ and } r(y) = s(\alpha)\}. \end{aligned}$$

Any two cylinder sets $Z(\alpha)$ and $Z(\beta)$ are either disjoint or one is a subset of the other. For example, $Z(\alpha) \subseteq Z(\beta)$ precisely when $\alpha = \beta\alpha'$ for some $\alpha' \in F$ with $r(\alpha') = s(\beta)$. The assumption that G is range finite implies that each $Z(\alpha)$ is a compact set. The topology on P is then locally compact, σ-compact, totally disconnected and Hausdorff. It coincides with the relative product topology obtained by viewing P as a subset of the infinite product of E with itself. Path space P will, in due course, be identified with the space of units for the groupoid $\mathcal{G}$.

The next step is to define an equivalence relation (*shift equivalence*) on P. Shift equivalence is the union of a sequence of relations, indexed by the integers. Let $x, y \in P$ and $k \in \mathbb{Z}$. If there is a positive integer N such that $x_{i+k} = y_i$ for all $i \geq N$, then we write $x \sim_k y$. We then say that x and y in P are *shift equivalent* if $x \sim_k y$ for some $k \in \mathbb{Z}$.

The groupoid is defined to be the set:

$$\mathcal{G} = \{(x, k, y) \mid x, y \in P, k \in \mathbb{Z}, x \sim_k y\}.$$

Elements (x, k, y) and (w, j, z) are composable if, and only if, $y = w$; when this is the case $(x, k, y) \cdot (y, j, z) = (x, k+j, z)$. Inversion is given by $(x, k, y)^{-1} = (y, -k, x)$. With these operations $\mathcal{G}$ is a groupoid. The units of $\mathcal{G}$ all have the form $(x, 0, x)$ for $x \in P$, allowing the identification of P with the space of units (which is also denoted by $\mathcal{G}^0$, as usual).

There is a natural topology which renders $\mathcal{G}$ a topological groupoid. A basis for this topology can be parameterized by pairs of finite paths α and β which satisfy

$s(\alpha) = s(\beta)$. For such α and β, let

$$\begin{aligned} Z(\alpha,\beta) &= \{(x,k,y) \mid x \in Z(\alpha), y \in Z(\beta), k = |\alpha| - |\beta|, \text{ and } x_{i+k} = y_i \text{ for } i > |\beta|\} \\ &= \{(\alpha z, |\alpha| - |\beta|, \beta z) \mid z \in P, r(z) = s(\alpha) = s(\beta)\}. \end{aligned}$$

We allow either α or β (or both) to be the empty paths. For example,

$$Z(\alpha,\emptyset) = \{(\alpha z, |\alpha|, z) \mid z \in P, r(z) = s(\alpha)\}.$$

Two basic open sets, $Z(\alpha,\beta)$ and $Z(\gamma,\delta)$ are either disjoint or one contains the other. For example, $Z(\alpha,\beta) \subseteq Z(\gamma,\delta)$ precisely when there is $\epsilon \in F$ such that $\alpha = \gamma\epsilon$ and $\beta = \delta\epsilon$. The following proposition is essentially quoted from [7].

Proposition 4.1. *The sets*

$$\{Z(\alpha,\beta) \mid \alpha,\beta \in F, s(\alpha) = s(\beta)\}$$

form a basis for a locally compact Hausdorff topology on $\mathcal{G}$. With this topology, $\mathcal{G}$ is a second countable, r-discrete locally compact groupoid in which each $Z(\alpha,\beta)$ (except possibly $Z(\emptyset,\emptyset)$) is a compact open $\mathcal{G}$-set. The product topology on the unit space P agrees with the topology it inherits by viewing it as the subset $\mathcal{G}^0 = \{(x,0,x) \mid x \in P\}$ of $\mathcal{G}$. The counting measures form a left Haar system for $\mathcal{G}$.

Kumjian, Pask, Raeburn, and Renault prove that the groupoid C*-algebra for $\mathcal{G}$ is isomorphic to the graph C*-algebra $C^*(G)$; this is done by identifying natural Cuntz–Krieger generators in $C^*(\mathcal{G})$ and proving universality. Recall that the groupoid C*-algebra is constructed by providing $C_c(\mathcal{G})$, the compactly supported continuous functions on $\mathcal{G}$, with a suitable (convolution style) multiplication, an involution, and a (universal) C*-norm and then completing the *-algebra. In particular, for each edge e, the set $Z(e,\emptyset) = \{(ez,1,z) \mid z \in P, r(z) = s(e)\}$ is compact and open; therefore its characteristic function $\chi_{Z(e,\emptyset)}$ may be viewed as an element of $C^*(\mathcal{G})$. Denote this element by S_e. A routine calculation shows that the initial space $S_e^*S_e$ is the characteristic function of $\{(x,0,x) \mid r(x) = s(e)\}$. Another routine calculation shows that for an edge f, $S_fS_f^* = \chi_{Z(f,f)}$. Now $Z(f,f)$ is a subset of $\{(x,0,x) \mid r(x) = s(e)\}$ exactly when $r(f) = s(e)$ and, in fact, $\{(x,0,x) \mid r(x) = s(e)\}$ is the union of all $Z(f,f)$ with $r(f) = s(e)$. Thus, the Cuntz–Krieger relations

$$S_e^*S_e = \sum_{r(f)=s(e)} S_fS_f^*$$

hold. Routine but tedious calculations show that $S_\alpha S_\beta^* = \chi_{Z(\alpha,\beta)}$. The following theorem consists of a combination of parts of Proposition 4.1 and Theorem 4.2 in [7]):

Theorem 4.2. *Let G be a range finite directed graph with no sources. With the notation above, $C^*(\mathcal{G})$ is generated by $\{S_e \mid e \in G\}$ and $C^*(\mathcal{G})$ is isomorphic to $C^*(G)$.*

Throughout the rest of this paper the graph C*-algebra–groupoid C*-algebra determined by the graph G will be denoted by $\mathcal{A}$.

5. The masa theorem

In the groupoid model there is a natural abelian subalgebra of $\mathcal{A}$: the functions supported on the space of units of $\mathcal{G}$. We shall denote this abelian algebra by $\mathcal{D}$. This is, of course, exactly the same abelian algebra as the one that appears in Section 3. Based on what happens for r-discrete principal groupoids and for the Cuntz groupoids which model the Cuntz algebras O_n, it might be suspected that $\mathcal{D}$ is always a masa in $\mathcal{A}$; however, this is not the case. Consider, for example, the graph which consists of a single vertex and a single edge e. Then there is only one infinite path, say $x = eee\dots$ and the unit space consists of the singleton $(x, 0, x)$. The whole groupoid may be identified with $\mathbb{Z}$: $\mathcal{G} = \{(x, k, x) \mid k \in \mathbb{Z}\}$ and $\mathcal{A} \cong C(\mathbb{T})$ while $\mathcal{D} \cong \mathbb{C}$.

For a more interesting example, let G consist of a single loop with n edges. So $E = \{e_1, e_2, \dots, e_n\}$ with $r(e_j) = s(e_{j-1})$ for $j = 2, \dots, n$ and $r(e_1) = s(e_n)$. In this case the graph C*-algebra is $M_n(C(\mathbb{T}))$. The algebra $\mathcal{D}$ corresponds to the diagonal matrices with scalar entries, which is not a masa.

We will prove that $\mathcal{D}$ is a masa in $\mathcal{A}$ if, and only if every loop has an entrance. This condition says that if the finite path $\alpha_1 \dots \alpha_n$ satisfies $r(\alpha_1) = s(\alpha_n)$, then there is an index j and an edge β such that $\beta \neq \alpha_j$ and $r(\beta) = r(\alpha_j)$. This condition was used earlier in the literature (expressed as "every loop has an exit", of course). In [6] and in [1], for example, it is shown that when this condition holds the C*-algebra generated by any system of Cuntz–Krieger partial isometries is isomorphic to the universal graph C*-algebra.

The isotropy group bundle of $\mathcal{G}$ is $\mathcal{G}^1 = \{(x, k, y) \mid x = y\}$. The space of units of $\mathcal{G}$ is $\mathcal{G}^0 = \{(x, 0, x) \mid x \in P\}$ The following lemma, combined with groupoid amenability and some results in [16], will yield the masa theorem.

Lemma 5.1. *Let G be a range finite directed graph with no sources. Let $\mathcal{G}$ be the associated groupoid. Then every loop in G has an entrance if, and only if, $\mathcal{G}^0$ is the interior of $\mathcal{G}^1$.*

Proof. Assume that every loop has an entrance. Let (x, k, x) be an element of $\mathcal{G}^1$ which is not in $\mathcal{G}^0$; in other words, assume that $k \neq 0$. We shall show that (x, k, x) can be approximated by elements of the complement of $\mathcal{G}^1$. Since $\mathcal{G}^0$ is open, this will show that $\mathcal{G}^0$ is the interior of $\mathcal{G}^1$.

Since $k \neq 0$, there is an integer N such that for $i \geq N$, $x_{i+k} = x_i$. Let $\beta = x_1 \dots x_{N-1}$, a finite path of length $N-1$ and let $\alpha = x_N \dots x_{N+k-1}$, a finite path of length k. The condition for shift equivalence assures that $x_{N+k} \dots x_{N+2k-1} = x_N \dots x_{N+k-1}$, etc. Thus, $x = \beta\alpha\alpha\alpha \dots$.

Since α can be concatenated with itself, α is a loop. Write $\alpha = \alpha_1 \dots \alpha_k$, where the $\alpha_i \in E$. Since every loop has an entrance, there is an edge y_j such that $y_j \neq \alpha_j$ and $r(y_j) = r(\alpha_j)$. Now let $y = y_j y_{j+1} \dots$ be an infinite path ending in the edge y_j. The assumption that the graph has no sources guarantees the existence of such an infinite path.

For each integer $p \geq 1$, let z^p be the infinite path $\beta\alpha \dots \alpha\alpha_1 \dots \alpha_{j-1} y$, where there are exactly p copies of α. If $k > 0$, then $z^{p+1} \sim_k z^p$ and if $k < 0$ then $z^p \sim_k z^{p+1}$. Now just observe that $z^p \neq z^{p+1}$ and that (z^{p+1}, k, z^p) or (z^p, k, z^{p+1}), as appropriate, converges to (x, k, x).

For the converse, assume that G has a loop α with no entrance. Let k be the length of this loop and let $x = \alpha\alpha\alpha\dots$. Then $(x, k, x) \in \mathcal{G}^1 \setminus \mathcal{G}^0$ and the singleton set $\{(x, k, x)\}$ is open in $\mathcal{G}$. Thus, $\mathcal{G}^0$ is not the interior of $\mathcal{G}^1$. □

Theorem 5.2. *Let G be a range finite directed graph with no sources. Let $\mathcal{G}$ be the associated groupoid. Then $\mathcal{D}$ is a masa in $\mathcal{A}$ if, and only if, every loop has an entrance.*

Proof. Results in [7] and [10] establish the amenability of $\mathcal{G}$. (This is proven for locally finite graphs in [7] and extended to range finite graphs — and beyond — in [10].) It follows that $C^*(\mathcal{G}) = C^*_{\mathrm{red}}(\mathcal{G})$. Proposition II.4.7 in [16] says that $\mathcal{D}$ is a masa in $C^*_{\mathrm{red}}(\mathcal{G})$ if, and only if, $\mathcal{G}^0$ is the interior of $\mathcal{G}^1$, so this, combined with Lemma 5.1 yields the theorem. □

6. The spectral theorem for bimodules. Part II

One of the most useful tools in the study of nonselfadjoint subalgebras of C*-algebras is the spectral theorem for bimodules of Muhly and Solel [9]. (See also [8] for a generalization due to Muhly, Qiu and Solel.) The theorem as it appears in these two references is not valid for graph C*-algebras; this section is devoted to the proof of a modified version of the spectral theorem for bimodules which is valid for a broad class of graph C*-algebras. The theorem as it appears here was proven in the context of the Cuntz algebras $\mathcal{O}_n$ in [2]; the proof of the general version is similar. We will give an extension of the spectral theorem for bimodules in Section 8.

Throughout this section, G denotes a range finite directed graph with no sources; $\mathcal{G}$ denotes the groupoid associated with G; and $\mathcal{A}$ is the C*-algebra constructed from G or $\mathcal{G}$. We shall need to use the convenient fact that elements of $\mathcal{A}$ can be identified with continuous functions on $\mathcal{G}$ which vanish at infinity. This is a consequence of the range finitness of G, which implies that the groupoid $\mathcal{G}$ is amenable ([7, Corollary 5.5] and [10, Theorem 4.2]). Amenability, in turn, implies that $C^*(\mathcal{G}) = C^*_{\mathrm{red}}(\mathcal{G})$ [16, p. 92]. Finally, Proposition II.4.2 in [16] allows us to identify the elements of $C^*(\mathcal{G})$ with (some of the) elements in $C_0(\mathcal{G})$, the continuous functions on $\mathcal{G}$ vanishing at infinity.

Definition 6.1. Let $\mathcal{B} \subseteq \mathcal{A}$ be a bimodule over $\mathcal{D}$. Define the *spectrum* of $\mathcal{B}$ to be

$$\sigma(\mathcal{B}) = \{(x, k, y) \in \mathcal{G} \mid \text{there is } f \in \mathcal{B} \text{ such that } f(x, k, y) \neq 0\}.$$

For any open subset P of $\mathcal{G}$, we let

$$A(P) = \{f \in \mathcal{A} \mid f(x, k, y) = 0 \text{ for all } (x, k, y) \notin P\}.$$

It is easy to check that $A(P)$, which consists of all functions in $\mathcal{A}$ which are supported on P, is a bimodule over $\mathcal{D}$ and that $\sigma(A(P)) = P$. It is also trivial to see that $\mathcal{B} \subseteq A(\sigma(\mathcal{B}))$. But it is not always true that $\mathcal{B} = A(\sigma(\mathcal{B}))$; a counterexample is given in [2]. Thus, the spectral theorem for bimodules for graph C*-algebras differs from the theorem for groupoid C*-algebras where the groupoid is r-discrete, amenable and principal [9].

In fact, the existence of a counterexample depends exactly on the presence of a loop in the graph.

Proposition 6.2. *Let G be a range finite directed graph with no sources. There is in $\mathcal{A}$ a bimodule $\mathcal{B}$ over $\mathcal{D}$ such that $\mathcal{B} \neq A(\sigma(\mathcal{B}))$ if, and only if, G has a loop.*

Proof. Suppose G has a loop $\alpha = \alpha_1\alpha_2 \dots \alpha_k$ with $s(\alpha_k) = s(\alpha) = r(\alpha) = r(\alpha_1)$. Let $x = \alpha\alpha\alpha\dots$ in $P(G)$. Write $S_\alpha = S_{\alpha_1} \dots S_{\alpha_k}$, $v = r(\alpha)$, and $\Phi = P_v + S_\alpha$. Let $\mathcal{B}$ denote the bimodule generated by Φ; this bimodule is the norm closure of all finite sums $\sum f_i \Phi g_i$, with $f_i, g_i \in \mathcal{D}$.

Now $\sigma(\Phi) = \sigma(P_v) \cup \sigma(S_\alpha)$ and

$$\begin{aligned}\sigma(P_v) &= \{(z, 0, z) \mid z \in P(G), r(z) = v\},\\ \sigma(S_\alpha) &= \{(\alpha z, k, z) \mid z \in P(G), r(z) = s(\alpha) = s(\alpha_k)\}.\end{aligned}$$

Since $(x, 0, x) \in \sigma(P_v)$ and $(x, k, x) \in \sigma(S_\alpha)$, $(x, 0, x)$ and (x, k, x) both lie in $\sigma(\Phi)$.

Viewing elements of $\mathcal{D}$ as functions on the groupoid supported on the unit space and using $\Phi = \chi_{\sigma(P_v)} + \chi_{\sigma(S_\alpha)} = \chi_{\sigma(P_v)\cup\sigma(S_\alpha)}$, we have

$$\begin{aligned}f\Phi(x, 0, x) &= f(x, 0, x)\Phi(x, 0, x) = f(x, 0, x),\\ f\Phi(x, k, x) &= f(x, 0, x)\Phi(x, k, x) = f(x, 0, x).\end{aligned}$$

Similarly, $\Phi f(x, 0, x) = f(x, 0, x) = \Phi f(x, k, x)$.

So, for any $f, g \in \mathcal{D}$, we have $f\Phi g(x, 0, x) = f\Phi g(x, k, x)$. The same equality is valid for sums and extends to the norm closure: if $h \in \mathcal{B}$, then $h(x, 0, x) = h(x, k, x)$. Since there are elements $h' \in A(\sigma(\mathcal{B}))$ with $h'(x, 0, x) \neq h'(x, k, x)$, this shows that $\mathcal{B} \neq A(\sigma(\mathcal{B}))$.

On the other hand, if G has no loops, $C^*(G)$ is AF and $\mathcal{G}$ is a principal grooupoid, so all bimodules satisfy $\mathcal{B} = A(\sigma(\mathcal{B}))$ by the Muhly-Solel spectral theorem for bimodules [9]. □

The spectral theorem for bimodules below provides two necessary and sufficient conditions for a bimodule to be determined by its spectrum. These are, in fact, the two equivalent conditions in Theorem 3.1. A third equivalent condition will be given in Section 8.

Theorem 6.3 (Spectral Theorem for Bimodules). *Let G be a range finite directed graph with no sources. Let $\mathcal{B} \subseteq \mathcal{A}$ be a bimodule over $\mathcal{D}$. Then the following statements are equivalent*:

(1) $\mathcal{B} = A(\sigma(\mathcal{B}))$.
(2) *$\mathcal{B}$ is generated by the Cuntz–Krieger partial isometries which it contains.*
(3) *$\mathcal{B}$ is invariant under the gauge automorphisms.*

Proof. Since the equivalence of (2) and (3) has already been established in Theorem 3.1, it suffices to prove the equivalence of (1) and (2).

To show that (1) ⇒ (2) we need to show that whenever P is an open subset of $\mathcal{G}$, $A(P)$ is generated by the Cuntz–Krieger partial isometries which it contains. Let $\mathcal{B}$ be the bimodule which is generated by the Cuntz–Krieger partial isometries which are in $A(P)$. Clearly, $\mathcal{B} \subseteq A(P)$; we must show the reverse containment.

Assume that $S_\alpha S_\beta^* \in A(P)$. So $Z(\alpha, \beta) \subseteq P$. Let f be a continuous function with support in $Z(\alpha, \beta)$. Define $g\colon Z(\alpha, \alpha) \to \mathbb{C}$ by

$$g(\alpha\gamma, 0, \alpha\gamma) = f(\alpha\gamma, |\alpha| - |\beta|, \beta\gamma).$$

Then,

$$\begin{aligned} g \cdot S_\alpha S_\beta^*(x,k,y) &= g(x,0,x) S_\alpha S_\beta^*(x,k,y) \\ &= \begin{cases} g(x,o,x), & \text{if } x = \alpha\gamma, k = |\beta| - |\alpha|, y = \beta\gamma \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} f(x,k,y), & \text{if } x = \alpha\gamma, k = |\beta| - |\alpha|, y = \beta\gamma \\ 0, & \text{otherwise} \end{cases} \\ &= f(x,k,y). \end{aligned}$$

Since $g \in \mathcal{D}$ and $S_\alpha S_\beta^* \in \mathcal{B}$, we have $f \in \mathcal{B}$.

Thus, $\mathcal{B}$ contains any continuous function supported on a subset of the form $Z(\alpha,\beta) \subseteq P$. Any compact open subset of P can be written as a finite union of sets of the form $Z(\alpha,\beta)$, so $\mathcal{B}$ contains any $f \in \mathcal{A}$ which is supported on a compact open subset of P. But any compact subset of P is contained in a compact open subset of P, so $\mathcal{B}$ contains all f which are supported on a compact subset of P. These functions are dense in $A(P)$ in the C*-norm, so $A(P) \subseteq \mathcal{B}$.

To prove that (2) $\Rightarrow$ (1), assume that $\mathcal{B}$ is a bimodule over $\mathcal{D}$ which is generated by the Cuntz–Krieger partial isometries which it contains. We first show that $\sigma(\mathcal{B}) = \cup Z(\alpha,\beta)$, where the union is taken over all α,β such that $S_\alpha S_\beta^* \in \mathcal{B}$. Indeed, if p is in this union, then there is $S_\alpha S_\beta^* \in \mathcal{B}$ such that $S_\alpha S_\beta^*(p) = 1$, so $p \in \sigma(\mathcal{B})$. On the other hand, if p is not in the union, then $S_\alpha S_\beta^*(p) = 0$ for all $S_\alpha S_\beta^* \in \mathcal{B}$. Write $p = (x,k,y)$ and let f and g be in $\mathcal{D}$. Then

$$f \cdot S_\alpha S_\beta^* \cdot g(p) = f(x,0,x) S_\alpha S_\beta^*(x,k,y) g(y,0,y) = 0.$$

It follows that all elements of the bimodule generated by the Cuntz–Krieger partial isometries in $\mathcal{B}$ vanish at p. But the Cuntz–Krieger partial isometries generate $\mathcal{B}$ itself, so $p \notin \sigma(\mathcal{B})$.

We already know that $\mathcal{B} \subseteq A(\sigma(\mathcal{B}))$; to show the reverse containment it is sufficient (since any $A(P)$ is generated by the Cuntz–Krieger partial isometries which it contains) to show that if $S_\alpha S_\beta^* \in A(\sigma(\mathcal{B}))$ then $S_\alpha S_\beta^* \in \mathcal{B}$.

Let $S_\alpha S_\beta^* \in A(\sigma(\mathcal{B}))$, so that $Z(\alpha,\beta) \subseteq \sigma(\mathcal{B})$. Let $p \in Z(\alpha,\beta)$. Then there is $S_\nu S_\mu^* \in \mathcal{B}$ such that $p \in Z(\nu,\mu) \subseteq \sigma(\mathcal{B})$. Since $Z(\alpha,\beta) \cap Z(\nu,\mu) \neq \emptyset$, we have either $Z(\alpha,\beta) \subseteq Z(\nu,\mu)$ or $Z((\nu,\mu) \subseteq Z(\alpha,\beta)$.

If $Z(\alpha,\beta) \subseteq Z(\nu,\mu)$, then there is a finite path ϵ such that $\alpha = \nu\epsilon$ and $\beta = \mu\epsilon$. A routine calculation shows that $S_\alpha S_\beta^* = S_\alpha S_\alpha^* S_\nu S_\mu^* S_\beta S_\beta^*$. But $S_\alpha S_\alpha^*$ and $S_\beta S_\beta^*$ are in $\mathcal{D}$ and $S_\nu S_\mu^* \in \mathcal{B}$, so $S_\alpha S_\beta^* \in \mathcal{B}$, as desired.

Suppose, on the other hand, that for any point $p \in Z(\alpha,\beta)$, the set $Z(\nu,\mu)$ obtained as above is a subset of $Z(\alpha,\beta)$. Then these sets form an open cover for $Z(\alpha,\beta)$. Since $Z(\alpha,\beta)$ is compact, we can find a finite subcover. It is routine to arrange that this subcover is disjoint (without losing the property that the associated Cuntz–Krieger partial isometries are in $\mathcal{B}$). Thus, we can write $Z(\alpha,\beta) = \cup_{i=1}^n Z(\nu_i,\mu_i)$, a finite disjoint union with all $S_{\nu_i} S_{\mu_i}^* \in \mathcal{B}$. But then

$$S_\alpha S_\beta^* = \sum_{i=1}^n S_{\nu_i} S_{\mu_i}^*$$

and so $S_\alpha S_\beta^* \in \mathcal{B}$. $\square$

7. Cocycles

As usual G is a range finite directed graph with no sources. Most of our attention will be focused on cocycles defined on the associated Cuntz–Krieger groupoid $\mathcal{G}$.

A real valued 1-cocycle on $\mathcal{G}$ is a continuous function $c\colon \mathcal{G} \to \mathbb{R}$ which satisfies the cocycle condition

$$c(x,k,y) + c(y,l,z) = c(x,k+l,z),$$

for all composable pairs. It follows that $c(x,0,x) = 0$, for x in path space P, and that

$$c((x,k,y)^{-1}) = c(y,-k,x) = -c(x,k,y),$$

for all $(x,k,y) \in \mathcal{G}$. The set of all 1-cocylces forms a group under addition, denoted by $Z^1(\mathcal{G},\mathbb{R})$.

A simple example of a cocycle is the one given by the formula $c(x,k,y) = k$. This cocycle is intimately related to the gauge automorphisms: for any $f \in C_c(\mathcal{G})$, $\gamma_z(f)(x,k,y) = z^k f(x,k,y)$. More generally, any cocycle c gives rise to the one-paramenter automorphism group

$$\eta_z(f)(x,k,y) = z^{c(x,k,y)} f(x,k,y).$$

Each η_z is a $*$-automorphism of $C_c(\mathcal{G})$ onto itself; it is not hard to show that this automorphism preserves the C*-norm and so extends to an automorphism of $\mathcal{A}$ with the formula above.

Remark 7.1. For each point $(x,k,y) \in \mathcal{G}$, the map $f \mapsto f(x,k,y)$ on $C_c(\mathcal{G})$ is decreasing with respect to the $||\cdot||_\infty$ norm, and hence also decreasing with respect to the C*-norm ([16, Prop. II.4.1]). So these maps extend to continuous linear functionals on $\mathcal{A}$. If $f \in \mathcal{A}$, we consider all functions of the form $t \to \rho(\eta_t(f))$, where ρ is a linear functional of the type above. Given $f \in \mathcal{A}$, it is easy to check that $t \to \rho(\eta_t(f))$ is an H^∞-function on $\mathbb{R}$ for all linear functionals of this form if, and only if, f is supported on $\{(x,k,y) \in \mathcal{G} \mid c(x,k,y) \geq 0\}$. (Note: when we write η_t it refers, of course, to η_z for $z = e^{it}$; use of the real variable is more appropriate when discussing H^∞.)

To emphasize the connection with analyticity, consider the simplest possible graph: the graph G consisting of a single vertex and a single loop. The associated groupoid for G is the group of integers, $\mathbb{Z}$. The groupoid C*-algebra (well, really, the group C*-algebra) is isomorphic to $C(\mathbb{T})$. Briefly, $C_c(\mathbb{Z})$ is a *-algebra in which the multiplication is convolution and the C*-norm of a function $f \in C_c(\mathbb{Z})$ is the $\|\ \|_\infty$ norm of the funtion $\theta \mapsto \sum_n f(n)e^{in\theta}$, $\theta \in \mathbb{T}$. Therefore, $C^*(\mathbb{Z})$ is identified with the C_0 functions on $\mathbb{Z}$ which are the Fourier coefficients of functions in $C(\mathbb{T})$.

A cocycle on $\mathbb{Z}$ is determined by its value at 1; so the only one of interest is $c(n) = n$. Let α_t be the associated one-parameter family of automorphisms acting on $C^*(\mathbb{Z})$. Since $\alpha_t(f)(n) = e^{int}f(n)$, when transferred via the inverse Fourier transform to $C(\mathbb{T})$, the automorphism group acts by translation:

$$\alpha_t(\phi)(\theta) = \phi(\theta + t).$$

For each $n \in \mathbb{Z}$, let ρ_n be the linear functional on $C^*(\mathbb{Z})$ given by $f \mapsto f(n)$. Transferred to $C(\mathbb{T})$, this is $\phi \mapsto \int \phi(\theta)e^{in\theta}\,d\theta$, where $d\theta$ is normalized Lebesgue measure on $\mathbb{T}$. The closed linear span of the functionals of this type can be identified with the complex valued measures on $\mathbb{T}$ which are absolutely continuous with respect to Lebesgue measure; i.e., with $L^1(\mathbb{T}, d\theta)$. Thus, "evaluation functionals" do not span the dual space of $C(\mathbb{T})$.

All the same, if $\phi \in C(\mathbb{T})$ and $t \mapsto \rho_n(\alpha_t(\phi))$ is in $H^\infty(\mathbb{R})$ for all n, then $t \mapsto \rho(\alpha_t(\phi))$ is in $H^\infty(\mathbb{R})$ for all $\rho \in C(\mathbb{T})^*$. This happens exactly when the Fourier transform $\hat{\phi}$ is supported on $\mathbb{Z}^+ = c^{-1}[0,\infty)$. Thus the analytic algebra associated with the cocycle c is just the disk algebra, $A(\mathbb{D})$.

Definition 7.2. Let $\mathcal{S}\colon P \to P$ be the shift map; thus if $x = x_1x_2\ldots$ is a path with terminal edge x_1, $\mathcal{S}(x) = x_2x_3\ldots$.

Note that $\mathcal{S}$ is a continuous map, in fact it is a local homeomorphism.

Let $\mathcal{G}(k)$ denote $\{(x,l,y) \in \mathcal{G} \mid l = k\}$ and $C(P)$ denote the space of continuous functions on P. We now give an example of a class of cocycles.

Example 7.3. Let $f \in C(P)$ and define c on $\mathcal{G}(k)$ $(k > 0)$ by

$$c(x,k,y) = \sum_{j=0}^{k-1} f(\mathcal{S}^j x) + \sum_{j=k}^{\infty}[f(\mathcal{S}^j x) - f(\mathcal{S}^{j-k}y)]$$

for $(x,k,y) \in \mathcal{G}(k)$. Observe that the infinite sum has only finitely many nonzero terms. For $k = 0$, set $c(x,0,y) = \sum_{j=0}^{\infty}[f(\mathcal{S}^j x) - f(\mathcal{S}^j y)]$, and for k negative set $c(x,k,y) = -c(y,-k,x)$.

To verify the cocycle condition, let (x,k,y), $(y,l,z) \in \mathcal{G}$ with $k,l \geq 0$. Then

$$\begin{aligned} c(x,k,y) + c(y,l,z) &= \sum_{j=0}^{k-1} f(\mathcal{S}^j x) + \sum_{j=k}^{\infty}[f(\mathcal{S}^j x) - f(\mathcal{S}^{j-k}y)] \\ &\quad + \sum_{i=0}^{l-1} f(\mathcal{S}^i y) + \sum_{i=l}^{\infty}[f(\mathcal{S}^i y) - f(\mathcal{S}^{i-l}z)] \\ &= \sum_{j=0}^{k+l-1} f(\mathcal{S}^j x) + \sum_{j=k+l}^{\infty}[f(\mathcal{S}^j x) - f(\mathcal{S}^{j-k-l}z)] \\ &= c(x,k+l,z). \end{aligned}$$

The other cases are similar. Finally, observe that if f is continuous on P, then the cocycle c is continuous in the topology on $\mathcal{G}$.

Note that the cocycle $c(x,k,z) = k$, which generates the gauge automorphisms, is produced by the constant function $f(x) = 1$.

Theorem 7.4. *Let G be a range finite directed graph with no sources. Let $\mathcal{G}$ be the associated Cuntz–Krieger groupoid and P the path space of G, identified with the unit space $\mathcal{G}^0$. Then there is a bijection $C(P) \longleftrightarrow Z^1(\mathcal{G},\mathbb{R})$ given as follows: for $f \in C(P)$, let c_f denote the cocycle constructed in Example 7.3. For $c \in Z^1$, let $f_c(x) = c(x,1,\mathcal{S}x)$. Then the two maps are inverses of each other: $f = f_{c_f}$ and $c = c_{f_c}$.*

Proof. Let $c \in Z^1(\mathcal{G},\mathbb{R})$ be given, and $(x,0,y) \in \mathcal{G}(0)$. Note that $(\mathcal{S}x,0,\mathcal{S}y)$ also belongs to $\mathcal{G}(0)$. From the cocycle condition we have

$$c(x,1,\mathcal{S}y) = c(x,1,\mathcal{S}x) + c(\mathcal{S}x,0,\mathcal{S}y) = c(x,0,y) + c(y,1,\mathcal{S}y). \tag{7.1}$$

With $f(x) = f_c(x) = c(x, 1, \mathcal{S}x)$, $f \in C(P)$, and we can rewrite Equation (7.1) as

$$f(x) - f(y) = c(x, 0, y) - c(\mathcal{S}x, 0, \mathcal{S}y). \tag{7.2}$$

Replacing x, y with $\mathcal{S}^j x, \mathcal{S}^j y$ and summing over $j = 0, 1, \ldots$ we obtain

$$c(x, 0, y) = \sum_{j=0}^{\infty} [f(\mathcal{S}^j x) - f(\mathcal{S}^j y)].$$

Since for sufficiently large j, $\mathcal{S}^j x = \mathcal{S}^j y$, the sum above is actually finite.

Now let $(x, k, y) \in \mathcal{G}(k)$ with $k > 0$. Then $(\mathcal{S}^k x, 0, y) \in \mathcal{G}(0)$, and we have

$$\begin{aligned} c(x, k, y) &= \sum_{j=0}^{k-1} c(\mathcal{S}^j x, 1, \mathcal{S}^{j+1} x) + c(\mathcal{S}^k x, 0, y) \\ &= \sum_{j=0}^{k-1} f(\mathcal{S}^j x) + \sum_{j=k}^{\infty} [f(\mathcal{S}^j x) - f(\mathcal{S}^{j-k} y)] \\ &= c_f(x, k, y). \end{aligned}$$

The case $k < 0$ is similar.

Conversely, given $f \in C(P)$, define the cocycle c as in Example 7.3. But then $c(x, 1, \mathcal{S}x) = f(x)$, so $f = f_{c_f}$. □

Remark 7.5. If G is a finite directed graph, then $\mathcal{G}(0)$ is the AF-groupoid associated with the stationary Bratteli diagram which at level n has a copy of the vertices V, and admits an edge from vertex v at level n to vertex w at level $n+1$ if, and only if, G has an edge from v to w. The restriction of a cocycle c on the groupoid $\mathcal{G}$ to the subgroupoid $\mathcal{G}(0)$ gives a cocycle on the AF groupoid $\mathcal{G}(0)$. This class of cocycles has not been systematically studied. It is, however, a proper subclass of $Z^1(\mathcal{G}(0), \mathbb{R})$, as we shall see in the subsection on integer-valued cocycles.

Proposition 7.6. *Let f and f_n ($n \in \mathbb{N}$) be continuous functions on P. Then $f_n \to f$ uniformly on compact subsets of P if, and only if, $c_{f_n} \to c_f$ uniformly on compact subsets of $\mathcal{G}$.*

Proof. Suppose $f_n \to f$ uniformly on each of the sets $Z(\alpha)$, $\alpha \in F$. Given $\alpha, \beta \in F$ and the basic compact open set $Z(\alpha, \beta) \subset \mathcal{G}(k)$ where $k = |\alpha| - |\beta| \geq 0$, we have, for $(x, k, y) \in Z(\alpha, \beta)$,

$$c_{f_n}(x, k, y) - c_f(x, k, y) =$$
$$\sum_{j=0}^{k-1} [f_n(\mathcal{S}^j x) - f(\mathcal{S}^j x)] + \sum_{j=k}^{|\alpha|} ([f_n(\mathcal{S}^j x) - f(\mathcal{S}^j x)] - [f_n(\mathcal{S}^{j-k} y) - f(\mathcal{S}^{j-k} y)])$$

and this converges uniformly to zero on $Z(\alpha, \beta)$. A similar argument applies when $|\alpha| - |\beta| < 0$.

For the converse, suppose that $Z(\alpha) \subset P$. Write $\alpha = x_1 x_2 \ldots x_n$, and let β be the empty string if $|\alpha| = 1$ and $\beta = x_2 x_3 \ldots x_n$ otherwise. Note that $x \in Z(\alpha)$ if, and only if, $(x, 1, \mathcal{S}x) \in Z(\alpha, \beta)$. Since $f(x) = c_f(x, 1, \mathcal{S}x)$ and c_{f_n} converges uniformly to c_f on $Z(\alpha, \beta)$, it follows that f_n converges uniformly to f on $Z(\alpha)$. □

Proposition 7.7. *Let $f \in C(P)$ and let c_f be the corresponding cocycle on $\mathcal{G}$. Then f is locally constant on P if, and only if, c_f is locally constant on $\mathcal{G}$.*

Proof. Assume c_f is locally constant. Given $x \in P$, let $Z(\alpha, \beta)$ be a neighborhood of $(x, 1, \mathcal{S}x)$ on which c_f is constant. Since $f(u) = c_f(u, 1, \mathcal{S}u)$ for all $u \in P$, it follows that f is constant on $Z(\alpha)$.

Suppose now that f is locally constant, and let $(x, k, y) \in \mathcal{G}$ be given. We suppose that $k \geq 0$ (the case $k < 0$ is analogous). There is $n \geq k$ such that $\mathcal{S}^j x = \mathcal{S}^{j-k} y$ for all $j \geq n$. Therefore we can write

$$c_f(x, k, y) = \sum_{j=0}^{k-1} f(\mathcal{S}^j x) + \sum_{j=k}^{n} [f(\mathcal{S}^j x) - f(\mathcal{S}^{j-k} y)].$$

For p chosen sufficiently large, if we let $\alpha = x_1 \dots x_p$ and $\beta = y_1 \dots y_{p-k}$, then $x_i = y_{i-k}$, for all $i \geq p+1$; $Z(\mathcal{S}^j(\alpha))$ is a clopen neighborhood of $\mathcal{S}^j(x)$ on which f is constant, for $j = 1, \dots, n$; and $Z(\mathcal{S}^j(\beta))$ is a clopen neighborhood of $\mathcal{S}^j(y)$ on which f is constant, for $j = 1, \dots, n-k$. Then $(x, k, y) \in Z(\alpha, \beta)$ and c_f is constant on $Z(\alpha, \beta)$. □

Remark 7.8. Proposition 7.7 applies, in particular, whenever f has finite range.

Definition 7.9. Let $Z^1_0(\mathcal{G}, \mathbb{R})$ denote the subset of $Z^1(\mathcal{G}, \mathbb{R})$ consisting of those cocycles c which vanish precisely on the unit space $\mathcal{G}^0$.

Remark 7.10. Every cocycle in $Z^1(\mathcal{G}, \mathbb{R})$ necessarily vanishes on $\mathcal{G}^0$. Also, note that Z^1_0 is not a subgroup of Z^1; indeed, $0 \notin Z^1_0$.

7.1. Bounded cocycles. In the context of AF algebras and their groupoids, bounded cocycles are of special interest due to the connection between bounded cocycles and reflexive subalgebras of AF algebras (cf. [11]). Thus, it is natural to investigate the role of bounded cocycles on Cuntz–Krieger groupoids.

A point $x \in P$ is *periodic* if $\mathcal{S}^k x = x$ for some $k > 0$. Note that the existence of a periodic point in P is equivalent to the existence of a loop in the graph G. Recall from [6] that G has no loops if, and only if, $\mathrm{C}^*(G)$ is an AF-algebra.

Proposition 7.11. *Let G be a range finite directed graph with no sources. Then $Z^1_0(\mathcal{G}, \mathbb{R})$ contains no bounded cocycle if, and only if, G contains a loop.*[1]

Proof. First, assume that G contains a loop. Then there is a periodic point, say $x = x_1 \dots x_p x_1 \dots x_p x_1 \dots$, in P. Let $c \in Z^1_0(\mathcal{G}, \mathbb{R})$ and let $f(y) = c(y, 1, \mathcal{S}y)$, $y \in P$. For $k \geq 1$, we have

$$\begin{aligned} c(x, kp, \mathcal{S}^{kp} x) = {} & f(x) + f(\mathcal{S}x) + \dots + f(\mathcal{S}^{p-1} x) \\ & + f(\mathcal{S}^p x) + f(\mathcal{S}(\mathcal{S}^p x)) + \dots + f(\mathcal{S}^{p-1}(\mathcal{S}^p x)) \\ & + \cdots \\ & + f(\mathcal{S}^{(k-1)p} x) + f(\mathcal{S}^{(k-1)p}(\mathcal{S}x)) + \dots + f(\mathcal{S}^{(k-1)p}(\mathcal{S}^{p-1} x)). \end{aligned}$$

Using $\mathcal{S}^p x = x$, this reduces to

$$c(x, kp, \mathcal{S}^{kp} x) = k[f(x) + f(\mathcal{S}x) + \dots + f(\mathcal{S}^{p-1} x)], \quad \text{for all } k \geq 1.$$

But $\sum_{j=0}^{p-1} f(\mathcal{S}^j x) = c(x, p, \mathcal{S}^p x) \neq 0$ since $(x, p, \mathcal{S}^p x) = (x, p, x) \notin \mathcal{G}_0$. As k is arbitrary, c is unbounded.

[1]The implication $Z^1_0(\mathcal{G}, \mathbb{R})$ contains no bounded cocycle $\Rightarrow$ G contains a loop is due to Allan Donsig. The authors thank Donsig for giving permission to use this result here.

Now assume that G contains no loop. Let $a\colon G \to \mathbb{R}^+$ be a function with the property that for each edge e, $T_e = \{f \in G \mid a(f) > a(e)\}$ is finite and

$$a(e) > \sum_{f \notin T_e} a(f).$$

(This is easily done after G is arranged as a sequence.)

Now define a continuous, locally constant function $f\colon C(P) \to \mathbb{R}$ by $f(x) = a(x_1)$, for $x = x_1x_2\ldots$. In other words, f has the value $a(x_1)$ on $Z(x_1) \subseteq C(P)$. Let c be the cocycle associated with f, as in Example 7.3. Clearly, c is bounded. Since any cocycle vanishes on the unit space, we just need to show that c is nonzero off the unit space.

Because G has no loops, there are no points of the form $(x,k,x) \in \mathcal{G}$ with $k \neq 0$. Therefore, we just need to show that $c(x,k,y) \neq 0$ whenever $x \neq y$. Thanks to the cocycle property, we may without loss of generality assume that $k \geq 0$. Write

$$\begin{aligned} x &= x_1 \ldots x_{p+k} z_1 z_2 \ldots, \\ y &= y_1 \ldots y_p z_1 z_2 \ldots. \end{aligned}$$

We then have

$$\begin{aligned} c(x,k,y) &= \sum_{j=0}^{k-1} f(\mathcal{S}^j x) + \sum_{j=k}^{\infty} [f(\mathcal{S}^j x) - f(\mathcal{S}^{j-k} y)] \\ &= \sum_{j=1}^{p+k} a(x_j) - \sum_{j=1}^{p} a(y_j). \end{aligned}$$

Since G has no loops, a given edge e may appear at most once in each of the paths $x_1 \ldots x_{p+k}$ and $y_1 \ldots y_p$. Some edges may appear in both paths, but then the terms cancel. Since $x \neq y$, there is an edge e amongst $x_1, \ldots, x_{p+k}, y_1, \ldots, y_p$ which appears once only and for which $a(e)$ is maximal; the summation property for a then guarantees that $c(x,k,y) \neq 0$. □

Recall that a graph is *transitive* if there is a path from any vertex to any other vertex. If a directed graph G is finite and transitive, it satisfies Cuntz and Krieger's condition for $C^*(G)$ to be simple.

Proposition 7.12. *Let G be a finite, transitive, directed graph, and let c be a bounded cocycle on the AF groupoid $\mathcal{G}(0)$. Suppose that $C^*(\mathcal{G}(0))$ is simple. Then c extends to a cocyle on $\mathcal{G}$. Furthermore, if c vanishes precisely on the unit space $\mathcal{G}^0$, then the extension can be chosen to vanish precisely on $\mathcal{G}^0$.*

Proof. By [16, p. 112], since $\mathrm{C}^*(\mathcal{G}(0))$ is simple and c is bounded, c is a coboundary: that is, there is a continuous function b on P so that $c(x,0,y) = b(x) - b(y)$.

Choose a point $x_0 \in P$, and let $[x_0]$ denote the equivalence class of x_0 in $\mathcal{G}(0)$: $[x_0] = \{y \in P \mid (y,0,x_0) \in \mathcal{G}(0)\}$. Since $\mathrm{C}^*(\mathcal{G}(0))$ is simple, it follows from [16] that the equivalence class of any point is dense; thus $[x_0]$ is dense.

We shall construct $f \in C(P)$ such that c_f extends c. Begin by setting $f(x_0) = 0$. For $x \in [x_0]$, define

$$f(x) = c(x,0,x_0) - c(\mathcal{S}x, 0, \mathcal{S}x_0).$$

The cocycle property for c shows that if $x, y \in [x_0]$, then

$$f(x) - f(y) = c(x, 0, y) - c(\mathcal{S}x, 0, \mathcal{S}y). \tag{7.3}$$

Since P is compact (because G is finite), Hausdorff and first countable, it is metrizable. Let ρ be a metric for P. Since a continuous function on a compact metric space is uniformly continuous, both b and $b \circ \mathcal{S}$ are uniformly continuous. Thus, given $\epsilon > 0$ there is a clopen cover $\{Z(\alpha) \mid \alpha \in \mathbb{A}\}$ with $\mathbb{A}$ finite and such that for $x, y \in Z(\alpha)$, both $|b(x) - b(y)| < \frac{\epsilon}{2}$ and $|b \circ \mathcal{S}(x) - b \circ \mathcal{S}(y)| < \frac{\epsilon}{2}$.

Now for $x, y \in [x_0] \cap Z(\alpha)$ we have

$$\begin{aligned} |f(x) - f(y)| &= |c(x, 0, y) - c(\mathcal{S}x, 0, \mathcal{S}y)| \\ &= |(b(x) - b(y)) - (b \circ \mathcal{S}(x) - b \circ \mathcal{S}(y))| \\ &\leq |b(x) - b(y)| + |b \circ \mathcal{S}(x) - b \circ \mathcal{S}(y)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

As f is uniformly continuous on a dense subset, it admits a continuous extension to P, also denoted by f. Note that since $\{(x, 0, y) \mid x, y \in [x_0]\}$ is dense in $\mathcal{G}(0)$, it follows that Equation (7.3) holds for $x, y \in P$. By the same argument used in the proof of Theorem 7.4,

$$c(x, 0, y) = \sum_{j=0}^{\infty} [f(\mathcal{S}^j x) - f(\mathcal{S}^j y)].$$

It is now immediate that the cocycle c_f (see Example 7.3) extends c.

Fix $k \in \mathbb{Z}$; we claim that c_f is bounded on each $\mathcal{G}(k)$. Of course if $k = 0$ this holds by assumption. If $k > 0$ and $(x, k, y) \in \mathcal{G}(k)$ we can write

$$\begin{aligned} c_f(x, k, y) &= \sum_{j=0}^{k-1} f(\mathcal{S}^j x) + \sum_{j=k}^{\infty} [f(\mathcal{S}^j x) - f(\mathcal{S}^{j-k} y)] \\ &= \sum_{j=0}^{k-1} f(\mathcal{S}^j x) + c(\mathcal{S}^k x, 0, y). \end{aligned}$$

Since f is bounded, k is fixed, and c is bounded on $\mathcal{G}(0)$, c is bounded on $\mathcal{G}(k)$. From the cocycle property it follows that c is bounded on $\mathcal{G}(k)$, for k negative as well.

Finally, suppose that c vanishes precisely on $\mathcal{G}^0$. If L is a constant, $g \in C(P)$, and $g = f + L$, the cocycle c_g is related to c_f by $c_g(x, k, y) = c_f(x, k, y) + kL$, for $(x, k, y) \in \mathcal{G}(k)$. In particular, c_f and c_g agree on $\mathcal{G}(0)$, and $c_g(x, 1, y) = c_f(x, 1, y) + L$, for $(x, 1, y) \in \mathcal{G}(1)$. Since c_f is bounded on $\mathcal{G}(1)$, we can pick L sufficiently large so that c_g is positive on $\mathcal{G}(1)$. For $k > 1$, any element of $\mathcal{G}(k)$ is a product of k elements in $\mathcal{G}(1)$, so that c_g is positive on $\mathcal{G}(k)$. For $k < 0$, the cocycle property guarantees that c_g is negatve on $\mathcal{G}(k)$. Hence, the equation $c_g = 0$ has exactly the same solutions as $c = 0$ on $\mathcal{G}(0)$, namely $c_g = 0$ precisely on the unit space $\mathcal{G}^0$. □

Remark 7.13. In case G is the directed graph with a single vertex and n edges (i.e., loop edges), $\mathcal{G}$ is the Cuntz groupoid and $\mathcal{G}(0)$ is the UHF(n^∞) groupoid. In this case the hypotheses of Proposition 7.12 are satisfied, and hence any bounded cocycle which vanishes precisely on the unit space extends to a cocycle on $\mathcal{G}$ vanishing precisely on $\mathcal{G}^0$. In particular, this applies to the refinment cocycle on the UHF(n^∞) groupoid.

7.2. Integer-valued cocycles. Among the most studied classes of cocycles on AF groupoids are the integer-valued cocycles, due to their connection with dynamical systems. Thus it is natural to examine integer-valued cocycles on Cuntz–Krieger groupoids.

Lemma 7.14. *Let $\mathcal{G}$ be a Cuntz–Krieger groupoid, and suppose $\mathcal{G}(0)$ has a dense equivalence class.* (*In particular, that will be the case when* $\mathrm{C}^*(\mathcal{G}(0))$ *is simple.*) *Suppose c is an integer valued cocycle defined on $\mathcal{G}(0)$. Then:*

(1) *c admits an extension to a cocycle on $\mathcal{G}$ if, and only if, c admits an extension to an integer valued cocycle on $\mathcal{G}$.*
(2) *If c vanishes precisely on the unit space $\mathcal{G}^0$, then c admits an extension to $\mathcal{G}$ with this property if, and only if, c admits an integer valued extension to $\mathcal{G}$ vanishing precisely on $\mathcal{G}^0$.*

Proof. By Theorem 7.4 any extension of c to $\mathcal{G}$ is of the form c_f, for some $f \in C(P)$. Furthermore, by Equation (7.2) in Theorem 7.4 we have, for any $(x,0,y) \in \mathcal{G}$,

$$c(x,0,y) = f(x) - f(y) + c(\mathcal{S}x, 0, \mathcal{S}y). \tag{7.4}$$

Let y be chosen to have a dense equivalence class in $\mathcal{G}(0)$; i.e., so that $\{x \mid (x,0,y) \in \mathcal{G}(0)\}$ is dense in P. Replacing f by $g = f + L$ for a constant L, we can assume that $g(y) = 0$. The values of c_f and c_g agree on $\mathcal{G}(0)$. But then from Equation (7.4) we have that

$$g(x) = c(x,0,y) - c(\mathcal{S}x, 0, \mathcal{S}y),$$

so that g is $\mathbb{Z}$-valued on a dense set, hence $\mathbb{Z}$-valued on P. This completes the first statement.

The proof of the second statement is achieved, *mutatis mutandis*, as in the last part of the proof of Proposition 7.12. □

Remark 7.15. Let G be a range finite directed graph with no sources. Any compact, open subset of P is a finite union of cylinder sets $Z(\alpha)$. The sets $Z(\alpha)$ can be taken disjoint. Suppose further that G is a finite graph, so that P is compact, and let $f \in C(P)$ be a function which assumes only finitely many values. For each t in the range of f, $f^{-1}(t)$ is a compact open subset of P, and hence can be written as a finite, disjoint union of cylinder sets. It follows that there is a positive integer N such that, for all $x \in P$, the value of f at x depends only on the first N 'coordinates' of x.

Theorem 7.16. *Let G be a finite, transitive, directed graph containing at least two distinct simple loops. Suppose that c is a $\mathbb{Z}$-valued cocycle defined on the AF subgroupoid $\mathcal{G}(0) \subset \mathcal{G}$, which vanishes precisely on the unit space $\mathcal{G}^0$. Then c has no extension to a $\mathbb{Z}$-valued cocycle on $\mathcal{G}$.*

If $\mathcal{G}(0)$ has a dense equivalence class, then c admits no extension to $\mathcal{G}$.

Proof. Let the two simple loops be denoted α and β. Each of α and β contains an edge not in the other. By transitivity, there is a path γ with $r(\gamma) = r(\alpha) = s(\alpha)$ and $s(\gamma) = r(\beta) = s(\beta)$, and another path γ' with $r(\gamma') = r(\beta)$, $s(\gamma') = r(\alpha)$.

Denote the loop $\gamma\beta\gamma'$ by β. β may no longer be simple, but it contains an edge not in α, and both loops α and β have the same initial and terminal vertex. Suppose α has k edges, and β has l edges. If $k \neq l$, we can replace α by $\alpha^l = \alpha \dots \alpha$ (l concatenations), and replace β by β^k. Changing notation and denoting the new loops by α and β, neither may be simple, but both now have the same number of edges, and β contains an edge not in α.

We suppose c is extendible; hence, there is a function $f \in C(P)$ such that the cocycle c is the restriction of c_f to $\mathcal{G}(0)$. Thus,

$$c(x,0,y) = \sum_{j=0}^{\infty}[f(\mathcal{S}^j x) - f(\mathcal{S}^j y)], \text{ for all } (x,0,y) \in \mathcal{G}(0).$$

Since c_f is assumed to be $\mathbb{Z}$-valued, it follows from $f(x) = c_f(x,1,\mathcal{S}x)$ that f is $\mathbb{Z}$-valued. As P is compact, f takes on only finitely many values, and hence the value of f at $x \in P$ depends only on some initial path of x: $\exists N \in \mathbb{Z}^+$ such that if η is a finite path, $|\eta| \geq N$ and $z, z' \in P$ with $s(\eta) = r(z) = r(z')$, then $f(\eta z) = f(\eta z')$. (Cf. Remark 7.15.)

Say $\alpha = e_1 \dots e_k$, $\beta = f_1 \dots f_k$, $e_i, f_i \in E$. By increasing N if necessary, we may assume $k|N$ and that $\ell = N/k > 1$. Define points $x, y \in P$ by

$$x = \alpha^\ell \alpha\beta\alpha^\ell\beta^\infty \text{ and } y = \alpha^\ell\beta\alpha\alpha^\ell\beta^\infty.$$

For $z \in P$, let $(z)_N$ denote the truncation of z: $(z)_N = z_1 \dots z_N$. For $n \in \mathbb{Z}^+$, let m, p be determined by the Euclidean algorithm: $n = mk + p$ with $0 \leq p < k$.

Observe that if $m = 0$ (so $n = p$), then

$$(\mathcal{S}^n x)_N = e_{p+1} \dots e_k \alpha^{\ell-1} e_1 \dots e_p,$$
$$(\mathcal{S}^n y)_N = e_{p+1} \dots e_k \alpha^{\ell-1} f_1 \dots f_p.$$

If $m = 1$ (so $n = k + p$),

$$(\mathcal{S}^n x)_N = e_{p+1} \dots e_k \alpha^{\ell-1} f_1 \dots f_p,$$
$$(\mathcal{S}^n y)_N = e_{p+1} \dots e_k \alpha^{\ell-2} \beta e_1 \dots e_p.$$

For any $1 < m \leq \ell - 1$, with α^0 understood to be the empty string,

$$(\mathcal{S}^n x)_N = e_{p+1} \dots e_k \alpha^{\ell-m-1} \alpha\beta\alpha^{m-2} e_1 \dots e_p,$$
$$(\mathcal{S}^n y)_N = e_{p+1} \dots e_k \alpha^{\ell-m-1} \beta\alpha\alpha^{m-2} e_1 \dots e_p.$$

For $m = \ell$,

$$(\mathcal{S}^n x)_N = e_{p+1} \dots e_k \beta\alpha^{\ell-2} e_1 \dots e_p,$$
$$(\mathcal{S}^n y)_N = f_{p+1} \dots f_k \alpha^{\ell-1} e_1 \dots e_p.$$

For $m = \ell + 1$,

$$(\mathcal{S}^n x)_N = f_{p+1} \dots f_k \alpha^{\ell-1} e_1 \dots e_p,$$
$$(\mathcal{S}^n y)_N = e_{p+1} \dots e_k \alpha^{\ell-1} e_1 \dots e_p.$$

Note that for $m \geq \ell + 2$, $\mathcal{S}^n x = \mathcal{S}^n y$. Also, viewing n as a function of m with p constant, observe that $(\mathcal{S}^{n(m)} y)_N = (\mathcal{S}^{n(m+1)} x)_N$ for $0 \leq m \leq \ell$ and that

$(\mathcal{S}^{n(0)}x)_N = (\mathcal{S}^{n(\ell+1)}y)_N$. It follows that

$$\sum_{n=0}^{\infty}[f(\mathcal{S}^n x) - f(\mathcal{S}^n y)] = 0.$$

In other words, $c_f(x,0,y) = 0$; this is impossible since, with $x \neq y$, $(x,0,y)$ is not a unit and $c_f(x,0,y) = c(x,0,y)$ was assumed to vanish only on the unit space.

If now $\mathcal{G}(0)$ contains a dense orbit, then the second statement of the theorem follows immediately from the second statement of Lemma 7.14. □

Remark 7.17. Lemma 7.14 and Theorem 7.16 together show that the standard cocycle on UHF(n^∞) (viewed as the core AF-subalgebra of the Cuntz algebra O_n) has no extension to a cocycle on O_n.

7.3. The 'analytic' subalgebra associated with a cocycle. Let $c \in Z^1(\mathcal{G}, \mathbb{R})$, and set

$$A(c) = \{f \in C^*(\mathcal{G}) \mid f \text{ is supported on the set } c^{-1}([0,\infty))\}.$$

From Remark 7.1, $f \in A(c)$ if, and only if, the maps $t \to \rho(\eta_t(f))$ is an H^∞ function (for ρ, η as in Remark 7.1). Thus $A(c)$ is also written as $H^\infty(c)$.

Clearly, any point in the interior of $c^{-1}([0,\infty))$ lies in the spectrum of $A(c)$. On the other hand, since the spectrum of any bimodule is open, it follows that $\sigma(A(c))$ is the interior of $c^{-1}([0,\infty))$.

Of particular interest are the cocycles $c \in Z^1_0$, that is, those which vanish precisely on the unit space, for in that case we have $\sigma(A(c)) = c^{-1}([0,\infty))$. If, furthermore, the directed graph G satisfies the condition that every loop has an entrance, then $\mathrm{C}^*(\mathcal{G}^0)$ is a masa in $\mathrm{C}^*(\mathcal{G})$ and $A(c)$ is triangular (since $A(c) \cap A(c)^* = \mathrm{C}^*(\mathcal{G}^0)$). Furthermore, $c^{-1}([0,\infty))$ clopen also implies that $A(c) + A(c)^*$ is dense in $\mathrm{C}^*(\mathcal{G})$. Indeed, if χ_1 is the characteristic function of $c^{-1}([0,\infty))$ and χ_2 is the characteristic function of $c^{-1}((-\infty,0))$ then any $f \in C_c(\mathcal{G})$ can be written as $f = f\chi_1 + f\chi_2$.

8. The spectral theorem for bimodules. Part III

In this section we extend the spectral theorem for bimodules to show that the condition of invariance under the gauge automorphisms can be replaced by invariance under the automorphism group associated with an 'arbitrary' locally constant cocycle (satisfying a mild constraint). As usual, we assume that the graph G is range finite and has no sources.

Remark 8.1. If K is any compact subset of the groupoid $\mathcal{G}$, then as Banach spaces $C(K) \subset \mathcal{A}$. Since $C(K)$ is complete in both the C*-and supremum norms, these norms are equivalent on $C(K)$.

Let $Z(\alpha,\beta)$ be a basic open set in $\mathcal{G}$. We define a partial homeomorphism τ on P with $\mathrm{dom}(\tau) = \{\alpha z \in P \mid r(z) = s(\alpha) = s(\beta)\}$ by $\tau(\alpha z) = \beta z$. By definition, $Z(\alpha,\beta) \subset \mathcal{G}(k)$, where $k = |\alpha| - |\beta|$.

Notation. With τ as above, denote

$$\mathcal{G}(\tau) = \{(x,\ell,y) \in \mathcal{G} : \; y = \tau(x)\}.$$

We refer to $\mathcal{G}(\tau)$ as the $\mathcal{G}$-*graph* of τ.

Note that the graph of τ could contain points (x, ℓ, y) with $\ell \neq k$. For example, if α is a loop and $\ell = |\alpha^2| - |\beta|$, then $(\alpha^\infty, \ell, \beta\alpha^\infty)$ also lies in $\mathcal{G}(\tau)$.

Notation. For $f \in C^*(\mathcal{G})$, we let f_τ denote the restriction of f to the $\mathcal{G}$-graph of the partial homeomorphism τ. Viewing f as a function on the groupoid $\mathcal{G}$, the restriction is well-defined as a function on $\mathcal{G}$.

Given $f \in \mathcal{A}$, it is not clear that the restriction f_τ also belongs to $\mathcal{A}$, much less that if f belongs to a norm-closed $\mathcal{D}$-bimodule $\mathcal{B}$, then f_τ also belongs to $\mathcal{B}$. Our first goal is to verify these statements.

By [7, Cor. 5.5], path space P is metrizable. Fix a metric on P. With τ as above, the domain of τ is the open compact neighborhood $Z(\alpha) \subset P$. For each $x \in \operatorname{dom}(\tau)$, let $U_n(x)$ be a clopen neighborhood centered at x with radius at most $1/n$. By compactness, there is a finite subcover, $U_n(x_1), \dots, U_n(x_{r_n})$ of $\operatorname{dom}(\tau)$. Let $U_{n,1} = U_n(x_1)$ and $U_{n,j} = U_n(x_j) \setminus \cup_{i=1}^{j-1} U_n(x_i)$ for $j = 2, \dots, r_n$. Thus the sets $U_{n,j}$ form a disjoint clopen cover of $Z(\alpha) = \operatorname{dom}(\tau)$. Let $\chi_{n,j}$ denote the characteristic function of $U_{n,j}$. Define $\Psi_n \colon \mathcal{A} \to \mathcal{A}$ by

$$\Psi_n(f) = \sum_{j=1}^{r_n} \chi_{n,j} \cdot f \cdot (\chi_{n,j} \circ \tau^{-1}).$$

(We identify any function $g \in C_0(P)$ with a function, also denoted by g, on $\mathcal{G}$ by using the natural identification of P with the unit space of $\mathcal{G}$. The extended function g vanishes at any point (x, k, y) for which $k \neq 0$ or $x \neq y$. This function is in $\mathcal{A}$; in fact, it is in $\mathcal{D}$.)

We will show that the sequence $\{\Psi_n(f)\}_{n=1}^\infty$ converges to f_τ, for any $f \in \mathcal{A}$. If f happens to be supported on some compact subset $K \subset \mathcal{G}$, then $\Psi_n(f)$ is also supported on K. Furthermore, if $(x, k, \tau(x)) \in K \cap \mathcal{G}(\tau))$, then $\Psi_n(f)(x, k, \tau(x)) = f(x, k, \tau(x))$ while for $(x, k, y) \notin \mathcal{G}(\tau)$ we have $\Psi_n(f)(x, k, y) \to 0$. The convergence is uniform on compact subsets and hence uniform on $\mathcal{G}$. By Remark 8.1, $\Psi_n(f) \to f_\tau$ in the C*-norm. This, of course, shows tht $f_\tau \in \mathcal{A}$, at least when f is compactly supported.

To handle the general case, we need to observe that each Ψ_n is norm decreasing. Indeed, Ψ_n has the form $f \mapsto \sum p_n f q_n$ where the sum is finite and each of $\{p_n\}$ and $\{q_n\}$ is a family of mutually orthogonal projections. Maps of this form on a C*-algebra are always contractive.

Lemma 8.2. *For $f \in \mathcal{A}$, $\Psi_n(f)$ converges to f_τ. In particular, $f_\tau \in \mathcal{A}$ and $||f_\tau|| \leq ||f||$. If f belongs to a norm closed $\mathcal{D}$-bimodule $\mathcal{B}$, then so does f_τ.*

Proof. We already know that $\Psi_n(f)$ converges pointwise to f_τ on $\mathcal{G}$. Given $\epsilon > 0$, let $g \in \mathcal{A}$ have compact support, with $||f - g|| < \epsilon$. Since g is compactly supported, $\Psi_n(g) \to g_\tau$ in C*-norm. Hence, there is a positive integer N such that $||\Psi_n(g) - g_\tau|| < \epsilon$ for all $n \geq N$. Then, since Ψ_n is contractive,

$$||\Psi_n(f) - \Psi_m(f)|| \leq ||\Psi_n(f - g)|| + ||\Psi_n(g) - \Psi_m(g)|| + ||\Psi_m(g - f)|| < 4\epsilon$$

for $n, m \geq N$. Thus $\{\Psi_n(f)\}_{n=1}^\infty$ has a limit, which must agree with its pointwise limit, f_τ. Hence $f_\tau \in \mathcal{A}$ and

$$||f_\tau|| \leq \lim_{n\to\infty} ||\Psi_n(f)|| \leq ||f||.$$

Finally, it is clear that if f belongs to a norm closed $\mathcal{D}$-bimodule $\mathcal{B}$, then so does each $\Psi_n(f)$; therefore $f_\tau \in \mathcal{B}$. $\square$

In Theorems 3.1 and 6.3 we proved that a closed $\mathcal{D}$-bimodule is determined by its spectrum if, and only if, it is invariant under the gauge automorphisms. As noted in Section 7, the gauge automorphisms arise in a natural way from the cocycle $c(x,k,y) = k$ on $\mathcal{G}$. In Theorem 8.3 below we show that a closed $\mathcal{D}$-bimodule is determined by its spectrum if, and only if, it is invariant under that one-parameter automorphism group associated with a locally constant cocycle c for which $c^{-1}(0) \subseteq \mathcal{G}(0)$. For a locally constant cocycle c, $c^{-1}(0) \subseteq \mathcal{G}(0)$ if, and only if, the fixed point algebra for this one-parameter automorphism group is contained in the core AF algebra.

As we saw in Theorem 7.4, continuous cocycles are in one-to-one correspondence with continuous functions on path space. Proposition 7.7 showed that locally constant cocycles arise from locally constant functions on path space.

Suppose that $\mathcal{B}$ is a norm closed $\mathcal{D}$-bimodule. It is automatic that $\mathcal{B} \subseteq A(\sigma(\mathcal{B}))$; if every Cuntz–Krieger partial isometry $S_\alpha S_\beta^*$ in $A(\sigma(\mathcal{B}))$ lies in $\mathcal{B}$, then $A(\sigma(\mathcal{B})) \subseteq \mathcal{B}$ and $\mathcal{B}$ is determined by its spectrum. Since we are using the groupoid model, $S_\alpha S_\beta^*$ is the characteristic function of the basic open subset $Z(\alpha,\beta)$ of $\mathcal{G}$.

A simple observation is useful in the proof of Theorem 8.3 below. Suppose that for each $(x,k,y) \in \sigma(\mathcal{B})$ there is a basic neighborhood $Z(\alpha,\beta)$ of (x,k,y) such that $S_\alpha S_\beta^* \in \mathcal{B}$. Then it follows that every Cuntz–Krieger partial isometry in $A(\sigma(\mathcal{B}))$ is in $\mathcal{B}$. Indeed, if $S_\alpha S_\beta^* \in \mathcal{B}$ and $Z(\gamma,\delta) \subseteq Z(\alpha,\beta)$, then $S_\gamma S_\delta^*$ can be obtained from $S_\alpha S_\beta^*$ by left and right multiplication by projections in $\mathcal{D}$ (use the range projections for S_γ and S_δ); therefore $S_\gamma S_\delta^* \in \mathcal{B}$ also. If $Z(\nu,\mu)$ is an arbitrary basic open subset of $\sigma(\mathcal{B})$, then by hypothesis, it can be covered by sets $Z(\alpha,\beta)$ for which $S_\alpha S_\beta^* \in \mathcal{B}$. Since $Z(\nu,\mu)$ is compact, there is a finite subcover. The observation about subsets allows us to find a finite subcover of disjoint sets of the form $Z(\gamma,\delta)$ with $S_\gamma S_\delta^* \in \mathcal{B}$. It now follows that $S_\nu S_\mu^*$ is a finite sum of elements of $\mathcal{B}$ and so is in $\mathcal{B}$ itself.

Let c be a real valued cocycle on $\mathcal{G}$. Recall that the associated one-parameter automorphism group on $\mathcal{A}$ is defined by

$$\eta_z(f)(x,k,y) = z^{c(x,k,y)} f(x,k,y), \quad \text{all } z \in \mathbb{T}.$$

(To avoid ambiguity, when $z = e^{it}$ with $0 \le t < 2\pi$ and a is a real number, we take $z^a = e^{ita}$.)

Theorem 8.3. *Let $\mathcal{B} \subseteq \mathcal{A}$ be a norm closed $\mathcal{D}$-bimodule and let c be a locally constant cocycle on $\mathcal{G}$ such that the fixed point algebra of the associated one-parameter automorphism group η is contained in the core AF algebra. Then $\mathcal{B} = A(\sigma(\mathcal{B}))$ if, and only if, $\mathcal{B}$ is invariant under η.*

Proof. If $f \in A(\sigma(\mathcal{B}))$ then f is supported on $\sigma(\mathcal{B})$; clearly each $\eta_z(f)$ is also supported on $\sigma(\mathcal{B})$. Thus, $\mathcal{B} = A(\sigma(\mathcal{B}))$ trivially implies that $\mathcal{B}$ is invariant under the η_z.

Now assume that $\mathcal{B}$ is invariant under the η_z. By the observations preceding the theorem, it suffices to prove that for each point $(x,k,y) \in \sigma(\mathcal{B})$, there is a basic open neighborhood $Z(\alpha,\beta)$ of (x,k,y) for which $S_\alpha S_\beta^* \in \mathcal{B}$.

Given $(x_0, k_0, y_0) \in \sigma(\mathcal{B})$ there is an element $f \in \mathcal{B}$ and a basic neighborhood $Z(\alpha, \beta)$ such that c is constant on $Z(\alpha, \beta)$ and $f(x, k, y) \neq 0$ for all $(x, k, y) \in Z(\alpha, \beta)$. We will show that $S_\alpha S_\beta^* \in \mathcal{B}$.

Let τ be the partial homeomorphism on P with $\mathrm{dom}(\tau) = Z(\alpha)$, $\mathrm{ran}(\tau) = Z(\beta)$ given by $\tau(\alpha z) = \beta z$, for all $\alpha z \in Z(\alpha)$. By Lemma 8.2, $f_\tau \in \mathcal{B}$.

Let $a = c(x_0, k_0, y_0)$. Define $E\colon \mathcal{A} \to \mathcal{A}$ by

$$E(g)(x, k, y) = \int_{\mathbb{T}} z^{-a} \eta_z(g)(x, k, y)\, dz.$$

The integration is with respect to normalized Lebesgue measure on $\mathbb{T}$. Each η_z is isometric, so E is contractive.

Since $f_\tau \in \mathcal{B}$, $E(f_\tau) \in \mathcal{B}$ also. When $(x, k_0, y) \in Z(\alpha, \beta)$, we have

$$z^{-a} \eta_z(f_\tau)(x, k_0, y) = f_\tau(x, k_0, y).$$

If $(x, k_0, y) \notin Z(\alpha, \beta)$, then $z^{-a} \eta_z(f_\tau)(x, k_0, y) = f_\tau(x, k_0, y)$ again holds, since both sides of the equation equal 0. Thus, $E(f_\tau)(x, k_0, y) = f_\tau(x, k_0, y)$, for all $(x, k_0, y) \in \mathcal{G}$. When $k \neq k_0$, then

$$\begin{aligned} c(x, k, y) - a &= c(x, k, y) - c(x, k_0, y) \\ &= c(x, k, y) + c(y, -k_0, x) \\ &= c(x, k - k_0, y) \neq 0. \end{aligned}$$

(The inequality follows from the fact that $c^{-1}(0) \subseteq \mathcal{G}(0)$.) Therefore $a \neq c(x, k, y)$ and the integrand in $E(f_\tau)(x, k, y)$ is a nonzero power of z multiplied by $f_\tau(x, k, y)$. It follows that $|E(f_\tau)(x, k, y)| < |f_\tau(x, k, y)|$ with the ratio between the two numbers dependent only on (x, k, y).

If we now let E^n be the n-fold composition of E with itself, we have

$$E^n(f_\tau)(x, k_0, y) = f_\tau(x, k_0, y)$$

for all $(x, k_0, y) \in Z(\alpha, \beta)$ and $E^n(f_\tau)(x, k, y) \to 0$ otherwise.

Let $f_{Z(\alpha,\beta)}$ denote the restriction of f_τ to $Z(\alpha, \beta)$. If f_τ has compact support, then $E^n(f_\tau) \to f_{Z(\alpha,\beta)}$ uniformly and (by Remark 8.1) in C*-norm as well. It follows in this case that $f_{Z(\alpha,\beta)} \in \mathcal{B}$.

For the general case, let $\epsilon > 0$ and let $g \in \mathcal{A}$ have compact support and satisfy $\|f - g\| < \epsilon$. Then g_τ has compact support and $\|f_\tau - g_\tau\| \leq \|f - g\| < \epsilon$ (Lemma 8.2). With $g_{Z(\alpha,\beta)}$ the restriction of g to $Z(\alpha, \beta)$, we know that there is $N \in \mathbb{N}$ such that $\|E^n(g_\tau - g_{Z(\alpha,\beta)})\| < \epsilon$ for all $n \geq N$. Therefore, when $n, m \geq N$,

$$\begin{aligned} \|E^n(f_\tau) - E^m(f_\tau)\| &\leq \|E^n(f_\tau - g_\tau)\| + \|E^n(g_\tau) - E^m(g_\tau)\| + \|E^m(g_\tau - f_\tau)\| \\ &< 4\epsilon. \end{aligned}$$

Thus, even when f_τ is not compactly supported, $E^n(f_\tau)$ is convergent in C*-norm; the limit must agree with the pointwise limit $f_{Z(\alpha,\beta)}$. Since each $E^n(f_\tau) \in \mathcal{B}$, we obtain $f_{Z(\alpha,\beta)} \in \mathcal{B}$.

Now define a continuous, compactly supported function h on P by

$$h(x) = \begin{cases} \frac{1}{f(x, k_0, y)}, & x \in \mathrm{dom}(\tau), \\ 0, & \text{otherwise.} \end{cases}$$

Extending h to all of $\mathcal{G}$ by taking it to have value 0 off P, we view h as an element of $\mathcal{D}$. But now $S_\alpha S_\beta^* = h f_{Z(\alpha,\beta)} \in \mathcal{B}$. This completes the proof. $\square$

9. Nest subalgebras of graph C*-algebras

An additional structure on on G — a total ordering of the edges — leads in a natural way to nest subalgebras of $\mathcal{A}$. Arbitrary total orders on E appear to be too general, so we consider orders on E which are compatible with a total ordering on V. Given a total order on V and, for each $v \in V$, a total order on $\{e \mid r(e) = v\}$, we can then define an order on E in which two edges with the same range are ordered by the order on $\{e \mid r(e) = v\}$ and two edges with different ranges are ordered by the order on V. We could, of course, use the sources instead of the ranges, or even combine the two; but it is orders compatible with the ranges which are most suitable for the algebras which we shall study. There is a way to rephrase the definition of the orders we study; we use this for the formal definition:

Definition 9.1. An *ordered graph* is a directed graph G together with a total order $\preceq$ on E which satisfies the property that, for each $v \in V$, $\{e \mid r(e) = v\}$ is an interval in the order on E.

Throughout this section we assume that G is a finite ordered graph. We use the order on the graph to define a nest of projections in $\mathcal{D}$; the associated nest subalgebra of $\mathcal{A}$ is the object of study.

For each k, the (left to right) lexicographic order gives a total order on F_k, the set of paths of length k. (The lexicographic order is based on the order on E.) We denote this order by $\preceq$. For each finite path $\alpha = \alpha_1 \dots \alpha_k$, let R_α denote the range of the partial isometry $S_\alpha = S_{\alpha_1} \dots S_{\alpha_k}$. $\{R_\alpha \mid |\alpha| = k\}$ is an orthogonal set of projections which sum to the identity. This set inherits a total order from the lexicographic order on F_k. We shall use the notation $\ll$ for the strict variant of this total order.

Let $\mathcal{N}_k$ be the nest whose atoms are the R_α with $|\alpha| = k$, taken in the order above. Projections in $\mathcal{N}_k$ have the form $\sum R_\alpha$, summed over initial segments in the order $\ll$.

Let $\alpha = \alpha_1 \dots \alpha_k \in F_k$. Write $\{e \in E \mid r(e) = s(\alpha_k)\}$ as $\{e_1, \dots, e_p\}$ with $e_1 \prec e_2 \prec \dots \prec e_p$. Then $\{\alpha e_1, \dots, \alpha e_p\}$ forms an order interval in F_{k+1}. For each path α, $R_\alpha = \sum R_{\alpha e}$, where the sum is over all edges e such that $r(e) = s(\alpha)$. It follows that $\mathcal{N}_k \subseteq \mathcal{N}_{k+1}$, for all k. Let $\mathcal{N} = \bigcup \mathcal{N}_k$ and

$$\operatorname{Alg} \mathcal{N} = \{A \in \mathcal{A} \mid AP = PAP \text{ for all } P \in \mathcal{N}\}.$$

Note that, for fixed k and p, $\mathcal{N} = \bigcup_n \mathcal{N}_{p+nk}$. Consequently, to prove that an element A of the graph C*-algebra is in $\operatorname{Alg}\mathcal{N}$ it suffices to prove that $A \in \operatorname{Alg}\mathcal{N}_{p+nk}$, for all n.

Definition 9.2. We shall refer to $\mathcal{N}$ and $\operatorname{Alg}\mathcal{N}$ as the *nest* and the *nest algebra* induced by the order $\preceq$ on E.

Remark 9.3. The material in this section was inspired by, and is an extension of, the work on the Volterra subalgebra of the Cuntz C*-algebra found in [12] and [2]. The Cuntz algebra O_n is the graph C*-algebra for a graph with one vertex and n loops. By symmetry, the choice of order on the n loops is immaterial . (Indeed, it is not hard to find a unitary in O_n which conjugates the generators in one order into the generators in another order.)

There is a natural representation of O_n acting on $L^2[0,1]$. For each $k = 1, \dots, n$, let S_k be the isometry on $L^2[0,1]$ associated with the affine, order preserving map from $[0,1]$ onto the interval $\left[\dfrac{k-1}{n}, \dfrac{k}{n}\right]$. If the n loops in the graph for O_n are $e_1, \ldots, e_n$ in order, then $S_1, \ldots, S_n$ are the corresponding generating isometries. The nest $\mathcal{N}$ then consists of the projections which correspond to the intervals of the form $\left[0, \dfrac{j}{n^k}\right]$, where j and k are nonnegative integers. This nest is strongly dense in the Volterra nest (which consists of projections corresponding to intervals $[0,t]$, $0 \le t \le 1$). In this representation, the nest algebra $\operatorname{Alg}\mathcal{N}$ is exactly the intersection of O_n with the usual Volterra nest algebra acting on $L^2[0,1]$.

Observe that $\operatorname{Alg}\mathcal{N}$ is invariant under the gauge automorphisms. Indeed, if $z \in \mathbb{T}$, $A \in \operatorname{Alg}\mathcal{N}$ and $P \in \mathcal{N}$, then, since P is in the fixed point algebra of the gauge automorphisms, $P^\perp \eta_z(A) P = \eta_z(P^\perp A P) = 0$. Thus, $\eta_z(A) \in \operatorname{Alg}\mathcal{N}$, for all $A \in \operatorname{Alg}\mathcal{N}$, $z \in \mathbb{T}$.

By the spectral theorem for bimodules (Theorem 6.3), $\operatorname{Alg}\mathcal{N}$ is the closed linear span of the Cuntz–Krieger partial isometries which it contains. We will now characterize the Cuntz–Krieger partial isometries $S_\alpha S_\beta^*$ in $\operatorname{Alg}\mathcal{N}$ in terms of the properties of the finite paths α and β. This, in turn, will enable us to give a description of the spectrum $\sigma(\operatorname{Alg}\mathcal{N})$.

Definition 9.4. A path α is *s-minimal* if $\alpha \preceq \beta$ whenever β is a path with $|\beta| = |\alpha|$ and $r(\beta) = s(\alpha)$. α is *s-maximal* if $\beta \preceq \alpha$ whenever β is a path with $|\beta| = |\alpha|$ and $r(\beta) = s(\alpha)$.

Remark 9.5. In a Cuntz algebra O_n, finite paths are essentially finite sequences from the integers $\{1, \ldots, n\}$. A finite path α is s-minimal if $\alpha_i = 1$ for all i and s-maximal if $\alpha_i = n$ for all i.

Proposition 9.6. *$S_\alpha \in \operatorname{Alg}\mathcal{N} \iff \alpha$ is s-minimal.*

Proof. Suppose that α is not s-minimal. Then there is a path β with $|\beta| = |\alpha|$, $r(\beta) = s(\alpha)$, and $\beta \prec \alpha$. With k the common degree of α and β, R_α and R_β are atoms from $\mathcal{N}_k$ and $R_\beta \ll R_\alpha$. Now, S_α is nonzero on R_β (since $r(\beta) = s(\alpha)$) and so $R_\alpha S_\alpha R_\beta \neq 0$. But then $S_\alpha \notin \operatorname{Alg}\mathcal{N}_k$ and hence $S_\alpha \notin \operatorname{Alg}\mathcal{N}$.

Now suppose that α is s-minimal. We distinguish two cases. First assume that $r(\alpha) \neq s(\alpha)$; i.e., α is not a loop. Then the initial space Q_α is a sum of atoms of the form R_β, where $|\beta| = |\alpha|$ and $\alpha \prec \beta$. If we let P be the smallest projection in $\mathcal{N}_k$ such that $R_\alpha \le P$, then each of the R_β in the sum for Q_α is orthogonal to P. Therefore, $S_\alpha = P S_\alpha P^\perp$ and $S_\alpha \in \mathcal{N}$.

Next assume that α is a loop. Then the initial space Q_α can be written as a sum $R_\alpha + \sum R_\beta$, where the β in the sum run over paths with $|\beta| = |\alpha|$, $r(\beta) = s(\alpha)$, and $\alpha \prec \beta$. Since $S_\alpha R_\beta = R_\alpha S_\alpha R_\beta$ for each such β, each $S_\alpha R_\beta \in \operatorname{Alg}\mathcal{N}$.

It remains to show that $S_\alpha R_\alpha \in \operatorname{Alg}\mathcal{N}$. Let k be the degree of α. As noted above, it is sufficient to show that $S_\alpha R_\alpha \in \operatorname{Alg}\mathcal{N}_{nk}$ for each positive integer n. Let P be a projection in $\mathcal{N}_{nk}$. If $P \perp R_\alpha$, then $S_\alpha R_\alpha P = 0$ and $S_\alpha R_\alpha$ trivially leaves P invariant. P is also trivially left invariant if $R_\alpha \le P$. This leaves the case in which $0 < PR_\alpha < R_\alpha$. To show that P is invariant under $S_\alpha R_\alpha$, it suffices to prove $S_\alpha R_\alpha P \in \operatorname{Alg}\mathcal{N}_{nk}$.

The projection PR_α can be written as a sum of atoms R_β (from $\mathcal{N}_{nk}$) where the β run through an interval in the order on F_{nk}. Let β be one of these paths. Write $\beta = \beta_1 \ldots \beta_n$, where each β_i has length k. We need to show that $S_\alpha R_\beta$ has range contained in atoms whose indices precede or equal β. The range of $S_\alpha R_\beta$ is $R_{\alpha\beta}$, which is a subprojection of $R_{\alpha\beta_1\dots\beta_{n-1}}$. The assumption that α is s-minimal implies that $\alpha \preceq \beta_1$. If $\alpha \prec \beta_1$, then $R_{\alpha\beta_1\dots\beta_{n-1}} \ll R_{\beta_1\dots\beta_n} = R_\beta$. On the other hand, if $\beta_1 = \alpha$, then $r(\beta_2) = s(\beta_1) = s(\alpha) = r(\alpha)$ and $\alpha \preceq \beta_2$ (again use α is s-minimal).

Once again, if $\alpha \prec \beta_2$ then $R_{\alpha\beta_1\dots\beta_{n-1}} = R_{\alpha\alpha\beta_2\dots\beta n-1} \ll R_{\alpha\beta_2\dots\beta_n} = R_{\beta_1\dots\beta_n}$. Continuing in this fashion, we see that if any of the β_i are unequal to α, then $S_\alpha R_\beta \in \operatorname{Alg}\mathcal{N}_{nk}$. Finally, if all $\beta_i = \alpha$, then S_α maps $R_{\alpha\dots\alpha}$ into itself, so again $S_\alpha R_\beta = S_\alpha R_{\alpha\dots\alpha} \in \operatorname{Alg}\mathcal{N}_{nk}$. From this it follows that $S_\alpha R_\alpha P \in \operatorname{Alg}\mathcal{N}_{nk}$ and the proposition is proven. □

Proposition 9.7. *Let α and β be two paths of equal length with $s(\alpha) = s(\beta)$ (so that $S_\alpha S_\beta^* \neq 0$). Then $S_\alpha S_\beta^* \in \operatorname{Alg}\mathcal{N}$ if, and only if, $\alpha \preceq \beta$.*

Proof. If $\alpha = \beta$, then $S_\alpha S_\beta^*$ is a projection in the canonical diagonal of the graph C*-algebra. Since the nest $\mathcal{N}$ is also in this diagonal, $S_\alpha S_\beta^* \in \operatorname{Alg}\mathcal{N}$. Note that $S_\alpha S_\beta^* = R_\alpha S_\alpha S_\beta^* R_\beta$. If $\alpha \prec \beta$, then $R_\alpha \ll R_\beta$ and $S_\alpha S_\beta^* \in \operatorname{Alg}\mathcal{N}$. If $\beta \prec \alpha$ then $R_\beta \ll R_\alpha$ and $S_\alpha S_\beta^* \notin \operatorname{Alg}\mathcal{N}$. □

Proposition 9.8. *Let α and β be two paths with $0 < k = |\beta| < |\alpha|$ and $s(\alpha) = s(\beta)$. Write $\alpha = \delta\gamma$ where $|\delta| = |\beta|$ and $|\gamma| = |\alpha| - |\beta|$. Then $S_\alpha S_\beta^* \in \operatorname{Alg}\mathcal{N}$ if, and only if, one of the following two conditions holds*:

(1) *$\delta \prec \beta$.*
(2) *$\delta = \beta$ and γ is s-minimal.*

Proof. The initial space for $S_\alpha S_\beta^*$ is the final space for S_β, namely R_β. The range space is $R_\alpha = R_{\delta\gamma}$, which is a subprojection of R_δ. If $\delta \prec \beta$ then $R_\delta \ll R_\beta$ as atoms from $\mathcal{N}_k$ and $S_\alpha S_\beta^* \in \operatorname{Alg}\mathcal{N}$. And if $\beta \prec \delta$ then $R_\beta \ll R_\delta$ and $S_\alpha S_\beta^* \notin \operatorname{Alg}\mathcal{N}$.

Assume that $\delta = \beta$, so that $\alpha = \beta\gamma$. Observe that $r(\gamma) = s(\beta)$ and that $s(\beta) = s(\alpha) = s(\gamma)$. Thus, $r(\gamma) = s(\gamma)$ and γ is a loop.

Now the initial space for S_β^* is R_β and the final space is the initial space for S_α. Therefore, $S_\alpha S_\beta^*$ maps R_β onto $R_\alpha = R_{\beta\gamma}$, which is a subprojection of R_β. So $S_\alpha S_\beta^*$ trivially leaves invariant any projection which contains R_β or is orthogonal to R_β.

Assume that γ is s-minimal. Let $t = |\gamma|$. It is sufficient to prove that, for any positive integer n, $S_\alpha S_\beta^* \in \operatorname{Alg}\mathcal{N}_{k+nt}$. By the preceding paragraph, it is enough to look at the action of $S_\alpha S_\beta^*$ on atoms from $\mathcal{N}_{k+nt}$ which are subprojections of R_β. Each of these atoms has the form $R_{\beta\eta_1\dots\eta_n}$, where $|\eta_i| = t$, for all i. Now S_β^* maps $R_{\beta\eta_1\dots\eta_n}$ onto $R_{\eta_1\dots\eta_n}$ and so $S_\alpha S_\beta^*$ maps $R_{\beta\eta_1\dots\eta_n}$ onto $R_{\beta\gamma\eta_1\dots\eta_n}$. The latter is a subprojection of $R_{\beta\gamma\eta_1\dots\eta_{n-1}}$, which is an atom from $\mathcal{N}_{k+nt}$. So all we have to do is to prove that $R_{\beta\gamma\eta_1\dots\eta_{n-1}}$ precedes or equals $R_{\beta\eta_1\dots\eta_n}$ in the ordering for atoms from $\mathcal{N}_{k+nt}$.

If every $\eta_i = \gamma$, this is trivial. Otherwise, let η_j be the first η which is unequal to γ. If $j = 1$, then $r(\eta_1) = s(\gamma)$. If $j > 1$, then $r(\eta_j) = s(\eta_{j-1}) = s(\gamma)$. Since γ is s-minimal, $\gamma \prec \eta_j$. But then $R_{\beta\gamma\eta_1\dots\eta_{n-1}} \ll R_{\beta\eta_1\dots\eta_n}$.

It remains to show that if γ is not s-minimal then $S_\alpha S_\beta^* \notin \operatorname{Alg}\mathcal{N}$. Suppose that η is a path with $|\eta| = |\gamma|$, $r(\eta) = s(\gamma) = s(\beta)$ and $\eta \prec \gamma$. Then $\beta\eta \prec \beta\gamma$ and $R_{\beta\eta} \ll R_{\beta\gamma}$. Now S_β^* maps $R_{\beta\eta}$ into the initial space for $S_{\beta\gamma} = S_\alpha$ and $S_\alpha S_\beta^*$ maps $R_{\beta\eta}$ into a subprojection of $R_{\beta\gamma}$; thus $S_\alpha S_\beta^* \notin \mathcal{N}$. □

Corollary 9.9. *Suppose that γ is a path such that $S_\gamma \in \operatorname{Alg}\mathcal{N}$. Then, for any β, $S_\beta S_\gamma S_\beta^* \in \operatorname{Alg}\mathcal{N}$.*

Proof. By Proposition 9.6, γ is s-minimal. If $s(\gamma) \neq s(\beta)$ then $S_\beta S_\gamma S_\beta^* = 0 \in \operatorname{Alg}\mathcal{N}$. Otherwise, condition (2) of Proposition 9.8 yields the corollary. □

The next two propositions can be proven with arguments analogous to the ones used in Proposition 9.6 and Proposition 9.8. However, a shortcut is available. If we reverse the order on paths of length k and therefore reverse the order on the corresponding atoms, we obtain the nest $\mathcal{N}^\perp$ instead. Since a path is s-minimal with respect to the reversed order if, and only if, it is s-maximal with respect to the original order and since $\operatorname{Alg}\mathcal{N}^\perp = (\operatorname{Alg}\mathcal{N})^*$, Proposition 9.10 and Proposition 9.11 are immediate consequences of Proposition 9.6 and Proposition 9.8.

Proposition 9.10. $S_\beta^* \in \operatorname{Alg}\mathcal{N} \iff \beta$ *is s-maximal.*

Proposition 9.11. *Let α and β bet two paths with $0 < |\alpha| < |\beta|$ and $s(\alpha) = s(\beta)$. Write $\beta = \delta\gamma$ where $|\delta| = |\alpha|$ and $|\gamma| = |\beta| - |\alpha|$. Then $S_\alpha S_\beta^* \in \operatorname{Alg}\mathcal{N}$ if, and only if, one of the following two conditions holds*:

(1) $\alpha \prec \delta$.
(2) $\alpha = \delta$ *and γ is s-maximal.*

The following theorem summarizes the sequence of propositions above:

Theorem 9.12. *Let G be a finite, ordered graph and let $\mathcal{N}$ be the associated nest. A Cuntz–Krieger partial isometry $S_\alpha S_\beta^*$ lies in $\operatorname{Alg}\mathcal{N}$ if, and only if, one of the following conditions holds*:

(1) $|\alpha| = |\beta|$ *and* $\alpha \preceq \beta$.
(2) $\alpha = \delta\gamma$ *with* $|\delta| = |\beta|$ *and* $\delta \prec \beta$.
(3) $\alpha = \beta\gamma$ *and γ is s-minimal.*
(4) $\beta = \delta\gamma$ *with* $|\delta| = |\alpha|$ *and* $\alpha \prec \delta$.
(5) $\beta = \alpha\gamma$ *and γ is s-maximal.*

We can now characterize the points $(x, k, y) \in \mathcal{G}$ that are in the spectrum of $\operatorname{Alg}\mathcal{N}$. Note that path space P is totally ordered by the lexicographic order based on the total order on E; once again we let $\preceq$ denote this order. (We will, in fact, need to compare x and y only when x and y are shift equivalent.)

Theorem 9.13. *Let G be a finite ordered graph and let $\mathcal{N}$ be the associated nest. A point $(x, k, y) \in \mathcal{G}$ is in $\sigma(\operatorname{Alg}\mathcal{N})$ if, and only if, one of the following conditions holds*:

(1) $x \prec y$.
(2) $x = y$ *and* $k = 0$.
(3) $x = y$, $k > 0$ *and* $x = \beta\gamma\gamma\gamma\dots$ *where γ is s-minimal and* $|\gamma| = k$.
(4) $x = y$, $k < 0$ *and* $x = \alpha\gamma\gamma\gamma\dots$ *where γ is s-maximal and* $|\gamma| = -k$.

Proof. The proof is, of course, based on the fact that $\sigma(\operatorname{Alg}\mathcal{N})$ is the union of the sets $Z(\alpha, \beta)$ with α and β satisfying any of the five conditions in the Theorem 9.12.

First suppose that $(x, k, y) \in \mathcal{G}$ and $x \prec y$ in the lexicographic order. We can then find α and β satisfying one of conditions (1), (2), or (4) (depending on the value of k) in Theorem 9.12, so that $(x, k, y) \in Z(\alpha, \beta) \subseteq \sigma(\operatorname{Alg}\mathcal{N})$. Thus

$\{(x, k, y) \in \mathcal{G} \mid x \prec y\} \subseteq \sigma(\operatorname{Alg}\mathcal{N})$. Equally well, if α and β satisfy (1) (with $\alpha \neq \beta$), (2), or (4), and $(x, k, y) \in Z(\alpha, \beta)$, then $x \prec y$.

By condition (1) in Theorem 9.12, any set of the form $Z(\alpha, \alpha)$ is contained in $\sigma(\operatorname{Alg}\mathcal{N})$; thus $(x, 0, x) \in \sigma(\operatorname{Alg}\mathcal{N})$, for all x.

Now suppose that $(x, k, y) \in Z(\alpha, \beta)$ when $\alpha = \beta\gamma$ and γ is s-minimal. (So $k > 0$.) Then

$$x = \beta_1 \dots \beta_n \gamma_1 \dots \gamma_k z_1 z_2 \dots \text{ and}$$
$$y = \beta_1 \dots \beta_n z_1 z_2 \dots .$$

Since γ is s-minimal and $z_1 \dots z_k$ is a finite path whose range is the source of γ, we have $\gamma \preceq z_1 \dots z_k$. If $\gamma \prec z_1 \dots z_k$, then $x \prec y$. So, suppose that $\gamma = z_1 \dots z_k$. Now $z_{k+1} \dots z_{2k}$ is a finite path whose range is the source of γ and so $\gamma \preceq z_{k+1} \dots z_{2k}$. Again, if $\prec$ holds, then $x \prec y$; otherwise $z_{k+1} \dots z_{2k} = \gamma$. It is now clear that an induction argument shows that either $x \prec y$ or $x = y$ has the form $\beta\gamma\gamma\gamma \dots$, where γ is s-minimal. The points (x, k, y) with $x \prec y$ have already been covered by the previous discussion, so the new points in $\sigma(\operatorname{Alg}\mathcal{N})$ are the ones of the form (x, k, x) where $x = \beta\gamma\gamma\gamma \dots$, $k = |\gamma|$, and γ is s-minimal. Any point of $\mathcal{G}$ of this form is in a suitable $Z(\alpha, \beta)$ and so is in $\sigma(\operatorname{Alg}\mathcal{N})$.

We can analyze $Z(\alpha, \beta)$ when $\beta = \alpha\gamma$ and γ is s-maximal in a similar way. If $(x, k, y) \in Z(\alpha, \beta)$ and $x \neq y$ then $x \prec y$. If $x = y$ then $-k = |\beta| - |\alpha| = |\gamma|$ and $x = \alpha\gamma\gamma\gamma \dots$ with γ s-maximal. Any point with this form is in $\sigma(\operatorname{Alg}\mathcal{N})$.

All that remains is to note that if $y \prec x$ then $(x, k, y) \notin \sigma(\operatorname{Alg}\mathcal{N})$. $\square$

We next determine the spectrum of the (Jacobson) radical of a nest subalgebra of a graph C*-algebra. Since the radical is invariant under automorphisms, we know that it is determined by its spectrum. Analogy with the case of upper triangular matrices and with refinement subalgebras of AF C*-algebras suggests that the spectrum of the radical consists of those points (x, k, y) in $\sigma(\operatorname{Alg}\mathcal{N})$ with $x \prec y$ (condition (1) in Theorem 9.13). Indeed:

Proposition 9.14. *The set $R = \{(x, k, y) \in \sigma(\operatorname{Alg}\mathcal{N}) \mid x \prec y\}$ is the spectrum of the radical of $\operatorname{Alg}\mathcal{N}$. Consequently, $A(R)$ is the radical of $\operatorname{Alg}\mathcal{N}$.*

Proof. Theorem 6.3 implies that the second statement follows from the first. Temporarily, let R_0 denote the spectrum of the radical of $\operatorname{Alg}\mathcal{N}$. We need to prove that $R_0 = R$.

We first show that $R \subseteq R_0$. Let $(x, k, y) \in R$, so that $x \prec y$. Choose finite strings α and β such that $s(\alpha) = s(\beta)$, $\alpha \prec \beta$ and $(x, k, y) \in \sigma(S_\alpha S_\beta^*)$. ($\alpha$ and β need not have the same length; by $\alpha \prec \beta$ we simply mean that there is an index j such that $\alpha_i = \beta_i$ for $i < j$ and $\alpha_j \prec \beta_j$.) Now, the range projection for $S_\alpha S_\beta^*$ is contained in the atom $R_{\alpha_1 \dots \alpha_j}$ and the initial projection is contained in $R_{\beta_1 \dots \beta_j}$. Since $R_{\alpha_1 \dots \alpha_j} \ll R_{\beta_1 \dots \beta_j}$, it follows that if P is the smallest projection in $\mathcal{N}$ which contains $R_{\alpha_1 \dots \alpha_j}$, then $R_{\beta_1 \dots \beta_j}$ is orthogonal to P. Thus, $S_\alpha S_\beta^* = P S_\alpha S_\beta^* P^\perp$ and $S_\alpha S_\beta^*$ lies in the radical of $\operatorname{Alg}\mathcal{N}$. Therefore $(x, k, y) \in R_0$ and $R \subseteq R_0$.

To complete the proof, we need to show that any point of $\sigma(\operatorname{Alg}\mathcal{N})$ which satisfies conditions (2), (3) or (4) is not in R_0. For points of the form $(x, 0, x)$ this is trivial — they are in the suppport set of a nonzero projection and the radical contains no nonzero projections. The arguments for points which satisfy conditions (3) and (4) are very similar, so we just treat the first of these two cases.

Assume $k > 0$, $x = \delta\gamma\gamma\gamma\dots$, $k = |\gamma|$, and γ is s-minimal. Suppose that $(x,k,x) \in R_0$. Since R_0 is open, there is a positive integer n so that if

$$\begin{aligned} \alpha &= \delta\gamma\dots\gamma \qquad (n+1 \text{ copies of } \gamma) \\ \beta &= \delta\gamma\dots\gamma \qquad\quad (n \text{ copies of } \gamma) \end{aligned}$$

then $Z(\alpha,\beta) \subseteq R_0$. Since $r(\gamma) = s(\delta)$ and $r(\gamma) = s(\gamma)$, it follows from the Cuntz–Krieger relations that $S_\gamma S_\gamma^* \le S_\delta^* S_\delta$ and $S_\gamma S_\gamma^* \le S_\gamma^* S_\gamma$. Using this, and the fact that α has one more copy of γ than β has, we obtain

$$S_\beta^* S_\alpha = S_\gamma^* \dots S_\gamma^* S_\delta^* S_\delta S_\gamma \dots S_\gamma = S_\gamma.$$

and, hence, $(S_\alpha S_\beta^*)^2 = S_\alpha S_\gamma S_\beta^*$. Since $s(\beta) = s(\gamma)$ and $r(\gamma) = s(\alpha)$, $(S_\alpha S_\beta^*)^2 \neq 0$. Note that $\alpha\gamma$ has the same form as α except that there are now $n+2$ γ's feeding into δ.

The same considerations as above show that if $\alpha^{(p)} = \delta\gamma\dots\gamma$ with $n+p$ copies of γ, then $S_\beta^* S_{\alpha^{(p)}} = S_\gamma \dots S_\gamma$ (p copies of γ) and the square $(S_{\alpha^{(p)}} S_\beta^*)^2$ is a nonzero partial isometry and so has norm 1. At this point it is now a simple matter to show that $\|(S_\alpha S_\beta^*)^k\| = 1$ for all k and therefore that $S_\alpha S_\beta^*$ is not quasi-nilpotent. But then $S_\alpha S_\beta^*$ is not in the radical of $\operatorname{Alg}\mathcal{N}$, contradicting $Z(\alpha,\beta) \subseteq R_0$. Thus any point in the spectrum of $\operatorname{Alg}\mathcal{N}$ which satisfies condition (3) of Theorem 9.13 is not in R_0. As mentioned earlier, points satisfying condition (4) are handled similarly. $\square$

It was shown in [12] that the radical of the Volterra subalgebra of the Cuntz algebra is the closed commutator ideal of the Volterra subalgebra. This result extends to graph C*-algebras. In the proposition below, we let $\mathcal{C}$ denote the closed ideal generated by the commutators of $\operatorname{Alg}\mathcal{N}$. The proof differs from the one in [12], which does not use groupoid techniques.

Proposition 9.15. *The radical of* $\operatorname{Alg}\mathcal{N}$ *is equal to the closed commutator ideal* $\mathcal{C}$.

Proof. As usual, we view all elements of $\mathcal{A}$ as functions on $\mathcal{G}$. The multiplication in $\mathcal{A}$ is then given by a convolution formula. By Proposition 9.14,

$$R = \{(x,k,y) \in \sigma(\operatorname{Alg}\mathcal{N}) \mid x \prec y\}$$

is the spectrum of the radical and $A(R)$ is the radical of $\operatorname{Alg}\mathcal{N}$.

Let $f, g \in \operatorname{Alg}\mathcal{N}$. If we show that $[f,g](x,k,x) = 0$ whenever $(x,k,x) \in \sigma(\operatorname{Alg}\mathcal{N})$ then $[f,g] \in A(R)$ and $\mathcal{C} \subseteq A(R)$. Now

$$f \cdot g(x,k,x) = \sum f(x,i,u) g(u,k-i,x)$$

where the sum is taken over all $i \in \mathbb{Z}$ and $u \in P$ for which (x,i,u) and $(u,k-i,x)$ lie in $\sigma(\operatorname{Alg}\mathcal{N})$. This requires both $x \preceq u$ and $u \preceq x$, so the only possibility for u is $u = x$. If we make the change of variable $j = k - i$, then

$$\begin{aligned} f \cdot g(x,k,x) &= \sum_i f(x,i,x) g(x,k-i,x) \\ &= \sum_j f(x,k-j,x) g(x,j,x) \\ &= g \cdot f(x,k,x). \end{aligned}$$

Thus, $[f,g] = fg - gf$ vanishes at all points in $\sigma(\mathrm{Alg}\,\mathcal{N})$ of the form (x,k,x) and so is supported on R. This shows that $[f,g] \in A(R)$ and it follows immediately that $\mathcal{C} \subseteq A(R)$.

To show that $A(R) \subseteq \mathcal{C}$, it suffices, by the spectral theorem for bimodules, to show that each Cuntz–Krieger partial isometry from $A(R)$ is in $\mathcal{C}$. If $S_\alpha S_\beta^* \in A(R)$, then there is j such that $\alpha_i = \beta_i$ for $i < j$ and $\alpha_j \prec \beta_j$. The range projection for $S_\alpha S_\beta^*$ is a subprojection of $R_{\alpha_1\ldots\alpha_j}$ and the initial projection is a subprojection of $R_{\beta_1\ldots\beta_j}$. Let P be the smallest projection in $\mathcal{N}$ which contains $R_{\alpha_1\ldots\alpha_j}$. Since $R_{\alpha_1\ldots\alpha_j} \ll R_{\beta_1\ldots\beta_j}$, $R_{\beta_1\ldots\beta_j} \perp P$ and $S_\alpha S_\beta^* = PS_\alpha S_\beta^* P^\perp$. This implies that

$$S_\alpha S_\beta^* = PS_\alpha S_\beta^* - S_\alpha S_\beta^* P = [P, S_\alpha S_\beta^*] \in \mathcal{C}$$

and the proposition is proven. □

If we let $D = \{(x,0,x) \mid x \in P\}$, then $\mathcal{D} = A(D) = \mathrm{Alg}\,\mathcal{N} \cap (\mathrm{Alg}\,\mathcal{N})^*$. Since $D \cup R$ is a proper subset of $\sigma(\mathrm{Alg}\,\mathcal{N})$, it follows that the norm closure of $A(D)+A(R)$ is a proper subset of $\mathrm{Alg}\,\mathcal{N}$. Thus, $\mathrm{Alg}\,\mathcal{N}$ does not have a radical plus diagonal decomposition. Furthermore, since $\sigma(\mathrm{Alg}\,\mathcal{N}) \cup \sigma(\mathrm{Alg}\,\mathcal{N})^{-1}$ is a proper subset of $\mathcal{G}$, $\mathrm{Alg}\,\mathcal{N} + (\mathrm{Alg}\,\mathcal{N})^*$ is not norm dense in $\mathcal{A}$. This says that $\mathrm{Alg}\,\mathcal{N}$ is "non-Dirichlet." When every loop has an entrance, $\mathcal{D}$ is a masa in $\mathcal{A}$ and $\mathrm{Alg}\,\mathcal{N}$ is triangular, but not strongly maximal triangular. However, we do have the following proposition:

Proposition 9.16. *Assume that G is a finite graph in which every loop has an entrance.* $\mathrm{Alg}\,\mathcal{N}$ *is maximal triangular in $\mathcal{A}$.*

Proof. Since $\mathrm{Alg}\,\mathcal{N} \cap (\mathrm{Alg}\,\mathcal{N})^* = \mathcal{D}$ is a masa, $\mathrm{Alg}\,\mathcal{N}$ is a triangular subalgebra of $\mathcal{A}$. Assume that $\mathrm{Alg}\,\mathcal{N} \subseteq \mathcal{T} \subset \mathcal{A}$ and that $\mathcal{T}$ is triangular. It follows that $\mathcal{T} \cap \mathcal{T}^* = \mathcal{D}$. Let $T \in \mathcal{T}$. Let P be a projection in $\mathcal{N}$. Since $PTP^\perp$ leaves invariant each projection in $\mathcal{N}$, $PTP^\perp \in \mathrm{Alg}\,\mathcal{N}$. Since $\mathcal{N} \subseteq \mathcal{T}$, $P^\perp TP \in \mathcal{T}$. It follows that $S = PTP^\perp + P^\perp TP$ is a selfadjoint element of $\mathcal{T}$ and hence lies in $\mathcal{D}$. Since $\mathcal{D}$ is abelian and $P \in \mathcal{D}$, P commutes with S. Thus $0 = P^\perp SP$. But $P^\perp SP = P^\perp TP$, so $P^\perp TP = 0$. Thus T leaves invariant each projection in $\mathcal{N}$ and so must be an element of $\mathrm{Alg}\,\mathcal{N}$. This shows that $\mathcal{T} \subseteq \mathrm{Alg}\,\mathcal{N}$ and $\mathrm{Alg}\,\mathcal{N}$ is maximal triangular. □

10. Normalizing partial isometries

In this section we characterize the partial isometries in a graph C*-algebra $\mathcal{A} = C^*(G)$ which normalize the canonical diagonal algebra $\mathcal{D}$. We assume throughout that G is a countable range finite directed graph with no sources such that each loop has an entrance. In particular, by Theorem 5.2, this ensures that $\mathcal{D}$ is a masa. The characterization of the $\mathcal{D}$-normalizing partial isometries will be applied in Section 11 to show that gauge invariant triangular subalgebras are classified by their spectra.

Recall that a partial isometry v is $\mathcal{D}$-normalizing if $v^*\mathcal{D}v \subseteq \mathcal{D}$ and $v\mathcal{D}v^* \subseteq \mathcal{D}$. We write $N_{\mathcal{D}}(\mathcal{B})$ for the set of all $\mathcal{D}$-normalizing partial isometries in a subset $\mathcal{B}$. Also we say that $v_1 + \cdots + v_n$ is an orthogonal sum of partial isometries if the set of initial projections $v_i^*v_i$, and also the set of final projections $v_iv_i^*$, consists of pairwise orthogonal projections.

In Theorem 10.1 we show that $\mathcal{D}$-normalizing partial isometries are, modulo coefficients from $\mathcal{D}$, orthogonal sums of Cuntz–Krieger partial isometries; moreover they are characterized by a property which is preserved by isometric isomorphism. The equivalence of (1) and (2) in the case of Cuntz algebras was obtained in [14, Lemma 5.4], where it formed the basis for the calculation of normalizing partial isometry homology groups of various triangular subalgebras.

Theorem 10.1. *Let v be a partial isometry in $\mathcal{A}$. Then the following assertions are equivalent*:

(1) *v is a $\mathcal{D}$-normalizing partial isometry.*
(2) *v is an orthogonal sum of a finite number of partial isometries of the form $dS_\alpha S_\beta^*$, where $d \in \mathcal{D}$.*
(3) *For all projections $p, q \in \mathcal{D}$, the norm $\|qvp\|$ is equal to 0 or 1.*

This theorem is in complete analogy with the following counterpart for AF C*-algebras. (See [13] or [15, Lemma 5.5 and Proposition 7.1].) The Cuntz–Krieger partial isometries play the same role for graph C*-algebras as systems of matrix units do for AF C*-algebras.

Theorem 10.2. *Let $\mathcal{B}$ be an AF C*-algebra with finite-dimensional subalgebra chain $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \dots$ and masas $\mathcal{C}_k \subseteq \mathcal{B}_k$ such that $N_{\mathcal{C}_k}(\mathcal{B}_k) \subseteq N_{\mathcal{C}_{k+1}}(\mathcal{B}_{k+1})$, for all k. Suppose also that the union of the $\mathcal{B}_k$ is dense in $\mathcal{B}$. Then the closed union $\mathcal{C}$ of the masas $\mathcal{C}_k$ is a masa in $\mathcal{B}$ and the following assertions are equivalent for a partial isometry v in $\mathcal{B}$:*

(1) *v is a $\mathcal{C}$-normalizing partial isometry.*
(2) *$v = cu$ where $c \in \mathcal{C}$ and $u \in N_{\mathcal{C}_k}(\mathcal{B}_k)$, for some k.*
(3) *For all projections $p, q \in \mathcal{C}$, the norm $\|qvp\|$ is equal to 0 or 1.*

Theorem 10.2 will be used in the proof of Theorem 10.1 to show that if v is in $N_{\mathcal{D}}(\mathcal{A})$ then so too is its AF part $v_0 = \Phi_0(v)$. We also require the following two lemmas:

Lemma 10.3. *Let α, β be paths of the same length and let $e = S_\alpha S_\beta^*$ be a nonzero partial isometry in the AF subalgebra $\mathcal{F}$ of $C^*(G)$. For each positive integer k there exist nonzero projections q, p with $q = epe^*$, such that for all paths γ with length at most k, and for all $S_\lambda S_\mu^*$ in $\mathcal{F}$ with $|\lambda| = |\mu| \leq k$, we have*

$$q(S_\gamma S_\lambda S_\mu^*)p = q(S_\lambda S_\mu^* S_\gamma)p = 0.$$

Proof. Note that if we verify the lemma for an integer k, then we have verified it for all integers less than k; thus we may increase a value for k if needed. This, together with the hypothesis that every loop has an entrance, allows us to choose a path $\pi = f_{2k} \dots f_1$ of length $2k$ and a path $w = w_1 w_2 \dots$ of length at least k such that:

(1) $r(\pi) = r(f_{2k}) = s(\alpha) = s(\beta)$.
(2) $r(w) = s(\pi) = s(f_1)$.
(3) For every integer d with $1 \leq d \leq k$, $f_d \dots f_1 \neq w_1 \dots w_d$.

Indeed, the assumption that there are no sources allows us to choose the path π with range vertex equal to $s(\alpha) = s(\beta)$. Possibly, we can choose π so that $r(f_d) \neq s(f_1)$ for $d = 1, \dots, k$. In this case, any extension $w = w_1 \dots w_k$ works, since $r(w_1 \dots w_d) = r(w_1) = s(f_1) \neq r(f_d) = r(f_d \dots f_1)$. On the other hand, if we must back into a loop then (increasing k if necessary), we can arrange that $f_k \dots f_1$

consists of multiple repeats of a single simple loop. By choosing w so that w_1 is an entrance to the loop, we guarantee that $f_d \dots f_1 \neq w_1 \dots w_d$ for all $d \leq k$.

Now let

$$p = S_{\beta\pi w} S^*_{\beta\pi w} \text{ and } q = S_{\alpha\pi w} S^*_{\alpha\pi w} = epe^*.$$

Let $\gamma = \gamma_1 \dots \gamma_d$ be a path with $1 \leq d = |\gamma| \leq k$. We first show that $qS_\gamma p = 0$. If not, then

$$S_{\alpha\pi w} S^*_{\alpha\pi w} S_\gamma S_{\beta\pi w} S^*_{\beta\pi w} \neq 0.$$

Now, recall that for any edges e and f, $S^*_e S_f = 0$ except when $e = f$ (the ranges of the generating partial isometries S_e are pairwise orthogonal) and that, if g is an edge with $r(g) = s(e)$ then $S^*_e S_e S_g = S_g$ (from the Cuntz–Krieger relations). Consequently, the edges in the finite path $\alpha\pi w$ match the edges (reading from left to right) in the path $\gamma\beta\pi w$. Since the length of π is at least twice the length of γ, the cancellations into $\gamma\beta\pi w$ bring us d edges into w; this forces $f_d \dots f_1 = w_1 \dots w_d$. But this contradicts the choice of π and w.

Insertion of $S_\lambda S^*_\mu$ either before or after S_γ does not change the result: there are at most k cancellations from S^*_μ, which cannot affect the second half of S_π since $|\pi| = 2k$, and the cancelled partial isometries are replaced by partial isometries from S_λ. Thus the general result holds. □

In Lemma 10.4, $B^*(G)$ is the (nonclosed) algebra generated by the Cuntz–Kreiger partial isometries and the maps Φ_m are as defined in Section 2.

Lemma 10.4. *Let $a \in B^*(G)$ and let $e = S_\alpha S^*_\beta$ be a partial isometry in the AF subalgebra $\mathcal{F}$ of $C^*(G)$. Then there exist projections p and $q = epe^*$ such that $qap = q\Phi_0(a)p$.*

Proof. By the observations in Section 2, $a - \Phi_0(a)$ can be written as a finite linear combination of terms of one of the two forms: $S_\gamma S_\lambda S^*_\mu$ and $S_\lambda S^*_\mu S^*_\gamma$, where $|\gamma| \geq 1$ and $|\lambda| = |\mu|$. An application of Lemma 10.3 gives projections p_1 and q_1 such that $q_1 = ep_1e^*$ and $q_1 S_\gamma S_\lambda S^*_\mu p_1 = 0$ for all terms of this type in the linear combination. Now let $f = (ep_1)^*$ and apply Lemma 10.3 again to obtain projections $q \leq q_1$ and $p \leq p_1$ with $fqf^* = p$ and $pS_\gamma S_\mu S^*_\lambda q = 0$ for all terms of the second type in the linear combination for $a - \Phi_0(a)$. It now follows that $qS_\lambda S^*_\mu S^*_\gamma p = 0$ and $qS_\gamma S_\lambda S^*_\mu p = 0$ for all the terms; hence $qap = q\Phi_0(a)p$. □

Proof of Theorem 10.1. The implications (2) $\Longrightarrow$ (1) and (1) $\Longrightarrow$ (3) are routine.

It remains prove that (3) $\Longrightarrow$ (2): Let v be a partial isometry in $\mathcal{A}$ which satisfies condition (3). We claim first that $\Phi_0(v)$ is a $\mathcal{D}$-normalizing partial isometry. If not, then, since $\Phi_0(v)$ is in the AF subalgebra $\mathcal{F}$, we can use Theorem 10.2 to find a partial isometry $e = S_\alpha S^*_\beta$ in $\mathcal{F}$ and a $\delta > 0$ such that, for any pair $p \leq e^*e$, $q = epe^*$ we have

$$\delta \leq \|q\Phi_0(v)p\| \leq 1 - \delta.$$

(This is easy to do using the function representation of v_0 on the AF subgroupoid and knowledge of the form of v_0 — that it is not an element of $\mathcal{D}$ times a matrix unit.)

Now let $v' \in B^*(G)$ be such that $\|v'-v\| < \delta/2$. By Lemma 10.4, there exist nonzero projections p and q with $q = epe^*$ such that $q\Phi_0(v')p = qv'p$. Since $\|qvp\|$ is either 0 or 1, either $\|qv'p\| \le \delta/2$ or $1-\delta/2 \le \|qv'p\|$. Since $\|\Phi_0(v')-\Phi_0(v)\| < \delta/2$ (Φ_0 is contractive), this implies that either $\|q\Phi_0(v)p\| < \delta$ or $1-\delta < \|q\Phi_0(v)p\|$, a contradiction. Thus, the 0-order term in the 'Fourier' series for v is $\mathcal{D}$-normalizing. It follows from Theorem 10.2 that $\Phi_0(v)$ has the form required in condition (2).

Now suppose that $m > 0$. If $|\nu| = m$ and $|\lambda| - |\mu| = m$, then the product $S_\nu^* S_\lambda S_\mu^*$ is either zero or of the form $S_{\lambda_1} S_\mu^*$ with $|\lambda_1| = |\mu|$. It follows that $S_\nu^*\Phi_m(v) = \Phi_0(S_\nu^* v)$. Since v satisfies condition (3), so does $S_\nu^* v$; the argument above shows that $S_\nu^*\Phi_m(v)$ is $\mathcal{D}$-normalizing and has the required form (condition (2)). This, in turn, implies that $S_\nu S_\nu^* \Phi_m(v)$ is $\mathcal{D}$-normalizing and has the required form for every path ν with length m. Consequently, $\Phi_m(v)$ satisfies condition (2). In a similar fashion, we can show that when $m < 0$, $\Phi_m(v)$ satisfies condition (2) (consider adjoints, for example).

Now, if w is a partial isometry and $ww^*xw^*w \neq 0$ then $\|w + ww^*xw^*w\| > 1$. From this observation and the Cesaro convergence, it follows that the operators $\Phi_m(v)$ are nonzero for only finitely many values of m and that v is the orthogonal sum of these operators. Thus v has the form required in condition (2). □

11. Triangular subalgebras determine their spectrum

In this section we show that the gauge invariant triangular subalgebras of certain graph C*-algebras are classified by the isomorphism type of their spectra. We assume throughout the section that G_1 and G_2 are countable range finite directed graphs with no sources and that each loop has an entrance.

Theorem 11.1. *For $i = 1, 2$, let $\mathcal{T}_i$ be a triangular subalgebra of $\mathcal{A}_i$ with diagonal $\mathcal{D}_i$ such that $\mathcal{T}_i$ generates $\mathcal{A}_i$ as a C^*-algebra and $\mathcal{T}_i$ is invariant under the gauge automorphisms (so that $\mathcal{T}_i = A(\mathcal{P}_i)$, where $\mathcal{P}_i = \sigma(\mathcal{T}_i)$). Then the following statements are equivalent:*

(1) *$\mathcal{T}_1$ and $\mathcal{T}_2$ are isometrically isomorphic operator algebras.*
(2) *There is a groupoid isomorphism $\gamma\colon \mathcal{G}_1 \to \mathcal{G}_2$ with $\gamma(\mathcal{P}_1) = \mathcal{P}_2$.*

Proof. If γ has the properties of (2) it is plain that there is a C*-algebra isomorphism $C^*(\mathcal{G}_1) \to C^*(\mathcal{G}_2)$ which restricts to an isometric isomorphism $\mathcal{T}_1 \to \mathcal{T}_2$. Assume then that $\Gamma\colon \mathcal{T}_1 \to \mathcal{T}_2$ is an isometric isomorphism. In view of the norm characterization of normalizing partial isometries, Theorem 10.1 condition (3), we have $\Gamma(N_{\mathcal{D}_1}(\mathcal{T}_1)) = N_{\mathcal{D}_2}(\mathcal{T}_2)$. Moreover, since $\Gamma(\mathcal{D}_1) = \mathcal{D}_2$, the groupoid support of $\Gamma(du)$ with $d \in \mathcal{D}_1$ and u in $N_{\mathcal{D}_1}(\mathcal{T}_1)$ is independent of d if d is a partial isometry and $d^*du = u$. Reciprocally, in view of Theorem 10.1 condition (2), normalizing partial isometries are determined by their groupoid support, up to a diagonal multiplier. Thus Γ induces a map

$$\widetilde{\gamma} : \{\mathrm{supp}(u) : u \in N_{\mathcal{D}_1}(\mathcal{T}_1)\} \to \{\mathrm{supp}(v) : v \in N_{\mathcal{D}_2}(\mathcal{T}_2)\}.$$

Each point g in $\mathcal{P}_1$ is an intersection of the sets $Z(\alpha, \beta)$ that contain it and so by local compactness $\widetilde{\gamma}$ defines a bijection $\gamma\colon \mathcal{G}_1 \to \mathcal{G}_2$ and this map, in turn, induces $\widetilde{\gamma}$. In particular, γ is a homeomorphism.

Note now that γ is a semigroupoid map. To see this observe first that if u_1 and u_2 are normalizing partial isometries in $\mathcal{T}_1$ with support sets U_1 and U_2 then u_1u_2

has support set $U_1 \cdot U_2$. Thus, if g_1 and g_2 are composable elements in $\mathcal{P}_1$ and

$$\{g_1\} = \cap_{n=1}^{\infty} U_n, \quad \{g_2\} = \cap_{n=1}^{\infty} V_n,$$

where U_n, V_n are supports of normalizing partial isometries, then

$$\{g_1 \cdot g_2\} = \cap_{n=1}^{\infty} U_n \cdot V_n.$$

Thus

$$\{\gamma(g_1 \cdot g_2)\} = \cap_{n=1}^{\infty} \widetilde{\gamma}(U_n \cdot V_n) = \cap_{n=1}^{\infty} \widetilde{\gamma}(U_n) \cdot \widetilde{\gamma}(V_n)$$

and this last set is the singleton set $\{\gamma(g_1) \cdot \gamma(g_2)\}$.

We now extend γ to a map from $\mathcal{G}_1$ to $\mathcal{G}_2$. Note first that since, by hypothesis, $\mathcal{T}_1$ generates $\mathcal{A}_1$ as a C*-algebra, the sets

$$U = U_1 \cdot U_2^{-1} \cdot U_3 \cdot U_4^{-1} \cdot \dots \cdot U_{2n}^{-1},$$

where $U = \mathrm{supp}(u_i)$ and $u_i \in N_{\mathcal{D}_1}(\mathcal{T}_1)$, have union equal to $\mathcal{G}_1$. Indeed, approximate u in $N_{\mathcal{D}_1}(\mathcal{A}_1)$ by a polynomial p in the generators and their adjoints,

$$p = \sum_m \sum_{|\lambda|-|\mu|=m} a_{\lambda\mu} S_\lambda S_\mu^*$$

and it follows that a point g in $\mathrm{supp}(u)$ must lie in the support of some $S_\lambda S_\mu^*$ in the sum. Extend γ to $\mathcal{G}_1$ by setting

$$\gamma(g_1 \cdot g_2^{-1} \cdot \dots \cdot g_{2n}^{-1}) = \gamma(g_1) \cdot \gamma(g_2^{-1}) \cdot \dots \cdot \gamma(g_{2n}^{-1}).$$

This is well-defined and onto, since $\mathcal{T}_2$ generates $\mathcal{A}_2$, and so, as before, this extension is a groupoid isomorphism. □

It is clear that the proof method above simplifies to give the following equivalence between the isomorphism type of the pair $(C^*(G), \mathcal{D})$ and the isomorphism type of the groupoid $\mathcal{G}$. (Compare, for example, [15, Theorem 7.5].)

Theorem 11.2. *The following statements are equivalent*:

(1) *There is a C^*-algebra isomorphism $\Gamma : C^*(G_1) \to C^*(G_2)$ with $\Gamma(\mathcal{D}_1) = \mathcal{D}_2$, where each $\mathcal{D}_i$ is the canonical abelian diagonal subalgebra of $C^*(G_i)$.*
(2) *There is a groupoid isomorphism $\gamma \colon \mathcal{G}_1 \to \mathcal{G}_2$.*

Similarly the proof above extends with only trivial changes to give an equivalence between the isomorphism type of the pair $(\mathcal{B}, \mathcal{D})$, consisting of a gauge invariant subalgebra $\mathcal{B}$ containing the diagonal $\mathcal{D}$, and the isomorphism type of the spectrum of $\mathcal{B}$.

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