New York Journal of Mathematics

New York J. Math. **15** (2009) 169–198.

Representations of higher rank graph algebras

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Abstract. Let $\mathbb{F}_\theta^+$ be a k-graph on a single vertex. We show that every irreducible atomic $*$-representation is the minimal $*$-dilation of a group construction representation. It follows that every atomic representation decomposes as a direct sum or integral of such representations. We characterize periodicity of $\mathbb{F}_\theta^+$ and identify a symmetry subgroup H_θ of $\mathbb{Z}^{\mathsf{k}}$. If this has rank s, then $C^*(\mathbb{F}_\theta^+) \cong C(\mathbb{T}^s) \otimes \mathfrak{A}$ for some simple C*-algebra $\mathfrak{A}$.

Contents

1. Introduction

There has been a lot of recent interest in the structure of operator algebras associated to graphs (see [16]). Kumjian and Pask [11] have introduced the

Received March 4, 2009.

Mathematics Subject Classification. 47L55, 47L30, 47L75, 46L05.

Key words and phrases. higher rank graph, aperiodicity condition, atomic representations, dilation.

First author partially supported by an NSERC grant. Second author partially supported by the Fields Institute.

ISSN 1076-9803/09

notion of higher rank graphs, which have a much more involved combinatorial structure. The C*-algebras of higher rank graphs are widely studied [6, 7, 12, 13, 17, 18, 19, 20]. Kribs and Power [10] initiated the study of the associated nonself-adjoint algebras. Power [15] began a detailed study of these operator algebras associated to higher rank graphs with a single vertex. This effort was continued with the authors of this paper [2, 3, 4] with a detailed analysis of rank 2 graphs on a single vertex. Two important accomplishments there were a complete structure theory for the atomic $*$-representations of the 2-graph, and a characterization of periodicity leading to the structure of the 2-graph C*-algebra in the periodic case.

The purpose of this paper is to extend those results to the case of k-graphs on a single vertex. These objects form an interesting class of semigroups with cancellation and unique factorization. The combinatorial structure of a k-graph $\mathbb{F}_\theta^+$ is much more difficult to classify for $\mathsf{k} \geq 3$, but we do show that there are lots of examples.

The goal is to describe the structure of the associated C*-algebra $\mathrm{C}^*(\mathbb{F}_\theta^+)$. It was shown by Kumjian and Pask [11] that this C*-algebra is simple when the k-graph satisfies an aperiodicity condition, and the converse was established by Robertson and Sims [19]. When this condition fails and $\mathsf{k} = 2$, we established in [4] the more detailed structure that $\mathrm{C}^*(\mathbb{F}_\theta^+) \simeq \mathrm{C}(\mathbb{T}) \otimes \mathfrak{A}$ where $\mathfrak{A}$ is a simple C*-algebra. When $\mathsf{k} \geq 3$, there is a symmetry group isomorphic to $\mathbb{Z}^s$ for some integer $s \leq \mathsf{k}$. We show that $\mathrm{C}^*(\mathbb{F}_\theta^+) \simeq \mathrm{C}(\mathbb{T}^s) \otimes \mathfrak{A}$, where $\mathfrak{A}$ is again a simple C*-algebra.

The first main result concerns atomic $*$-representations. The $*$-representations of $\mathbb{F}_\theta^+$ are the row isometric representations which are defect free (see the next section for definitions). These are the representations of the semigroup which yield $*$-representations of the associated C*-algebra $\mathrm{C}^*(\mathbb{F}_\theta^+)$. An important class of such representations (atomic representations) have the additional property that there is an orthonormal basis which is permuted, up to scalars of modulus 1, by the isometries which are the images of elements of $\mathbb{F}_\theta^+$.

The analysis of these representations relies on dilation theory. Every defect free, row contractive representation dilates to a unique minimal $*$-dilation [3, 20]. So one can understand a $*$-representation by understanding its restriction to a coinvariant cyclic subspace. The key to our analysis is to show that there is a natural family of atomic defect free representations modelled on the representations of an abelian group of rank k. In the rank 2 case [2], a detailed case by case analysis led to the conclusion that every irreducible atomic $*$-representation is the dilation of one of these group constructions. In this paper, we provide a direct argument that avoids the case by case approach. So it sheds new light even when $\mathsf{k} = 2$.

Every k-graph $\mathbb{F}_\theta^+$ has a faithful $*$-representation of inductive type, i.e., an inductive limit of copies of the left regular representation of $\mathbb{F}_\theta^+$. This representation has a natural symmetry group $H_\theta \leq \mathbb{Z}^{\mathsf{k}}$. The graph is aperiodic if and only if $H_\theta = \{0\}$. In general, this is a free group of rank $s \le \mathsf{k}$. Building on the detailed structure of periodic 2-graphs in [4], we show that the centre of $\mathrm{C}^*(\mathbb{F}_\theta^+)$ is isomorphic to $\mathrm{C}(\mathbb{T}^s)$. This leads to our decomposition of $\mathrm{C}^*(\mathbb{F}_\theta^+)$ as a tensor product $\mathrm{C}^*(\mathbb{F}_\theta^+) \cong \mathrm{C}(\mathbb{T}^s) \otimes \mathfrak{A}$.

2. Background

Kumjian and Pask [11] define a k-graph as a small category Λ with a degree map $\deg : \Lambda \to \mathbb{N}^\mathsf{k}$ satisfying the *factorization property*: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^\mathsf{k}$ with $\deg(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $\deg(\mu) = m$ and $\deg(\nu) = n$. It is perhaps more convenient to consider Λ as a directed graph in which the vertices have degree 0 and the edges are graded by their (nonzero) degree, which takes values in $\mathbb{N}^\mathsf{k}$, and satisfy the unique factorization property above.

We are restricting our attention in this paper to k-graphs on a single vertex. In this case, every path has the same source and range vertex, and hence any two paths can be composed. So in this case, the k-graph is a semigroup. The unique factorization property implies that the semigroup has cancellation.

Let ε_i for $1 \le i \le \mathsf{k}$ be the standard generators for $\mathbb{Z}^\mathsf{k}$. The generators of $\mathbb{F}_\theta^+$ are the paths of degree ε_i, for $1 \le i \le \mathsf{k}$. Let m_i denote the number of edges of degree ε_i, which we label $e^i_\mathfrak{s}$ for $\mathfrak{s} \in \mathbf{m}_i = \{1, 2, \dots, m_i\}$. There are no commutation relations amongst the set $\{e^i_1, \dots, e^i_{m_i}\}$. However the factorization property implies that each product $e^i_\mathfrak{s} e^j_\mathfrak{t}$ also factors as $e^j_{\mathfrak{t}'} e^i_{\mathfrak{s}'}$ for some pair of edges. The uniqueness of the factorization implies that there is a permutation θ_{ij} in $S_{\mathbf{m}_i \times \mathbf{m}_j}$ so that

$$e^i_\mathfrak{s} e^j_\mathfrak{t} = e^j_{\mathfrak{t}'} e^i_{\mathfrak{s}'} \quad \text{where} \quad \theta_{ij}(\mathfrak{s}, \mathfrak{t}) = (\mathfrak{s}', \mathfrak{t}').$$

The family $\theta = \{\theta_{ij} : 1 \le i < j \le \mathsf{k}\}$ determines the k-graph $\mathbb{F}_\theta^+$, which is the semigroup generated by $\{e^i_\mathfrak{s} : 1 \le i \le \mathsf{k},\ 1 \le \mathfrak{s} \le m_i\}$ subject to these relations. The degree map sends a word $w \in \mathbb{F}_\theta^+$ to $\deg(w) \in \mathbb{N}_0^\mathsf{k}$ which counts the number of terms from each family $\{e^i_1, \dots, e^i_{m_i}\}$.

Unfortunately, not every family of permutations θ yields a k-graph. There are evidently issues about associativity of the product and uniqueness of the factorization. For $\mathsf{k} = 2$, every permutation yields a 2-graph; but this is not true for $\mathsf{k} \ge 3$. See, for example, [7, 17]. Fowler and Sims [7] showed that for $\mathsf{k} \ge 3$, θ determines a k-graph if and only if every three sets of generators satisfy the following *cubic condition* showing that a word of degree $(1, 1, 1)$ is well-defined. You should interpret the following identities by noting that each equality follows from a series of three uses of the commutation relations to reverse the order of the three terms. There are two orders in which this can be accomplished, and the end result must be the same.

$$\begin{aligned} e^i_{\mathfrak{t}_1} e^j_{\mathfrak{t}_2} e^k_{\mathfrak{t}_3} &= e^i_{\mathfrak{t}_1} e^k_{\mathfrak{t}'_3} e^j_{\mathfrak{t}'_2} = e^k_{\mathfrak{t}''_3} e^i_{\mathfrak{t}'_1} e^j_{\mathfrak{t}'_2} = e^k_{\mathfrak{t}''_3} e^j_{\mathfrak{t}''_2} e^i_{\mathfrak{t}''_1} \\ e^i_{\mathfrak{t}_1} e^j_{\mathfrak{t}_2} e^k_{\mathfrak{t}_3} &= e^j_{\mathfrak{t}_{2'}} e^i_{\mathfrak{t}_{1'}} e^k_{\mathfrak{t}_3} = e^j_{\mathfrak{t}_{2'}} e^k_{\mathfrak{t}_{3'}} e^i_{\mathfrak{t}_{1''}} = e^k_{\mathfrak{t}_{3''}} e^j_{\mathfrak{t}_{2''}} e^i_{\mathfrak{t}_{1''}} \end{aligned}$$

$$\text{implies} \quad e^i_{\mathfrak{t}''_1} = e^i_{\mathfrak{t}_{1''}},\ e^j_{\mathfrak{t}''_2} = e^j_{\mathfrak{t}_{2''}} \text{ and } e^k_{\mathfrak{t}''_3} = e^k_{\mathfrak{t}_{3''}}.$$

Hence, for $\mathsf{k} \geq 3$, $\mathbb{F}^+_\theta$ is a k-graph if and only if the restriction of $\mathbb{F}^+_\theta$ to every triple family of edges $\{e^i_{\mathfrak{s}},\ e^j_{\mathfrak{t}},\ e^k_{\mathfrak{u}}\}$ is a 3-graph.

We can consider each permutation θ_{ij} as a permutation of $\prod_{i=1}^{\mathsf{k}} \mathbf{m}_i$ which fixes the coordinates except for i, j, on which it acts as θ_{ij}. With this abuse of notation, one can rephrase the cubic condition as:

$$\theta_{ij}\theta_{ik}\theta_{jk} = \theta_{jk}\theta_{ik}\theta_{ij} \quad \text{for all} \quad 1 \leq i < j < k \leq \mathsf{k}.$$

This will facilitate calculations.

We provide a few examples.

Example 2.1. Power [15] showed that there are nine 2-graphs with $m_1 = m_2 = 2$ up to isomorphism. In [4], we showed that only two of these are periodic (defined in the next section). These are the flip algebra in which $\theta(\mathfrak{s}, \mathfrak{t}) = (\mathfrak{t}, \mathfrak{s})$ and the square algebra given by the permutation $\big((1,1), (1,2), (2,2), (2,1)\big)$.

A more typical example of a 2-graph is the forward 3-cycle semigroup given by the permutation $\big((1,1), (1,2), (2,1)\big)$. Curiously, the reverse 3-cycle semigroup arising from the 3-cycle $\big((1,1), (2,1), (1,2)\big)$ yields a 2-graph which is not isomorphic to the forward 3-cycle semigroup.

Example 2.2. Let $m_i = n$ for all $1 \leq i \leq \mathsf{k}$ and θ_{ij} be the transposition $\theta_{ij}(\mathfrak{s}, \mathfrak{t}) = (\mathfrak{t}, \mathfrak{s})$. Equivalently, this means that $e^i_{\mathfrak{s}} e^j_{\mathfrak{t}} = e^j_{\mathfrak{s}} e^i_{\mathfrak{t}}$ for all i, j and all $\mathfrak{s}, \mathfrak{t}$. It is readily calculated that

$$\theta_{ij}\theta_{ik}\theta_{jk}(\mathfrak{s}, \mathfrak{t}, \mathfrak{u}) = (\mathfrak{u}, \mathfrak{t}, \mathfrak{s}) = \theta_{jk}\theta_{ik}\theta_{ij}(\mathfrak{s}, \mathfrak{t}, \mathfrak{u}).$$

Thus this is a k-graph.

Example 2.3. Let $m_1 = m_2 = m_3 = 2$. Let $\theta_{13} = \theta_{23}$ be the forward 3-cycle $\big((1,1), (1,2), (2,1)\big)$ and let θ_{12} be the flip. Observe that $\theta_{13}(\mathfrak{s}, \mathfrak{t}) = (\mathfrak{t}, \mathfrak{s} + \mathfrak{t})$ where addition is calculated in $\mathbb{Z}/2\mathbb{Z}$. Thus θ yields a 3-graph because

$$\theta_{12}\theta_{13}\theta_{23}(\mathfrak{s}, \mathfrak{t}, \mathfrak{u}) = (\mathfrak{u}, \mathfrak{t} + \mathfrak{u}, \mathfrak{s} + \mathfrak{t} + \mathfrak{u}) = \theta_{23}\theta_{13}\theta_{12}(\mathfrak{s}, \mathfrak{t}, \mathfrak{u}).$$

Example 2.4. Let $m_1 = m_2 = m_3 = 2$. Let $\theta_{13} = \theta_{23}$ be the square algebra which can be written $\theta_{13}(\mathfrak{s}, \mathfrak{t}) = (\mathfrak{t}, \mathfrak{s} + 1)$, and let θ_{12} be the flip. Then θ determines a 3-graph since

$$\theta_{12}\theta_{13}\theta_{23}(\mathfrak{s}, \mathfrak{t}, \mathfrak{u}) = (\mathfrak{u}, \mathfrak{t} + 1, \mathfrak{s} + 1) = \theta_{23}\theta_{13}\theta_{12}(\mathfrak{s}, \mathfrak{t}, \mathfrak{u}).$$

2.5. Representations. Now consider the representations of $\mathbb{F}_\theta^+$, by which we mean a homomorphism of $\mathbb{F}_\theta^+$ into $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. A *(partially) isometric representation* of $\mathbb{F}_\theta^+$ is a semigroup homomorphism $\sigma : \mathbb{F}_\theta^+ \to \mathcal{B}(\mathcal{H})$ whose range consists of (partial) isometries on $\mathcal{H}$.

Call σ *row contractive* if the operator $[\sigma(e_1^i) \cdots \sigma(e_{m_i}^i)]$, considered as an operator from $\mathcal{H}^{(m_i)}$ to $\mathcal{H}$, is a contraction for $1 \le i \le \mathsf{k}$. Likewise σ is *row isometric* if these row operators are isometries. A row contractive representation is *defect free* if

$$\sum_{\mathfrak{s}=1}^{m_i} \sigma(e_{\mathfrak{s}}^i)\sigma(e_{\mathfrak{s}}^i)^* = I \quad \text{for all} \quad 1 \le i \le \mathsf{k}.$$

A row isometric defect free representation is a $*$*-representation* of $\mathbb{F}_\theta^+$.

The row isometric condition is equivalent to saying that the $\sigma(e_{\mathfrak{s}}^i)$'s are isometries with pairwise orthogonal range for each $1 \le i \le \mathsf{k}$. If σ is row isometric and defect free, then the sum of these ranges is the whole space.

The most basic example of an isometric representation of $\mathbb{F}_\theta^+$ is the *left regular representation* λ. This is defined on $\ell^2(\mathbb{F}_\theta^+)$ with orthonormal basis $\{\xi_w : w \in \mathbb{F}_\theta^+\}$ given by $\lambda(v)\xi_w = \xi_{vw}$. Each $\lambda(e_{\mathfrak{s}}^i)$ is an isometry. Because the factorization of an element in $\mathbb{F}_\theta^+$ can begin with a unique element of $\{e_1^i, \dots, e_{m_i}^i\}$ if it has any of these elements as factors, it is clear that the ranges of $\lambda(e_{\mathfrak{s}}^i)$ are pairwise orthogonal for $1 \le \mathfrak{s} \le m_i$. Hence this is a row isometric representation. However it is also evident that it is not defect free since the range of each $\sigma(e_{\mathfrak{s}}^i)$ is orthogonal to ξ_w if w is any path containing none of the edges $e_{\mathfrak{s}}^i$, such as the empty path, or a path only in the other generators $e_{\mathfrak{t}}^j$ for $j \ne i$.

One forms a $*$-algebra $\mathcal{A}$ generated by $\mathbb{F}_\theta^+$ subject to the relations implicit in $*$-representations that each $e_{\mathfrak{s}}^i$ is an isometry, i.e., $e_{\mathfrak{s}}^{i*} e_{\mathfrak{s}}^i = 1$, and the defect free condition $\sum_{\mathfrak{s}=1}^{m_i} e_{\mathfrak{s}}^i e_{\mathfrak{s}}^{i*} = 1$ for $1 \le i \le \mathsf{k}$. It is an easy exercise to see that $\mathcal{A}$ is the span of words of the form uv^* for $u, v \in \mathbb{F}_\theta^+$. This follows from the identity

$$e_{\mathfrak{s}}^{i*} e_{\mathfrak{t}}^j = e_{\mathfrak{s}}^{i*} e_{\mathfrak{t}}^j \sum_{\mathfrak{r}=1}^{m_i} e_{\mathfrak{r}}^i e_{\mathfrak{r}}^{i*} = \sum_{\mathfrak{r}=1}^{m_i} e_{\mathfrak{s}}^{i*} e_{\mathfrak{r}'}^i e_{\mathfrak{t}_r}^j e_{\mathfrak{r}}^{i*} = \sum_{\mathfrak{r}=1}^{m_i} \delta_{\mathfrak{s}\mathfrak{r}'} e_{\mathfrak{t}_r}^j e_{\mathfrak{r}}^{i*}.$$

Every $*$-representation π of $\mathbb{F}_\theta^+$ extends to a representation of $\mathcal{A}$.

The k-graph C*-algebra $\mathrm{C}^*(\mathbb{F}_\theta^+)$ is the universal C*-algebra for $*$-representations of $\mathbb{F}_\theta^+$. This is the completion of $\mathcal{A}$ with respect to the norm

$$\|A\| = \sup\{\|\pi(A)\| : \pi \text{ is a } *\text{-representation}\}.$$

This is the C*-algebra generated by $\mathbb{F}_\theta^+$ with the universal property that every $*$-representation of $\mathbb{F}_\theta^+$ extends uniquely to a $*$-representation of $\mathrm{C}^*(\mathbb{F}_\theta^+)$.

The universal property of $\mathrm{C}^*(\mathbb{F}_\theta^+)$ yields a family of *gauge automorphisms*. For any character φ in the dual group $\widehat{\mathbb{Z}^{\mathsf{k}}} \cong \mathbb{T}^{\mathsf{k}}$ of $\mathbb{Z}^{\mathsf{k}}$, consider the representation $\gamma_\varphi(w) = \varphi(\deg(w))w$. This is evidently an automorphism of $\mathbb{T}\mathbb{F}_\theta^+$. So by the universal property of $\mathrm{C}^*(\mathbb{F}_\theta^+)$, it extends to a $*$-automorphism of $\mathrm{C}^*(\mathbb{F}_\theta^+)$, which we also denote γ_φ.

Integration over the $\mathbb{T}^{\mathsf{k}}$ yields a faithful expectation

$$\Phi(X) = \int_{\mathbb{T}^{\mathsf{k}}} \gamma_\varphi(X)\, d\varphi.$$

Checking this map on words uv^*, one readily sees that $\Phi(uv^*) = \delta_{k0} uv^*$ where $k = \deg(uv^*) := \deg(u) - \deg(v)$. Therefore

$$\mathfrak{F} := \operatorname{Ran} \Phi = \operatorname{span}\{uv^* : \deg(uv^*) = 0\}.$$

Kumjian and Pask [11] show that this is an AF-algebra. In our case of a single vertex, it is the UHF algebra for the supernatural number $\prod_{i=1}^{\mathsf{k}} m_i^\infty$. In particular, it is simple.

2.6. Dilations. Dilation theory is generally in the realm of nonself-adjoint operator algebras, not C*-algebras. However there is a natural operator algebra $\mathcal{A}_\theta$ associated to $\mathbb{F}_\theta^+$ generated by the left regular representation $\lambda(\mathbb{F}_\theta^+)$ as a subalgebra of $\mathcal{B}(\ell^2(\mathbb{F}_\theta^+))$ that plays a critical role here. The reason is that it naturally generates $\mathrm{C}^*(\mathbb{F}_\theta^+)$ and this C*-algebra is the C*-envelope of $\mathcal{A}_\theta$, so that the maximal representations of $\mathcal{A}_\theta$ are precisely the $*$-representations of $\mathrm{C}^*(\mathbb{F}_\theta^+)$, which in turn are associated to the defect free row isometric representations of $\mathbb{F}_\theta^+$. The advantage of working in this context is that one can show that defect free representations of $\mathbb{F}_\theta^+$ extend to representations of $\mathcal{A}_\theta$. In turn, these representations have a unique minimal dilation to a representation which extends to a $*$-representation of $\mathrm{C}^*(\mathbb{F}_\theta^+)$. As we shall show, this allows us to examine the structure of $*$-representations by focussing on a smaller much more tractable representation of $\mathbb{F}_\theta^+$. We briefly review the relevant ideas of dilation theory required in this context.

If σ is a representation of an operator algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, we say that a representation ρ on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ is a dilation of σ if $\sigma(a) = P_{\mathcal{H}}\rho(a)|_{\mathcal{H}}$ for all $a \in \mathcal{A}$. This implies that $\mathcal{H}$ is semiinvariant, i.e., $\mathcal{H} = \mathcal{M}_1 \ominus \mathcal{M}_2$ for two invariant subspaces $\mathcal{M}_2 \subset \mathcal{M}_1$. Arveson's dilation theory [1], extended by Hamana [8], shows that $\mathcal{A}$ sits in a canonical C*-algebra known as its C*-envelope, $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$. This is determined by the universal property that whenever $j : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a completely isometric isomorphism, there is a unique $*$-homomorphism of $\mathrm{C}^*(j(\mathcal{A}))$ onto $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$ which extends j^{-1}.

A recent proof of Hamana's Theorem by Dritschel and McCullough [5] shows that every completely contractive representation ρ of $\mathcal{A}$ dilates to a maximal dilation σ, in the sense that any further dilation τ of σ always has the form $\tau \simeq \sigma \oplus \pi$ for another representation π. Moreover, these maximal representations of $\mathcal{A}$ are precisely those representations which extend to a $*$-representation of $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$.

The operator algebra that figures here is the nonself-adjoint unital operator algebra $\mathcal{A}_\theta$ defined above. There is no simple criterion on a representation of $\mathbb{F}_\theta^+$ which is equivalent to it having a completely contractive extension to $\mathcal{A}_\theta$. A necessary condition is that the representation be row contractive because

$$\big\|\big[\sigma(e_1^i)\ \ldots\ \sigma(e_{m_i}^i)\big]\big\| \le \big\|\big[\lambda(e_1^i)\ \ldots\ \lambda(e_{m_i}^i)\big]\big\| = 1.$$

However it was shown in [3] that this is a strictly weaker condition than being completely contractive even for 2-graphs on one vertex.

Two results provide the information that we need, and they are both valid for arbitrary k-graphs, not just the single vertex case. The first is a result of Katsoulis and Kribs [9] on the C*-envelope of higher rank graph algebras.

Theorem 2.7 (Katsoulis–Kribs)**.** *The C*-envelope of $\mathcal{A}_\theta$ is $\mathrm{C}^*(\mathbb{F}_\theta^+)$.*

This implies that every completely contractive representation of $\mathcal{A}_\theta$ dilates to a $*$-representation of $\mathrm{C}^*(\mathbb{F}_\theta^+)$, and hence to a $*$-representation of $\mathbb{F}_\theta^+$.

In [3], we established a dilation theorem for a class of doubly generated operator algebras which includes the 2-graphs on one vertex. We showed, in particular, that every defect free representation dilates to a $*$-representation; and consequently, they yields completely contractive representations of $\mathcal{A}_\theta$. Using the Poisson transform defined by Popescu in [14], Skalski and Zacharias [20] studied the dilation theory of higher rank graphs in a very general context. In particular, their results include the following dilation theorem which is valid for all k.

Theorem 2.8 (Skalski–Zacharias)**.** *Every defect free, row contractive representation of $\mathbb{F}_\theta^+$ has a unique minimal $*$-dilation.*

Consequently, every defect free representation of $\mathbb{F}_\theta^+$ extends to a completely contractive representation of $\mathcal{A}_\theta$.

The algebra $\mathcal{A}_\theta$ will not play a direct role in the current paper, which is focussed on $*$-representations and the structure of $\mathrm{C}^*(\mathbb{F}_\theta^+)$. However, it lurks in the background because we use dilation theory to simplify the analysis of the representations.

3. Atomic representations

Atomic representations of 2-graphs are comprehensively studied in [2]. They provide a very interesting class of representations, and play an important role in the study of 2-graphs. We now introduce such representations of k-graphs.

Definition 3.1. A partially isometric representation is *atomic* if there is an orthonormal basis which is permuted, up to scalars, by each partial isometry.

That is, σ is atomic if there is an orthonormal basis $\{\xi_k : k \geq 1\}$ so that for each $w \in \mathbb{F}_\theta^+$, $\sigma(w)\xi_k = \alpha\xi_l$ for some l and some $\alpha \in \mathbb{T} \cup \{0\}$.

As in [2], we refer to the atomic partially isometric representations in this paper simply as atomic representations, and likewise we refer to the row contractive defect free atomic representations simply as defect free atomic representations.

Every atomic representation determines a graph with the standard basis vectors representing the vertices, and the partial isometries $\sigma(e^i_{\mathfrak{s}})$ determining directed edges: when $\sigma(e^i_{\mathfrak{s}})\xi_a = \xi_b$, we draw an edge from vertex a to vertex b labelled $e^i_{\mathfrak{s}}$. The graph is called connected if there is an undirected path from each vertex to every other. This graph contains all information about the representation except for the scalars of modulus one.

As in [3, Lemma 5.6], it is easy to see that the $*$-dilation of an atomic defect free representation remains atomic. For the convenience of the reader, we reproduce the proof here.

Lemma 3.2. *If σ is an atomic defect free representation, with respect to some basis of $\mathcal{H}_\sigma$, with minimal $*$-dilation π, then this basis extends to a basis for $\mathcal{H}_\pi$ making π an atomic $*$-representation.*

Proof. Let π be a minimal row isometric dilation of σ acting on $\mathcal{K}$. Consider the standard basis $\{\xi_k : k \geq 1\}$ for $\mathcal{H}$ with respect to which σ is atomic. We claim that the set of subspaces $\{\mathbb{C}\pi(x)\xi_k : k \geq 1, x \in \mathbb{F}_\theta^+\}$ forms an orthonormal family of 1-dimensional subsets spanning $\mathcal{K}$, with repetitions. Indeed, $\mathcal{H}$ is coinvariant and cyclic; so these sets span $\mathcal{K}$. It suffices to show that any two such sets, say $\mathbb{C}\pi(x_1)\xi_1$ and $\mathbb{C}\pi(x_2)\xi_2$, either coincide or are orthogonal.

Let $d = \deg(x_1) \vee \deg(x_2) \in \mathbb{N}_0^{\mathsf{k}}$. Since σ is defect free, there are unique basis vectors ζ_j and words y_j with $d(y_j) = d - \deg(x_j)$ so that $\mathbb{C}\sigma(y_j)\zeta_j = \mathbb{C}\xi_j$. Thus using $\mathbb{C}\zeta_j$ and the word x_jy_j, we may suppose that the two words have the same degree. For convenience of notation, we suppose that this has already been done.

Now two distinct words of the same degree have pairwise orthogonal ranges. Thus if $x_1 \neq x_2$, then $\mathbb{C}\pi(x_1)\xi_1$ and $\mathbb{C}\pi(x_2)\xi_2$ are orthogonal. On the other hand, if $x_1 = x_2$, then if $\mathbb{C}\xi_1 = \mathbb{C}\xi_2$, the images are equal; while if $\mathbb{C}\xi_1$ and $\mathbb{C}\xi_2$ are orthogonal, they remain orthogonal under the action of the isometry $\pi(x_1)$. $\square$

3.3. Inductive limit representations. In this subsection, we will define a whole family of $*$-dilations of λ which are, in fact, inductive limits of λ. They play a central role in what follows.

Arbitrarily choose an *infinite tail* τ of $\mathbb{F}_\theta^+$; that is, an infinite word in the generators which has infinitely many terms from each family $\{e^i_1, \ldots, e^i_{m_i}\}$. Such an infinite word can be factored so that these terms occur in succession as

$$\tau = e^{1}_{\mathsf{t}_{01}} \cdots e^{\mathsf{k}}_{\mathsf{t}_{0\mathsf{k}}} e^{1}_{\mathsf{t}_{11}} \cdots e^{\mathsf{k}}_{\mathsf{t}_{1\mathsf{k}}} \cdots = \tau_0 \tau_1 \tau_2 \dots$$

where $\tau_s = e^{1}_{\mathsf{t}_{s1}} \cdots e^{\mathsf{k}}_{\mathsf{t}_{s\mathsf{k}}}$ for $s \geq 0$. Let $\mathcal{F}_s = \mathcal{F} := \mathbb{F}_\theta^+$ for $s \geq 0$, viewed as discrete sets on which the generators of $\mathbb{F}_\theta^+$ act as injective maps by right multiplication, namely, $\rho(w)f = fw$ for all $f \in \mathcal{F}$. Consider $\rho_s = \rho(\tau_s)$ as a map from $\mathcal{F}_s$ into $\mathcal{F}_{s+1}$. Define $\mathcal{F}_\tau$ to be the injective limit set

$$\mathcal{F}_\tau = \varinjlim(\mathcal{F}_s, \rho_s).$$

Also let ι_s denote the injections of $\mathcal{F}_s$ into $\mathcal{F}_\tau$. Thus $\mathcal{F}_\tau$ may be viewed as the union of $\mathcal{F}_0, \mathcal{F}_1, \dots$ with respect to these inclusions.

The left regular action λ of $\mathbb{F}_\theta^+$ on itself induces corresponding maps on $\mathcal{F}_s$ by $\lambda_s(w)f = wf$. Observe that $\rho_s \lambda_s = \lambda_{s+1} \rho_s$. The injective limit of these actions is an action λ_τ of $\mathbb{F}_\theta^+$ on $\mathcal{F}_\tau$. Let λ_τ also denote the corresponding representation of $\mathbb{F}_\theta^+$ on $\ell^2(\mathcal{F}_\tau)$. That is, we let $\{\xi_f : f \in \mathcal{F}_\tau\}$ denote the orthonormal basis and set $\lambda_\tau(w)\xi_f = \xi_{wf}$. It is easy to see that this provides a defect free, isometric representation of $\mathbb{F}_\theta^+$; i.e., it is a $*$-representation.

Let $\tau^s = \tau_0 \tau_1 \dots \tau_s$ for $s \geq 0$. We may consider an element $w = \iota_s(v)$ as $w = v\tau^{s*}$. This makes sense in that $\xi_w = \lambda_\tau(v)\lambda_\tau(\tau^s)^*\xi_{\iota_0(\varnothing)}$. In particular, we have $\xi_\varnothing = \xi_{\iota_0(\varnothing)}$.

Since λ_τ is a $*$-dilation, it extends to a representation of $\mathrm{C}^*(\mathbb{F}_\theta^+)$ which we also denote by λ_τ. This is always a faithful representation. This is the analogue of [3, Theorem 3.6].

Theorem 3.4. *For any infinite tail τ, the representation λ_τ is faithful on $\mathrm{C}^*(\mathbb{F}_\theta^+)$.*

Proof. Because of the gauge invariance uniqueness theorem, it suffices to show that there are gauge automorphisms of $\mathrm{C}^*(\lambda_\tau(\mathbb{F}_\theta^+))$. This is accomplished by conjugation by a diagonal unitary. Given a word $w = v\tau^{s*} \in \mathcal{F}_\tau$, define $\deg(w) = \deg(v) - \deg(\tau^s)$. It is clear that this is well-defined and extends the degree map on $\mathbb{F}_\theta^+$. Given $\varphi \in \widehat{\mathbb{Z}^\mathsf{k}}$, define $U_\varphi = \operatorname{diag}\big(\varphi(\deg(w))\big)$ with respect to the basis $\{\xi_w : w \in \mathcal{F}_\tau\}$. Then for u in $\mathbb{F}_\theta^+$ and $w = v\tau^{s*}$ in $\mathcal{F}_\tau$,

$$\begin{aligned}
U_\varphi \lambda_\tau(u) U_\varphi^* \xi_w &= U_\varphi \lambda_\tau(u) \overline{\varphi(\deg(w))} \xi_w \\
&= U_\varphi \overline{\varphi(\deg(v) - \deg(\tau^s))} \xi_{uw} \\
&= \varphi(\deg(uv) - \deg(\tau^s)) \overline{\varphi(\deg(v) - \deg(\tau^s))} \xi_{uw} \\
&= \varphi(\deg(u)) \lambda_\tau(u) \xi_w.
\end{aligned}$$

Hence $U_\varphi \lambda_\tau(u) U_\varphi^* = \lambda_\tau(\gamma_\varphi(u))$. □

Because of the dilation theory, λ_τ is completely determined by any cyclic coinvariant subspace $\mathcal{H}$ of $\ell^2(\mathcal{F}_\tau)$ as the unique minimal $*$-dilation of this compression. We will describe such a subspace which will be convenient.

Observe that $\lambda_\tau(w)^*\xi_{\iota_0(\varnothing)}$ is nonzero if and only if w is an initial segment of τ after appropriate factorization. That is, given $n=(n_1,\dots,n_{\mathsf{k}})\in\mathbb{N}_0^{\mathsf{k}}$, one may factor τ in exactly one way so that $\tau=w_n\tau_n'$ in which w_n has degree n. Let $\zeta_{-n}:=\lambda_\tau(w_n)^*\xi_{\iota_0(\varnothing)}$ for $n\in\mathbb{N}_0^{\mathsf{k}}$. In particular, $\zeta_{(-s,\dots,-s)}=\xi_{\iota_s(\varnothing)}$. Then

$$\mathcal{H}_\tau=\operatorname{span}\{\zeta_n : n\in(-\mathbb{N}_0)^{\mathsf{k}}\}$$

is evidently a cyclic subspace because it contains each $\xi_{\iota_s(\varnothing)}$, and is coinvariant by construction.

Note that beginning at any of the standard basis vectors ξ_w, there will be some word v so that $\lambda_\tau(v)^*\xi_w$ is a basis vector ζ_n in $\mathcal{H}_\tau$. As the restriction of λ_τ to the cyclic subspace generated by ζ_n is unitarily equivalent to λ, it is easy to understand why the restriction of λ_τ to $\mathcal{H}_\tau$ determines the whole representation.

For each $n\in-\mathbb{N}_0^{\mathsf{k}}$, there are unique integers $\mathfrak{t}_n^i$ so that ζ_n is in the range of $\lambda_\tau(e^i_{\mathfrak{t}_n^i})$ for $1\le i\le\mathsf{k}$; that is, $\lambda_\tau(e^i_{\mathfrak{t}_n^i})\zeta_{n-\varepsilon_i}=\zeta_n$. Set $\Sigma(\tau,n)=(\mathfrak{t}_n^1,\dots,\mathfrak{t}_n^{\mathsf{k}})$. This determines the data set

$$\Sigma(\tau)=\{\Sigma(\tau,n) : n\in-\mathbb{N}_0^{\mathsf{k}}\}.$$

Definition 3.5. Two tails τ_1 and τ_2 are said to be *tail equivalent* if their data sets eventually coincide; i.e., there is $T\in-\mathbb{N}_0^{\mathsf{k}}$ so that

$$\Sigma(\tau_1,n)=\Sigma(\tau_2,n)\quad\text{for all}\quad n\le T.$$

We say that τ_1 and τ_2 are *p-shift tail equivalent* for some $p\in\mathbb{Z}^k$ if there is a $T\in\mathbb{N}_0^k$ so that

$$\Sigma(\tau_1,n)=\Sigma(\tau_2,n+p)\quad\text{for all}\quad n\le T.$$

Then τ_1 and τ_2 are *shift tail equivalent* if they are p-shift tail equivalent for some $p\in\mathbb{Z}^{\mathsf{k}}$.

Clearly, if τ_1 and τ_2 are shift tail equivalent, then λ_{τ_1} and λ_{τ_2} are unitarily equivalent.

We now introduce two important notions: the symmetry of τ and the aperiodicity condition of $\mathbb{F}_\theta^+$.

Definition 3.6. A tail τ is said to be *p-periodic* if $\Sigma(\tau,n)=\Sigma(\tau,n+p)$ for all $n\le 0\wedge -p$; and *eventually p-periodic* if τ is p-shift tail equivalent to itself. The *symmetry group* of τ is the subgroup of $\mathbb{Z}^{\mathsf{k}}$

$$H_\tau=\{p\in\mathbb{Z}^{\mathsf{k}} : \tau \text{ is eventually } p\text{-periodic}\}.$$

The *symmetry group* of $\mathbb{F}_\theta^+$ is defined by

$$H_\theta:=\cap_\tau H_\tau$$

as τ runs over all possible infinite tails of $\mathbb{F}_\theta^+$.

A tail τ is called *aperiodic* if $H_\tau=\{0\}$. The semigroup $\mathbb{F}_\theta^+$ is *aperiodic* if $H_\theta=\{0\}$. Otherwise we say that $\mathbb{F}_\theta^+$ is *periodic*.

Clearly if there is an aperiodic infinite tail τ, then $\mathbb{F}_\theta^+$ is aperiodic. The following result shows that our definition coincides with the Kumjian–Pask aperiodicity condition.

Proposition 3.7. *$\mathbb{F}_\theta^+$ has an infinite tail with $H_\tau = H_\theta$. In particular, when $\mathbb{F}_\theta^+$ is aperiodic, there is an aperiodic tail. More generally, we have $H_\theta \cap \mathbb{N}_0^{\mathsf{k}} = \{0\}$.*

Proof. For each p in $\mathbb{Z}^{\mathsf{k}} \setminus H_\theta$, there is a tail τ such that $p \notin H_\tau$. Hence there is some $n \in -\mathbb{N}_0^{\mathsf{k}}$ so that $n+p \in -\mathbb{N}_0^{\mathsf{k}}$ and $\Sigma(\tau,n) \neq \Sigma(\tau,n+p)$. Choose s so that $0 \geq n, n+p \geq (-s,\dots,-s)$. Then the finite initial segment $w_p = \tau_0 \dots \tau_s$ of τ already exhibits the lack of p-symmetry. So any infinite tail that contains w_p infinitely often can never exhibit p-symmetry for $n \leq T$. Form a tail τ by stringing together the words w_p, repeating each one infinitely often. By construction, $H_\tau \subset H_\theta$. The other inclusion is true by definition.

If $p \in \mathbb{N}_0^{\mathsf{k}} \setminus \{0\}$, it is easy to write down a finite sequence without p-symmetry. Splicing such words into τ as above shows that $H_\theta \cap \mathbb{N}_0^{\mathsf{k}} = \{0\}$. $\square$

4. A group construction

In this section, we describe a large family of defect free atomic representations with a very nice structure. By the dilation theorem, they encode a family of atomic $*$-representations which are obtained as the unique minimal $*$-dilations. The main result is that every irreducible atomic $*$-representation arises from this construction; and every atomic $*$-representation decomposes as a direct integral of the irreducible ones.

Let G be a finitely generated abelian group with k designated generators $\mathfrak{g}_1,\dots,\mathfrak{g}_{\mathsf{k}}$. Suppose that functions are given

$$\begin{aligned} \mathfrak{t}^i &: G \to \{1,\dots,m_i\}, \qquad \mathfrak{t}^i(g) =: \mathfrak{t}^i_g, \quad i=1,\dots,\mathsf{k}, \\ \alpha^i &: G \to \mathbb{T}, \qquad\qquad\qquad \alpha^i(g) =: \alpha^i_g, \quad i=1,\dots,\mathsf{k}. \end{aligned}$$

Consider a defect free atomic representation $\sigma : \mathbb{F}_\theta^+ \to \mathcal{B}(\ell^2(G))$ given by

$$\sigma(e^i_{\mathfrak{t}})\xi_{g-\mathfrak{g}_i} = \delta_{\mathfrak{t},\mathfrak{t}^i_g}\, \alpha^i_g\, \xi_g \quad \text{for} \quad i=1,\dots,\mathsf{k}.$$

This ensures that ξ_g is in the range of $\sigma(e^i_{\mathfrak{t}^i_g})$ for each $1 \leq i \leq \mathsf{k}$. In order for σ to be a representation, the commutation relations must be satisfied, namely

$$e^i_{\mathfrak{t}_g} e^j_{\mathfrak{t}_{g-\mathfrak{g}_i}} = e^j_{\mathfrak{t}_g} e^i_{\mathfrak{t}_{g-\mathfrak{g}_j}} \quad \text{and} \quad \alpha^i_{\mathfrak{t}_g} \alpha^j_{\mathfrak{t}_{g-\mathfrak{g}_i}} = \alpha^j_{\mathfrak{t}_g} \alpha^i_{\mathfrak{t}_{g-\mathfrak{g}_j}}$$

for all $g \in G$ and $1 \leq i < j \leq \mathsf{k}$. Such a representation will be called a *group construction representation.*

Example 4.1. Actually defining such relations might not be so easy. However in the case of $G = \mathbb{Z}^{\mathsf{k}}$ with the standard generators, we can obtain all such representations from infinite tails. Indeed, we saw that an infinite tail τ gives rise to an inductive limit representation λ_τ. We then identified a subspace $\mathcal{H}_\tau$ with basis $\{\zeta_n : n \in -\mathbb{N}_0^{\mathsf{k}}\}$. It is possible to continue this 'forward' to obtain a (noncanonical) representation modelled on the group $\mathbb{Z}^{\mathsf{k}}$.

This may be accomplished by extending τ arbitrarily to a doubly infinite word, say $\omega^*\tau$, where ω^* is a tail in reverse order. This will specify how we are allowed to move forward and stay within our subspace. If $n \in -\mathbb{N}^{\mathsf{k}}$, recall that there is a unique word w_n so that τ factors as $\tau = w_n\tau'$. Suppose that $m \in \mathbb{N}_0^{\mathsf{k}}$. Then there is a unique word v_m of degree m so that $\omega^* = \omega'^* v_m$. Let $\zeta_{n+m} = \lambda_\tau(v_m w_n)\zeta_n$. It is not difficult to verify that this is well-defined.

We identify $\mathcal{K} = \operatorname{span}\{\zeta_m : m \in \mathbb{Z}^{\mathsf{k}}\}$ with $\ell^2(\mathbb{Z}^{\mathsf{k}})$. It is easy to see that $\mathcal{K}$ is a coinvariant subspace; and it is cyclic because $\mathcal{H}_\tau$ is cyclic. Let σ be the compression of λ_τ to this subspace. Then we have a representation of group type with all constants $\alpha_g^i = 1$.

It turns out that the scalars α_g^i are not difficult to control. It was shown in [2, Theorem 5.1] that the representation is unitarily equivalent to another group construction representation in which these constants are independent of $g \in G$. The proof is essentially the same as the 2-graph case. So we state it without proof.

For each group G, there is a canonical homomorphism κ of $\mathbb{Z}^{\mathsf{k}}$ onto G sending the standard generators ε_i to $\mathfrak{g}_i$ for $1 \le i \le \mathsf{k}$. Let K be the kernel of κ, so that $G \cong \mathbb{Z}^{\mathsf{k}}/K$.

Theorem 4.2. *Let $G = \mathbb{Z}^{\mathsf{k}}/K$ as above, and let σ be a group construction representation. Then σ is unitarily equivalent to another group construction representation with the same functions $\mathfrak{t}^i$ and constant functions α^i.*

In fact, as for 2-graphs, the constants determine a unique character ψ of K. If a path in the graph is a loop returning to the vertex where it started, then it determines a unique word in the generators and their adjoints whose degree d belongs to K. The scalar multiple of the vertex vector obtained by application of the partial isometry is $\psi(d)$. The choice of the scalars α^i is determined by choosing an extension φ of ψ to a character on $\mathbb{Z}^{\mathsf{k}}$; and they are given by $\alpha_i = \varphi(\varepsilon_i)$. The character φ is determined by ψ and a character χ of G; and the constants α^i can be replaced by $\alpha^i\chi(\mathfrak{g}_i)$. We will not need this detailed information, so we refer the interested reader to the proof in [2] in the 2-graph case.

There are two useful notions of symmetry for these group constructions.

Definition 4.3. A group construction representation σ of $\mathbb{F}_\theta^+$ on $\ell^2(G)$ has a *full symmetry subgroup* $H \le G$ if H is the largest subgroup of G such that $\mathfrak{t}_{g+h}^i = \mathfrak{t}_g^i$ and $\alpha_{g+h}^i = \alpha_g^i$ for all $g \in G$ and $h \in H$ and $1 \le i \le \mathsf{k}$.

We define the *symmetry group* $H_\sigma \le G$ to be the largest subgroup $H \le G$ for which there is some $T = (T_1, \dots, T_{\mathsf{k}}) \in \mathbb{Z}^{\mathsf{k}}$ such that $\mathfrak{t}_{g+h}^i = \mathfrak{t}_g^i$ and

$\alpha^i_{g+h} = \alpha^i_g$ for all $g \in G$ and $h \in H$ and $1 \le i \le \mathsf{k}$ with both $g, g+h \in G_T = \big\{ \sum n_i \mathfrak{g}_i : n_i \le T_i \big\}$.

We also will say that two group constructions σ and τ on $\ell^2(G)$ given by functions $\mathfrak{s}^i_g$, α^i_g and $\mathfrak{t}^i_g$, β^i_g respectively are *tail equivalent* if $\mathfrak{t}^i_g = \mathfrak{s}^i_g$ and $\beta^i_g = \alpha^i_g$ for all $g \in G_T$ and $1 \le i \le \mathsf{k}$.

Note that the symmetry group H_σ is well-defined because an increasing sequence of subgroups of a finitely generated abelian group G is eventually constant.

The significance of tail equivalence is that two tail equivalent representations will have unitarily equivalent $*$-dilations. Indeed, by the Skalski–Zacharias dilation theorem, each has a unique $*$-dilation. However, both must coincide with the unique $*$-dilation of their common restriction to $\ell^2(G_T)$.

We wish to show that these group constructions may be obtained by dilating certain partial group constructions, preserving the symmetry. In particular, we will show that a group construction σ with symmetry group H is tail equivalent to one with full symmetry group H.

Theorem 4.4. *Let G be an abelian group with generators $\mathfrak{g}_1, \dots, \mathfrak{g}_\mathsf{k}$. Given $T = (T_1, \dots, T_\mathsf{k})$ with $T_i \in \mathbb{Z} \cup \{\infty\}$, let $G_T = \big\{ \sum n_i \mathfrak{g}_i : n_i \le T_i \big\}$. Suppose that σ is a representation of $\mathbb{F}^+_\theta$ on $\ell^2(G_T)$ which is determined by functions $\mathfrak{t}^i : G_T \to \mathbf{m}_i$ and $\alpha^i : G_T \to \mathbb{T}$ so that*

$$\sigma(e^i_\mathfrak{t})\xi_{g-\mathfrak{g}_i} = \delta_{\mathfrak{t},\mathfrak{t}^i_g}\, \alpha^i_g\, \xi_g \quad \text{for} \quad i = 1, \dots, \mathsf{k} \text{ and } g \in G_T.$$

Then σ may be dilated to a group construction representation on $\ell^2(G)$.

Proof. Consider dilations to $\ell^2(G_{T'})$ for $T' \ge T$. Among these dilations, select one which is maximal in the sense that it cannot be dilated to a representation on a larger subset of this type. There is no loss of generality in assuming that this set is G_T itself. We will show that $G_T = G$.

Indeed, otherwise there is some $T_i < \infty$. By the Skalski–Zacharias dilation theorem, there is a $*$-dilation $\widetilde{\sigma}$ of σ on a Hilbert space $\mathcal{H}$ containing $\ell^2(G_T)$, and by Lemma 3.2, this representation is atomic.

Assume first that every $T_i < \infty$. Set $a = \sum_{i=1}^{\mathsf{k}} T_i \mathfrak{g}_i$; and arbitrarily pick some i. Let $T' = T + \varepsilon_i$, so

$$G_{T'} = G_T + \mathfrak{g}_i = G_T \cup (a + \mathfrak{g}_i + G_0) = G_T \,\dot\cup\, (a + \mathfrak{g}_i + S)$$

where $S = \big\{ \sum_{j \ne i} n_j \mathfrak{g}_j : n_j \le 0 \big\}$. We will extend σ to $\ell^2(G_{T'})$. To this end, arbitrarily select $p_a \in \mathbf{m}_i$. Identify $\xi_{a+\mathfrak{g}_i}$ with $\widetilde{\sigma}(e^i_{p_a})\xi_a$ in $\mathcal{H}$; and set $\mathfrak{t}^i_a = p_a$ and $\alpha^i_a = 1$. For each $s = \sum_{j \ne i} n_j \mathfrak{g}_j$ in S, $a + s \in G_T$ and there is a unique word $w \in \mathbb{F}^+_\theta$ of degree $(|n_1|, \dots, |n_\mathsf{k}|)$ so that $\sigma(w)\xi_{a+s} = \alpha\xi_a$ for some $\alpha \in \mathbb{T}$. Factor $e^i_{p_a} w = w' e^i_{p_{a+s}}$. Then identify $\xi_{a+s+\mathfrak{g}_i}$ with $\widetilde{\sigma}(e^i_{p_{a+s}})\xi_{a+s}$ in $\mathcal{H}$; and set $\mathfrak{t}^i_{a+s} = p_{a+s}$ and $\alpha^i_{a+s} = 1$.

When $n_j \neq 0$, we need to define $\mathfrak{t}^j_{a+s+\mathfrak{g}_i}$. Observe that $a+s+\mathfrak{g}_j \in G_T$. If $\mathfrak{t}^j_{a+s} = q$, we can factor $w = ve^j_q$. Now

$$\sigma(w)\xi_{a+s} = \sigma(v)\alpha^j_{a+s}\xi_{a+s+\mathfrak{g}_j} = \alpha\xi_a.$$

Factor $e^i_{p_a}v = v'e^i_{p_{a+s+\mathfrak{g}_j}}$ as before; and factor $e^i_{p_{a+s+\mathfrak{g}_j}}e^j_q = e^j_{q'}e^i_{p'}$. Then we have

$$v'e^j_{q'}e^i_{p'} = v'e^i_{p_{a+s+\mathfrak{g}_j}}e^j_q = e^i_{p_a}ve^j_q = e^i_{p_a}w = w'e^i_{p_{a+s}} = v'e^j_{q'}e^i_{p_{a+s}}.$$

Therefore $p' = p_{a+s}$ and

$$\begin{aligned}\widetilde{\sigma}(e^j_{q'})\xi_{a+s+\mathfrak{g}_i} &= \widetilde{\sigma}(e^j_{q'}e^i_{p_{a+s}})\xi_{a+s} = \widetilde{\sigma}(e^i_{p_{a+s+\mathfrak{g}_j}}e^j_q)\xi_{a+s}\\ &= \widetilde{\sigma}(e^i_{p_{a+s+\mathfrak{g}_j}})\alpha^j_{a+s}\xi_{a+s+\mathfrak{g}_j} = \alpha^j_{a+s}\xi_{a+s+\mathfrak{g}_j+\mathfrak{g}_i}.\end{aligned}$$

So we set $\mathfrak{t}^j_{a+s+\mathfrak{g}_i} = q'$ and $\alpha^j_{a+s+\mathfrak{g}_i} = \alpha^j_{a+s}$. It now follows that we have defined a dilation of σ to $\ell^2(G_{T'})$. This contradicts the maximality of G_T.

Now consider the case in which $T_j = \infty$ for $j \in J$, for some subset $J \subset \{1,\dots,\mathsf{k}\}$. If $G_T \neq G$, then there is some i with $T_i < \infty$. Let $a_k \in \mathbb{N}_0^{\mathsf{k}}$ where $a^i_k = T_i$ when $T_i < \infty$ and $a^j_k = k$ when $j \in J$. Arguing as above, we can construct a dilation σ_k to $\ell^2(A_k)$ where $A_k = G_T \,\dot\cup\, (a+\mathfrak{g}_i+S)$. If we look at the action of σ_k at ξ_{a_l} for $0 \le l \le k$, then one sees that the value of $p_{a_l} = \mathfrak{t}^i_{a_l}$ takes some value infinitely often. Using a diagonal argument, we may drop to a subsequence so that the values of p_{a_l} are constant for a sequence σ_{k_s}. Hence it is apparent that one can define a representation σ' on $\ell^2(A_\infty)$, where $A_\infty = \bigcup_k A_k = G_T + \mathfrak{g}_i = G_{T'}$ which extends σ. Since G_T was presumed to be maximal, we obtain $G_T = G$ as desired. □

Corollary 4.5. *Let G be an abelian group with generators $\mathfrak{g}_1,\dots,\mathfrak{g}_\mathsf{k}$. Suppose that σ is a representation of $\mathbb{F}^+_\theta$ on $\ell^2(G_T)$ determined by functions $\mathfrak{t}^i : G_T \to \mathbf{m}_i$ and $\alpha^i : G_T \to \mathbb{T}$ satisfying*

$$\sigma(e^i_\mathfrak{t})\xi_{g-\mathfrak{g}_i} = \delta_{\mathfrak{t},\mathfrak{t}^i_g}\,\alpha^i_g\,\xi_g \quad \textit{for} \quad i = 1,\dots,\mathsf{k} \textit{ and } g \in G_T.$$

Moreover, suppose that H is a subgroup of G such that

$$\mathfrak{t}^i_g = \mathfrak{t}^i_{g+h} \textit{ and } \alpha^i_g = \alpha^i_{g+h} \quad \textit{whenever} \quad g, g+h \in G_T.$$

Then σ may be dilated to a group construction representation on $\ell^2(G)$ with full symmetry group H.

Proof. Using the symmetry group H, it is routine to extend the representation to $\ell^2(G_T+H)$ with the same symmetry. Then by collapsing the cosets of H to single points, we obtain a representation τ of G/H on $\ell^2((G_T+H)/H)$. By Theorem 4.4, there is a dilation τ' of τ on $\ell^2(G/H)$. Unfolding this yields a representation on $\ell^2(G)$ with full symmetry group H which dilates σ. □

Corollary 4.6. *Suppose that σ is a group construction representation of $\mathbb{F}_\theta^+$ on $\ell^2(G)$ with symmetry group H. Then σ is tail equivalent to another group construction representation τ which has full symmetry group H.*

Proof. There is a G_T so that σ compressed to $\ell^2(G_T)$ has symmetry group H. The previous corollary shows how to dilate this to τ with full symmetry group H. Evidently, τ is tail equivalent to σ. $\square$

Corollary 4.7. *Suppose that τ is an infinite tail with a symmetry group $H < \mathbb{Z}^{\mathsf{k}}$. Then there is a group construction representation of $\mathbb{F}_\theta^+$ on $\ell^2(\mathbb{Z}^{\mathsf{k}})$ with symmetry group H with minimal $*$-dilation λ_τ.*

Proof. By definition of the symmetry group of τ, there is a coinvariant subspace $\mathcal{H}$ spanned by $\{\zeta_n : n \le T\}$ which has H symmetry. By Corollary 4.5, this dilates to a group construction σ on $\mathbb{Z}^{\mathsf{k}}$ with full symmetry group H. The minimal $*$-dilation of σ is a minimal $*$-dilation of $P_{\mathcal{H}}\lambda_\tau|_{\mathcal{H}}$—which is unique; and hence this dilation is λ_τ. $\square$

5. Decomposing atomic representations

We are now prepared to prove that every atomic $*$-representation may be decomposed as a direct sum of $*$-representations obtained from dilating the group construction. In [2], this was established for rank 2 graphs by a detailed case by case analysis. So our new proof provides insight into that case as well.

It is evident that the span of basis vectors in any connected component of the graph is a reducing subspace; and every atomic representation decomposes as a direct sum of connected ones. So in our analysis of atomic representations, it suffices to consider the case of a connected graph.

Proposition 5.1. *Let σ be an atomic $*$-representation with connected graph. For any standard basis vector η, there is a unique infinite tail $\tau = \tau_0\tau_1\tau_2\dots$ such that η is in the range of $\sigma(\tau_0\dots\tau_n)$ for all $n \ge 0$. The shift-tail equivalence class of τ is independent of the choice of η.*

Proof. Since each basis vector is in the range of $\sigma(e^i_j)$ for exactly one $j \in \mathbf{m}_i$, the existence and uniqueness of τ is clear. For any two basis vectors η_1 and η_2, there is a basis vector η and words u_1 and u_2 so that $\sigma(u_i)\eta = \eta_i$. If τ is the infinite tail obtained for η, it follows that the tails τ_i obtained for η_i are $\tau_i = u_i\tau$. So they are $\deg(u_2) - \deg(u_1)$ shift tail equivalent. $\square$

As a consequence, we can define the symmetry group of σ in an unambiguous way.

Definition 5.2. Define the *symmetry group* of an atomic $*$-representation with connected graph as the symmetry group H_τ of any tail τ derived from any standard basis vector.

Theorem 5.3. *Every atomic $*$-representation σ of $\mathbb{F}_\theta^+$ with a connected graph is obtained as the minimal $*$-dilation of a group construction representation ρ. Moreover, the symmetry group of σ coincides with the full symmetry group of ρ.*

Proof. Let σ be an atomic $*$-representation with connected graph and standard basis $\{\eta_j : j \geq 0\}$. Start with an arbitrary basis vector, say η_0. There is a unique infinite tail $\tau = \tau_0\tau_1\tau_2\dots$ with the degree of each τ_i equal to $(1, 1, \dots, 1)$ such that η_0 is in the range of $\sigma(\tau_0 \dots \tau_n)$ for all $n \geq 0$. Let λ_τ be the inductive limit representation determined by τ. The standard basis for $\ell^2(\mathcal{F}_\tau)$ will be denoted $\{\xi_w : w \in \mathcal{F}_\tau\}$.

Let the symmetry group of λ_τ be $H < \mathbb{Z}^{\mathsf{k}}$. By Corollary 4.7, there is a group construction representation μ on $\ell^2(\mathbb{Z}^{\mathsf{k}})$ with full symmetry group H which has λ_τ as its unique minimal $*$-dilation. The basis $\{\zeta_n : n \in \mathbb{Z}^{\mathsf{k}}\}$ is identified with a subset of the basis of $\ell^2(\mathcal{F}_\tau)$ generating a coinvariant subspace.

There is a canonical map θ of $\{\xi_w : w \in \mathcal{F}_\tau\}$ onto the basis $\{\eta_j\}$ which intertwines the actions of λ_τ and σ in the sense

$$\sigma(v)\theta(\xi_w) = \theta(\xi_{vw}) \quad \text{for all} \quad v \in \mathbb{F}_\theta^+ \text{ and } w \in \mathcal{F}_\tau.$$

Indeed, every element of $\mathcal{F}_\tau$ may be factored as $w = u\tau^{s*}$ for some $s \geq 0$ and $u \in \mathbb{F}_\theta^+$. Then we define $\theta(\xi_w) = \sigma(u)\sigma(\tau^s)^*\eta_0$. In particular, θ carries $\{\zeta_n : n \in \mathbb{Z}^{\mathsf{k}}\}$ onto the basis of a coinvariant subspace of σ.

Suppose that $\eta_j = \theta(\zeta_m) = \theta(\zeta_n)$ for $n, m \in \mathbb{Z}^{\mathsf{k}}$. Set $l = m \wedge n \in \mathbb{Z}^{\mathsf{k}}$. Then there are unique words $u, v \in \mathbb{F}_\theta^+$ of degrees $m - l$ and $n - l$ respectively so that

$$\lambda_\tau(u)\zeta_l = \zeta_m \quad \text{and} \quad \lambda_\tau(v)\zeta_l = \zeta_n.$$

Hence $\lambda_\tau(u)\lambda_\tau(v)^*\zeta_n = \zeta_m$. Therefore $\sigma(u)\sigma(v)^*\eta_j = \eta_j$. Conversely, if $\sigma(u)\sigma(v)^*\eta_j = \eta_j$ and $\theta(\zeta_n) = \eta_j$, then setting $m - n = \deg(u) - \deg(v)$, we obtain $\theta(\zeta_m) = \eta_j$.

Observe that $K_j = \{m - n : \eta_j = \theta(\zeta_m) = \theta(\zeta_n)\}$ is a subgroup of H. Indeed, it is clear that if $\theta(\zeta_m) = \theta(\zeta_n)$, then the infinite tails obtained by pulling back from the vectors ζ_m and ζ_n both coincide with the tail obtained by pulling back from η_j. Hence $m - n \in H$. Consequently, if $\deg(uv^*) = m - n$ such that $\lambda_\tau(u)\lambda_\tau(v)^*\zeta_n = \zeta_m$ and $\theta(\zeta_l) = \eta_j$, it follows that $\lambda_\tau(u)\lambda_\tau(v)^*\zeta_l = \zeta_{l+m-n}$. So if $k_i = m_i - n_i \in K_j$, and $\lambda_\tau(u_i)\lambda_\tau(v_i)^*\zeta_{n_i} = \zeta_{m_i}$, then

$$\lambda_\tau(u_2)\lambda_\tau(v_2)^*\lambda_\tau(u_1)\lambda_\tau(v_1)^*\zeta_l = \zeta_{l+k_1+k_2}.$$

So $k_1 + k_2 \in K_j$. Also, $-k \in K_j$ because $\lambda_\tau(v)\lambda_\tau(u)^*\zeta_m = \zeta_n$.

Next we note that the subgroups K_j are ordered by inclusion. That is, if $m < n$, and $\theta(\zeta_m) = \eta_i$ and $\theta(\zeta_n) = \eta_j$, then $K_j \leq K_i$. This follows since there is a word $w \in \mathbb{F}_\theta^+$ so that $\lambda_\tau(w)\zeta_m = \zeta_n$. Consequently, $\sigma(w)\eta_i = \eta_j$. If $k \in K_j$ and uv^* is the word of degree k such that $\sigma(u)\sigma(v)^*\eta_j = \eta_j$. There are unique words u', v', w' in $\mathbb{F}_\theta^+$ with the same degrees as u, v, w so that

$$\eta_j = \sigma(u)\sigma(v)^*\sigma(w)\eta_i = \sigma(w')\sigma(u')\sigma(v')^*\eta_i.$$

It follows that $\sigma(u')\sigma(v')^*\eta_i$ is the unique basis vector obtained by pulling back from η_j by $\deg(w') = \deg(w)$ steps. Hence $w' = w$ and $\sigma(u')\sigma(v')^*\eta_i = \eta_i$. So $\deg(u') - \deg(v') = \deg(u) - \deg(v) = k$ belongs to K_i.

An increasing sequence of subgroups of $\mathbb{Z}^{\mathsf{k}}$ is eventually constant. So there is a subgroup $K \leq H$ so that $K_j = K$ for all η_j in a coinvariant subspace $\mathcal{L}$ which is the image under θ of $\{\zeta_m : m \in G_T\}$ for some $T \in \mathbb{Z}^{\mathsf{k}}$. Consider the representation κ on $\ell^2((G_T + K)/K)$ induced by λ_τ obtained by collapsing cosets of K. The induced map $\widetilde{\theta}$ of $(G_T + K)/K$ onto the basis of $\mathcal{L}$ is injective, and yields a unitary equivalence. By Corollary 4.5, there is a group construction representation ρ on $\ell^2(\mathbb{Z}^k/K)$ with symmetry group H/K that dilates κ. The minimal $*$-dilation of ρ is unitarily equivalent to the minimal $*$-dilation of $P_{\mathcal{L}}\sigma|_{\mathcal{L}}$, namely σ. □

Now we consider irreducibility. Theorem 5.3 shows that it suffices to consider group construction representations. Thus the result we want follows from [2, Theorem 5.6] where it is established for the $\mathsf{k} = 2$ case, but the proof is not dependent on that restriction. A group construction on $\ell^2(G)$ with symmetry group $H \neq \{0\}$ decomposes as a direct integral or sum over the dual group $\hat{H}$ of a family of group constructions on $\ell^2(G/H)$ with identical functions $\mathfrak{t}^i$ but with different constants parameterized by $\hat{H}$.

Theorem 5.4. *An atomic $*$-representation σ with connected graph is irreducible if and only if its symmetry group is trivial.*

In general, if σ is the $*$-dilation of a group construction on $\ell^2(G)$ with symmetry group H, then σ decomposes as a direct sum or direct integral over $\hat{H}$ of the $*$-dilations of irreducible group constructions on $\ell^2(G/H)$.

The import is that a complete set of the irreducible atomic $*$-representations can be obtained by taking the inductive representation for each infinite word τ, determining the symmetry group H, and using this to construct a family of irreducible group construction representations on $\ell^2(\mathbb{Z}^{\mathsf{k}}/H)$ with different constants indexed by $\hat{H}$.

6. Finitely correlated atomic representations

A representation is *finitely correlated* if it has a finite-dimensional coinvariant cyclic subspace. By the previous section, this representation must decompose as a direct sum of group constructions—and these will necessarily be finite groups in this case.

As in the 2-graph case, the group constructions on finite groups are particularly tractable. Moreover, it is possible to provide a simple condition that determines the possible group constructions on the product groups

$G = \mathcal{C}_{n_1} \times \cdots \times \mathcal{C}_{n_\mathsf{k}}$. Every finite abelian group with k generators is a quotient of $\mathcal{C}_{n_1} \times \cdots \times \mathcal{C}_{n_\mathsf{k}}$, where n_i is the order of $\mathfrak{g}_i$.

The first lemma depends only on two families of generators at a time. So it is immediate from [2, Lemma 6.1].

Lemma 6.1. *Let σ be a representation of $\mathbb{F}_\theta^+$. Suppose that there are words e_u^i and e_v^j and a unit vector ξ such that*

$$\sigma(e_u^i)\xi = \alpha\xi \quad \text{and} \quad \sigma(e_v^j)\xi = \beta\xi$$

for some $|\alpha| = |\beta| = 1$. Then $e_u^i e_v^j = e_v^j e_u^i$.

Corollary 6.2. *If σ is a group construction representation of $\mathbb{F}_\theta^+$ on a finite group $G = \mathcal{C}_{n_1} \times \cdots \times \mathcal{C}_{n_\mathsf{k}}/H$, then there are unique words $u_i \in \mathbf{m}_i^*$ with $|u_i| = n_i$ such that $\sigma(e_{u_i}^i)\xi_0 = \alpha_i\xi_0$ where $|\alpha_i| = 1$ for $1 \le i \le \mathsf{k}$; and the $\{e_{u_i}^i\}$ all commute.*

Just as in the 2-graph case [2], the converse is also true.

Theorem 6.3. *Let $e_{u_i}^i$ be commuting words of lengths $|u_i| = n_i$, and let $\alpha_i \in \mathbb{T}$ for $1 \le i \le \mathsf{k}$. Then there is a unique group construction representation σ of $\mathbb{F}_\theta^+$ on $G = \mathcal{C}_{n_1} \times \cdots \times \mathcal{C}_{n_\mathsf{k}}$ such that $\sigma(e_{u_i}^i)\xi_0 \in \mathbb{C}\xi_0$ for $1 \le i \le \mathsf{k}$ and $\alpha_g^i = \alpha_i$ for all $g \in G$.*

Proof. It is clear that the constants α_i pose no additional complication, so we will ignore them and assume that $\alpha_i = 1$ for all i.

As in [2, Lemma 6.2], the commutation relations of the k-graph completely determine the functions $\mathfrak{t}^i$ on G. We will sketch the ideas. Write

$$u_i = j_0 j_{n_i-1} \dots j_2 j_1 =: \mathfrak{t}_0^i \mathfrak{t}_{(n_i-1)\mathfrak{g}_i}^i \dots \mathfrak{t}_{2\mathfrak{g}_i}^i \mathfrak{t}_{\mathfrak{g}_i}^i.$$

Then from the 2-graph case, one uses the commutation relations to factor $e_{u_j}^j e_{u_i}^i$ in the form $e_v^j e_{u_{is}}^i e_w^j$, where $|v| = n_j - s$ and $|w| = s$. Consider the first factorization as two loops, first moving in the $\mathfrak{g}_i$ direction starting at ξ_0 and returning to ξ_0, followed by the loop in the $\mathfrak{g}_j$ direction around to ξ_0 again. The second factorization moves first in the $\mathfrak{g}_j$ direction to $\xi_{s\mathfrak{g}_j}$, then around a loop in the $\mathfrak{g}_i$ direction through vectors $\xi_{s\mathfrak{g}_j+t\mathfrak{g}_i}$ back to $\xi_{s\mathfrak{g}_j}$ again, and then continuing on in the $\mathfrak{g}_j$ direction through the $\xi_{s'\mathfrak{g}_j}$ to ξ_0. In this way we obtain the functions $\mathfrak{t}_{s\mathfrak{g}_j+t\mathfrak{g}_i}^i$.

One by one, introduce the next term $e_{u_k}^k$ and partially commute through in order to define $\mathfrak{t}_g^i$ for all $g \in G$. The fact that $\mathbb{F}_\theta^+$ is a k-graph means that there is unique factorization, and so the result is independent of the order in which this calculation is performed. The fact that the words commute is exactly what is required in order that one returns to the original word $e_{u_i}^i$ when one loops around, passing this word past $e_{u_j}^j$. $\square$

Example 6.4. Choose the 3-graph in Example 2.3. Let $u = 112$. Then e_u, f_u, g_u are mutually commuting. Using the construction of the proof in Theorem 6.3, we obtain a 27-dimensional finitely correlated representation

σ of $\mathbb{F}_\theta^+$. It is not hard to see that σ has a nontrivial symmetry group. Moreover, σ can be decomposed into a direct sum of three-dimensional atomic representations of the following form:

$$\rho(e_1)\xi_1 = \omega_1\xi_2,\ \rho(e_1)\xi_2 = \omega_1\xi_3,\ \rho(e_2)\xi_3 = \omega_1\xi_1;$$
$$\rho(f_1)\xi_1 = \omega_2\xi_3,\ \rho(f_1)\xi_3 = \omega_2\xi_2,\ \rho(f_2)\xi_2 = \omega_2\xi_1;$$
$$\rho(g_1)\xi_1 = \omega_3\xi_3,\ \rho(g_1)\xi_3 = \omega_3\xi_2,\ \rho(g_2)\xi_2 = \omega_3\xi_1.$$

where ω_i are cube roots of unity.

We now show that there are many finitely correlated representations.

Theorem 6.5. *There are irreducible finite-dimensional defect free representations of $\mathbb{F}_\theta^+$ of arbitrarily large dimension.*

Proof. We begin with arbitrary words $u_i \in \mathbf{m}_i^*$ and consider $a_0 = e^1_{u_1}$ and $b_0 = e^2_{u_2} e^3_{u_3} \dots e^{\mathsf{k}}_{u_{\mathsf{k}}}$. The technique from [2, Proposition 6.7] produces a pair of commuting words as follows. Consider the 2-graph consisting of families

$$E = \{e^1_u : |u| = |u_1|\} \text{ and } F = \{w \in \mathbb{F}_\theta^+ : \deg(w) = (0, |u_2|, \dots, |u_{\mathsf{k}}|)\}.$$

There is a permutation $\widetilde{\theta}$ of $E \times F$ which determines the commutation relation $e^1_u w = w' e^1_{u'}$ via $\widetilde{\theta}(e^1_u, w) = (e^1_{u'}, w')$. There is a cycle beginning with (a_0, b_0), namely

$$(a_0, b_0), (a_1, b_1), \dots, (a_{n-1}, b_{n-1}), (a_n, b_n) = (a_0, b_0).$$

Then $a_i b_i = b_{i+1} a_{i+1}$ for $i \in \mathbb{Z}/n\mathbb{Z}$. Hence $a := a_{n-1} \dots a_1 a_0$ commutes with $b := b_0 b_1 \dots b_{n-1}$.

Now we can factor $b = dc$ where $\deg(c) = (0, n|u_2|, 0, \dots, 0)$ and $\deg(d) = (0, 0, n|u_3|, \dots, n|u_{\mathsf{k}}|)$. At this point, we can move to the inductive step. Suppose that we have commuting words $a^i_0 = e^i_{v_i}$ for $1 \le i \le s$ and $c_0 d_0$ where $c_0 = e^{s+1}_{v_{s+1}}$ and d_0 is a word in the remaining variables. Consider the rank $s+2$-graph $\mathbb{F}^+$ with generators

$$E_i = \{e^i_u : |u| = |v_i|\}, 1 \le i \le s+1, \text{ and } F = \{w : \deg(w) = \deg(d_0)\}.$$

We wish to build a defect free representation σ of $\mathbb{F}^+$ on $\ell^2(\mathbb{Z})$ so that for each basis vector δ_k, there are unique elements $a^i_k \in E_i$ so that

$$\sigma(a^i_k)\delta_k = \delta_k \quad \text{for} \quad 1 \le i \le s \text{ and } k \in \mathbb{Z}$$

and words $c_k \in E_{s+1}$ and $d_k \in F$ so that

$$\sigma(c_k)\delta_k = \delta_{k+1} \quad \text{and} \quad \sigma(d_k)\delta_{k+1} = \delta_k \quad \text{for} \quad k \in \mathbb{Z}.$$

It is clear that for such a representation to exist, certain commutation relations must hold. Starting at $k = 0$, we must define c_k and d_k by the rules

$$c_k d_k = d_{k+1} c_{k+1} \quad \text{for} \quad k \in \mathbb{Z}$$

and define a^i_{k+1} by the rules

$$c_k a^i_k = a^i_{k+1} c_k \quad \text{for} \quad k \in \mathbb{Z}.$$

We must also verify that $\{a^i_k : 1 \le i \le s\}$ commute, and

$$d_k a^i_{k+1} = a^i_k d_k \quad \text{for} \quad k \in \mathbb{Z}.$$

To see that this does follow, observe that since a^i_0 commutes with $d_0 c_0$, we have $c_0 a^i_0 = a^i_1 c'$; and hence

$$d_0 a^i_1 c' = d_0 c_0 a^i_0 = a^i_0 d_0 c_0 = d' a' c_0.$$

Thus by unique factorization, $d' = d_0$, $c' = c_0$ and $a' = a^i_1$. That is, $c_0 a^i_0 = a^i_1 c_0$ and $a^i_0 d_0 = d_0 a^i_1$. Also

$$a^i_1 (c_0 d_0) = c_0 a^i_0 d_0 = (c_0 d_0) a^i_1$$

and

$$a^i_1 a^j_1 c_0 = c_0 a^i_0 a^j_0 = c_0 a^j_0 a^i_0 = a^j_0 a^i_0 c_0.$$

So $\{a^i_1 : 1 \le i \le s\} \cup \{d_1 c_1 = c_0 d_0\}$ is a commuting family. The relations now follow by recursion.

As before, we see that the map taking (a^i_k, c_k, d_k) to $(a^i_{k+1}, c_{k+1}, d_{k+1})$ results from an application of a permutation of $E_1 \times \cdots \times E_{s+1} \times F$. Hence there is an integer n so that $(a^i_n, c_n, d_n) = (a^i_0, c_0, d_0)$. Therefore, identifying δ_{n+j} with δ_j, we can wrap this sequence into a finite-dimensional representation on $\operatorname{span}\{\delta_k : 0 \le k < n\}$. In particular, the words which fix δ_0 must commute by Lemma 6.1. So $\{e^i_{v_i} : 1 \le i \le s\}$ together with $e^{s+1}_{v_{s+1}} := c_{n-1} c_{n-2} \dots c_1 c_0$ and $d = d_0 d_1 \dots d_{n-1}$ form a commuting family.

Repeated application of this technique produces k commuting words. By Theorem 6.3, this gives rise to a defect free finitely correlated representation on $\ell^2(G)$ where G is a finite product group. This may have nontrivial symmetry. The first word had the form $e^1_{u'u_1}$ where u_1 was arbitrary. We can ensure that this word has no small periods; for example, take any word of length N starting with a 1 followed by N 2's. Then this cannot be the power of a word of length less than $2N$. So even once we have quotiented out by the symmetry group to obtain an irreducible representation, we have dimension at least $2N$. $\square$

7. Periodicity

In this section, we examine the periodic case in more detail. This builds on the detailed analysis of periodicity in 2-graphs in [4].

In the case of 2-graphs, periodicity is a very special property that requires rather stringent structural properties. In particular, [4, Theorem 3.1] shows that if a 2-graph $\mathbb{F}^+_\theta$ is periodic, then $H_\theta = \mathbb{Z}(a, -b)$ for some $a, b > 0$. Moreover, this occurs if and only if there is a bijection $\gamma : \mathbf{m}^a_1 \to \mathbf{m}^b_2$ so that

$$e^1_u e^2_v = e^2_{\gamma(u)} e^1_{\gamma^{-1}(v)} \quad \text{for all} \quad u \in \mathbf{m}^a_1 \text{ and } v \in \mathbf{m}^b_2.$$

Equivalently, the 2-graph with generators $E_1 = \{e^1_u : u \in \mathbf{m}^a_1\}$ and $E_2 = \{e^2_v : v \in \mathbf{m}^b_2\}$ is just a flip algebra.

The following theorem is the appropriate generalization of this result for higher rank graphs.

Theorem 7.1. *Let $\mathbb{F}_\theta^+$ be a k-graph, and let a_i, b_j be positive integers and $c_k = 0$ for $1 \le i \le p < j \le p+q < k \le \mathsf{k}$. Then the following conditions are equivalent for $\pi =: (a_1, \dots, a_p, -b_{p+1}, \dots, -b_{p+q}, 0, \dots, 0)$:*

(i) *$\mathbb{F}_\theta^+$ is π-periodic.*
(ii) *Every tail is π-periodic.*
(iii) *There is a bijection $\gamma : E \to F$, where*

$$E = \left\{ \prod_{i=1}^{p} e_{u_i}^i : u_i \in \mathbf{m}_i^{a_i} \right\} \text{ and } F = \left\{ \prod_{j=p+1}^{p+q} e_{v_j}^j : v_j \in \mathbf{m}_j^{b_j} \right\},$$

such that

$$(\dagger) \qquad ef = \gamma(e)\gamma^{-1}(f) \quad \text{for all} \quad e \in E \text{ and } f \in F;$$

and

$$e\tau = \gamma(e)\tau \quad \text{for every infinite tail } \tau \text{ and } e \in E.$$

Moreover, if $p + q = \mathsf{k}$, this condition on the tails is automatic.

Proof. If $\mathbb{F}_\theta^+$ is π-periodic, then by definition, every infinite tail τ is eventually π-periodic. Suppose that there were an infinite tail τ which is not π-periodic. Then some initial segment of τ, say τ_1, fails to be π-periodic. Therefore any infinite tail which contains the sequence τ_1 infinitely often is not eventually π-periodic, contrary to hypothesis. Hence (ii) holds. Clearly (ii) implies (i).

Suppose that (ii) holds. The 2-graph $\mathfrak{G}^+$ with generators from E and F has the property that every infinite tail is $(1, -1)$-periodic. By [4, Theorem 3.1], there is a bijection γ of E onto F such that $ef = \gamma(e)\gamma^{-1}(f)$ for all $e \in E$ and $f \in F$. If τ is any infinite tail, take any $e \in E$ and $f \in F$, and consider the tail $\tau' = fe\tau = \gamma^{-1}(f)\gamma(e)\tau$. Since τ' is π-periodic, one obtains identical tails by deleting an initial word of degree $(a_1, \dots, a_p, 0, \dots, 0)$ or an initial word of degree $(0, \dots, 0, b_{p+1}, \dots, b_{p+q}, 0, \dots, 0)$. That is $e\tau = \gamma(e)\tau$. When $p+q = \mathsf{k}$, the π-periodicity of τ is equivalent to the $(1, -1)$-periodicity of an infinite tail in $\mathfrak{G}^+$; and this follows from the existence of γ by [4, Theorem 3.1].

Finally, suppose that (iii) holds. We may factor an arbitrary infinite word τ as $\tau = fe\tau'$ for some $e \in E$ and $f \in F$. So $\tau = \gamma^{-1}(f)\gamma(e)\tau'$. To check π-periodicity of τ, we need to compare $\gamma(e)\tau$ with $e\tau$. These are equal by (iii). So (ii) holds. □

Periodicity yields nontrivial elements in the centre of $\mathrm{C}^*(\mathbb{F}_\theta^+)$ just as in the 2-graph case [4, Lemma 5.4].

Corollary 7.2. *Let $\mathbb{F}_\theta^+$ be a k-graph with π-periodicity. Using the notation from Theorem 7.1, define $W = \sum_{e \in E} \gamma(e)e^*$ in $\mathrm{C}^*(\mathbb{F}_\theta^+)$. Then W is a* *unitary in the centre of* $\mathrm{C}^*(\mathbb{F}_\theta^+)$ *satisfying* $We = \gamma(e)$ *for all* $e \in E$*; and* W *is a sum of terms of degree* $-\pi$*. Also* $W = e^*\gamma(e)$ *for any* $e \in E$*.*

Proof. Since any inductive representation σ is faithful by Theorem 3.4, we can compute within $\mathcal{B}(\mathcal{H}_\sigma)$. Take any standard basis vector ξ and consider the infinite tail τ obtained by pulling back from ξ; i.e., τ is the unique infinite tail such that ξ is in the range of $\sigma(w)$ whenever $\tau = w\tau'$. Factor $\tau = e\tau'$ for some $e \in E$. Let $\zeta = \sigma(e)^*\xi$ and $\xi' = \sigma(\gamma(e))\zeta$. Then

$$\xi' = \sigma(\gamma(e))\sigma(e)^*\xi = \sigma(W)\xi.$$

The infinite tail obtained by pulling back from ξ' is evidently $\gamma(e)\tau'$, which equals τ by Theorem 7.1(iii).

Now let $g = e^i_j$ be any generator of $\mathbb{F}_\theta^+$. We will show that $\sigma(W)$ commutes with $\sigma(g)^*$. Note that ξ is in the range of $\sigma(g)$ if and only if τ factors as $g\tau''$ (when one uses the commutation relations to move the first $e^i_{j'}$ term to the initial position). If ξ is not in the range of $\sigma(g)$, then neither is ξ', and so

$$\sigma(W)\sigma(g)^*\xi = 0 = \sigma(g)^*\xi' = \sigma(g)^*\sigma(W)\xi.$$

If ξ is in the range of $\sigma(g)$, there is a unique word $e' \in E$ so that $\tau = ge'\widetilde{\tau}$. Also factor $ge' = eg'$ for some $g' = e^i_{j'}$. Let $\eta = \sigma(g)^*\xi$, $\eta' = \sigma(g)^*\xi'$ and $\zeta' = \sigma(e')^*\eta = \sigma(g')^*\zeta$. Now $\tau = g\gamma(e')\widetilde{\tau}$ by Theorem 7.1(iii). Thus ξ' is in the range of $\sigma(g\gamma(e'))$, say $\xi' = \sigma(g\gamma(e'))\widetilde{\zeta}$. Now $\gamma(e)g' = \widetilde{g}\gamma(\widetilde{e})$ for some $\widetilde{g} = e^i_{\widetilde{j}}$ and $\widetilde{e} \in E$. But then

$$\sigma(g\gamma(e'))\widetilde{\zeta} = \xi' = \sigma(\gamma(e)g')\zeta' = \sigma(\widetilde{g}\gamma(\widetilde{e}))\zeta'.$$

It follows that $\widetilde{g} = g$, $\widetilde{e} = e'$ and $\widetilde{\zeta} = \zeta'$. Therefore

$$\sigma(g)^*\sigma(W)\xi = \sigma(g)^*\xi' = \eta' = \sigma(\gamma(e'))\sigma(e')^*\eta = \sigma(W)\sigma(g)^*\xi.$$

We conclude that W commutes with g^* for every generator of $\mathbb{F}_\theta^+$.

It is evident that $\sigma(W)$ carries the range of each $\sigma(e)$ for $e \in E$ onto the range of $\sigma(\gamma(e))$. As the ranges of $\sigma(e)$ for $e \in E$ are pairwise orthogonal and sum to the whole space, as do the ranges of $\sigma(\gamma(e))$, it follows that $\sigma(W)$ is unitary. Thus it commutes with all of $\mathrm{C}^*(\mathbb{F}_\theta^+)$; i.e., it is in the centre. The identity $We = \gamma(e)$ for $e \in E$ is clear. Also by construction,

$$\deg(\gamma(e)e^*) = \deg(\gamma(e) - \deg(e) = -\pi.$$

Now $eW = We = \gamma(e)$, and hence $W = e^*\gamma(e)$. □

We now take a closer look at this for $\mathsf{k} = 3$. The permutation θ_{12} which define the relations between $\{e^1_i\}$ and $\{e^2_j\}$ determines a function $\widetilde{\theta}_{12}$ from $\mathbf{m}_1^* \times \mathbf{m}_2^*$ so that $\widetilde{\theta}_{12}(u, v) = (u', v')$ when $e^1_u e^2_v = e^2_{v'} e^1_{u'}$.

Proposition 7.3. *Let* $\mathbb{F}_\theta^+$ *be a* 3*-graph with generating sets* $\{e_i : i \in \mathbf{l}\}$*,* $\{f_j : j \in \mathbf{m}\}$ *and* $\{g_k : k \in \mathbf{n}\}$*. Suppose that* $\mathbb{F}_\theta^+$ *has* $(a, b, -c)$ *symmetry,* *where $a, b, c \in \mathbb{N}$; and let $\gamma : \mathbf{l}^a \times \mathbf{m}^b \to \mathbf{n}^c$ satisfy (†), i.e.,*

$$e_{u_0} f_{v_0} g_{\gamma(u_1,v_1)} = g_{\gamma(u_0,v_0)} e_{u_1} f_{v_1} \quad \textit{for all} \quad (u_i, v_i) \in \mathbf{l}^a \times \mathbf{m}^b.$$

Let $\delta = \gamma\widetilde{\theta}_{12}^{-1}$. Then for all $(u_i, v_i) \in \mathbf{l}^a \times \mathbf{m}^b$,

$$e_{u_0} g_{\delta(u_1,v_0)} = g_{\gamma(u_0,v_0)} e_{u_1} \quad \textit{and} \quad f_{v_0} g_{\gamma(u_1,v_1)} = g_{\delta(u_1,v_0)} f_{v_1};$$

and conversely these relations imply that $\mathbb{F}_\theta^+$ is $(a, b, -c)$-periodic. Moreover,

$$f_{v_0} e_{u_0} g_{\delta(u_1,v_1)} = g_{\delta(u_0,v_0)} f_{v_1} e_{u_1}.$$

Proof. There is a word $w \in \mathbf{n}^c$ so that

$$g_{\gamma(u_0,v_0)} e_{u_1} f_{v_1} = e_{u_0} g_w f_{v_1} = e_{u_0} f_{v_0} g_{\gamma(u_1,v_1)}.$$

Since $g_{\gamma(u_0,v_0)} e_{u_1} = e_{u_0} g_w$, w depends only on u_0, u_1, v_0; and since

$$g_w f_{v_1} = f_{v_0} g_{\gamma(u_1,v_1)},$$

w depends only on u_1, v_0, v_1. Therefore w is a function $w = \delta(u_1, v_0)$. That is,

$$e_{u_0} g_{\delta(u_1,v_0)} = g_{\gamma(u_0,v_0)} e_{u_1} \quad \text{and} \quad f_{v_0} g_{\gamma(u_1,v_1)} = g_{\delta(u_1,v_0)} f_{v_1}.$$

Hence

$$f_{v_0} e_{u_0} g_{\delta(u_1,v_1)} = f_{v_0} g_{\gamma(u_0,v_1)} e_{u_1} = g_{\delta(u_0,v_0)} f_{v_1} e_{u_1}.$$

Fix (u_0, v_0) and (u_1, v_1). Let $\widetilde{\theta}_{12}^{-1}(u_i, v_i) = (u_i', v_i')$ for $i = 0, 1$; so that $f_{v_i} e_{u_i} = e_{u_i'} f_{v_i'}$. Also let $\delta^{-1}\gamma(u_1', v_1') = (u, v)$. Then

$$\begin{aligned} g_{\gamma\widetilde{\theta}_{12}^{-1}(u_0,v_0)} f_{v_1} e_{u_1} &= g_{\gamma(u_0',v_0')} e_{u_1'} f_{v_1'} = e_{u_0'} f_{v_0'} g_{\gamma(u_1',v_1')} \\ &= f_{v_0} e_{u_0} g_{\delta(u,v)} = f_{v_0} g_{\gamma(u_0,v)} e_u \\ &= g_{\delta(u_0,v_0)} f_v e_u. \end{aligned}$$

Hence $\delta(u_0, v_0) = \gamma\widetilde{\theta}_{12}^{-1}(u_0, v_0)$.

The converse is straightforward. By Theorem 7.1 (iii), the tail condition is automatic. Hence $\mathbb{F}_\theta^+$ is $(a, b, -c)$-periodic. □

Proposition 7.3 allows us to define some examples of periodic 3-graph algebras.

Example 7.4. Suppose that $\theta_{12} = \mathrm{id}$; i.e., the e_i's commute with the f_j's. Also suppose that $n = lm$ and fix a bijection $\gamma : \mathbf{l} \times \mathbf{m} \to \mathbf{n}$. By Proposition 7.3, we require $\delta = \gamma$. So define relations

$$e_i g_{\gamma(i',j')} = g_{\gamma(i,j')} e_{i'} \quad \text{and} \quad f_j g_{\gamma(i',j')} = g_{\gamma(i',j)} f_{j'}.$$

It is easy to check the cubic condition—so this determines a 3-graph $\mathbb{F}_\theta^+$. By Proposition 7.3, $\mathbb{F}_\theta^+$ is $(1, 1, -1)$-periodic. Its symmetry group is exactly $\mathbb{Z}(1, 1, -1)$.

Example 7.5. Suppose that $l = m$, $n = m^2$ and θ_{12} is the transposition: $\theta_{12}(i,j) = (j,i)$; i.e., $e_i f_j = f_i e_j$. Then we again identify a bijection $\gamma : \mathbf{l} \times \mathbf{m} \to \mathbf{n}$ and motivated by Proposition 7.3, define $\delta(i,j) = \gamma(j,i)$. Then define the commutation relations

$$e_i g_{\gamma(j,k)} = g_{\gamma(i,j)} e_k \quad \text{and} \quad f_i g_{\gamma(j,k)} = g_{\gamma(i,j)} f_k.$$

This is easily seen to be a 3-graph $\mathbb{F}_\theta^+$. By construction, it has $(1,1,-1)$-periodicity.

The first two variables form a 2-graph with $(1,-1)$-periodicity. Since $(1,-1,0)$ has a 0, it is convenient to look for $(3,1,-2)$-periodicity instead. One readily computes that

$$\begin{aligned} e_{i_1} e_{i_2} e_{i_3} f_j g_{\gamma(i_1',i_2')} g_{\gamma(i_3',j')} &= e_{i_1} e_{i_2} g_{\gamma(i_3,j)} e_{i_1'} f_{i_2'} g_{\gamma(i_3',j')} \\ &= e_{i_1} g_{\gamma(i_2,i_3)} e_j g_{\gamma(i_1',i_2')} e_{i_3'} f_{j'} \\ &= g_{\gamma(i_1,i_2)} e_{i_3'} g_{\gamma(j,i_1')} e_{i_2'} e_{i_3'} f_{j'} \\ &= g_{\gamma(i_1,i_2)} g_{\gamma(i_3,j)} e_{i_1'} e_{i_2'} e_{i_3'} f_{j'}. \end{aligned}$$

Thus the function $\Gamma : \mathbf{l}^3 \times \mathbf{m} \to \mathbf{n}^2$ by $\Gamma(i_1,i_2,i_3,j) = (\gamma(i_1,i_2), \gamma(i_3,j))$ is a bijection satisfying (†).

Thus the symmetry group is

$$H = \mathbb{Z}(1,1,-1) + \mathbb{Z}(3,1,-2) = \mathbb{Z}(1,-1,0) + \mathbb{Z}(1,1,-1).$$

Example 7.6. A different example can be defined when $l = m = n = 2$. Let $\mathbb{F}_\theta^+$ be the 3-graph with $\mathbb{F}_{\theta_{12}}^+$ being the flip algebra, and $\mathbb{F}_{\theta_{13}}^+$, $\mathbb{F}_{\theta_{23}}^+$ each being the square algebra. A straightforward computation shows that $\mathbb{F}_\theta^+$ has the symmetry group $H = \mathbb{Z}(1,-1,0) + \mathbb{Z}(2,0,-2)$.

Example 7.7. In [4], many examples of periodic 2-graphs were exhibited, some with surprisingly high order of periodicity. It is easy to combine a number of 2-graphs together with other variables by making them commute. So suppose that $\mathbb{F}_{\theta_{2i-1,2i}}^+$ are 2-graphs with symmetry groups $H_i = \mathbb{Z}(a_i,-b_i)$ for $1 \le i \le s$, and let $\mathbb{F}_\theta^+$ be any k-graph with symmetry group $H \le \mathbb{Z}^k$. Form a $2s+k$-graph $\mathfrak{G}_\theta^+ = \prod_{i=1}^s \mathbb{F}_{\theta_{2i-1,2i}}^+ \times \mathbb{F}_\theta^+$ by declaring that variables in the different factors of the product commute. Then it is routine to check that this is a $2s+k$–graph with symmetry group $H_\theta = \prod_{i=1}^s H_i \times H \le \mathbb{Z}^{2s+k}$.

8. The structure of graph C*-algebras

Kumjian and Pask [11] showed that $\mathrm{C}^*(\mathbb{F}_\theta^+)$ is simple if $\mathbb{F}_\theta^+$ is aperiodic. Robertson and Sims [19] proved the converse. In [4], we showed that in the case of a periodic 2-graph $\mathbb{F}_\theta^+$ on a single vertex, one has the more precise description $\mathrm{C}^*(\mathbb{F}_\theta^+) \cong \mathrm{C}(\mathbb{T}) \otimes \mathfrak{A}_1$ for some simple C*-algebra $\mathfrak{A}_1$. We wish to extend this result to k-graphs (on a single vertex). The proof will follow the method of [4] of constructing two approximately inner expectations.

Let H_θ be the symmetry group of $\mathbb{F}_\theta^+$. Then $H_\theta \cong \mathbb{Z}^s$ for some $s \le \mathsf{k}$. Let $h_1, \dots, h_s$ be a set of free generators, and set $\vec{h} = (h_1, \dots, h_s)$. By Corollary 7.2, each $h \in H_\theta$ determines a unitary operator W_h in the centre $\mathcal{Z}$ of $\mathrm{C}^*(\mathbb{F}_\theta^+)$. It turns out that this map is a group homomorphism—but we will not establish that at this time. Instead we define $W_i = W_{h_i}$ and use these as generators for an abelian algebra which will eventually turn out to be all of $\mathcal{Z}$. For $n = (n_1, \dots, n_s) \in \mathbb{Z}^s$, write $W^n = \prod_{i=1}^s W_i^{n_i}$ and $n \cdot \vec{h} = \sum_{i=1}^s n_i h_i$.

Let $G_\theta = \mathbb{Z}^{\mathsf{k}}/H_\theta$, and let π be the quotient map. Recall that every character $\varphi \in \widehat{\mathbb{Z}^{\mathsf{k}}} \cong \mathbb{T}^{\mathsf{k}}$ determines a gauge automorphism γ_φ of $\mathrm{C}^*(\mathbb{F}_\theta^+)$ such that $\gamma_\varphi(w) = \varphi(\deg w) w$ for all $w \in \mathbb{F}_\theta^+$. Each character $\psi \in \widehat{G_\theta}$ determines the character $\psi\pi$ on $\mathbb{Z}^{\mathsf{k}}$ which takes the value 1 on H_θ. Let γ_ψ denote the corresponding gauge automorphism. We will define an expectation on $\mathrm{C}^*(\mathbb{F}_\theta^+)$ which respects the symmetry of $\mathbb{F}_\theta^+$:

$$\Psi(X) = \int_{\widehat{G_\theta}} \gamma_\psi(X)\, d\psi.$$

Lemma 8.1. *The joint spectrum of* $(W_1, \dots, W_s)$ *is* $\mathbb{T}^s$*; and*

$$\mathrm{C}^*(\mathfrak{F}, W_1, \dots, W_s) \cong \mathrm{C}(\mathbb{T}^s) \otimes \mathfrak{F} \cong \mathrm{C}(\mathbb{T}^s, \mathfrak{F}).$$

Proof. If χ is any character of H_θ, there is a character φ in $\widehat{\mathbb{Z}^{\mathsf{k}}}$ which extends χ. Therefore

$$\gamma_\varphi(W^n) = \varphi(n \cdot \vec{h}) W^n = \chi(n) W^n.$$

Thus γ_φ restricts to an automorphism of $\mathrm{C}^*(W_1, \dots, W_s)$ and for any $\lambda_i \in \mathbb{T}$, there is some φ so that $\gamma_\varphi(W_i) = \lambda_i W_i$ for $1 \le i \le s$. So the joint spectrum $\sigma(W_1, \dots, W_s)$ is invariant under the transitive action of the torus. Hence $\sigma(W_1, \dots, W_s) = \mathbb{T}^s$ and $\mathrm{C}^*(W_1, \dots, W_s) \cong \mathrm{C}(\mathbb{T}^s)$.

There is a canonical map of the tensor product $\mathrm{C}(\mathbb{T}^s) \otimes \mathfrak{F} \cong \mathrm{C}(\mathbb{T}^s, \mathfrak{F})$ onto $\mathrm{C}^*(\mathfrak{F}, W_1, \dots, W_s)$ which sends the constant functions onto $\mathfrak{F}$ and sends z_i to W_i. Since $\mathfrak{F}$ is simple, the kernel consists of all functions vanishing on some closed subset of $\mathbb{T}^s$. However, since the joint spectrum of $(W_1, \dots, W_s)$ is all of $\mathbb{T}^s$, this set must be empty; and this map is an isomorphism. $\square$

Theorem 8.2. *The map* Ψ *is a faithful, completely positive, approximately inner expectation onto* $\mathrm{C}^*(\mathfrak{F}, W_1, \dots, W_s)$.

Proof. Since Ψ is the average of automorphisms, it is clearly faithful and completely positive.

Let $w = uv^*$, where $u, v \in \mathbb{F}_\theta^+$, be a typical word in $\mathrm{C}^*(\mathbb{F}_\theta^+)$. Then $\deg(w) = \deg(u) - \deg(v)$ is a homomorphism into $\mathbb{Z}^{\mathsf{k}}$. The kernel, the words of zero degree, generate $\mathfrak{F}$ as a C^*-algebra. If $\deg(w) = h \in H_\theta$, then $\pi \deg(w) = 0$; so for any $\psi \in \widehat{G_\theta}$, we have $\gamma_\psi(w) = w$. Therefore $\Psi(w) = w$.

On the other hand, if $\deg(w) \notin H_\theta$, then $\pi \deg(w) = g \neq 0$. Therefore

$$\Psi(w) = \int_{\widehat{G_\theta}} \psi(g)\, d\psi \ w = 0.$$

By Corollary 7.2, W_i is a sum of words of degree $-h_i$. Hence W^n is a sum of words of degree $-n \cdot \vec{h}$. Thus if $\deg(w) = h = n \cdot \vec{h} \in H_\theta$, then wW^n is a sum of words of degree 0, and so lies in $\mathfrak{F}$. Therefore $w = FW^{*n}$ belongs to $\mathrm{C}^*(\mathfrak{F}, W_1, \dots, W_s)$. Conversely, this C*-algebra is spanned by terms of the form uv^*W^n where $\deg(uv^*) = 0$ and $n \in \mathbb{Z}^s$. We have seen that $\Psi(uv^*W^n) = uv^*W^n$. Thus Ψ is an expectation onto $\mathrm{C}^*(\mathfrak{F}, W_1, \dots, W_s)$.

Lastly, we will construct a sequence of isometries $V_n \in \mathrm{C}^*(\mathbb{F}_\theta^+)$ so that $\lim_{n\to\infty} V_n^* X V_n = \Psi(X)$ for all $X \in \mathrm{C}^*(\mathbb{F}_\theta^+)$.

By Proposition 3.7, there is an infinite tail τ with $H_\tau = H_\theta$. Therefore if $u, v \in \mathbb{F}_\theta^+$ and $\deg(uv^*) \notin H_\theta$, then $u\tau \neq v\tau$. It follows that for some initial segment τ_0 of τ, $\tau_0^* u^* v \tau_0 = 0$. So for each integer $n \geq 1$, select τ_n so that $\tau_n^*(u^*v)\tau_n = 0$ whenever $\deg(uv^*) \notin H_\theta$ and

$$\deg(u) \vee \deg(v) \leq \mathbf{n} := (n, n, \dots, n).$$

Let $\mathcal{S}_n = \{x \in \mathbb{F}_\theta^+ : \deg(x) = \mathbf{n}\}$. Define an isometry in $\mathrm{C}^*(\mathbb{F}_\theta^+)$ by

$$V_n = \sum_{x \in \mathcal{S}_n} x \tau_n x^*.$$

Suppose that $uv^* \in \mathfrak{F}$; so $\deg(u) = \deg(v)$. For n sufficiently large, $\deg(u) \leq \mathbf{n}$. Then uv^* can be written as a sum of words of the same form with $\deg(u) = \deg(v) = \mathbf{n}$. Recall that when $\deg(x) = \deg(u)$, then $x^*u = \delta_{x,u}$. Therefore

$$V_n^*(uv^*)V_n = \sum_{x \in \mathcal{S}_n} \sum_{y \in \mathcal{S}_n} x\tau_n^*(x^*u)(v^*y)\tau_n y^* = u\tau_n^*\tau_n v^* = uv^*.$$

It follows that $V_n^*(uv^*)V_n = uv^*$ whenever $\deg(u) = \deg(v) \leq \mathbf{n}$.

Next suppose that $\deg(uv^*) = h = m \cdot \vec{h} \in H_\theta$. Then $W^m(uv^*)$ belongs to $\mathbb{F}_\theta^+$. By the previous paragraph, for n sufficiently large,

$$\begin{aligned} V_n^*(uv^*)V_n &= V_n^* W^{*m}(W^m uv^*)V_n = W^{*m} V_n^*(W^m uv^*)V_n \\ &= W^{*m}(W^m uv^*) = uv^*. \end{aligned}$$

Finally, suppose that $\deg(uv^*) \notin H_\theta$. For sufficiently large n, we have $\deg(u) \vee \deg(v) \leq \mathbf{n}$. Hence if $x, y \in \mathcal{S}_n$, then $x^*(uv^*)y$ is either 0 or it has the form ab^* where $\deg(a) \vee \deg(b) \leq \mathbf{n}$ and

$$\deg(a) - \deg(b) = \deg(u) - \deg(v) \notin H_\theta.$$

Consequently,

$$x\tau_n^* x^*(uv^*)y\tau_n y^* = x(\tau_n^* ab^* \tau_n)y^* = 0.$$

Summing over $\mathcal{S}_n \times \mathcal{S}_n$ yields $V_n^*(uv^*)V_n = 0$.

We have shown that

$$\lim_{n\to\infty} V_n^*(uv^*)V_n = \Psi(uv^*)$$

for all words uv^*. Thus this limit extends to all of $\mathrm{C}^*(\mathbb{F}_\theta^+)$. $\square$

The next step is to localize at the point $\mathbf{1} = (1, \dots, 1)$ in $\sigma(W_1, \dots, W_s)$. Essentially this is evaluation at $\mathbf{1}$.

Theorem 8.3. *Let q be the quotient map of $\mathrm{C}^*(\mathbb{F}_\theta^+)$ onto*

$$\mathfrak{A} := \mathrm{C}^*(\mathbb{F}_\theta^+)/\langle W_1 - I, \dots, W_s - I\rangle.$$

Let $\varepsilon_{\mathbf{1}}$ be evaluation at $\mathbf{1}$ in $\mathrm{C}(\mathbb{T}^s, \mathfrak{F})$. Then there is a faithful, completely positive, approximately inner expectation Ψ_1 of $\mathfrak{A}$ onto $\mathfrak{F}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{C}^*(\mathbb{F}_\theta^+) \arrow[r, "q"] \arrow[d, "\Psi"'] & \mathfrak{A} \arrow[d, "\Psi_1"] \\ \mathrm{C}(\mathbb{T}^s, \mathfrak{F}) \arrow[r, "\varepsilon_{\mathbf{1}}"'] & \mathfrak{F}. \end{tikzcd}$$

Moreover, $\mathfrak{A}$ is a simple C-algebra.*

Proof. For each $\psi \in \widehat{G_\theta}$, we have $\gamma_\psi(W_i) = W_i$. Hence γ_ψ carries the ideal $\langle W_1 - I, \dots, W_s - I\rangle$ onto itself. Therefore it induces an automorphism $\dot\gamma_\psi$ of $\mathfrak{A}$; and $\dot\gamma_\psi q = q\gamma_\psi$. Define a map on $\mathfrak{A}$ by

$$\Psi_1(A) = \int_{\widehat{G_\theta}} \dot\gamma_\psi(A)\, d\psi \quad \text{for} \quad A \in \mathfrak{A}.$$

Then it follows that $\Psi_1 q = q\Psi = \varepsilon_{\mathbf{1}}\Psi$ because the restriction of q to $\mathrm{C}^*(\mathfrak{F}, W_1, \dots, W_s) \cong \mathrm{C}(\mathbb{T}^s, \mathfrak{F})$ is evidently $\varepsilon_{\mathbf{1}}$.

Since Ψ_1 is an average of automorphisms, it is a faithful, completely positive map into $\mathfrak{F}$. Now q is an isomorphism on $\mathfrak{F}$; so we may identify $q\mathfrak{F}$ in $\mathfrak{A}$ with $\mathfrak{F}$. For any $F \in \mathfrak{F}$,

$$\Psi_1(F) = \varepsilon_{\mathbf{1}}(\Psi(F)) = \varepsilon_{\mathbf{1}}(F) = F.$$

Thus Ψ_1 is an expectation.

Let $\dot V_n = q(V_n)$. Ψ_1 is approximately inner because if $A = qX \in \mathfrak{A}$,

$$\Psi_1(A) = q\Psi(X) = \lim_{n\to\infty} q(V_n^* X V_n) = \lim_{n\to\infty} \dot V_n^* A \dot V_n.$$

The fact that $\mathfrak{A}$ is simple now follows. Indeed, if $\mathfrak{J}$ is a nonzero ideal of $\mathfrak{A}$, then it contains a nonzero positive element A. Since Ψ_1 is faithful, $\Psi_1(A) \neq 0$. Also because Ψ_1 is approximately inner, $\Psi_1(A)$ belongs to $\mathfrak{F} \cap \mathfrak{J}$. Since $\mathfrak{F}$ is simple, we conclude that $\mathfrak{F} \cap \mathfrak{J} = \mathfrak{F}$ contains the identity; and hence $\mathfrak{J} = \mathfrak{A}$. $\square$

Remark 8.4. The expectation $\Psi_1 q = \varepsilon_{\mathbf{1}}\Psi$ of $\mathrm{C}^*(\mathbb{F}_\theta^+)$ onto $\mathfrak{F}$ is not the expectation Φ obtained by integrating over all gauge automorphisms. Indeed $\Phi(W_i) = 0$ while $\Psi_1 q(W_i) = \Psi_1(I) = I$.

We are now ready to obtain the structure of $\mathrm{C}^*(\mathbb{F}_\theta^+)$.

Theorem 8.5. *Suppose that the symmetry group H_θ of $\mathbb{F}_\theta^+$ has rank s. Then*

$$\mathrm{C}^*(\mathbb{F}_\theta^+) \cong \mathrm{C}(\mathbb{T}^s) \otimes \mathfrak{A}.$$

Proof. For $z = (z_1, \dots, z_s) \in \mathbb{T}^s$, let $\mathfrak{J}_z = \langle W_1 - z_1 I, \dots, W_s - z_s I\rangle$ and set $\mathfrak{A}_z := \mathrm{C}^*(\mathbb{F}_\theta^+)/\mathfrak{J}_z$. Let $\chi_z \in \widehat{H_\theta}$ be the character $\chi_z(h_i) = z_i$, and let φ be any extension to a character on $\widehat{\mathbb{Z}^{\mathsf{k}}}$. Since $\deg(W_i) = -h_i$, we have $\gamma_\varphi(W_i) = \bar{z}_i W_i$. Therefore $\gamma_\varphi(W_i - I) = \bar{z}_i(W_i - z_i I)$. So γ_φ carries the ideal $\mathfrak{J}_{\mathbf{1}}$ onto $\mathfrak{J}_z$. It follows that $\mathfrak{A}_z \cong \mathfrak{A}$ via the automorphism $\dot{\gamma}_\varphi$.

Moreover, the lifting from χ to φ can be done in a locally continuous way. That is, for any $z \in \mathbb{T}^s$, there is a neighbourhood of z on which we can select a continuous lifting. It follows that the map taking z to $q_z(X) = q(\gamma_\varphi(X))$ is a continuous function into $\mathfrak{A}$. This provides a $*$-homomorphism Θ of $\mathrm{C}^*(\mathbb{F}_\theta^+)$ into $\mathrm{C}(\mathbb{T}^s, \mathfrak{A})$.

The restriction of Θ to $\mathrm{C}(\mathbb{T}^s, \mathfrak{F})$ is readily seen to be the identity map. In particular, this restriction is an isomorphism. By Theorem 8.3, it follows that $\Theta\Psi = (\Psi_1 \otimes \mathrm{id})\Theta$. The left-hand side is faithful, and therefore Θ must be an monomorphism.

If $A \in \mathfrak{A}$ and $qX = A$, then for any $f \in \mathrm{C}(\mathbb{T}^s)$,

$$\Theta(f(W_1, \dots, W_s)X) = f(z_1, \dots, z_s)A.$$

These functions span $\mathrm{C}(\mathbb{T}^s, \mathfrak{A})$, and therefore Θ is surjective. This establishes the desired isomorphism. □

As an immediate consequence, we obtain the following characterization of the simplicity of k-graph C*-algebras, which was proved in full generality by Kumjian–Pask [11] (sufficiency) and Robertson–Sims [19] (necessity) using completely different approaches.

Corollary 8.6. $\mathrm{C}^*(\mathbb{F}_\theta^+)$ *is simple if and only if* $\mathbb{F}_\theta^+$ *is aperiodic.*

Another immediate consequence is a description of the centre of $\mathrm{C}^*(\mathbb{F}_\theta^+)$.

Corollary 8.7. *The center of* $\mathrm{C}^*(\mathbb{F}_\theta^+)$ *is* $\mathrm{C}^*(W_1, \dots, W_s) \cong \mathrm{C}(\mathbb{T}^s)$.

Corollary 7.2 defined an element W_h in $\mathcal{Z}$ for any $h \in H_\theta$. If $h = (a_1, \dots, a_n)$, then we set $E = \big\{ \prod e^i_{u_i} : a_i > 0,\ u_i \in \mathbf{m}_i^{a_i} \big\}$ and $F = \big\{ \prod e^i_{v_i} : a_i < 0,\ u_i \in \mathbf{m}_i^{-a_i} \big\}$. There is a bijection γ of E onto F so that $ef = \gamma(e)\gamma^{-1}(f)$ for $e \in E$ and $f \in F$. The central unitary W_h is then defined by the formula $W_h = \sum_{e \in E} \gamma(e)e^*$. This is the sum of words of degree $-h$.

Write $h = n \cdot \vec{h}$. In $\mathrm{C}^*(W_1, \dots, W_s)$, the unitary W^n also has $\deg(W^n) = -h$. Therefore, in the identification of $\mathrm{C}^*(W_1, \dots, W_s)$ with $\mathrm{C}(\mathbb{T}^s)$ which sends W_i to z_i, both W_h and W^n are sent to a scalar multiple of z^n. However, it is clear from the commutation relations in $\mathbb{F}_\theta^+$ that multiplying out the product $W_1^{n_1} \dots W_s^{n_s}$ will be a sum of words of the form uv^* with no scalars, as is W_h. So they must be equal.

Since $\mathcal{Z} = \mathrm{C}^*(W_1, \dots, W_s) = \mathrm{span}\{W^n : n \in \mathbb{Z}^s\}$, we deduce:

Corollary 8.8. *The map taking $h \in H_\theta$ to W_h is a group homomorphism into the unitary group of $\mathcal{Z}$; and $\mathcal{Z} = \mathrm{span}\{W_h : h \in H_\theta\}$.*

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