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Distribution of the linear flow length in a honeycomb in the small-scatterer limit

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Abstract. We study the statistics of the linear flow in a punctured honeycomb lattice, or equivalently the free motion of a particle on a regular hexagonal billiard table, with holes of equal size at the corners, obeying the customary reflection rules. In the small-scatterer limit we prove the existence of the limiting distribution of the free path length with randomly chosen origin of the trajectory, and explicitly compute it.

Contents

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1. Introduction

From the regular hexagon of unit size, remove circular holes of small radius $\varepsilon>0$ centered at the vertices, obtaining the billiard table H_ε of area $|H_\varepsilon|=\frac{3\sqrt{3}}{2}-2\pi\varepsilon^2$. For each pair $(\mathbf{x},\omega)\in H_\varepsilon\times[0,2\pi]$, consider a point particle moving at unit speed on a linear trajectory, with specular reflections when reaching the boundary. The time $\tau_\varepsilon^{\mathrm{hex}}(\mathbf{x},\omega)$ it takes the particle to reach one of the holes is called the *free path length* (or *first exit time*). Equivalently, one can consider the unit honeycomb tessellation of the Euclidean plane, with "fat points" (obstacles or scatterers) of radius ε centered at the vertices $m\mathbf{e}_1+n\mathbf{j}$, $m\not\equiv n \pmod 3$, of the lattice $\Lambda_6=\mathbb{Z}\mathbf{e}_1+\mathbb{Z}\mathbf{j}=\mathbb{Z}^2\begin{pmatrix}1 & 0\\ 1/2 & \sqrt{3}/2\end{pmatrix}$, $\mathbf{j}=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, and a particle moving at unit speed and velocity ω on a linear trajectory, until it hits one of the obstacles (see Figure 1). If the initial position $\mathbf{x}$ is always chosen in a fundamental domain, the first hitting time coincides with $\tau_\varepsilon^{\mathrm{hex}}(\mathbf{x},\omega)$. This paper is concerned with estimating, as $\varepsilon\to 0^+$, the probability that $\varepsilon\tau_\varepsilon^{\mathrm{hex}}(\mathbf{x},\omega)>\xi$:

$$\mathbb{P}_\varepsilon^{\mathrm{hex}}(\xi)=\frac{1}{2\pi|H_\varepsilon|}\left|\left\{(\mathbf{x},\omega)\in H_\varepsilon\times[0,2\pi]:\varepsilon\tau_\varepsilon^{\mathrm{hex}}(\mathbf{x},\omega)>\xi\right\}\right|, \tag{1.1}$$

$\xi\in[0,\infty)$. We will prove that $\Phi^{\mathrm{hex}}(\xi)=\lim_{\varepsilon\to0^+}\mathbb{P}_\varepsilon^{\mathrm{hex}}(\xi)$ exists for all $\xi\geqslant 0$, and show how to explicitly compute this quantity.

The square lattice analog of estimating (1.1) has a long history, originating in the work of H. A. Lorentz [15] and G. Pólya [20]. In the two-dimensional situation a complete solution was given in [7], confirming the conjectural formulas from [12]. One technical tool of [7] consists of a certain three-strip partition of the unit square, associated with the continued fraction decomposition of the slope of the trajectory. Introduced in [1], this was first employed for deriving quantitative results on the statistics of the free path length in the periodic Lorentz gas in [10, 13]. The second important tool was the use of Weil estimates for Kloosterman sums, first employed in the study of problems of this type in [2, 3, 4]. A different approach [17, 18] transfers the general problem about the statistics of the free path length in $\mathbb{R}^d$ modulo a covolume one Euclidean lattice, into a problem concerning dynamics on the space of covolume one lattices (or affine lattices) in $\mathbb{R}^d$. This led to two important results [17]: the existence of the limiting distribution in the small-scatterer limit in *any* dimension $d \geqslant 2$, and the independence of the limiting distribution of the choice of the initial point when this does not belong to $\mathbb{Q}^d$. The explicit two-dimensional formulae from [12, 4, 7] can be recovered on this way in the more general form provided by the *collision operator* [19] (see also [11, 8] for calculation of the collision operator using continued fractions). However, it is still challenging to obtain explicit formulae for the limiting distribution in dimension $d \geqslant 3$, or even in dimension $d = 2$ when the initial point belongs to $\mathbb{Q}^2$. A more detailed history and presentation of the literature and of various ideas and tools involved in this and related problems, as well as a description of recent developments in the study of the periodic Lorentz gas, is provided in [14, 16].

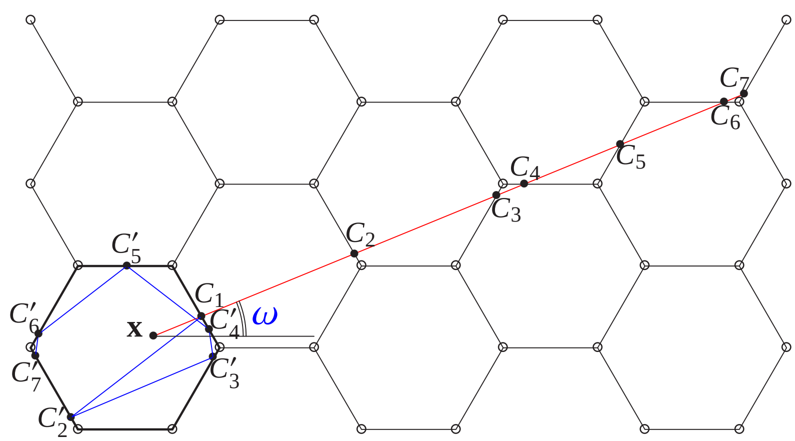

FIGURE 1. The free path in a hexagonal billiard and respectively in a hexagonal lattice

The case of the honeycomb is manifestly more involved. The version of this problem where the initial point is chosen to be the center of the hexagon has been solved in [5]. The main additional difficulty arises from the absence of a theory of continued fractions in the case of the hexagonal tessellation. To bypass this obstacle, we shall deform this tessellation, as in [5], into $\mathbb{Z}^2_{(3)} = \{(m,n) \in \mathbb{Z}^2, m \not\equiv n \pmod 3\}$. The three-strip partition of the unit square employed in the situation of the square lattice [10, 7], or equivalently the corresponding tiling of $\mathbb{R}^2$ shown in Figure 6, will be useful here. However, the presence of certain (mod 3) constraints translates here in the existence of a positive proportion of angles with very long trajectories. This leads to a large number of (nonredundant) cases that have to be analyzed individually. The main result is

Theorem 1. *There exists a decreasing continuous function* $\Phi^{\mathrm{hex}} : [0,\infty) \to (0,\infty)$, $\Phi^{\mathrm{hex}}(0) = 1$, $\Phi^{\mathrm{hex}}(\infty) = 0$, *such that for any* $\delta > 0$, *as* $\varepsilon \to 0^+$,

$$\mathbb{P}_\varepsilon^{\mathrm{hex}}(\xi) = \Phi^{\mathrm{hex}}(\xi) + O_\delta\big(\varepsilon^{1/8-\delta}\big), \quad \forall \xi \geqslant 0,$$

uniformly for ξ *in compact subsets of* $[0,\infty)$. *Moreover, there exist constants* $C_1, C_2 > 0$ *such that*

$$\frac{C_1}{\xi} \leqslant \Phi^{\mathrm{hex}}(\xi) \leqslant \frac{C_2}{\xi}, \quad \forall \xi \in [1,\infty). \tag{1.2}$$

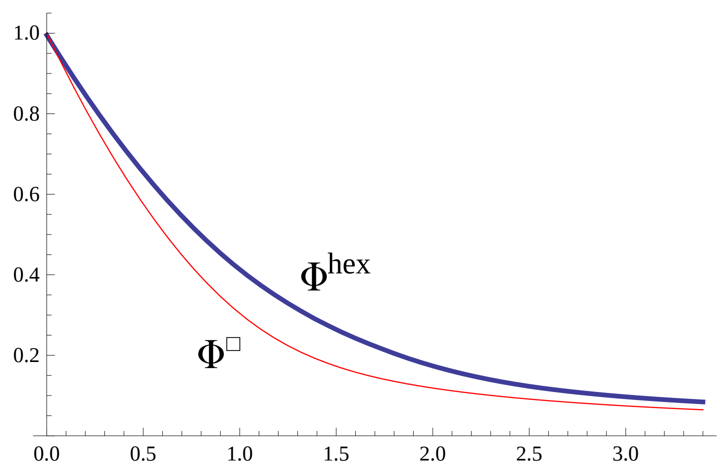

FIGURE 2. The limiting repartition functions Φ^{hex} and $\Phi^{\square}$

Estimate (1.2) is discussed in Remark 5.3. The repartition function Φ^{hex} can be explicitly computed as

$$\Phi^{\mathrm{hex}}(\xi) = \frac{4}{\pi^2}\, G\left(\frac{2\xi}{\sqrt{3}}\right) = \frac{4}{\pi^2} \sum_{k=0}^{40} G_k\left(\frac{2\xi}{\sqrt{3}}\right), \tag{1.3}$$

where[1]

$$G_0(\xi) = \int_0^1 du \left(\int_{1-u}^1 dw\ F_{(0.1)}(\xi; u, w) + \int_1^\infty dw\ F_{(0.2)}(\xi; u, w) \right), \tag{1.4}$$

[1]To keep notation short denote throughout $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, $x_+ = \max\{x, 0\}$.

with

$$F_{(0.1)}(\xi;u,w) = \frac{(w+u-1)^2(2w+u)}{w^2(w+u)^2}(u-\xi)_+ + \frac{(w+u-1)^2}{w(w+u)^2}(w-\xi)_+ + \frac{w+u-1}{w(w+u)}\left(\frac{2-u-w}{w+u}+\frac{1-u}{w}\right)(w+u-\xi)_+,$$

$$F_{(0.2)}(\xi;u,w) = \frac{1-u}{(w-u)w}\left(\frac{w+u-1}{w}+\frac{w-1}{w-u}\right)(u-\xi)_+ + \frac{(1-u)^2}{(w-u)^2w}(w-\xi)_+ + \frac{(1-u)^2}{(w-u)w^2}(w+u-\xi)_+.$$

(1.5) $$G_1(\xi) = \frac{1}{8}\int_0^1 du \int_{2-u}^{\infty} dw\, F_{(1.1)}(\xi;u,w), \quad \text{with}$$

$$F_{(1.1)}(\xi;u,w) = \frac{(1-u)^2}{(w-u)^2w}\Big((w+u-\xi)_+\wedge u + (w+u-\xi)_+\wedge w\Big) + \frac{1-u}{(w-u)w}\left(\frac{w+u-2}{w-u}+\frac{w+2u-2}{w}\right)\cdot(2u-\xi)_+\wedge u + \frac{(1-u)^2}{(w-u)w^2}\Big((w+2u-\xi)_+\wedge u + (w+2u-\xi)_+\wedge(w+u)\Big).$$

(1.6) $$G_2(\xi) = \frac{1}{8}\int_0^1 du \int_1^{2-u} dw\, F_{(1.2.1)}(\xi;u,w), \quad \text{with}$$

$$F_{(1.2.1)}(\xi;u,w) = \frac{(2-u-w)^2}{4(w-u)w^2}\Big((2w+u-\xi)_+\wedge w + (2w+u-\xi)_+\wedge(w+u)\Big) + \frac{2-u-w}{2(w-u)w}\left(\frac{2w+u-2}{2w}+\frac{w-1}{w-u}\right)\cdot\Big((w+u-\xi)_+\wedge u + (w+u-\xi)_+\wedge w\Big) + \frac{(2-u-w)^2}{2(w-u)^2w}\cdot(2w-\xi)_+\wedge w.$$

(1.7) $$G_3(\xi) = \frac{1}{8}\int_0^1 du \int_1^{2-u} dw\, F_{(1.2.2)}(\xi;u,w), \quad \text{with}$$

$$\begin{aligned}
&F_{(1.2.2)}(\xi;u,w)\\
&= \frac{(2-u-w)^2}{4w^2(w+u)}\Big((2w+u-\xi)_+\wedge w+(2w+u-\xi)_+\wedge(w+u)\Big)\\
&\quad+\frac{2-u-w}{2w(w+u)}\left(\frac{w+u-1}{w+u}+\frac{2w+u-2}{2w}\right)\\
&\qquad\cdot\Big((w+u-\xi)_+\wedge u+(w+u-\xi)_+\wedge w\Big)\\
&\quad+\frac{(2-u-w)^2}{2w(w+u)^2}\cdot(2w+2u-\xi)_+\wedge(w+u).
\end{aligned}$$

$$G_4(\xi)=\frac{1}{8}\int_0^1 du\int_{1-u}^{2-2u} dw\, F_{(1.2.3)}(\xi;u,w),\quad \text{with} \tag{1.8}$$

$$\begin{aligned}
&F_{(1.2.3)}(\xi;u,w)\\
&= \frac{(w+u-1)^2}{w(w+u)^2}\Big((w+u-\xi)_+\wedge u+(w+u-\xi)_+\wedge w\Big)\\
&\quad+\frac{w+u-1}{w(w+u)}\left(\frac{2-u-w}{w+u}+\frac{2-2u-w}{w}\right)\cdot(2w+2u-\xi)_+\wedge(w+u)\\
&\quad+\frac{(w+u-1)^2}{w^2(w+u)}\Big((w+2u-\xi)_+\wedge u+(w+2u-\xi)_+\wedge(w+u)\Big).
\end{aligned}$$

$$G_5(\xi)=\frac{1}{8}\int_0^1 du\int_{2-2u}^{2-u} dw\, F_{(1.2.4)}(\xi;u,w),\quad \text{with} \tag{1.9}$$

$$\begin{aligned}
F_{(1.2.4)}(\xi;u,w) &= \frac{(w+2u-2)^2}{4u^2w}\Big((w+u-\xi)_+\wedge u+(w+u-\xi)_+\wedge w\Big)\\
&\quad+\frac{w+2u-2}{2uw}\left(\frac{2-u-w}{2u}+\frac{1-u}{w}\right)\\
&\qquad\cdot\Big((w+2u-\xi)_+\wedge u+(w+2u-\xi)_+\wedge(w+u)\Big)\\
&\quad+\frac{(w+2u-2)^2}{2uw^2}\cdot(2u-\xi)_+\wedge u.
\end{aligned}$$

$$G_6(\xi)=\frac{1}{8}\int_0^1 du\int_{2-2u}^{2-u} dw\, F_{(1.2.5)}(\xi;u,w),\quad \text{with} \tag{1.10}$$

$$
\begin{aligned}
&F_{(1.2.5)}(\xi;u,w)\\
&= \frac{(2-u-w)^2}{4u^2(w+u)}\Big((w+2u-\xi)_+\wedge u + (w+2u-\xi)_+\wedge(w+u)\Big)\\
&\quad + \frac{2-u-w}{2u(w+u)}\left(\frac{w+u-1}{w+u}+\frac{w+2u-2}{2u}\right)\\
&\qquad \cdot\Big((w+u-\xi)_+\wedge u + (w+u-\xi)_+\wedge w\Big)\\
&\quad + \frac{(2-u-w)^2}{2u(w+u)^2}\cdot(2w+2u-\xi)_+\wedge(w+u).
\end{aligned}
$$

(1.11) $$G_7(\xi) = \frac{1}{8}\int_0^1 du \int_1^{2-u} dw\, F_{(2.1)}(u,w), \quad \text{with}$$

$$F_{(2.1)}(\xi;u,w) = \frac{2-u-w}{(w-u)(w+u)}\left(\frac{w+u-1}{w+u}+\frac{w-1}{w-u}\right)\cdot(w+u-\xi)_+\wedge u.$$

(1.12) $$\begin{aligned}G_8(\xi) &= \frac{1}{8}\int_0^1 du \int_{(\xi\vee 1)\wedge 2-u}^{2-u} dw\, F_{(2.2.1)}(\xi;u,w)\\ &\quad + \frac{1}{8}\int_0^1 du\int_{\xi\vee 2-u}^{\infty} dw\, F_{(2.2.2)}(\xi;u,w), \quad \text{with}\end{aligned}$$

$$
\begin{aligned}
F_{(2.2.1)}(\xi;u,w) &= \frac{(w+u-1)^2}{w(w+u)}\left(\frac{w+u}{w}-\frac{\xi}{w+u}\right),\\
F_{(2.2.2)}(\xi;u,w) &= \frac{(1-u)^2(w+u-\xi)}{(w-u)^2 w}\\
&\quad + \frac{(1-u)u}{(w-u)w}\left(\frac{w+u-1}{w}+\frac{w+u-2}{w-u}\right).
\end{aligned}
$$

(1.13) $$
\begin{aligned}
&G_9(\xi)\\
&= \frac{1}{8}\sum_{1\leqslant N<\xi}\int_0^1 du\int_{[\xi-u,\xi]\cap[N,N+1-u]} dw\, F_{(N;2.3.1)}(\xi;u,w)\\
&\quad + \frac{1}{8}\sum_{1\leqslant N<\xi}\int_0^1 du\int_{[\xi-u,\xi]\cap[N+1-u,N+1]} dw\, G_{(N;2.3.1)}(\xi;u,w)\\
&\quad + \frac{1}{8}\sum_{1\leqslant N<\xi}\int_0^1 du\int_{[\xi-u,\xi]\cap[N+1,N+2-u]} dw\, H_{(N;2.3.1)}(\xi;u,w),
\end{aligned}
$$

with

$$\begin{aligned} F_{(N;2.3.1)}(\xi;u,w) &= \frac{(w+u-\xi)(w-N)^2}{(w-Nu)^2 w}, \\ G_{(N;2.3.1)}(\xi;u,w) &= \frac{w+u-\xi}{(w+u)^2 w}, \\ H_{(N;2.3.1)}(\xi;u,w) &= \frac{(w+u-\xi)(N+2-u-w)}{\big(w-(N+1)u\big)(w+u)} \\ &\quad \cdot \left(\frac{1}{w+u} + \frac{w-(N+1)}{w-(N+1)u} \right). \end{aligned}$$

$$\begin{aligned} (1.14)\quad G_{10}(\xi) &= \frac{1}{8} \sum_{1\leqslant N<\xi} \int_0^1 du \int_{[\xi,\xi+u]\cap[N+1,N+2]} dw\, F_{(N;2.3.2)}(\xi;u,w) \\ &\quad + \frac{1}{8} \sum_{1\leqslant N<\xi} \int_0^1 du \int_{[\xi,\xi+u]\cap[N+2,\infty)} dw\, G_{(N;2.3.2)}(\xi;u,w), \quad \text{with} \end{aligned}$$

$$\begin{aligned} F_{(N;2.3.2)}(\xi;u,w) &= \frac{w-(N+1)}{\big(w-(N+1)u\big)w} \cdot \left(2 - \frac{\xi}{w} - \frac{\xi(1-u)}{w-(N+1)u} \right), \\ G_{(N;2.3.2)}(\xi;u,w) &= \frac{1-u}{\big(w-(N+2)u\big)\big(w-(N+1)u\big)} \\ &\quad \cdot \left(2 - \frac{\xi(1-u)}{w-(N+2)u} - \frac{\xi(1-u)}{w-(N+1)u} \right). \end{aligned}$$

$$\begin{aligned} (1.15)\quad G_{11}(\xi) =& \frac{1}{8} \sum_{n=2}^{\infty} \int_{1-\frac{1}{n-1}}^{1} du \int_{(n-1)(1-u)}^{1} dw\, F_{(n;2.4.1)}(\xi;u,w) \\ &+ \frac{1}{8} \sum_{n=2}^{\infty} \int_0^{1-\frac{1}{n-1}} du \int_{(n-1)(1-u)}^{n(1-u)} dw\, F_{(n;2.4.1)}(\xi;u,w) \\ &+ \frac{1}{8} \sum_{n=2}^{\infty} \int_{1-\frac{1}{n-1}}^{1-\frac{1}{n}} du \int_{1}^{n(1-u)} dw\, F_{(n;2.4.1)}(\xi;u,w), \quad \text{with} \end{aligned}$$

$$\begin{aligned} F_{(n;2.4.1)}(\xi;u,w) &= \frac{\big(w-(n-1)(1-u)\big)^2}{w\big(2w+(n-1)u\big)^2} \cdot (2w+nu-\xi)_+ \wedge w \\ &\quad + \frac{w-(n-1)(1-u)}{w\big(2w+(n-1)u\big)} H_{(n;2.4.1)}(\xi;u,w), \quad \text{with} \end{aligned}$$

$$
\begin{aligned}
&H_{(n;2.4.1)}(\xi;u,w)\\
&= \left(\frac{1-u}{w} - \frac{n(1-u)-w}{w}\right)\cdot\big(2w+(n+1)u-\xi\big)_+ \wedge (w+u)\\
&\quad + \left(\frac{1+n(1-u)-w}{2w+(n-1)u} + \frac{n(1-u)-w}{w}\right)\cdot(2w+nu-\xi)_+ \wedge (w+u).
\end{aligned}
$$

$$
G_{12}(\xi) = \frac{1}{8}\sum_{n=1}^{\infty}\int_0^1 du \int_{n(1-u)\vee 1}^{n(1-u)+1} dw\, F_{(n;2.4.2)}(\xi;u,w), \tag{1.16}
$$

$$
G_{13}(\xi) = \frac{1}{8}\sum_{n=2}^{\infty}\int_0^1 du \int_{n(1-u)\wedge 1}^{1} dw\, F_{(n;2.4.2)}(\xi;u,w), \quad \text{with} \tag{1.17}
$$

$$
\begin{aligned}
&F_{(n;2.4.2)}(\xi;u,w) =\\
&\frac{2-u}{\big(2w+(n-1)u\big)(2w+nu)}\left(\frac{1+(n+1)(1-u)-w}{2w+nu} + \frac{1+n(1-u)-w}{2w+(n-1)u}\right)\\
&\quad\cdot\big(2w+(n+1)u-\xi\big)_+ \wedge (w+u)\\
&+ \frac{2-u}{\big(2w+(n-1)u\big)(2w+nu)}\left(\frac{w-(n-1)(1-u)}{2w+(n-1)u} + \frac{w-n(1-u)}{2w+nu}\right)\\
&\quad\cdot(2w+nu-\xi)_+ \wedge w\\
&+ \frac{\big(w-n(1-u)\big)^2}{(w+nu)(2w+nu)^2}\Big(\big(2w+(n+1)u-\xi\big)_+ \wedge w - (2w+nu-\xi)_+ \wedge w\Big)\\
&- \frac{\big(1+n(1-u)-w\big)^2}{(w+nu)\big(2w+(n-1)u\big)^2}\\
&\quad\cdot\Big((2w+(n+1)u-\xi)_+ \wedge (w+u) - (2w+nu-\xi)_+ \wedge (w+u)\Big).
\end{aligned}
$$

$$
G_{14}(\xi) = \frac{1}{8}\int_0^1 du \int_{1-u}^{1} dw\, F_{(2.4.2.3)}(\xi;u,w), \quad \text{with} \tag{1.18}
$$

$$
\begin{aligned}
F_{(2.4.2.3)}(\xi;u,w) = {}& \frac{w+u-1}{(w+u)(2w+u)}\left(\frac{3-2u-w}{2w+u} + \frac{2-u-w}{w+u}\right)\\
&\cdot(2w+2u-\xi)_+ \wedge (w+u)\\
&+ \frac{(w+u-1)^2}{(w+u)^2(2w+u)}\cdot(2w+u-\xi)_+ \wedge w\\
&+ \frac{(w+u-1)^2}{(w+u)(2w+u)^2}\cdot(2w+2u-\xi)_+ \wedge w.
\end{aligned}
$$

$$
G_{15}(\xi) = \frac{1}{8}\sum_{n=0}^{\infty}\int_0^1 du \int_{n(1-u)+1}^{(n+1)(1-u)+1} dw\, F_{(n;2.4.3)}(\xi;u,w), \quad \text{with} \tag{1.19}
$$

$$
\begin{aligned}
&F_{(n;2.4.3)}(\xi;u,w)\\
&= \frac{\big(1+(n+1)(1-u)-w\big)^2}{(w-u)(2w+nu)^2}\cdot\big(2w+(n+1)u-\xi\big)_+\wedge(w+u)\\
&\quad+\frac{\big(1+(n+1)(1-u)-w\big)^2}{(w-u)^2(2w+nu)}\cdot(2w+nu-\xi)_+\wedge w\\
&\quad+\frac{1+(n+1)(1-u)-w}{(w-u)(2w+nu)}\left(\frac{w-n(1-u)}{2w+nu}+\frac{w-n(1-u)-1}{w-u}\right)\\
&\qquad\cdot\big(2w+(n+1)u-\xi\big)_+\wedge w.
\end{aligned}
$$

$$
\begin{aligned}
G_{16}(\xi) =&\frac{1}{8}\int_0^1 du\int_{1-u}^{2-u} dw\, F_{(3.1.1)}(\xi;u,w)\\
&+\frac{1}{8}\int_0^1 du\int_{2-u}^{\infty} dw\, F_{(3.1.2)}(\xi;u,w),\quad\text{with}
\end{aligned}
\tag{1.20}
$$

$$
\begin{aligned}
F_{(3.1.1)}(\xi;u,w) &= \frac{(w+u-1)^2}{(w+u)^2 w}\cdot(w+u-\xi)_+\wedge w,\\
F_{(3.1.2)}(\xi;u,v) &= \frac{(1-u)^2}{(w-u)^2 w}\cdot(w+u-\xi)_+\wedge w.
\end{aligned}
$$

$$
G_{17}(\xi) = \frac{1}{8}\int_0^1 du\int_{(1\vee(\xi-u))\wedge(2-u)}^{2-u} dw\, F_{(3.2)}(\xi;u,w),\quad\text{with}
\tag{1.21}
$$

$$
\begin{aligned}
F_{(3.2)}(\xi;u,w) = &\frac{2-u-w}{(w-u)(w+u)}\left(\frac{w+u-1}{w+u}+\frac{w-1}{w-u}\right)\\
&\cdot(w+u-\xi)_+\wedge w+\frac{(2-u-w)^2 w}{(w-u)^2(w+u)}.
\end{aligned}
$$

$$
\begin{aligned}
&G_{18}(\xi)\\
&\quad=\frac{1}{8}\sum_{1\leqslant N\leqslant\xi}\int_0^1 du\int_{\left[1+\frac{1-u}{N+1},1+\frac{1-u}{N}\right]\cap\left[\frac{\xi-u}{N+1},\frac{\xi-u}{N}\right]} dw\, F_{(N;3.3.1)}(\xi;u,w),
\end{aligned}
\tag{1.22}
$$

with

$$
F_{(N;3.3.1)}(\xi;u,w) = \frac{\big(1-u-N(w-1)\big)^2\big((N+1)w+u-\xi\big)}{(w-u)^2(Nw+u)}.
$$

$$
\begin{aligned}
&G_{19}(\xi)\\
&\quad=\frac{1}{8}\sum_{1\leqslant N\leqslant\xi}\int_0^1 du\int_{\left[1,1+\frac{1-u}{N+1}\right]\cap\left[\frac{\xi-u}{N+1},\frac{\xi-u}{N}\right]} dw\, F_{(N;3.3.2)}(\xi;u,w),\quad\text{with}
\end{aligned}
\tag{1.23}
$$

$$F_{(N;3.3.2)}(\xi;u,w)$$
$$= \frac{(N+1)w+u-\xi}{(Nw+u)\big((N+1)w+u\big)^2}$$
$$+ \frac{1-u-(N+1)(w-1)}{(w-u)\big((N+1)w+u\big)}\left(2-\frac{\xi}{(N+1)w+u}-\frac{\xi(w-1)}{w-u}\right).$$

$$G_{20}(\xi)=\frac{1}{8}\int_0^1 du\int_{2-u}^{\infty} dw\, F_{(3.4.1)}(\xi;u,w), \quad \text{with} \tag{1.24}$$

$$F_{(3.4.1)}(\xi;u,w)=\frac{(1-u)^2(2w-u)}{(w-u)^2w^2}\cdot\Big((w+2u-\xi)_+\wedge u-(2u-\xi)_+\wedge u\Big)$$
$$+\frac{1-u}{(w-u)w}\left(\frac{w+u-1}{w}+\frac{w-1}{w-u}\right)\cdot(2u-\xi)_+\wedge u$$
$$+\frac{(1-u)^2}{(w-u)w^2}\cdot(w+2u-\xi)_+\wedge(w+u).$$

$$G_{21}(\xi)=\frac{1}{8}\int_0^1 du\int_{2-2u}^{2-u} dw\, F_{(3.4.2)}(\xi;u,w), \quad \text{with} \tag{1.25}$$

$$F_{(3.4.2)}(\xi;u,w)$$
$$=\frac{w+2u-2}{2uw}\left(\frac{1-u}{w}+\frac{2-u-w}{2u}\right)\cdot(w+2u-\xi)_+\wedge(w+u)$$
$$+\frac{(w+2u-2)(w-2u+2)}{4uw^2}\cdot(w+2u-\xi)_+\wedge u$$
$$+\frac{(w+2u-2)^2}{2uw^2}\cdot(2u-\xi)_+\wedge u.$$

$$G_{22}(\xi)=\frac{1}{8}\int_{\frac{1}{2}}^1 du\int_{2-2u}^{1} dw\, F_{(3.4.3)}(\xi;u,w), \quad \text{with} \tag{1.26}$$

$$F_{(3.4.3)}(\xi;u,w)=\frac{(2-u-w)(3w+3u-2)}{4u(w+u)^2}\cdot(w+2u-\xi)_+\wedge u$$
$$+\frac{(2-u-w)^2}{2u(w+u)^2}\cdot(2w+2u-\xi)_+\wedge(w+u)$$
$$+\frac{(2-u-w)^2}{4u^2(w+u)}\cdot(w+2u-\xi)_+\wedge(w+u).$$

$$G_{23}(\xi)=\frac{1}{8}\int_0^{\frac{1}{3}} du\int_1^{\frac{3-u}{2}} dw\, F_{(3.4.4)}(\xi;u,w) \tag{1.27}$$
$$+\frac{1}{8}\int_{\frac{1}{3}}^{\frac{1}{2}} du\int_1^{2-2u} dw\, F_{(3.4.4)}(\xi;u,w), \quad \text{with}$$

$$F_{(3.4.4)}(\xi;u,w) = \frac{(w+u-1)^2(3w+2u)}{w^2(w+u)(w+2u)} \cdot (w+2u-\xi)_+ \wedge u + \left\{ \frac{2(w+u-1)}{w(w+2u)} \left(\frac{1-u}{w} + \frac{3-u-2w}{w+2u} \right) - \frac{(w+u-1)^2}{w^2(w+u)} \right\} \cdot (2w+2u-\xi)_+ \wedge (w+u) + \frac{(w+u-1)^2}{w^2(w+u)} \cdot (w+2u-\xi)_+ \wedge (w+u) + \frac{(w+u-1)^2}{(w+u)(w+2u)^2} \cdot (2w+2u-\xi)_+ \wedge u.$$

(1.28)
$$G_{24}(\xi) = \frac{1}{8} \int_{\frac{1}{3}}^{\frac{1}{2}} du \int_{2-2u}^{\frac{3-u}{2}} dw\, F_{(1;3.4.5)}(\xi;u,w) + \frac{1}{8} \int_{\frac{1}{2}}^{1} du \int_{u}^{\frac{3-u}{2}} dw\, F_{(1;3.4.5)}(\xi;u,w) + \frac{1}{8} \sum_{n=2}^{\infty} \int_0^1 du \int_1^{1+\frac{1-u}{n+1}} dw\, F_{(n;3.4.5)}(\xi;u,w), \quad \text{with}$$

$$F_{(n;3.4.5)}(\xi;u,w) = \frac{\big(n(w-1)+u\big)^2}{(nw+u)(nw+2u)^2} \cdot \big((n+1)w+2u-\xi\big)_+ \wedge u + \frac{\big(1-u-n(w-1)\big)^2}{(nw+u)\big((n-1)w+2u\big)^2} \cdot (nw+2u-\xi)_+ \wedge (w+u) + \left\{ \frac{2-w}{(nw+2u)\big((n-1)w+2u\big)} \left(\frac{(n-1)(w-1)+u}{(n-1)w+2u} + \frac{n(w-1)+u}{nw+2u} \right) - \frac{\big(n(w-1)+u\big)^2}{(nw+u)(nw+2u)^2} \right\} \cdot (nw+2u-\xi)_+ \wedge u + H_{(n;3.4.5)}(\xi;u,w) \cdot \big((n+1)w+2u-\xi\big)_+ \wedge (w+u), \quad \text{with}$$

$$H_{(n;3.4.5)}(\xi;u,w) = \frac{2-w}{(nw+2u)\big((n-1)w+2u\big)} \left(\frac{1-u-n(w-1)}{(n-1)w+2u} + \frac{1-u-(n+1)(w-1)}{nw+2u} \right) - \frac{\big(1-u-n(w-1)\big)^2}{(nw+u)\big((n-1)w+2u\big)^2}.$$

(1.29)
$$G_{25}(\xi) = \frac{1}{8} \int_0^{\frac{1}{3}} du \int_{\frac{3-u}{2}}^{2-2u} dw\, F_{(3.4.6)}(\xi;u,w), \quad \text{with}$$

$$
\begin{aligned}
&F_{(3.4.6)}(\xi;u,w))\\
&=\left\{\frac{(1-u)^2}{(w-u)w^2}-\frac{(w+u-1)^2}{(w+u)w^2}\right\}\cdot(2w+2u-\xi)_+\wedge(w+u)\\
&\quad+\left\{\frac{1-u}{(w-u)w}\left(\frac{w+u-1}{w}+\frac{w-1}{w-u}\right)-\frac{(2-w-u)^2}{(w-u)^2(w+u)}\right\}\\
&\qquad\cdot(w+2u-\xi)_+\wedge u\\
&\quad+\frac{(w+u-1)^2}{(w+u)w^2}\cdot(w+2u-\xi)_+\wedge(w+u)\\
&\quad+\frac{(2-w-u)^2}{(w-u)^2(w+u)}\cdot(2w+2u-\xi)_+\wedge u.
\end{aligned}
$$

(1.30)
$$
\begin{aligned}
G_{26}(\xi)=&\frac{1}{8}\int_0^{\frac{1}{3}}du\int_{2-2u}^{2-u}dw\,F_{(1;3.4.7)}(\xi;u,w)\\
&+\frac{1}{8}\int_{\frac{1}{3}}^{1}du\int_{\frac{3-u}{2}}^{2-u}dw\,F_{(1;3.4.7)}(\xi;u,w)\\
&+\frac{1}{8}\sum_{n=2}^{\infty}\int_0^1 du\int_{1+\frac{1-u}{n+1}}^{1+\frac{1-u}{n}}dw\,F_{(n;3.4.7)}(\xi;u,w),\quad\text{with}
\end{aligned}
$$

$$
\begin{aligned}
&F_{(n;3.4.7)}(\xi)\\
&=\frac{\big(1-u-n(w-1)\big)^2}{(w-u)(nw+2u)\big((n-1)w+2u\big)}\cdot\big((n+1)w+2u-\xi\big)_+\wedge(w+u)\\
&\quad+\left\{\frac{1-u-n(w-1)}{(w-u)\big((n-1)w+2u\big)}\left(\frac{(n-1)(w-1)+u}{(n-1)w+2u}+\frac{w-1}{w-u}\right)\right.\\
&\qquad\left.-\frac{\big(1-u-n(w-1)\big)^2}{(w-u)^2(nw+u)}\right\}\cdot(nw+2u-\xi)_+\wedge u\\
&\quad+\frac{\big(1-u-n(w-1)\big)^2}{(nw+u)\big((n-1)w+2u\big)^2}\cdot(nw+2u-\xi)_+\wedge(w+u)\\
&\quad+\frac{\big(1-u-n(w-1)\big)^2}{(w-u)^2(nw+u)}\cdot\big((n+1)w+2u-\xi\big)_+\wedge u.
\end{aligned}
$$

(1.31)
$$
G_{27}(\xi)=\frac{1}{8}\int_0^1 du\int_{2\vee(\xi-u)\vee(1+u)}^{\infty}dw\,F_{(4.1.1.1)}(\xi;u,w),\quad\text{with}
$$

$$
F_{(4.1.1.1)}(\xi)=\frac{(1-u)^2}{(w-2u)(w-u)^2}\cdot(w+u-\xi)_+\wedge w.
$$

$$(1.32)\quad G_{28}(\xi)=\frac{1}{8}\int_0^1 du\int_{(\xi\vee 2-u)\wedge 2}^{2} dw\, F_{(4.1.1.2.1)}(\xi;u,w) + \frac{1}{8}\int_0^1 du\int_{((\xi-u)\vee 1)\wedge(2-u)}^{2-u} dw\, F_{(4.1.1.2.2)}(\xi;u,w),\quad \text{with}$$

$$F_{(4.1.1.2.1)}(\xi;u,w)=\left(\frac{w+u-2}{2u(w-u)}\left(\frac{1-u}{w-u}+\frac{2-w}{2u}\right)+\frac{(2-w)^2}{4u^2w}\right) \cdot (w+u-\xi)\wedge w+\frac{(2-w)^2}{2uw},$$

$$F_{(4.1.1.2.2)}(\xi;u,w)=\frac{w-1}{w-u}\left(\frac{2-u-w}{w-u}+\frac{2-w}{w}\right)+\frac{(w-1)^2}{(w-u)^2w} \cdot (w+u-\xi)\wedge w.$$

$$(1.33)\quad G_{29}(\xi)=\frac{1}{8}\sum_{1\leqslant N\leqslant \xi}\int_0^{\xi\wedge 1} du\int_{\left[1,1+\frac{1-u}{N+1}\right]\cap\left[\frac{\xi-u}{N+1},\frac{\xi-u}{N}\right]} dw\, F_{(4.1.2)}(\xi;u,w),$$

with

$$F_{(4.1.2)}(\xi;u,w)=\frac{w-1}{w-u}\left(\frac{1-u}{w-u}+\frac{2-w}{w}\right)-\frac{(w-1)^2(\xi-u)}{(w-u)^2w}.$$

$$(1.34)\quad G_{30}(\xi) =\frac{1}{8}\sum_{1\leqslant N\leqslant \xi}\int_0^{\xi\wedge 1} du\int_{\left[1+\frac{1-u}{N+1},1+\frac{1-u}{N}\right]\cap\left[\frac{\xi-u}{N+1},\frac{\xi-u}{N}\right]} dw\, F_{(N;4.1.3)}(\xi;u,w),$$

with

$$F_{(N;4.1.3)}(\xi;u,w)=\frac{(2-w)^2\big((2N+1)w+3u-\xi\big)}{(Nw+2u)^2w} +\frac{\big((N+1)(w-1)+u-1\big)\big((N+1)w+u-\xi\big)}{(w-u)(Nw+2u)} \cdot\left(\frac{2-w}{Nw+2u}+\frac{1-u-N(w-1)}{w-u}\right).$$

$$(1.35)\quad G_{31}(\xi) =\frac{1}{8}\sum_{1\leqslant N\leqslant \xi}\int_0^{\xi\wedge 1} du\int_{\left[1+\frac{1-u}{N},2\right]\cap\left[\frac{\xi-u}{N+1},\frac{\xi-u}{N}\right]} dw\, F_{(N;4.1.4)}(\xi;u,w),$$

with

$$F_{(N;4.1.4)}(\xi;u,w) =\frac{(2-w)^2}{(Nw+2u)^2}\left(\frac{(N+1)w+u-\xi}{(N-1)w+2u}+\frac{(2N+1)w+3u-\xi}{w}\right).$$

$$G_{32}(\xi) = \frac{1}{8}\int_0^1 du \int_{2\wedge(u+\xi\vee 1)}^2 dw\, \frac{(2-w)^2}{(w-2u)w}. \tag{1.36}$$

(1.37) $G_{33}(\xi)$

$$= \frac{1}{8}\sum_{0\leqslant N\leqslant \xi}\int_0^1 du \int_{[1+u,2]\cap\left[\frac{\xi+u}{N+2},\frac{\xi+u}{N+1}\right]} dw\, F_{(N;4.2.2)}(\xi;u,w), \quad \text{with}$$

$$F_{(N;4.2.2)}(\xi;u,w) = \frac{(2-w)^2}{\big((N+2)w-2u\big)^2}\left(\frac{(N+2)w-u-\xi}{(N+1)w-2u} + \frac{(2N+4)w-3u-\xi}{w}\right).$$

$$G_{34}(\xi) = \frac{1}{8}\int_0^1 du \int_{1+u}^2 dw\, F_{(4.3.1)}(\xi;u,w), \quad \text{with} \tag{1.38}$$

$$F_{(4.3.1)}(\xi;u,w) = \frac{2-w}{(w-2u)w}\left(\frac{w-1}{w} + \frac{w-u-1}{w-2u}\right)\cdot (w-\xi)_+\wedge u.$$

$$\begin{aligned} G_{35}(\xi) =& \frac{1}{8}\int_0^1 du \int_1^2 dw\, F_{(4.3.2.1)}(\xi;u,w) \\ &+ \frac{1}{8}\int_0^1 du \int_2^\infty dw\, F_{(4.3.2.2)}(\xi;u,w), \quad \text{with} \end{aligned} \tag{1.39}$$

$$F_{(4.3.2.1)}(\xi;u,w) = \frac{(w-1)^2}{(w-u)w^2}\cdot (w-\xi)_+\wedge u,$$

$$F_{(4.3.2.2)}(\xi;u,w) = \frac{(1-u)^2}{(w-2u)^2(w-u)}\cdot (w-\xi)_+\wedge u.$$

$$\begin{aligned} G_{36}(\xi) =& \frac{1}{8}\int_0^1 du \int_{2-u}^2 dw\, F_{(4.3.3.1)}(\xi;u,w) \\ &+ \frac{1}{8}\int_0^1 du \int_2^\infty dw\, F_{(4.3.3.2)}(\xi;u,w), \quad \text{with} \end{aligned} \tag{1.40}$$

$$\begin{aligned} F_{(4.3.3.1)}(\xi;u,w) =& \frac{w+u-2}{2u(w-u)}\left(\frac{1-u}{w-u} + \frac{2-w}{2u}\right)\cdot (w+2u-\xi)_+\wedge u \\ &+ \frac{(w+u-2)^2}{2u(w-u)^2}\cdot (2u-\xi)_+\wedge u, \end{aligned}$$

$$\begin{aligned} F_{(4.3.3.2)}(\xi;u,w) =& \frac{(1-u)^2}{(w-2u)(w-u)^2}\cdot (w+2u-\xi)_+\wedge u \\ &+ \frac{1-u}{(w-2u)(w-u)}\left(\frac{w-2}{w-2u} + \frac{w+u-2}{w-u}\right) \\ &\cdot (2u-\xi)_+\wedge u. \end{aligned}$$

$$G_{37}(\xi) = \frac{1}{8}\sum_{n=1}^{\infty}\int_0^1 du \int_{1+\frac{1-u}{n+1}}^{1+\frac{1-u}{n}} dw\, F_{(n;4.3.4.1)}(\xi;u,w) + \frac{1}{8}\sum_{n=1}^{\infty}\int_0^1 du \int_{1+\frac{1-u}{n}}^{2} dw\, F_{(n;4.3.4.2)}(\xi;u,w), \quad \text{with} \tag{1.41}$$

$$\begin{aligned} & F_{(n;4.3.4.1)}(\xi;u,w) \\ & = \frac{\big((n+1)(w-1)+u-1\big)^2}{(w-u)^2(nw+2u)} \cdot (nw+2u-\xi)_+ \wedge u \\ & \quad + \frac{(n+1)(w-1)+u-1}{(w-u)(nw+2u)} \left(\frac{2-w}{nw+2u} + \frac{1-u-n(w-1)}{w-u} \right) \\ & \quad \cdot \big((n+1)w+2u-\xi\big)_+ \wedge u, \\ & F_{(n;4.3.4.2)}(\xi;u,w) \\ & = \frac{(2-w)^2}{(nw+2u)\big((n-1)w+2u\big)^2} \cdot (nw+2u-\xi)_+ \wedge u \\ & \quad + \frac{(2-w)^2}{(nw+2u)^2\big((n-1)w+2u\big)} \cdot \big((n+1)w+2u-\xi\big)_+ \wedge u. \end{aligned}$$

$$G_{38}(\xi) = \frac{1}{8}\int_0^1 du \int_1^2 dw\, F_{(4.4.1.1)}(\xi;u,w) + \frac{1}{8}\int_0^1 du \int_2^{\infty} dw\, F_{(4.4.1.2)}(\xi;u,w), \quad \text{with} \tag{1.42}$$

$$F_{(4.4.1.1)}(\xi;u,w) = \frac{(w-1)^2}{(w-u)w^2} \cdot (w-\xi)_+ \wedge (w-u),$$

$$F_{(4.4.1.2)}(\xi;u,w) = \frac{(1-u)^2}{(w-2u)^2(w-u)} \cdot (w-\xi)_+ \wedge (w-u).$$

$$G_{39}(\xi) = \frac{1}{8}\int_0^1 du \int_{1+u}^2 dw\, F_{(4.4.2)}(\xi;u,w), \quad \text{with} \tag{1.43}$$

$$F_{(4.4.2)}(\xi) = \frac{2-w}{(w-2u)w}\left(\frac{w-1}{w} + \frac{w-u-1}{w-2u} \right) \cdot (w-\xi)_+ \wedge (w-u).$$

$$G_{40}(\xi) = \frac{1}{8}\int_0^1 du \int_{1+u}^2 dw \sum_{n=0}^{\infty} F(n;4.4.3)(\xi;u,w), \quad \text{with} \tag{1.44}$$

$$\begin{aligned} & F_{(n;4.4.3)}(\xi;u,w) \\ & = \frac{(2-w)^2}{\big((n+1)w-2u\big)^2\big((n+2)w-2u\big)} \cdot \big((n+2)w-2u-\xi\big)_+ \wedge (w-u) \\ & \quad + \frac{(2-w)^2}{\big((n+1)w-2u\big)\big((n+2)w-2u\big)^2} \cdot \big((n+3)w-2u-\xi\big)_+ \wedge (w-u). \end{aligned}$$

To give an idea about the complexity of the problem, we notice that in the case of the square lattice only the term G_0 arises, with a different constant and no $\frac{2}{\sqrt{3}}$ scaling for ξ. Notice also that the limiting distribution $\Phi^{\square}$ satisfies (1.2).

In the case of a lattice it was actually proved in [17] that, for every $\mathbf{x} \in \mathbb{R}^2 \setminus \mathbb{Q}^2$, $\lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \big| \{ \omega \in [0, 2\pi) : \varepsilon \tau_\varepsilon^{\square}(\mathbf{x}, \omega) > \xi \} \big| = \Phi^{\square}(\xi)$. It would be interesting to know whether a similar result holds true in the case of the honeycomb for a generic choice of $\mathbf{x}$.

The analog problem about the free path length in a regular polygon with n sides ($n \neq 3, 4, 6$) seems to be out of reach at this time, due to lack of a tractable coding for the linear flow. In the case of the regular octogon the recent results in [21] may prove helpful.

2. Translating the problem to the square lattice with mod 3 constraints

For manifest symmetry reasons, it suffices to consider $\mathbf{x} \in H_\varepsilon$ and $\omega \in \big[0, \frac{\pi}{6}\big]$, or equivalently $t = \tan \omega \in \big[0, \frac{1}{\sqrt{3}}\big]$. We will simply write $\tau_\varepsilon^{\text{hex}}(\mathbf{x}, \omega) = \tau_\varepsilon^{\text{hex}}(\mathbf{x}, t)$. As in [5], consider the lattice $\mathbb{Z}^2 M_0$, $M_0 = \left(\begin{smallmatrix} 1 & 0 \\ 1/2 & \sqrt{3}/2 \end{smallmatrix} \right)$, and the linear transformation $T\mathbf{x} = \mathbf{x} M_0^{-1}$ on $\mathbb{R}^2$:

$$T(x, y) = \left(x - \frac{y}{\sqrt{3}}, \frac{2y}{\sqrt{3}} \right) = (x', y').$$

This maps the vertices $\big(q + \frac{a}{2}, \frac{a\sqrt{3}}{2}\big)$ of the grid of equilateral triangles of unit side onto the vertices (q, a) of the square lattice $\mathbb{Z}^2$. The vertices of the honeycomb are mapped exactly into $\mathbb{Z}^2_{(3)}$, the subset of elements of $\mathbb{Z}^2$ with $q \not\equiv a \pmod 3$ (see Figure 3). The points of the x-axis are fixed by T. The circular scatterers $S_{q,a,\varepsilon} = (x_0, y_0) + \varepsilon(\cos\theta, \sin\theta)$ with $(x_0, y_0) = \big(q + \frac{a}{2}, \frac{a\sqrt{3}}{2}\big)$ are mapped onto ellipsoidal scatterers $(x_0', y_0') + \varepsilon\big(\cos\theta - \frac{\sin\theta}{\sqrt{3}}, \frac{2\sin\theta}{\sqrt{3}}\big)$ centered at $(x_0', y_0') = (q, a) = T(x_0, y_0)$. The channel of width $w = 2\varepsilon$, bounded by the two lines of slope $t = \tan\omega$ and tangent to the circle $S_{q,a,\varepsilon}$, is mapped (see Figure 4) onto the channel of width $w' = 2\varepsilon' \cos\omega'$, bounded by the two lines tangent to the ellipse $T(S_{q,a,\varepsilon})$, of slope $t' = \tan\omega' = \Psi(t)$, where

$$\Psi : \left[0, \frac{1}{\sqrt{3}}\right] \to [0, 1], \quad t' = \Psi(t) = \frac{2t}{\sqrt{3} - t}, \quad t = \Psi^{-1}(t') = \frac{t'\sqrt{3}}{t' + 2}.$$

The intersection of these two channels and the x-axis is the horizontal segment centered at the origin, of length

$$\frac{2\varepsilon}{\sin\omega} = \frac{w}{\sin\omega} = \frac{w'}{\sin\omega'} = \frac{2\varepsilon'}{\tan\omega'}.$$

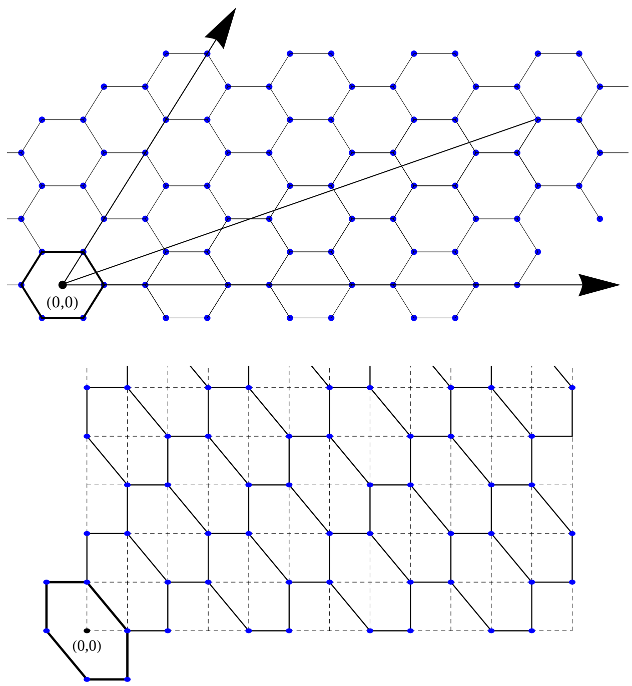

FIGURE 3. The free path length in the honeycomb and in the deformed honeycomb

In particular

$$\varepsilon' = \varepsilon'(\omega, \varepsilon) = \frac{\varepsilon \tan \omega'}{\sin \omega} = \frac{\varepsilon}{\cos(\pi/6 + \omega)}. \tag{2.1}$$

We can first replace each circular scatterer $S_{q,a,\varepsilon}$ by the segment $\tilde{S}_{q,a,\varepsilon}$ centered at (x_0, y_0), of slope $\frac{\pi}{3}$ and length $\frac{2\varepsilon}{\cos(\pi/6+\omega)} = 2\varepsilon'$ (see Figures 3 and 4). Indeed, this change will result in altering, for each ω, the free path length $\tau_\varepsilon^{\mathrm{hex}}(\mathbf{x}, t)$ to the free path length $\tilde{\tau}_\varepsilon^{\mathrm{hex}}(\mathbf{x}, t)$ corresponding to the latter model by a quantity lesser than $4\sqrt{3}\varepsilon^2$, which is insignificant for the final result.

Next we apply T to transport the problem from the honeycomb to the square lattice with congruence (mod 3) constraints (or in the opposite direction through T^{-1}). The unit regular hexagon H centered at the origin is mapped to the hexagon $T(H)$ which contains $(0, 0)$ in Figure 4. Actually it will be more convenient to replace $T(H)$ by the fundamental domain F consisting of the union of the square $[0, 1)^2$ and of its translates $[-1, 0) \times [0, 1)$ and $[0, 1) \times [-1, 0)$. Let $\varepsilon' = \varepsilon'(\omega, \varepsilon)$ be as in (2.1), $t' = \tan \omega'$ as above, and consider the vertical segment $V_{\varepsilon'} = \{0\} \times [-\varepsilon', \varepsilon']$. Consider

$$\tilde{q}_{\varepsilon'}^{\square}(\mathbf{x}', t') = \inf \left\{ n \in \mathbb{N}_0 : \mathbf{x}' + (n, nt') \in \mathbb{Z}^2_{(3)} + V_{\varepsilon'} \right\},$$

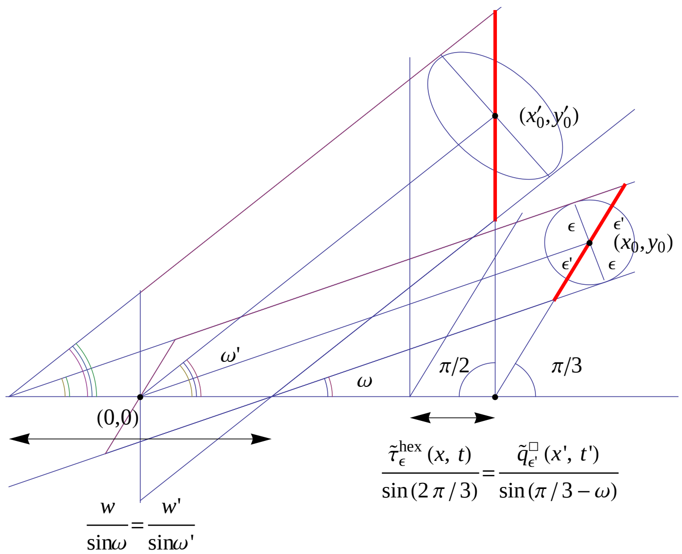

FIGURE 4. Change of scatterers under the linear transformation T

the *horizontal free path length* in the square lattice with vertical scatterers of (nonconstant) length $2\varepsilon'$ centered at points $(x_0', y_0') = (q, a) \in \mathbb{Z}^2_{(3)}$. Consider also $q^{\square}_{\varepsilon_0}(\mathbf{x}', t')$, the horizontal free path in the square lattice with vertical scatterers of *constant* length $2\varepsilon_0$ centered at points $(x_0', y_0') = (q, a) \in \mathbb{Z}^2_{(3)}$. Clearly $q^{\square}_{\varepsilon_+}(\mathbf{x}', t') \leqslant \tilde{q}^{\square}_{\varepsilon'}(\mathbf{x}', t') \leqslant q^{\square}_{\varepsilon_-}(\mathbf{x}', t')$ when t' belongs to an interval I' and $\varepsilon_- \leqslant \varepsilon' = \varepsilon'(t') \leqslant \varepsilon^+$, $\forall t' \in I'$.

For each angle ω' the transformation T maps H onto F and T^{-1} preserves the structure of channels in the corresponding three-strip partition from [1, 7, 10] (see also the expository paper [14]). Removal of vertical scatterers $V_{q,a,\varepsilon'} = T(\tilde{S}_{q,a,\varepsilon})$ with $q \equiv a \pmod 3$ in the $\mathbb{Z}^2_{(3)}$ picture results in dividing the corresponding channel of the three-strip partition from the square lattice model into several subchannels, and in the occurrence of longer trajectories associated with them. This is transported by T^{-1} back to the honeycomb model. The key observation here is that, by the Rule of Sines,

$$\frac{\tilde{\tau}^{\text{hex}}_{\varepsilon}(\mathbf{x}, t)}{\sin(2\pi/3)} = \frac{\tilde{q}^{\square}_{\varepsilon'}(T\mathbf{x}, t')}{\sin(\pi/3 - \omega)} = \frac{\tilde{q}^{\square}_{\varepsilon'}(T\mathbf{x}, t')}{\cos(\pi/6 + \omega)}.$$

This shows that
(2.2)
$$\tilde{\tau}^{\text{hex}}_{\varepsilon}(\mathbf{x}, t) > \frac{\xi}{\varepsilon} \quad \Longleftrightarrow \quad \tilde{q}^{\square}_{\varepsilon'}(T\mathbf{x}, t') > \frac{2\xi \cos(\pi/6 + \omega)}{\varepsilon\sqrt{3}} = \frac{\xi'}{\varepsilon'}, \quad \xi' := \frac{2\xi}{\sqrt{3}},$$

leading to

$$\chi_{\left(\frac{\xi}{\varepsilon},\infty\right)}\left(\tilde{\tau}_\varepsilon^{\rm hex}(\mathbf{x},t)\right) = \chi_{\left(\frac{\xi'}{\varepsilon'},\infty\right)}\left(\tilde{q}_{\varepsilon'}^{\square}(T\mathbf{x},t')\right), \quad \forall \mathbf{x}\in H, \forall t \in \left[0,\frac{1}{\sqrt{3}}\right].$$

For each interval $I=[\tan\omega_0,\tan\omega_1]\subseteq\left[0,\frac{1}{\sqrt{3}}\right]$ one has

$$\varepsilon_I^- := \frac{\varepsilon}{\cos(\pi/6+\omega_0)} \leqslant \varepsilon' = \frac{\varepsilon}{\cos(\pi/6+\omega)} \leqslant \varepsilon_I^+ := \frac{\varepsilon}{\cos(\pi/6+\omega_1)},$$
$$\varepsilon_I^\pm = \big(1+O|I|\big)\varepsilon.$$

Employing now (2.2) and the fact that $\varepsilon\mapsto\tilde{q}_\varepsilon^{\square}(T\mathbf{x},t')$ is nondecreasing, and taking

$$\xi_I^- := \frac{\xi'\varepsilon_I^-}{\varepsilon_I^+} \leqslant \xi' \leqslant \xi_I^+ := \frac{\xi'\varepsilon_I^+}{\varepsilon_I^-}, \quad \xi_I^\pm = \big(1+O(|I|)\big)\xi',$$

we infer

$$\begin{aligned}
&\chi_{\left(\frac{\xi_I^+}{\varepsilon_I^+},\infty\right)}\big(q_{\varepsilon_I^+}^{\square}(T\mathbf{x},t')\big) = \chi_{\left(\frac{\xi'}{\varepsilon_I^-},\infty\right)}\big(q_{\varepsilon_I^+}^{\square}(T\mathbf{x},t')\big) \leqslant \chi_{\left(\frac{\xi'}{\varepsilon'},\infty\right)}\big(q_{\varepsilon_I^+}^{\square}(T\mathbf{x},t')\big)\\
&\leqslant \chi_{\left(\frac{\xi'}{\varepsilon'},\infty\right)}\big(\tilde{q}_{\varepsilon'}^{\square}(T\mathbf{x},t')\big) = \chi_{\left(\frac{\xi}{\varepsilon},\infty\right)}\big(\tilde{\tau}_\varepsilon^{\rm hex}(\mathbf{x},t)\big) \leqslant \chi_{\left(\frac{\xi'}{\varepsilon'},\infty\right)}\big(q_{\varepsilon_I^-}^{\square}(T\mathbf{x},t')\big)\\
&\leqslant \chi_{\left(\frac{\xi'}{\varepsilon_I^+},\infty\right)}\big(q_{\varepsilon_I^-}^{\square}(T\mathbf{x},t')\big) = \chi_{\left(\frac{\xi_I^-}{\varepsilon_I^-},\infty\right)}\big(q_{\varepsilon_I^-}^{\square}(T\mathbf{x},t')\big).
\end{aligned}$$

Consider

$$\mathbb{G}_{I,\varepsilon}(\xi) := \int_{\Psi(I)} \frac{dt'}{t'^2+t'+1}\int_F d\mathbf{x}'\,\chi_{\left(\frac{\xi}{\varepsilon},\infty\right)}\big(q_\varepsilon^{\square}(\mathbf{x}',t')\big). \tag{2.3}$$

Applying the change of variable $(\mathbf{x}',t')=\big(T\mathbf{x},\Psi(t)\big)$ and employing (2.3), $d\mathbf{x}=\frac{\sqrt{3}}{2}d\mathbf{x}'$ and $\frac{dt}{t^2+1}=\frac{\sqrt{3}}{2}\cdot\frac{dt'}{t'^2+t'+1}$, we infer

$$\begin{aligned}
\frac{3}{4}\,\mathbb{G}_{I,\varepsilon_I^+}(\xi_I^+) &\leqslant \frac{3}{4}\int_{\Psi(I)}\frac{dt'}{t'^2+t'+1}\int_F d\mathbf{x}'\,\chi_{\left(\frac{\xi_I^+}{\varepsilon_I^+},\infty\right)}\big(q_{\varepsilon_I^+}^{\square}(\mathbf{x}',t')\big)\\
&\leqslant \widetilde{\mathbb{P}}_{I,\varepsilon}(\xi) := \iint_{H\times I}\chi_{\left(\frac{\xi}{\varepsilon},\infty\right)}\big(\tilde{\tau}_\varepsilon^{\rm hex}(\mathbf{x},\omega)\big)\,d\mathbf{x}d\omega\\
&= \int_I\frac{dt}{t^2+1}\int_H d\mathbf{x}\,\chi_{\left(\frac{\xi}{\varepsilon},\infty\right)}\big(\tilde{\tau}_\varepsilon^{\rm hex}(\mathbf{x},t)\big)\\
&\leqslant \frac{3}{4}\int_{\Psi(I)}\frac{dt'}{t'^2+t'+1}\int_F d\mathbf{x}'\,\chi_{\left(\frac{\xi_I^-}{\varepsilon_I^-},\infty\right)}\big(q_{\varepsilon_I^-}^{\square}(\mathbf{x}',t')\big)\\
&= \frac{3}{4}\,\mathbb{G}_{I,\varepsilon_I^-}(\xi_I^-).
\end{aligned}$$

To simplify notation, we simply denote $\mathbb{G}_{I,1/(2Q)}$ by $\mathbb{G}_{I,Q}$ throughout. We shall employ the following result, whose proof occupies the remaining part of the paper.

Theorem 2. *Let $c, c' > 0$ such that $c + c' < 1$. For every interval $I \subseteq [0,1]$ of length $|I| \asymp Q^c$, every $\xi \geqslant 0$ and $\delta > 0$, uniformly for ξ in compact subsets K of $[0,\infty)$,*

$$\mathbb{G}_{I,Q}(\xi) = \frac{2c_I}{\zeta(2)}\, G(\xi) + O_{\delta,K}\big(E_{c,c',\delta}(Q)\big), \tag{2.4}$$

where $G(\xi)$ is the 41 term sum described in (1.3) above, and

$$c_I = \int_I \frac{dt}{t^2+t+1}, \quad c_{[0,1]} = \frac{\pi}{3\sqrt{3}},$$

$$E_{c,c',\delta}(Q) = Q^{\max\{2c'-1/2,-c-c'\}+\delta}. \tag{2.5}$$

Proof of Theorem 1. Let $Q^- = Q_I^- := \Big\lfloor \frac{1}{2\varepsilon_I^-} \Big\rfloor + 1$ and $\frac{\varepsilon_I^-}{1+2\varepsilon_I^-} \leqslant \varepsilon^- := \frac{1}{2Q^-} \leqslant \varepsilon_I^-$. Then $q^{\square}_{\varepsilon_I^-}(\mathbf{x}',t') \leqslant q^{\square}_{\varepsilon^-}(\mathbf{x}',t')$, so

$$\chi_{(\xi_I^-/\varepsilon_I^-,\infty)}\big(q^{\square}_{\varepsilon_I^-}(\mathbf{x}',t')\big) \leqslant \chi_{(\xi_I^-/\varepsilon_I^-,\infty)}\big(q^{\square}_{\varepsilon_-}(\mathbf{x}',t')\big).$$

In a similar way, taking $Q^+ = Q_I^+ := \Big\lfloor \frac{1}{2\varepsilon_I^+} \Big\rfloor$, $\varepsilon_I^+ \leqslant \varepsilon^+ := \frac{1}{2Q^+} \leqslant \frac{\varepsilon_I^+}{1-2\varepsilon_I^+}$, we have $\chi_{(\xi_I^+/\varepsilon_I^+,\infty)}\big(q^{\square}_{\varepsilon_I^+}(\mathbf{x}',t')\big) \geqslant \chi_{(\xi_I^+/\varepsilon_I^+,\infty)}\big(q^{\square}_{\varepsilon^+}(\mathbf{x}',t')\big)$. On the other hand $\frac{\varepsilon^\pm}{\varepsilon_I^\pm} = 1 + O(\varepsilon)$, hence

$$\frac{\xi_I^\pm}{\varepsilon_I^\pm} = \frac{\xi_I^\pm}{\varepsilon^\pm} \cdot \frac{\varepsilon^\pm}{\varepsilon_I^\pm} = \big(1 + O(|I|)\big)\, \frac{\xi'}{\varepsilon^\pm}.$$

We now infer

$$\begin{aligned} \mathbb{G}_{I,Q^+}\Big(\big(1+O(|I|)\big)\xi'\Big) &\leqslant \frac{3}{4}\, \mathbb{G}_{I,\varepsilon_I^+}(\xi_I^+) \leqslant \widetilde{\mathbb{P}}_{I,\varepsilon}(\xi) \\ &\leqslant \frac{3}{4}\, \mathbb{G}_{I,\varepsilon_I^-}(\xi_I^-) \leqslant \frac{3}{4}\, \mathbb{G}_{I,Q^-}\Big(\big(1+O(|I|)\big)\xi'\Big). \end{aligned} \tag{2.6}$$

Now partition $\Big[0,\frac{1}{\sqrt{3}}\Big]$ into $N = [\varepsilon^{-c}]$ intervals $I_j = [\tan\omega_j, \tan\omega_{j+1}]$ of equal length $|I_j| = \frac{1}{N} \asymp \varepsilon^c$, with $0 < c < 1$ to be chosen later. As above, consider $Q_j^\pm = Q_{I_j}^\pm$. The intervals $\Psi(I_j)$ partition $[0,1]$ and $|\Psi(I_j)| \asymp \varepsilon^c$. Applying (2.6), Theorem 2 and the property of G of being Lipschitz on the compact K we infer

$$\begin{aligned} \mathbb{P}_\varepsilon^{\mathrm{hex}}(\xi) &= \frac{6}{\pi|H|} \sum_{j=1}^N \widetilde{\mathbb{P}}_{I_j,\varepsilon}(\xi) = \frac{\sqrt{3}}{\pi} \sum_{j=1}^N \mathbb{G}_{I_j,Q_j^\pm}(\xi) \\ &= \frac{2\sqrt{3}}{\pi\zeta(2)} \sum_{j=1}^N c_{I_j} G\bigg(\Big(1+O(|I|)\Big)\frac{2\xi}{\sqrt{3}}\bigg) + O_{\delta,K}\big(N\varepsilon^{-\max\{2c_1-1/2,-c-c_1\}}\big) \\ &= \frac{2c_{[0,1]}\sqrt{3}}{\pi\zeta(2)}\, G\left(\frac{2\xi}{\sqrt{3}}\right) + O_{\delta,K}\big(\varepsilon^{-c} + \varepsilon^{-\max\{c+2c_1-1/2,-c_1\}-\delta}\big). \end{aligned}$$

Taking $c = c_1 = \frac{1}{8}$, we find

$$\mathbb{P}_\varepsilon^{\mathrm{hex}}(\xi) = \frac{4}{\pi^2}\, G\left(\frac{2\xi}{\sqrt{3}}\right) + O_{\delta,K}\big(\varepsilon^{1/8-\delta}\big),$$

as stated in Theorem 1 and in (1.3). □

3. Some number theoretical estimates

In this section we review and prove some number theoretical estimates that will be further used to estimate certain sums over integer lattice points with congruence constraints. The principal Dirichlet character $\pmod{\ell}$ will be denoted by χ_0. The number of divisors of N is denoted by $\sigma_0(N)$.

Lemma 3.1 ([2, Lemma 2.2]). *For each function $f \in C^1[0,N]$ of total variation $T_0^N f$,*

$$\sum_{\substack{1\leqslant q\leqslant N\\ (q,\ell)=1}} f(q) = \frac{\varphi(\ell)}{\ell}\int_0^N f(x)\,dx + O\left((\|f\|_\infty + T_0^N f)\sigma_0(N)\right).$$

Lemma 3.2 ([5, Lemma 2.2]). *For each function $V \in C^1[0,N]$,*

$$\sum_{\substack{1\leqslant q\leqslant N\\ (q,\ell)=1}} \frac{\varphi(q)}{q}\, V(q) = C(\ell)\int_0^N V(x)\,dx + O_\ell\left((\|V\|_\infty + T_0^N V)\log N\right),$$

where

$$C(\ell) = \frac{\varphi(\ell)}{\zeta(2)\ell}\prod_{\substack{p\in\mathcal{P}\\ p\mid\ell}}\left(1-\frac{1}{p^2}\right)^{-1} = \frac{1}{\zeta(2)}\prod_{\substack{p\in\mathcal{P}\\ p\mid\ell}}\left(1+\frac{1}{p}\right)^{-1} = \frac{\varphi(\ell)}{\ell L(2,\chi_0)}.$$

We need a more precise form of Lemma 3.1, as follows:

Lemma 3.3. *Suppose that $(r,\ell)=1$. For each function $V \in C^1[0,N]$,*

$$\sum_{\substack{1\leqslant q\leqslant N\\ q\equiv r \pmod{\ell}}} \frac{\varphi(q)}{q}\, V(q) = \frac{C(\ell)}{\varphi(\ell)}\int_0^N V(x)\,dx + O_\ell\left((\|V\|_\infty + T_0^N V)\log N\right).$$

In particular

$$\sum_{\substack{1\leqslant q\leqslant N\\ q\equiv \pm 1 \pmod{3}}} \frac{\varphi(q)}{q}\, V(q) \sim \frac{3}{8\zeta(2)}\int_0^N V(x)\,dx,$$

$$\sum_{\substack{1\leqslant q\leqslant N\\ q\equiv 0 \pmod{3}}} \frac{\varphi(q)}{q}\, V(q) \sim \frac{1}{4\zeta(2)}\int_0^N V(x)\,dx.$$

Proof. When $(k,\ell)=1$, denote by $\bar{k}$ the multiplicative inverse of $k \pmod{\ell}$. Let $G=U(\mathbb{Z}/\ell\mathbb{Z})$ denote the multiplicative group of units of $\mathbb{Z}/\ell\mathbb{Z}$ and $\widehat{G}$ be the group of characters $\chi : G \to \mathbb{T}$, extended as multiplicative functions on $\mathbb{N}$. Set $V_d(x)=V(xd)$. By Schur's orthogonality relations for characters, for every $x,s\in\mathbb{N}$ with $(s,\ell)=1$,

$$\frac{1}{\varphi(\ell)}\sum_{\chi\in\widehat{G}}\chi(x)\chi(\bar{s})=\frac{1}{\varphi(\ell)}\sum_{\chi\in\widehat{G}}\chi(x)\,\overline{\chi(s)}=\begin{cases}1 & \text{if } x\equiv s \pmod{\ell},\\ 0 & \text{if } x\not\equiv s \pmod{\ell}.\end{cases} \tag{3.1}$$

Taking $s=r$ and summing over $x=q=md\leqslant N$, we infer by Möbius summation

$$\begin{aligned}\sum_{\substack{q\leqslant N\\ q\equiv r \ (\mathrm{mod}\ \ell)}}\frac{\varphi(q)}{q}V(q)&=\frac{1}{\varphi(\ell)}\sum_{q\leqslant N}\frac{\varphi(q)}{q}V(q)\sum_{\chi\in\widehat{G}}\chi(q)\chi(\bar{r})\\ &=\frac{1}{\varphi(\ell)}\sum_{q\leqslant N}\sum_{d\mid q}\frac{\mu(d)}{d}V(q)\sum_{\chi\in\widehat{G}}\chi(q)\chi(\bar{r})\\ &=\frac{1}{\varphi(\ell)}\sum_{d\leqslant N}\sum_{m=1}^{\lfloor N/d\rfloor}\sum_{\chi\in\widehat{G}}\frac{\mu(d)}{d}V_d(m)\chi(md)\chi(\bar{r})\\ &=\frac{1}{\varphi(\ell)}\sum_{\chi\in\widehat{G}}\chi(\bar{r})\sum_{d\leqslant N}\frac{\mu(d)\chi(d)}{d}\sum_{m=1}^{\lfloor N/d\rfloor}V_d(m)\chi(m).\end{aligned} \tag{3.2}$$

We split the inner sum above according to whether $\chi=\chi_0$ or $\chi\neq\chi_0$. Employing Lemma 3.1 for the function V_d, we find that the contribution of the former is

$$\begin{aligned}&\frac{1}{\varphi(\ell)}\sum_{d\leqslant N}\frac{\mu(d)\chi_0(d)}{d}\left(\frac{\varphi(\ell)}{\ell}\int_0^{\lfloor N/d\rfloor}V_d+O\big((\|V_d\|_\infty+T_0^{N/d}V_d)\log N\big)\right)\\ &=\frac{1}{\ell}\left(\sum_{d\leqslant N}\frac{\mu(d)\chi_0(d)}{d^2}\right)\int_0^N V+O_\ell\left((\|V\|_\infty+T_0^N V)\log N\right)\\ &=\left(\frac{1}{\ell L(2,\chi_0)}+O_\ell\Big(\frac{1}{N}\Big)\right)\int_0^N V+O_\ell\left((\|V\|_\infty+T_0^N V)\log N\right)\\ &=\frac{C(\ell)}{\varphi(\ell)}\int_0^N V+O_\ell\left((\|V\|_\infty+T_0^N V)\log N\right).\end{aligned}$$

When $\chi\neq\chi_0$, we find by partial summation and Pólya–Vinogradov (or a weaker inequality) that the innermost sum in (3.2) is $\ll_\ell T_0^N V+\|V\|_\infty$, so the total contribution of nonprincipal characters in (3.2) is $\ll_\ell \|V\|_\infty\log N$. ∎

Lemma 3.4 ([6, Proposition A4])**.** *Assume that $q \geqslant 1$ and h are integers, $\mathcal{I}$ and $\mathcal{J}$ are intervals of length less than q, and $f \in C^1(\mathcal{I} \times \mathcal{J})$. For each integer $T > 1$ and any $\delta > 0$*

$$\sum_{\substack{a \in \mathcal{I}, b \in \mathcal{J} \\ (a,q)=1 \\ ab \equiv h \pmod{q}}} f(a,b) = \frac{\varphi(q)}{q^2} \iint_{\mathcal{I} \times \mathcal{J}} f(x,y)\, dx\, dy + \mathcal{E},$$

with

$$\mathcal{E} \ll_\delta T^2 \|f\|_\infty q^{1/2+\delta} (h,q)^{1/2} + T \|\nabla f\|_\infty q^{3/2+\delta} (h,q)^{1/2} + \frac{\|\nabla f\|_\infty |\mathcal{I}|\, |\mathcal{J}|}{T},$$

where $\|f\|_\infty$ and $\|\nabla f\|_\infty$ are the sup-norm of f and respectively $\left|\frac{\partial f}{\partial x}\right| + \left|\frac{\partial f}{\partial y}\right|$ on $\mathcal{I} \times \mathcal{J}$.

For q, ℓ positive integers, denote

$$A(q,\ell) = \begin{cases} 1 & \text{if } (q,\ell) = 1, \\ \prod\limits_{\substack{p \in \mathcal{P} \\ p \mid (q,\ell)}} \left(1 - \frac{1}{p}\right)^{-1} & \text{if } (q,\ell) > 1. \end{cases}$$

Lemma 3.5. *Suppose that $(r,\ell) = 1$. For any interval $\mathcal{I}$, uniformly in $|\mathcal{I}|$,*

$$\sum_{\substack{x \in \mathcal{I},\, (x,q)=1 \\ x \equiv r \pmod{\ell}}} 1 = \frac{A(q,\ell)}{\ell} \cdot \frac{\varphi(q)}{q} |\mathcal{I}| + O\big(\sigma_0(q)\big).$$

Proof. Without loss of generality, take $\mathcal{I} = [1,N]$. As in the proof of Lemma 3.3, take $s = r$ and sum in (3.1) over $x \in \mathcal{I}$ with $(x,q) = 1$. By Möbius inversion with $x = md$, we infer

$$\begin{aligned}
\sum_{\substack{x \in \mathcal{I},\, (x,q)=1 \\ x \equiv r \pmod{\ell}}} 1 &= \frac{1}{\varphi(\ell)} \sum_{\substack{x \in \mathcal{I} \\ (x,q)=1}} \sum_{\chi \in \widehat{G}} \chi(x) \chi(\bar{r}) \\
&= \frac{1}{\varphi(\ell)} \sum_{\chi \in \widehat{G}} \sum_{x \in \mathcal{I}} \chi(x) \chi(\bar{r}) \sum_{\substack{d \mid q \\ d \mid x}} \mu(d) \\
&= \frac{1}{\varphi(\ell)} \sum_{d \mid q} \mu(d) \sum_{\substack{x \in \mathcal{I} \\ d \mid x}} \sum_{\chi \in \widehat{G}} \chi(x) \chi(\bar{r}) \\
&= \frac{1}{\varphi(\ell)} \sum_{d \mid q} \mu(d) \sum_{m=1}^{\lfloor N/d \rfloor} \sum_{\chi \in \widehat{G}} \chi(md) \chi(\bar{r}).
\end{aligned} \tag{3.3}$$

The contribution of $\chi = \chi_0$ to (3.3) is

$$\begin{aligned}(3.4)\qquad \frac{1}{\varphi(\ell)} \sum_{d|q} \mu(d)\chi_0(d) \sum_{\substack{1\leqslant m\leqslant \lfloor N/d\rfloor \\ (m,\ell)=1}} 1 &= \frac{1}{\varphi(\ell)} \sum_{d|q} \mu(d)\chi_0(d) \left(\frac{\varphi(\ell)}{\ell} \left(\frac{N}{d} + O(1) \right) \right) \\ &= \frac{|\mathcal{I}|}{\ell} \sum_{d|q} \frac{\mu(d)\chi_0(d)}{d} + O\big(\sigma_0(q)\big).\end{aligned}$$

In the contribution of nonprincipal characters to (3.3),

$$(3.5)\qquad \frac{1}{\varphi(\ell)} \sum_{d|q} \mu(d) \sum_{\substack{\chi\in\widehat{G} \\ \chi\neq\chi_0}} \chi(d)\chi(\bar{r}) \sum_{m=1}^{\lfloor N/d\rfloor} \chi(m),$$

the innermost sum is $\ll_\ell 1$ (by Pólya–Vinogradov or a weaker inequality), showing that the quantity in (3.5) is $\ll_\ell \sigma_0(q)$. The statement follows now because the sum in (3.4) is equal to $\sum_{d|q} \frac{\mu(d)}{d} = \frac{\varphi(q)}{q}$ when $(q,\ell)=1$, while when $(q,\ell)>1$, writing $q = p_1^{\alpha_1}\cdots p_r^{\alpha_r}\tilde{q}$ with $p_1,\dots,p_r$ prime divisors of ℓ and $(\tilde{q},\ell)=1$, this equals

$$\sum_{d|\tilde{q}} \frac{\mu(d)}{d} = \frac{\varphi(\tilde{q})}{\tilde{q}} = \frac{\varphi(q)}{q} \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right)^{-1} = A(q,\ell)\,\frac{\varphi(q)}{q}. \qquad \square$$

We also need a slight extension of Lemma 3.4. Suppose that $(r,\ell)=1$ and denote by $\bar{x}$ the multiplicative inverse of $x \pmod{\ell q}$ when $(x,\ell q)=1$. The Kloosterman type sums

$$\begin{aligned} K(m,n;\ell q) &:= \sum_{\substack{x \ (\mathrm{mod}\ \ell q) \\ (x,\ell q)=1}} e\left(\frac{mx+n\bar{x}}{\ell q}\right), \\ \widetilde{K}_r(m,n;\ell q) &:= \sum_{\substack{x \ (\mathrm{mod}\ \ell q) \\ (x,q)=1,\, x\equiv r (\mathrm{mod}\ \ell)}} e\left(\frac{mx+n\bar{x}}{\ell q}\right), \\ K_{\mathcal{I},r}(m,n;\ell q) &:= \sum_{\substack{x\in\mathcal{I},\,(x,q)=1 \\ x\equiv r \ (\mathrm{mod}\ \ell q)}} e\left(\frac{mx+n\bar{x}}{\ell q}\right), \end{aligned}$$

will be used to estimate

$$\begin{aligned}&\widetilde{N}_{q,\ell,r,h}(\mathcal{I}_1,\mathcal{I}_2) \\ &\qquad := \#\big\{(x,y)\in\mathcal{I}_1\times\mathcal{I}_2 : (x,q)=1, x\equiv r \ (\mathrm{mod}\ \ell), xy\equiv h \ (\mathrm{mod}\ \ell q)\big\}.\end{aligned}$$

Lemma 3.6. *When $(r,\ell)=1$, for any interval $\mathcal{I}$ of length less than q,*

$$|K_{\mathcal{I},r}(0,n;\ell q)| \ll_{\ell,\delta} (n,q)^{1/2} q^{1/2+\delta}.$$

Proof. We write

$$\begin{aligned} &K_{\mathcal{I},r}(0,n;\ell q) \\ &= \sum_{\substack{x\in\mathcal{I},\,(x,q)=1 \\ x\equiv r \pmod{\ell}}} e\left(\frac{n\bar{x}}{\ell q}\right) = \sum_{\substack{x \pmod{\ell q} \\ (x,q)=1,\, x\equiv r \pmod{\ell}}} e\left(\frac{n\bar{x}}{\ell q}\right) \sum_{y\in\mathcal{I}} \frac{1}{\ell q} \sum_{k \pmod{\ell q}} e\left(\frac{k(y-x)}{\ell q}\right) \\ &= \frac{1}{\ell q} \sum_{k \pmod{\ell q}} \widetilde{K}_r(-k,n;\ell q) \sum_{y\in\mathcal{I}} e\left(\frac{ky}{\ell q}\right), \end{aligned}$$

and

$$\begin{aligned} |\widetilde{K}_r(m,n;\ell q)| &= \left| \sum_{\substack{x \pmod{\ell q} \\ (x,\ell q)=1}} e\left(\frac{mx+n\bar{x}}{\ell q}\right) \frac{1}{\ell} \sum_{j \pmod{\ell}} e\left(\frac{j(x-r)}{\ell}\right) \right| \\ &= \frac{1}{\ell} \left| \sum_{j \pmod{\ell}} e\left(-\frac{jr}{\ell}\right) K(m+jq,n;\ell q) \right| \\ &\leqslant \max_j |K(m+jq,n;\ell q)|. \end{aligned}$$

Employing[2]

$$\left| \sum_{y\in\mathcal{I}} e\left(\frac{ky}{\ell q}\right) \right| \leqslant \min\left\{ |\mathcal{I}|+1, \frac{1}{2\left\|\frac{k}{\ell q}\right\|} \right\}, \tag{3.6}$$

$$|K(0,n;\ell q)| = |c_{\ell q}(n)| \leqslant (n,\ell q) \leqslant (n,q)\ell \ll_\ell (n,q)^{1/2} q^{1/2},$$

and the Weil estimate

$$|K(m+jq,n;\ell q)| \leqslant \sigma_0(\ell q)(m+jq,n,\ell q)^{1/2}(\ell q)^{1/2} \ll_{\ell,\delta} (n,q)^{1/2} q^{(1+\delta)/2},$$

we infer

$$\begin{aligned} |K_{\mathcal{I},r}(0,n;\ell q)| &\leqslant \frac{|\mathcal{I}|+1}{\ell q} |\widetilde{K}_r(0,n;\ell q)| + \frac{1}{\ell q} \sum_{\substack{k \pmod{\ell q} \\ k\neq 0}} \frac{|\widetilde{K}_r(-k,n;\ell q)|}{\left\|\frac{k}{\ell q}\right\|} \\ &\ll_{\ell,\delta} (n,q)^{1/2} q^{(1+\delta)/2} + \frac{1}{2\ell q} \cdot (n,q)^{1/2} q^{(1+\delta)/2} \ell q \log(\ell q) \\ &\ll_\ell (n,q)^{1/2} q^{1/2+\delta}. \end{aligned}$$

□

[2] Here $\|x\| = \operatorname{dist}(x,\mathbb{Z})$, $x\in\mathbb{R}$, and $c_q(n) = K(n,0;q) = K(0,n;q)$ is the Ramanujan sum.

Lemma 3.7. *Suppose that $(r,\ell)=1$. For any intervals $\mathcal{I}_1$ and $\mathcal{I}_2$ of length less than q, any integer h and any $\delta>0$,*

$$\widetilde{N}_{q,\ell,r,h}(\mathcal{I}_1,\mathcal{I}_2)=\frac{A(q,\ell)}{\ell^2}\cdot\frac{\varphi(q)}{q^2}\,|\mathcal{I}_1|\,|\mathcal{I}_2|+O_{\ell,\delta}\Big((h,q)^{1/2}q^{1/2+\delta}\Big).$$

Proof. We write

$$\widetilde{N}_{q,\ell,r,h}(\mathcal{I}_1,\mathcal{I}_2)=\sum_{\substack{x\in\mathcal{I}_1,\,y\in\mathcal{I}_2\\(x,q)=1,\,x\equiv r \pmod{\ell}}}\frac{1}{\ell q}\sum_{k \pmod{\ell q}} e\left(\frac{k(y-h\bar{x})}{\ell q}\right)=M+E,$$

where $\bar{x}$ denotes the multiplicative inverse of $x \pmod{\ell q}$ and

$$(3.7)\qquad E=\frac{1}{\ell q}\sum_{\substack{k \pmod{\ell q}\\k\neq 0}}\left(\sum_{y\in\mathcal{I}_2}e\left(\frac{ky}{\ell q}\right)\right)K_{\mathcal{I}_1,r}(0,-hk;\ell q),$$

$$\begin{aligned}M=&\frac{1}{\ell q}\sum_{\substack{x\in\mathcal{I}_1,\,y\in\mathcal{I}_2\\(x,q)=1,\,x\equiv r \pmod{\ell}}}1\\=&\frac{1}{\ell q}\left(\frac{A(q,\ell)}{\ell}\cdot\frac{\varphi(q)}{q}\,|\mathcal{I}_1|+O\big(\sigma_0(q)\big)\right)\big(|\mathcal{I}_2|+O(1)\big)\\=&\frac{A(q,\ell)}{\ell^2}\cdot\frac{\varphi(q)}{q^2}\,|\mathcal{I}_1|\,|\mathcal{I}_2|+O_\ell\big(\sigma_0(q)\big).\end{aligned}$$

From (3.6), Lemma 3.6 and $(hk,\ell q)\leqslant(h,q)(k,q)\ell$, we infer as in the proof of [6, Proposition A3]

$$\begin{aligned}(3.8)\qquad |E|&\ll_{\ell,\delta}q^{-1/2+\delta/2}\sum_{\substack{k \pmod{\ell q}\\k\neq 0}}\frac{(hk,\ell q)^{1/2}}{\big\|\frac{k}{\ell q}\big\|}\\&\ll_{\ell,\delta}q^{-1/2+\delta/2}(h,q)^{1/2}\sum_{\substack{k \pmod{\ell q}\\k\neq 0}}\frac{(k,q)^{1/2}}{\big\|\frac{k}{\ell q}\big\|}\\&\ll q^{(1+\delta)/2}(h,q)^{1/2}\sum_{1\leqslant k\leqslant(q-1)/2}\frac{(k,q)^{1/2}}{k}\ll_\delta (h,q)^{1/2}q^{1/2+\delta}.\end{aligned}$$

The statement follows now from (3.7) and (3.8). ☐

Lemma 3.8. *Suppose that $(r,\ell)=1$. Let $\mathcal{I}_1$, $\mathcal{I}_1$ be intervals of length less than q, $f\in C^1(\mathcal{I}_1\times\mathcal{I}_2)$, and $h\in\mathbb{Z}$. For any integer $T>1$ and any $\delta>0$,*

$$\sum_{\substack{x\in\mathcal{I}_1,y\in\mathcal{I}_2\\(x,q)=1,\,x\equiv r \pmod{\ell}\\xy\equiv h \pmod{\ell q}}}f(x,y)=\frac{A(q,\ell)}{\ell^2}\cdot\frac{\varphi(q)}{q^2}\iint_{\mathcal{I}_1\times\mathcal{I}_2}f(u,v)\,du\,dv+\mathcal{E},$$

with

$$\mathcal{E} \ll_{\ell,\delta} T^2\|f\|_\infty q^{1/2+\delta}(h,q)^{1/2} + T\|\nabla f\|_\infty q^{3/2+\delta}(h,q)^{1/2} + \frac{\|\nabla f\|_\infty |\mathcal{I}_1||\mathcal{I}_2|}{T}.$$

Proof. This plainly follows from Lemma 3.7 as in the proof of [3, Lemma 2.2]. □

Only the case $\ell = 3$ and $x \equiv r \not\equiv 0 \pmod 3$ is needed here, with main term

$$\sum_{\substack{x\in\mathcal{I}_1,\, y\in\mathcal{I}_2 \\ (x,q)=1,\, x\equiv r \ (\mathrm{mod}\ 3) \\ xy\equiv h \ (\mathrm{mod}\ 3q)}} f(x,y) \sim \frac{\varphi(q)}{q^2}\left(\iint_{\mathcal{I}_1\times\mathcal{I}_2} f(x,y)\,dx\,dy\right)\cdot \begin{cases} \frac{1}{9} & \text{if } 3\nmid q, \\ \frac{1}{6} & \text{if } 3 \mid q. \end{cases}$$

4. Coding the linear flow in $\mathbb{Z}^2_{(3)}$ and the three-strip partition

Consider c, c', δ, ξ and I as in the statement of Theorem 2 and $E_{c,c',\delta}(Q)$ as in (2.5). For (A_Q), (B_Q) sequences of real numbers, we write $A_Q \approxeq B_Q$ when $A_Q = B_Q + O_{\delta,\xi}\big(E_{c,c',\delta}(Q)\big)$, uniformly for ξ in compact subsets of $[0,\infty)$. Our primary aim is to estimate the quantity $\mathbb{G}_{I,Q}(\xi)$ from Theorem 2, associated with lattice points from $\mathbb{Z}^2_{(3)}$ with corresponding vertical scatterers of width $2\varepsilon = \frac{1}{Q}$, as $Q \to \infty$.

It is useful to recall first the approach and notation from [7]. $\mathcal{F}(Q)$ denotes the set of Farey fractions of order Q, consisting of rational numbers $\gamma = \frac{a}{q}$, $0 < a \leqslant q \leqslant Q$, with $(a,q) = 1$. The interval I will be first partitioned into intervals $I_\gamma = (\gamma, \gamma')$ with γ, γ' consecutive in $\mathcal{F}_I(Q) := \mathcal{F}(Q) \cap I$. Each interval I_γ is further partitioned into subintervals $I_{\gamma,k}$, $k \in \mathbb{Z}$, defined as

$$I_{\gamma,k} = (t_k, t_{k-1}], \quad I_{\gamma,0} = (t_0, u_0], \quad I_{\gamma,-k} = (u_{k-1}, u_k], \quad k \in \mathbb{N},$$

where

$$t_k = \frac{a_k - 2\varepsilon}{q_k}, \quad u_k = \frac{a'_k + 2\varepsilon}{q'_k}, \quad k \in \mathbb{N}_0,$$

$$q_k = q' + kq, \quad a_k = a' + ka, \quad q'_k = q + kq', \quad a'_k = a + ka', \quad k \in \mathbb{Z},$$

satisfy the fundamental relations

$$\begin{cases} a_{k-1}q_k - a_k q_{k-1} = 1 = a_{k-1}q - aq_{k-1}, \\ a'_k q'_{k-1} - a'_{k-1}q'_k = 1 = a'q'_{k-1} - a'_{k-1}q', \quad k \in \mathbb{Z}, \\ 2\varepsilon q_k \wedge 2\varepsilon q'_k \geqslant 2\varepsilon(q+q') > 1, \quad k \geqslant 1. \end{cases}$$

Consider also

$$t := \tan\omega \quad \text{and} \quad \gamma_k = \frac{a_k}{q_k}, \quad k \in \mathbb{N}.$$

As it will be seen shortly, the coding of the linear flow is considerably more involved than in the case of the square lattice. As a result our attempt of providing asymptotic results for the repartition of the free path length will require additional partitioning for each of the interval $I_{\gamma,k}$. For symmetry reasons[3] the mediant intervals $(\gamma,\gamma_1]$ and $(\gamma_1,\gamma']$ will contribute by the same amount to the main term, so we shall only consider $t\in(\gamma,\gamma_1]$, and redefine

$$I_{\gamma,0} := (t_0, t_{-1}], \qquad t_{-1} := \gamma_1 = \frac{a'+a}{q'+q}.$$

This explains the appearance of the factor 2 in formula (2.4).

As in [7, Section 3] we shall consider[4], when $t=\tan\omega\in I_{\gamma,k}$, $k\in\mathbb{N}_0$,

$$\begin{aligned} w_{\mathcal{A}_0}(t) &= a+2\varepsilon-qt = q(u_0-t),\\ w_{\mathcal{C}_k}(t) &= a_{k-1}-2\varepsilon-q_{k-1}t = q_{k-1}(t_{k-1}-t),\\ w_{\mathcal{B}_k}(t) &= q_k t - a_k + 2\varepsilon = q_k(t-t_k)\in[0,2\varepsilon], \end{aligned}$$

representing the widths of the bottom, center, and respectively top channels $\mathcal{A}_0,\mathcal{C}_k,\mathcal{B}_k$, of the three-strip partition of $[0,1)^2$ (see Figures 5 and 6). Clearly

$$\begin{cases} 2\varepsilon = w_{\mathcal{A}_0}(t)+w_{\mathcal{B}_k}(t)+w_{\mathcal{C}_k}(t), \\ 1 = q w_{\mathcal{A}_0}(t) + q_{k+1} w_{\mathcal{C}_k}(t) + q_k w_{\mathcal{B}_k}(t), \end{cases} \qquad \forall t\in I_{\gamma,k}.$$

Recall [7, 10] that in the case of the square lattice, the three weights corresponding to ω are given by

$$\begin{aligned} W_{\mathcal{A}_0}(t) &= (q-\xi Q)_+ w_{\mathcal{A}_0}(t),\\ W_{\mathcal{B}_k}(t) &= (q_k-\xi Q)_+ w_{\mathcal{B}_k}(t),\\ W_{\mathcal{C}_k}(t) &= (q_{k+1}-\xi Q)_+ w_{\mathcal{C}_k}(t). \end{aligned} \tag{4.1}$$

They reflect the area of the parallelogram of height given respectively by $w_{\mathcal{A}_0}$, $w_{\mathcal{C}_k}$ or $w_{\mathcal{B}_k}$, and length given by the distance from ξQ to the bottom of the corresponding subchannel (if ξQ is lesser than the total length of the subchannel).

The range for $q_k = q'+kq$, respectively q'_k, will be

$$q_k\in\mathcal{I}_{q,k} := \big(Q+(k-1)q, Q+kq\big], \quad \text{respectively} \quad q'_k\in\mathcal{I}_{q',k}.$$

Denote by $r,r_k,r',r'_k\in\mathbb{Z}/3\mathbb{Z}$ the remainders $\pmod 3$ of $q-a$, q_k-a_k, $q'-a'$, $q'_k-a'_k$ respectively. The equality $a'q-aq'=1$ shows that at most one element of the triple $(r,r_k,r_{k+1})\in(\mathbb{Z}/3\mathbb{Z})^3$ can be equal to zero. Similarly, at most one element of the triple (r',r'_k,r'_{k+1}) can be zero.

To ascertain the contribution of the slope $t=\tan\omega\in I_{\gamma,k}$ to $\mathbb{G}_{I,Q}(\xi)$, we should look at the tiling $\mathfrak{S}_\omega$ defined by the three-strip partition of $\mathbb{R}^2$ (shown in Figure 6), but also at $\mathfrak{S}_\omega^{\leftarrow}$, its left-horizontal translate by $(1,0)$, and at $\mathfrak{S}_\omega^{\downarrow}$, its down-vertical translate by $(0,1)$. Since the slits $(m,n)\in\mathbb{Z}^2$ with $m\equiv n \pmod 3$ are being removed, sinks are going to arise in the channels. This phenomenon will lead to frequent occurrence of trajectories much longer then in the case of the square lattice. A careful analysis of the bottom of

[3]Which are not geometrically obvious but become apparent after translating $\mathbb{G}_{I,Q}(\xi)$ into sums involving (sub)intervals (of) $I_{\gamma,k}$ and Farey fractions from $\mathcal{F}_I(Q)$.

[4]Here we use $t=\tan\omega$ as variable and use $\frac{dt}{t^2+t+1}$ instead of $d\omega$.

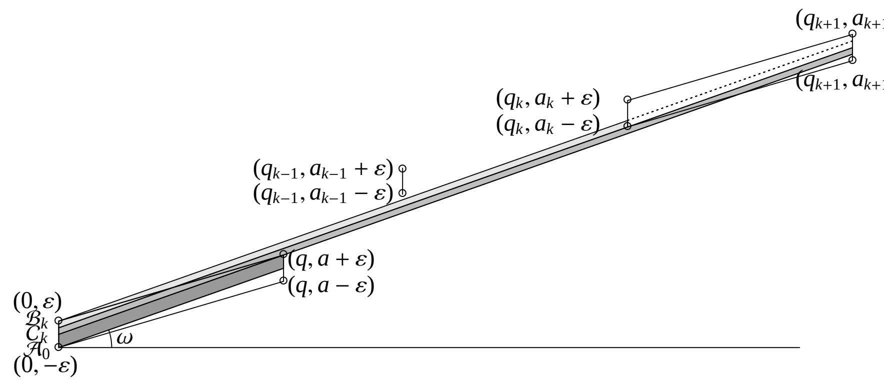

FIGURE 5. The three-strip partition of $\mathbb{R}^2/\mathbb{Z}^2$ when $t \in I_{\gamma,k}$

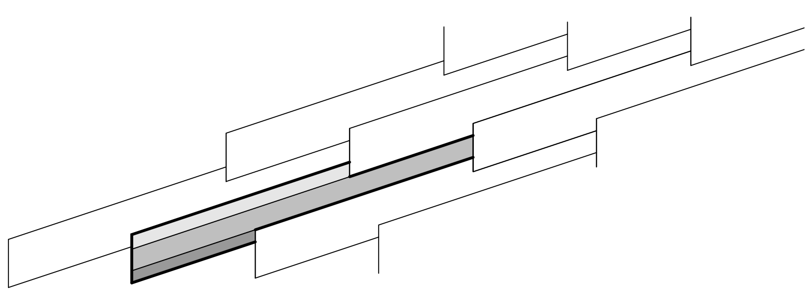

FIGURE 6. The tiling $\mathfrak{S}_\omega$ of the plane (shaded region represents $\mathbb{R}^2/\mathbb{Z}^2$)

the channels $\mathcal{A}_0$, $\mathcal{C}_k$ and $\mathcal{B}_k$ is required when the corresponding slit where trajectory ends in the case of the square lattice has been removed. Besides, there is a manifest difference between the three situations where the channel originates at $O = (0,0)$ (this will be referred to as $\mathcal{C}_O$ *contribution*), at $(-1,0)$ ($\mathcal{C}_\leftarrow$ *contribution*), or at $(-1,0)$ ($\mathcal{C}_\downarrow$ *contribution*), resulting from the different congruence conditions $\pmod 3$ satisfied by the centers of removed slits. This is shown in Figure 6, where the small circles centered at lattice points (m,n) with $m \not\equiv n \pmod 3$ represent the vertical slits of width $2\varepsilon = \frac{1}{Q}$. We were not able to spot any symmetry that would reduce the analysis to only one of these three types of channels. The contributions to Φ^{hex} of the five types of situations that we analyze seem to be quite different (see Figure 19).

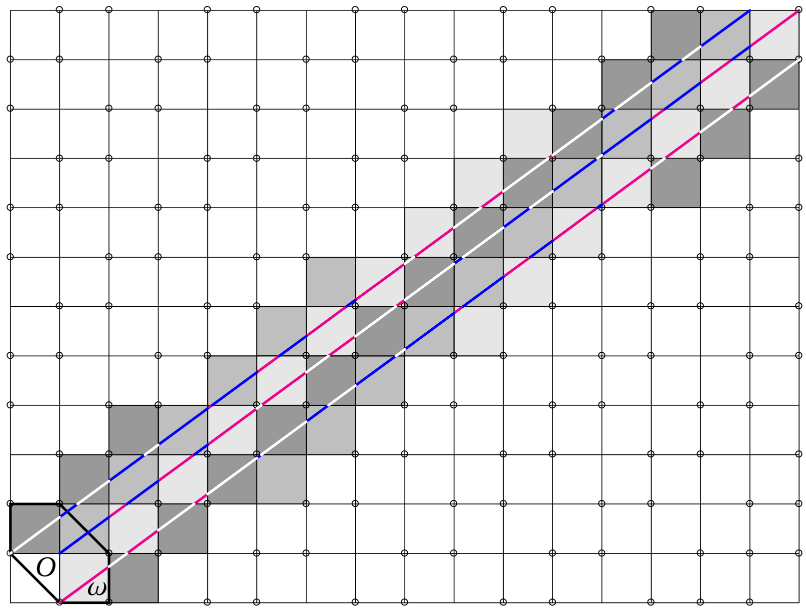

FIGURE 7. The channels $\mathcal{C}_0$, $\mathcal{C}_{\leftarrow}$ and $\mathcal{C}_{\downarrow}$

To attain a better visualization of the structure of channels, we shall represent the slope $\tan\omega$ horizontally. The possible situations are shown in Figure 8, where dotted lines indicate that the corresponding slit has been removed from $\mathfrak{S}_\omega$ in the case of $\mathcal{C}_O$, from $\mathfrak{S}_\omega^{\leftarrow}$ in the case of $\mathcal{C}_{\leftarrow}$ and respectively from $\mathfrak{S}_\omega^{\downarrow}$ in the case of $\mathcal{C}_{\downarrow}$.

To clarify the terminology, by "slit q_k" etc. we will mean the slit which is centered at some lattice point (q_k, m) and which intersects the channel that is analyzed (there is at most one such point for given q_k).

5. The contribution of channels whose slits are not removed

This resembles the situation of the square lattice and will be discussed in this section. The more intricate situation of the channels where bottom slits are removed will be analyzed in Sections 6-9. When the "first" slit (i.e. the one corresponding to q for $\mathcal{A}_0$, q_{k+1} for $\mathcal{C}_k$, respectively q_k for $\mathcal{B}_k$) is not removed, the table from Figure 8 shows that the corresponding weight is described in the table from Figure 9.

The weights $W_{\mathcal{A}_0}$, $W_{\mathcal{B}_k}$ and $W_{\mathcal{C}_k}$ are given by (4.1), as in the case of the square lattice. The cumulative contribution is

$$\mathbb{G}_{I,Q}^{(0)}(\xi) = \sum_{(\alpha,\beta)\in(\mathbb{Z}/3\mathbb{Z})^2\setminus\{(0,0)\}} \mathbb{G}_{I,Q,\alpha,\beta}^{(0)}(\xi),$$

with

$$\mathbb{G}^{(0)}_{I,Q,\alpha,\beta}(\xi) = \sum_{k=0}^{\infty} \sum_{\substack{\gamma\in\mathcal{F}_I(Q) \\ q-a\equiv\alpha \ (\mathrm{mod}\ 3) \\ q_k-a_k\equiv\beta \ (\mathrm{mod}\ 3)}} \int_{t_k}^{t_{k-1}} \frac{2\big((q-\xi Q)_+ w_{\mathcal{A}_0}(t) + (q_{k+1}-\xi Q)_+ w_{\mathcal{C}_k}(t) + (q_k-\xi Q)_+ w_{\mathcal{B}_k}(t)\big)\,dt}{t^2+t+1}. \tag{5.1}$$

Remark 5.1. Putting $x=q-a$, $y=q_k$, and employing

$$q_k - a_k = q_k - \frac{1+aq_k}{q} = \frac{xy-1}{q},$$

we see that the summation conditions in (5.1) are equivalent to $(x,q)=1$, $x\equiv\alpha \pmod 3$, and $xy\equiv \beta q+1 \pmod{3q}$. Note that $(\beta q+1,q)=1$, and sum as follows:

- When $\alpha\neq 0$, Lemma 3.8 may be applied (because $(x,3q)=1$) followed by Lemma 3.3.
- When $\alpha=0$, we have $\beta\neq 0$ and $x=3\tilde{x}$, $\tilde{x}\in\frac{q}{3}(1-I)$. Furthermore, $\beta q+1\equiv 0 \pmod 3$, so $q\equiv -\beta \pmod 3$ and one may first sum, as in Lemma 3.4, over $(\tilde{x},y)$ and the conditions

$$\begin{cases} q-a=x=3\tilde{x}, \quad \tilde{x}\in\frac{q}{3}(1-I), \quad (\tilde{x},q)=1, \\ q\equiv -\beta \pmod 3, \quad \tilde{x}y\equiv \frac{\beta q+1}{3} \pmod q, \quad y\in\mathcal{I}_{q,k}, \end{cases} \tag{5.2}$$

  then sum over $q\in[1,Q]$ with $q\equiv -\beta \pmod 3$ employing Lemma 3.3.

In all situations where sums over $\gamma\in\mathcal{F}_I(Q)$ with $q-a\equiv\alpha \pmod 3$ and $q_k-a_k\equiv\beta \pmod 3$, $(\alpha,\beta)\neq(0,0)$, have to be evaluated, the resulting constant will be $\frac{1}{8\zeta(2)}$.

The following elementary estimate will be used throughout.

Lemma 5.2. *For $b,c,h,\lambda,\mu\in\mathbb{R}$ such that $0\leqslant b,c,c+h\leqslant 1$ and $\lambda c+\mu=0$, there exist $\theta',\theta=\theta_{c,h}\in[-3,3]$ such that*

$$\int_c^{c+h} \frac{\lambda t+\mu}{t^2+t+1}\,dt = \frac{h^2\lambda}{2(b^2+b+1)} + h^3\theta(|\lambda|+|\mu|) + h^2\theta'|b-c||\lambda|.$$

Proof. This follows immediately applying Taylor's formula twice:

$$\int_c^{c+h}\frac{dt}{t^2+t+1} = \frac{h}{c^2+c+1} - \frac{h^2(2c+1)}{2(c^2+c+1)^2} + \xi' h^3,$$

$$\int_c^{c+h}\frac{t\,dt}{t^2+t+1} = \left(\frac{h}{c^2+c+1} - \frac{h^2(2c+1)}{2(c^2+c+1)^2}\right)c + \frac{h^2}{2(c^2+c+1)} + \xi'' h^3,$$

with $|\xi'|\leqslant 1$ and $|\xi''|\leqslant 2$, and employing

$$\left|\frac{1}{c^2+c+1} - \frac{1}{b^2+b+1}\right| \leqslant 3|b-c|. \qquad \square$$

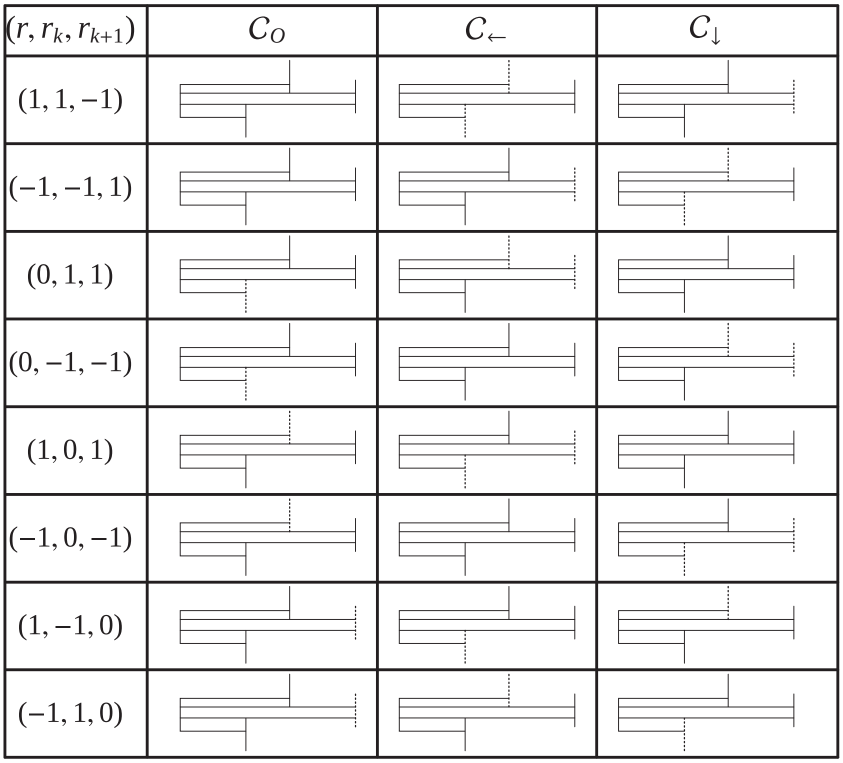

FIGURE 8. The removed slits for $\mathcal{C}_O$, $\mathcal{C}_\leftarrow$ and $\mathcal{C}_\downarrow$

Together with $0 < q_1 - Q \leqslant q \wedge q'$, Lemma 5.2 yields ($t_{-1} = \gamma_1$)

$$\begin{aligned}\int_{t_0}^{t_{-1}} \frac{w_{\mathcal{B}_0}(t)\,dt}{t^2+t+1} &= \int_{t_0}^{\gamma_1} \frac{(q't - a' + 2\varepsilon)}{t^2+t+1}\,dt \\ &= \frac{(q_1-Q)^2}{2Q^2 q' q_1^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{Q^5}\right).\end{aligned} \tag{5.3}$$

$$\begin{aligned}&\int_{t_0}^{t_{-1}} \frac{w_{\mathcal{C}_0}(t)\,dt}{t^2+t+1} \\ &= \int_{\gamma}^{\gamma_1} \frac{(qt-a)\,dt}{t^2+t+1} - \int_{\gamma}^{t_0} \frac{(qt-a)\,dt}{t^2+t+1} - \int_{t_0}^{\gamma_1} \frac{(q't-a'+2\varepsilon)\,dt}{t^2+t+1} \\ &= q\,\frac{(\gamma_1-\gamma)^2 - (t_0-\gamma)^2}{2(\gamma^2+\gamma+1)} - q'\frac{(\gamma_1-t_0)^2}{2(\gamma^2+\gamma+1)} \\ &\quad + O\left(q(\gamma_1-\gamma)^3 + q'(\gamma_1-t_0)^3 + q'(\gamma_1-t_0)^2(t_0-\gamma)\right) \\ &= \frac{q_1-Q}{2Q^2q'q_1(\gamma^2+\gamma+1)}\left(\frac{2Q-q_1}{q_1} + \frac{Q-q}{q'}\right) + O\left(\frac{1}{Q^3q^2}\right).\end{aligned} \tag{5.4}$$

(r, r_k, r_{k+1})	C_O	$C_{\leftarrow}$	$C_{\downarrow}$
$(1,1,-1)$	$W_{\mathcal{A}_0}+W_{\mathcal{C}_k}+W_{\mathcal{B}_k}$	$W_{\mathcal{C}_k}$	$W_{\mathcal{A}_0}+W_{\mathcal{B}_k}$
$(-1,-1,1)$	$W_{\mathcal{A}_0}+W_{\mathcal{C}_k}+W_{\mathcal{B}_k}$	$W_{\mathcal{A}_0}+W_{\mathcal{B}_k}$	$W_{\mathcal{C}_k}$
$(0,1,1)$	$W_{\mathcal{C}_k}+W_{\mathcal{B}_k}$	$W_{\mathcal{A}_0}$	$W_{\mathcal{A}_0}+W_{\mathcal{C}_k}+W_{\mathcal{B}_k}$
$(0,-1,-1)$	$W_{\mathcal{C}_k}+W_{\mathcal{B}_k}$	$W_{\mathcal{A}_0}+W_{\mathcal{C}_k}+W_{\mathcal{B}_k}$	$W_{\mathcal{A}_0}$
$(1,0,1)$	$W_{\mathcal{A}_0}+W_{\mathcal{C}_k}$	$W_{\mathcal{B}_k}$	$W_{\mathcal{A}_0}+W_{\mathcal{C}_k}+W_{\mathcal{B}_k}$
$(-1,0,-1)$	$W_{\mathcal{A}_0}+W_{\mathcal{C}_k}$	$W_{\mathcal{A}_0}+W_{\mathcal{C}_k}+W_{\mathcal{B}_k}$	$W_{\mathcal{B}_k}$
$(1,-1,0)$	$W_{\mathcal{A}_0}+W_{\mathcal{B}_k}$	$W_{\mathcal{C}_k}+W_{\mathcal{B}_k}$	$W_{\mathcal{A}_0}+W_{\mathcal{C}_k}$
$(-1,1,0)$	$W_{\mathcal{A}_0}+W_{\mathcal{B}_k}$	$W_{\mathcal{A}_0}+W_{\mathcal{C}_k}$	$W_{\mathcal{C}_k}+W_{\mathcal{B}_k}$

FIGURE 9. The contribution of channels whose slits are not removed

$$\begin{aligned}
\int_{t_0}^{t_{-1}} \frac{w_{\mathcal{A}_0}(t)\,dt}{t^2+t+1} &= \int_{t_0}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt - \int_{\gamma_1}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt \\
&= q\,\frac{(u_0-t_0)^2-(u_0-\gamma_1)^2}{2(\gamma^2+\gamma+1)} \\
&\quad + O\left(q(u_0-t_0)^3+q(u_0-t_0)^2(u_0-\gamma)\right) \\
&= \frac{(q_1-Q)^2(q_1+q')}{2Q^2q'^2q_1^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{Q^3qq'}\right).
\end{aligned} \tag{5.5}$$

For every $k \geqslant 1$, we find

$$\begin{aligned}
\int_{t_k}^{t_{k-1}} \frac{w_{\mathcal{B}_k}(t)\,dt}{t^2+t+1} &= \int_{t_k}^{t_{k-1}} \frac{q_kt-a_k+2\varepsilon}{t^2+t+1}\,dt \\
&= \frac{(Q-q)^2}{2Q^2q_{k-1}^2q_k(\gamma^2+\gamma+1)} + O\left(\frac{1}{qq_{k-1}^2q_k^2}\right).
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
\int_{t_k}^{t_{k-1}} \frac{w_{\mathcal{C}_k}(t)\,dt}{t^2+t+1} &= \int_{t_k}^{t_{k-1}} \frac{a_{k-1}-2\varepsilon-q_{k-1}t}{t^2+t+1}\,dt \\
&= \frac{(Q-q)^2}{2Q^2q_{k-1}q_k^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{qq_{k-1}^2q_k^2}\right).
\end{aligned} \tag{5.7}$$

$$(5.8)\qquad \int_{t_k}^{t_{k-1}} \frac{2\varepsilon\, dt}{t^2+t+1} = \frac{t_{k-1}-t_k}{Q(\gamma^2+\gamma+1)} + O\left(\varepsilon(t_{k-1}-t_k)(t_k-\gamma)+\varepsilon(t_{k-1}-t_k)^2\right) = \frac{Q-q}{Q^2 q_{k-1}q_k(\gamma^2+\gamma+1)} + O\left(\frac{1}{Qqq_{k-1}q_k^2}\right).$$

$$(5.9)\qquad \int_{t_k}^{t_{k-1}} \frac{w_{\mathcal{A}_0}(t)\, dt}{t^2+t+1} = \int_{t_k}^{t_{k-1}} \frac{2\varepsilon - w_{\mathcal{B}_k}(t)-w_{\mathcal{C}_k}(t)}{t^2+t+1}\, dt = \frac{Q-q}{2Q^2 q_{k-1}q_k(\gamma^2+\gamma+1)}\left(\frac{q_{k+1}-Q}{q_k}+\frac{q_k-Q}{q_{k-1}}\right) + O\left(\frac{1}{qq'q_{k-1}q_k^2}\right).$$

The total contribution of the error terms from (5.3)–(5.7) and (5.9) to $\mathbb{G}^{(0)}_{I,Q}(\xi)$ is

$$\ll \sum_{\gamma\in\mathcal{F}_I(Q)} \frac{1}{(qq')^2} + \sum_{k=1}^{\infty}\sum_{\gamma\in\mathcal{F}_I(Q)}\left(\frac{1}{q'q_{k-1}q_k^2}+\frac{1}{qq_{k-1}^2q_k}\right)$$
$$\leqslant \sum_{\gamma\in\mathcal{F}_I(Q)} \frac{1}{(qq')^2} + \sum_{\gamma\in\mathcal{F}_I(Q)} \frac{1}{qq'}\sum_{k=1}^{\infty}\frac{1}{q_{k-1}q_k} = 2\sum_{\gamma\in\mathcal{F}_I(Q)}\frac{1}{(qq')^2}$$
$$\leqslant \frac{2}{Q}\sum_{\gamma\in\mathcal{F}_I(Q)}\frac{1}{qq'} \leqslant \frac{2}{Q}\left(|I|+\frac{2}{Q}\right) \leqslant \frac{6|I|}{Q}.$$

When summing over a family of intervals I that partition $[0,1]$, this adds up to $O(Q^{-1})$. Thus all error terms above can be discarded below. We emphasize that, since the contribution of each interval $I_{\gamma,k}$ is $\leqslant |I_{\gamma,k}|$, we can remove one element of $\mathcal{F}_I(Q)$ for every I.

Applying Remark 5.1 to the inner sum in (5.1) with $x=q-a\in q(1-I)$, $y=q_k\in\mathcal{I}_{q,k}$, and employing formulas (5.3)–(5.9), we find

$$(5.10)\qquad \mathbb{G}^{(0)}_{I,Q,\alpha,\beta}(\xi) = \sum_{q=1}^{Q}\sum_{k=1}^{\infty}\sum_{\substack{x\in q(1-I),\, y\in\mathcal{I}_{q,k}\\ x\equiv\alpha \ (\mathrm{mod}\ 3)\\ xy\equiv\beta q+1\ (\mathrm{mod}\ 3q)}} f_q(x,y) + \sum_{q=1}^{Q}\sum_{\substack{x\in q(1-I),\, y\in(Q-q,Q]\\ x\equiv\alpha\ (\mathrm{mod}\ 3)\\ xy\equiv\beta q+1\ (\mathrm{mod}\ 3q)}} g_q(x,y),$$

with C^1 functions f_q and g_q on $\mathbb{R}\times\left(\mathbb{R}_+\setminus\{q,\xi Q-q,\xi Q\}\right)$ given by

$$f_q(x,y)=\frac{q^2}{q^2+(q-x)^2}\,F_q(y),\qquad g_q(x,y)=\frac{q^2}{q^2+(q-a)^2}\,G_q(y),\quad \text{with}$$

$$\begin{aligned}F_q(y) =&\frac{Q-q}{Q^2(y-q)y}\left(\frac{y+q-Q}{y}+\frac{y-Q}{y-q}\right)(q-\xi Q)_+\\&+\frac{(Q-q)^2}{Q^2(y-q)y}\left(\frac{(y-\xi Q)_+}{y-q}+\frac{(y+q-\xi Q)_+}{y}\right),\\G_q(y) =&\frac{y+q-Q}{Q^2y(y+q)}\left(\frac{2Q-q-y}{y+q}+\frac{Q-q}{y}\right)(y+q-\xi Q)_+\\&+\frac{(y+q-Q)^2}{Q^2y(y+q)^2}\left(\frac{2y+q}{y}\,(q-\xi Q)_+ +(y-\xi Q)_+\right).\end{aligned}$$

The innermost sums in (5.10) will be estimated employing Lemmas 3.4 or 3.8. We first need to bound $\|f_q\|_\infty$, $\|\nabla f_q\|_\infty$ on $q(1-I)\times\mathcal{I}_{q,k}$, $k\geqslant 1$, and respectively $\|g_q\|_\infty$, $\|\nabla g_q\|_\infty$ on $q(1-I)\times(Q-q,Q]$. From $y\geqslant Q$ in the first case and $y+q-Q\leqslant Q\leqslant Q\leqslant y+q\leqslant 2Q$ in the second one, we find for all $y\in\mathcal{I}_{q,k}$, $k\geqslant 1$:

$$\begin{aligned}0\leqslant f_q(x,y)\leqslant F_q(y)&\leqslant\frac{Q-q}{Q^2(y-q)y}\left(q+(Q-q)\left(2+\frac{q}{y-q}+\frac{q}{y}\right)\right)\\&\leqslant\frac{4(Q-q)}{Q(y-q)y},\end{aligned}\tag{5.11}$$

$$|F_q'(y)|\leqslant\frac{16(\xi+1)(Q-q)}{Q(y-q)^2y},\tag{5.12}$$

and for all $y\in(Q-q,Q]$:

$$0\leqslant g_q(x,y)\leqslant G_q(y)\leqslant\frac{2(y+q)}{Q^2(y+q)}+\frac{(2y+q)q+y^2}{Q^2(y+q)^2}=\frac{3}{Q^2},\tag{5.13}$$

$$|G_q'(y)|\leqslant\frac{20}{Qy^2}.\tag{5.14}$$

From (5.11) we infer

$$\sum_{k=1}^\infty\|F_q\|_\infty\leqslant\frac{2(Q-q)}{Q}\sum_{k=1}^\infty\frac{1}{\big(Q+(k-2)q\big)\big(Q+(k-1)q\big)}=\frac{4}{Qq},$$

which leads in turn to

$$\sum_{q=1}^Q q^{1/2+\delta}\sum_{k=1}^\infty\|f_q\|_\infty\leqslant\frac{2}{Q}\sum_{q=1}^Q q^{-1/2+\delta}\leqslant 8Q^{1/2+\delta}.$$

From (5.12) we infer

$$\begin{aligned}\sum_{k=1}^\infty\|F_q'\|_\infty&\leqslant\frac{16(\xi+1)(Q-q)}{Q}\sum_{k=1}^\infty\frac{1}{\big(Q+(k-2)q+1\big)^2\big(Q+(k-1)q+1\big)}\\&\leqslant\frac{16(\xi+1)}{Q}\sum_{k=1}^\infty\frac{1}{\big(Q+(k-2)q+1\big)\big(Q+(k-1)q+1\big)}\leqslant\frac{16(\xi+1)}{Qq(Q-q+1)},\end{aligned}$$

which leads to

$$\sum_{k=1}^{\infty} \|\nabla f_q\|_\infty \leqslant \sum_{k=1}^{\infty} \left(\frac{2}{q} \, \|F_q\|_\infty + \|F_q'\|_\infty \right) \leqslant \frac{24(\xi+1)}{q^2(Q-q+1)},$$

and finally gives

$$\sum_{q=1}^{Q} q^{3/2+\delta} \sum_{k=1}^{\infty} \|\nabla f_q\|_\infty \ll \sum_{1\leqslant q\leqslant \frac{Q}{2}} \frac{q^{-1/2+\delta}}{Q-q+1} + \sum_{\frac{Q}{2}\leqslant q\leqslant Q} \frac{q^{-1/2+\delta}}{Q-q+1} \leqslant \frac{2}{Q}\sum_{q=1}^{Q} q^{-1/2+\delta} + \left(\frac{Q}{2}\right)^{-1/2+\delta} \sum_{n=1}^{Q} \frac{1}{n} \ll Q^{-1/2+\delta}\log Q, \tag{5.15}$$

$$\sum_{q=1}^{Q} q^2 \sum_{k=1}^{\infty} \|\nabla f_q\|_\infty \ll_\xi \sum_{q=1}^{Q} \frac{1}{Q-q+1} \ll \log Q. \tag{5.16}$$

Taking $T = [Q^{c'}]$, we infer upon (5.15) and (5.16) with $E_{c,c',\delta}(Q)$ as in Theorem 2,

$$\sum_{q=1}^{Q}\sum_{k=1}^{\infty}\left(Tq^{1/2+\delta}\Big(T\|f_q\|_\infty + q\|\nabla f_q\|_\infty\Big) + \frac{q|I||\mathcal{I}_{q,k}|}{T}\,\|\nabla f_q\|_\infty\right) \ll_{\xi,\delta} T^2Q^{-1/2+\delta} + TQ^{-1/2+\delta}\log Q + |I|T^{-1}\log Q \ll E_{c,c',\delta}(Q),$$

while (5.13) and (5.14) yield

$$\sum_{q=1}^{Q}\left(Tq^{1/2+\delta}\Big(T\|g_q\|_\infty + q\|\nabla g_q\|_\infty\Big) + \frac{q^2|I|}{T}\,\|\nabla g_q\|_\infty\right) \ll_\delta E_{c,c',\delta}(Q).$$

When $\alpha \neq 0$, Lemma 3.8 applies to the innermost sums in (5.10). The two error estimates above hold uniformly for ξ in compact subsets of $[0,\infty)$, yielding

$$\begin{aligned} \mathbb{G}^{(0)}_{I,Q,\alpha,\beta}(\xi) &\approxeq \sum_{\substack{1\leqslant q\leqslant Q\\ 3\nmid q}} \frac{\varphi(q)}{9q^2}\cdot qc_I\left(\sum_{k=1}^{\infty}\int_{\mathcal{I}_{q,k}} F_q(y)\,dy + \int_{Q-q}^{Q} G_q(y)\,dy\right) \\ &\approxeq \sum_{\substack{1\leqslant q\leqslant Q\\ 3\mid q}} \frac{\varphi(q)}{6q^2}\cdot qc_I\left(\sum_{k=1}^{\infty}\int_{\mathcal{I}_{q,k}} F_q(y)\,dy + \int_{Q-q}^{Q} G_q(y)\,dy\right). \end{aligned} \tag{5.17}$$

Applying Lemma 3.3 with $\ell = 3$ to the sum over q in (5.17) and making the substitution $(q, y) = (Qu, Qw)$, we gather that $\mathbb{G}^{(0)}_{I,Q,\alpha,\beta}(\xi)$ is

$$\begin{aligned}\approxeq \left(\frac{3}{4}\cdot\frac{1}{9}+\frac{1}{4}\cdot\frac{1}{6}\right)\frac{c_I}{\zeta(2)} \\ \cdot \int_0^1 du \left(\int_{1-u}^1 dw\, F_{(0.1)}(\xi;u,w) + \int_1^\infty dw\, F_{(0.2)}(\xi;u,w)\right),\end{aligned}$$

with $F_{(0.1)}$ and $F_{(0.2)}$ as in (1.4). The total contribution of cases $\alpha \neq 0$ to $\mathbb{G}^{(0)}_{I,Q}(\xi)$ is obtained by multiplying the quantity above by 6.

When $\alpha = 0$ and $\beta = \pm 1$, we sum as above with summation conditions in the inner sum given by (5.2) and get, employing Lemmas 3.8 and 3.2,

$$\begin{aligned}&\mathbb{G}^{(0)}_{I,Q,0,1}(\xi) + \mathbb{G}^{(0)}_{I,Q,0,-1}(\xi) \\ &\approxeq \sum_{\substack{1\leqslant q\leqslant Q\\(q,3)=1}} \frac{\varphi(q)}{q^2}\cdot\frac{qc_I}{3}\left(\int_Q^\infty F_q(y)\,dy + \int_{Q-q}^Q G_q(y)\,dy\right) \\ &\approxeq \frac{c_I}{4\zeta(2)}\int_0^1 du\left(\int_{1-u}^1 dw\, F^{(0.1)}(\xi;u,w) + \int_1^\infty dw\, F^{(0.2)}(\xi;u,w)\right).\end{aligned}$$

Applying Lemma 3.2 to the estimate below, we eventually find

$$\mathbb{G}^{(0)}_{I,Q}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_0(\xi), \tag{5.18}$$

with $G_0(\xi)$ as in (1.4).

Remark 5.3. For each angle ω, the weight $W_{\gamma,k}(t) = W^{\mathrm{hex}}_{\gamma,k}(t)$ is clearly no larger than the weight $W^{\square}_{\gamma,k}(t)$ from the situation where no slit is being removed (and which corresponds, up to some scaling, to the case of the unit square).[5] Since the corresponding limiting distribution $\Phi^{\square}$ satisfies (1.2) [9, 7, 12], it follows that Φ^{hex} satisfies the second inequality in (1.2). The first inequality in (1.2) follows for instance from

$$\begin{aligned}G_*(\xi) &:= \int_0^1 du \int_1^\infty dw\, \frac{(1-u)^2}{(w-u)^2 w}(w-\xi)_+ \\ &\geqslant \int_0^1 du\, (1-u)^2 \int_\xi^\infty \frac{v\, dv}{(v+\xi)^3} = \frac{3}{16\xi}.\end{aligned}$$

6. Channels with removed slits. The case $r = r_k = \pm 1$

Assume first that $r = r_k = 1$. Then $r_{k+1} = r + r_k = -1 \pmod 3$. One has to analyze the $\mathcal{C}_{\leftarrow}$ and the $\mathcal{C}_{\downarrow}$ contributions. We shall sum as in Remark 5.1

[5]Note that because of the scaling of ξ this does not imply $\Phi^{\mathrm{hex}}(\xi) \leqslant \Phi^{\square}(\xi)$ (see also Figure 2).

above, with $x = q - a \in q(1-I)$, $\alpha = 1$, $y = q_k \in \mathcal{I}_{q,k}$, $\beta = 1$, considering

$$\sum{}^{*} = \sum_{\substack{\gamma \in \mathcal{F}_I(Q) \\ q-a \equiv 1 \ (\mathrm{mod}\ 3) \\ q_k - a_k \equiv 1 \ (\mathrm{mod}\ 3)}}^{*} .$$

The table in Figure 8 shows that the weights $W_{\gamma,k}(t)$ do coincide for $(\mathcal{C}_{\leftarrow}, r = r_k = 1)$ and for $(\mathcal{C}_{\downarrow}, r = r_k = -1)$, so they do for $(\mathcal{C}_{\downarrow}, r = r_k = 1)$ and $(\mathcal{C}_{\leftarrow}, r = r_k = -1)$. This eventually shows that the corresponding contribution for $r = r_k = -1$ has the same main term and error terms as the one for $r = r_k = 1$ (we just need to replace β by $-\beta$ and α by $-\alpha$, which will produce the same main term and error size). As a result, we shall only take $r = r_k = 1$ and double the total contribution in the sequel.

6.1. The $\mathcal{C}_{\leftarrow}$ contribution. The slits q and q_k are removed, while $2q$, q_{k+1}, q_{k+2}, $2q_k$ and $q_k + q_{k+1}$ are not, because $2r = r_{k+1} = 2r_k = 2$, $r_{k+2} = r_k + r_{k+1} = 0 \not\equiv 1 \pmod 3$. Denote by $T(\bullet)$, respectively $B(\bullet)$, the height of the top, respectively bottom, of the slit $\bullet$ with respect to the top of the strip $\mathcal{B}_k$, with positive downwards direction. Since $B(q_{k+1}) = 2w_{\mathcal{B}_k} + w_{\mathcal{C}_k} < T(2q) = 2(w_{\mathcal{B}_k} + w_{\mathcal{C}_k}) < B(q_{k+2}) = 3w_{\mathcal{B}_k} + 2w_{\mathcal{C}_k}$ and $B(2q_k) = w_{\mathcal{B}_k} - w_{\mathcal{A}_0} - w_{\mathcal{C}_k} < T(q_{k+1} + q_k) = w_{\mathcal{B}_k} - w_{\mathcal{A}_0}$, the slits $2q$, q_{k+2}, q_{k+1}, $q_{k+1} + q_k$ and $2q_k$ lock all channels $\mathcal{B}_k$, $\mathcal{C}_k$ and $\mathcal{A}_0$. Two cases arise:

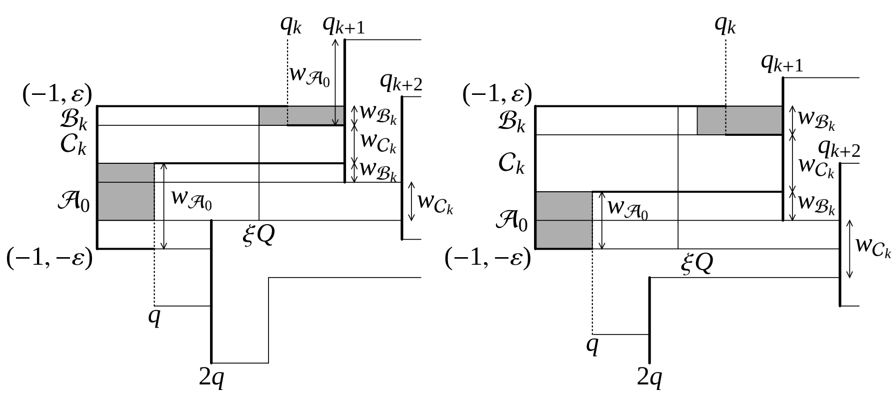

FIGURE 10. The case $t \in I_{\gamma,k}$, $(\mathcal{C}_{\leftarrow}, r = r_k = 1)$, and $w_{\mathcal{B}_k} + w_{\mathcal{C}_k} < w_{\mathcal{A}_0}$ respectively $w_{\mathcal{B}_k} + w_{\mathcal{C}_k} > w_{\mathcal{A}_0}$

6.1.1. $w_{\mathcal{B}_k} < w_{\mathcal{A}_0}$ ($\iff t = \tan\omega < \gamma_{k+1}$). In this case $\mathcal{B}_k$ is locked by the slit q_{k+1}, while $\mathcal{A}_0$ is locked by the slits q_{k+1}, q_{k+2} and $2q$. Two subcases arise:

(I) $w_{\mathcal{B}_k} + w_{\mathcal{C}_k} < w_{\mathcal{A}_0}$ ($\iff t < \frac{a+\varepsilon}{q}$). Then $B(q_{k+1}) < T(2q) < 2\varepsilon$, so $\mathcal{A}_0$ is locked by the slits q_{k+1}, q_{k+2} and $2q$ (see left-hand side of Figure 10).

The widths of the relevant three subchannels of $\mathcal{A}_0$ are (from bottom to top) $2\varepsilon - T(2q) = 2(a+\varepsilon-qt)$, $w_{\mathcal{C}_k}$ and $w_{\mathcal{B}_k}$, so $W_{\gamma,k}(t)$ is given by[6]

$$\text{(6.1)}\quad W^{(1)}_{\gamma,k}(t) = 2(a+\varepsilon-qt)\cdot(2q-\xi Q)_+\wedge q + w_{\mathcal{C}_k}(t)\cdot(q_{k+2}-\xi Q)_+\wedge q + w_{\mathcal{B}_k}(t)\cdot(q_{k+1}-\xi Q)_+\wedge q + w_{\mathcal{B}_k}(t)\cdot(q_{k+1}-\xi Q)_+\wedge q_k.$$

(II) $w_{\mathcal{B}_k} < w_{\mathcal{A}_0} < w_{\mathcal{B}_k} + w_{\mathcal{C}_k}$ ($\Longleftrightarrow t > \frac{a+\varepsilon}{q}$). Then $B(q_{k+1}) = 2w_{\mathcal{B}_k} + w_{\mathcal{C}_k} < 2\varepsilon < T(2q) = 2w_{\mathcal{B}_k} + 2w_{\mathcal{C}_k}$, so $\mathcal{A}_0$ is locked by the slits q_{k+1} and q_{k+2} (see right-hand side of Figure 10). The widths of the relevant two subchannels of $\mathcal{A}_0$ are (from bottom to top) $2\varepsilon - B(q_{k+1}) = a_{k+1} - q_{k+1}t$ and $w_{\mathcal{B}_k}$, hence in this case

$$\text{(6.2)}\quad \begin{aligned} W_{\gamma,k}(t) &= W^{(2)}_{\gamma,k}(t) \\ &= (a_{k+1}-q_{k+1}t)\cdot(q_{k+2}-\xi Q)_+\wedge q \\ &\quad + w_{\mathcal{B}_k}(t)\cdot(q_{k+1}-\xi Q)_+\wedge q + w_{\mathcal{B}_k}(t)\cdot(q_{k+1}-\xi Q)_+\wedge q_k. \end{aligned}$$

6.1.2. $w_{\mathcal{A}_0} < w_{\mathcal{B}_k}$ ($\Longleftrightarrow t > \gamma_{k+1}$). In this case $\mathcal{A}_0$ is locked by the slits q_{k+1} and $\mathcal{B}_k$ by q_{k+1}, $q_{k+1}+q_k$ and $2q_k$. Two subcases arise:

(III) $w_{\mathcal{A}_0} < w_{\mathcal{B}_k} < w_{\mathcal{A}_0} + w_{\mathcal{C}_k}$ ($\Longleftrightarrow \gamma_{k+1} < t < \frac{a_k-\varepsilon}{q_k}$). Then $B(2q_k) < 0 < T(q_{k+1})$, so $\mathcal{B}_k$ is locked by the slits q_{k+1} and $q_{k+1}+q_k$. The widths of the relevant two subchannels of $\mathcal{B}_k$ are $w_{\mathcal{A}_0}$ and $T(q_{k+1}) = q_{k+1}t - a_{k+1}$, hence in this case

$$\text{(6.3)}\quad \begin{aligned} W_{\gamma,k}(t) &= W^{(3)}_{\gamma,k}(t) \\ &= w_{\mathcal{A}_0}(t)\cdot(q_{k+1}-\xi Q)_+\wedge q + w_{\mathcal{A}_0}(t)\cdot(q_{k+1}-\xi Q)_+\wedge q_k \\ &\quad + (q_{k+1}t-a_{k+1})\cdot(q_{k+1}+q_k-\xi Q)_+\wedge q_k. \end{aligned}$$

(IV) $w_{\mathcal{A}_0} + w_{\mathcal{C}_k} < w_{\mathcal{B}_k}$ ($\Longleftrightarrow \frac{a_k-\varepsilon}{q_k} < t$). Then $0 < B(2q_k) < T(q_{k+1})$, so $\mathcal{B}_k$ is locked by the slits q_{k+1}, $q_{k+1}+q_k$ and $2q_k$. The widths of the relevant three subchannels of $\mathcal{B}_k$ are $w_{\mathcal{A}_0}$, $T(q_{k+1}) - B(2q_k) = w_{\mathcal{C}_k}$ and $B(2q_k) = 2(q_k t - a_k + \varepsilon)$, showing that in this case

$$\text{(6.4)}\quad \begin{aligned} W_{\gamma,k}(t) &= W^{(4)}_{\gamma,k}(t) \\ &= w_{\mathcal{A}_0}(t)\cdot(q_{k+1}-\xi Q)_+\wedge q + w_{\mathcal{A}_0}(t)\cdot(q_{k+1}-\xi Q)_+\wedge q_k \\ &\quad + w_{\mathcal{C}_k}(t)\cdot(q_{k+1}+q_k-\xi Q)_+\wedge q_k + 2(q_k t - a_k + \varepsilon) \\ &\qquad \cdot(2q_k-\xi Q)_+\wedge q_k. \end{aligned}$$

6.2. The $\mathcal{C}_\downarrow$ contribution. Since $r=1, r_k=1, r_k+r_{k+1} = r+r_{k+1} = 0 \not\equiv 1 \pmod 3$, none of the corresponding slits is being removed. Moreover, $2r_{k+1} = 1 \not\equiv 0 \pmod 3$, thus the slit $2q_{k+1}$ is not removed either, and the slits $q_{k+2} = q + q_{k+1}$, $q_k + q_{k+1}$ and $2q_{k+1}$ lock the central channel (see Figure 11). Furthermore, ordering $w_{\mathcal{B}_k}$, $B(q_k+q_{k+1}) = 2w_{\mathcal{B}_k} - w_{\mathcal{A}_0}$,

[6] Only $\mathcal{A}_0$ and $\mathcal{B}_k$ have to taken into account here because $\mathcal{C}_k$ has been already considered in the previous section.

$w_{\mathcal{B}_k} + w_{\mathcal{C}_k}$ and $T(q_{k+2}) = 2w_{\mathcal{B}_k} + w_{\mathcal{C}_k} - w_{\mathcal{A}_0}$, we find as in Subsection 6.1 that the corresponding weight $W_{\gamma,k}(t)$ is given by[7]

$$\begin{cases} (w_{\mathcal{B}_k} - w_{\mathcal{A}_0})(q_k + q_{k+1} - \xi Q)_+ \wedge q_{k+1} & \\ \quad +(w_{\mathcal{A}_0} + w_{\mathcal{C}_k} - w_{\mathcal{B}_k})(2q_{k+1} - \xi Q)_+ \wedge q_{k+1} & \text{if } w_{\mathcal{A}_0} < w_{\mathcal{B}_k} < w_{\mathcal{A}_0} + w_{\mathcal{C}_k}, \\ w_{\mathcal{C}_k}(q_k + q_{k+1} - \xi Q)_+ \wedge q_{k+1} & \text{if } w_{\mathcal{A}_0} + w_{\mathcal{C}_k} < w_{\mathcal{B}_k}, \\ (w_{\mathcal{A}_0} - w_{\mathcal{B}_k})(q_{k+2} - \xi Q)_+ \wedge q_{k+1} & \\ \quad +(w_{\mathcal{B}_k} + w_{\mathcal{C}_k} - w_{\mathcal{A}_0})(2q_{k+1} - \xi Q)_+ \wedge q_{k+1} & \text{if } w_{\mathcal{B}_k} < w_{\mathcal{A}_0} < w_{\mathcal{B}_k} + w_{\mathcal{C}_k}, \\ w_{\mathcal{C}_k}(q_{k+2} - \xi Q)_+ \wedge q_{k+1} & \text{if } w_{\mathcal{B}_k} + w_{\mathcal{C}_k} < w_{\mathcal{A}_0}. \end{cases}$$

Equivalently, if we set

$$\begin{aligned} W^{(5)}_{\gamma,k}(t) &:= w_{\mathcal{C}_k}(t) \cdot (q_{k+2} - \xi Q)_+ \wedge q_{k+1} \\ W^{(6)}_{\gamma,k}(t) &:= (a_{k+1} - q_{k+1}t) \cdot (q_{k+2} - \xi Q)_+ \wedge q_{k+1} \\ &\qquad + 2(qt - a - \varepsilon) \cdot (2q_{k+1} - \xi Q)_+ \wedge q_{k+1} \\ W^{(7)}_{\gamma,k}(t) &:= (q_{k+1}t - a_{k+1}) \cdot (q_k + q_{k+1} - \xi Q)_+ \wedge q_{k+1} \\ &\qquad + 2(a_k - \varepsilon - q_k t) \cdot (2q_{k+1} - \xi Q)_+ \wedge q_{k+1} \\ W^{(8)}_{\gamma,k}(t) &:= w_{\mathcal{C}_k}(t) \cdot (q_k + q_{k+1} - \xi Q)_+ \wedge q_{k+1}, \end{aligned}$$

then $W_{\gamma,k}(t)$ is given by

$$\begin{cases} W^{(5)}_{\gamma,k}(t) & \text{if } t < \frac{a+\varepsilon}{q} \wedge \gamma_{k+1}, \\ W^{(6)}_{\gamma,k}(t) & \text{if } \frac{a+\varepsilon}{q} < t < \gamma_{k+1}, \\ W^{(7)}_{\gamma,k}(t) & \text{if } \gamma_{k+1} < t < \frac{a_k - \varepsilon}{q_k}, \\ W^{(8)}_{\gamma,k}(t) & \text{if } \frac{a_k-\varepsilon}{q_k} \vee \gamma_{k+1} < t. \end{cases} \tag{6.5}$$

6.3. Estimating the total contribution.

6.3.1. $q_{k+1} > 2Q$. In this case $k \geqslant 1$ and $t_{k-1} < \frac{a_k - \varepsilon}{q_k} < \gamma_{k+1} < \frac{a+\varepsilon}{q}$. The cumulative contribution of $\mathcal{C}_\leftarrow$ and $\mathcal{C}_\downarrow$ when $r = r_k = 1$ or $r = r_k = -1$ and arising from (6.1)–(6.5), is given by

$$\begin{aligned} \mathbb{G}^{(1.1)}_{I,Q}(\xi) &= 2 \sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_k \in \mathcal{I}_{q,k} \\ q_{k+1} > 2Q}} \int_{t_k}^{t_{k-1}} \frac{W^{(1)}_{\gamma,k}(t) + W^{(5)}_{\gamma,k}(t)}{t^2 + t + 1}\, dt \\ &= \mathbb{G}^{(1.1.1)}_{I,Q}(\xi) + \mathbb{G}^{(1.1.2)}_{I,Q}(\xi) + \mathbb{G}^{(1.1.3)}_{I,Q}(\xi), \end{aligned}$$

[7] Only the contribution of $\mathcal{C}_k$ needs to be taken into account here.

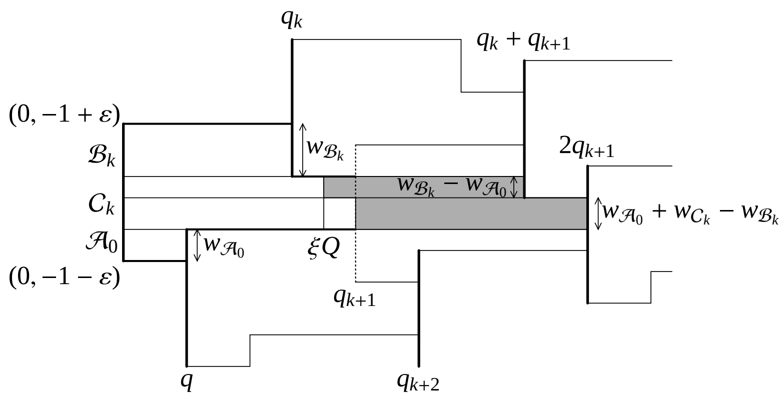

FIGURE 11. The case $t \in I_{\gamma,k}$, $(\mathcal{C}_\downarrow, r = r_k = 1)$, $w_{\mathcal{A}_0} < w_{\mathcal{B}_k} < w_{\mathcal{A}_0} + w_{\mathcal{C}_k}$

with

$$\mathbb{G}_{I,Q}^{(1.1.1)}(\xi) = 2\sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_k \in \mathcal{I}_{q,k} \\ q_{k+1} > 2Q}} \Big((q_{k+1} - \xi Q)_+ \wedge q + (q_{k+1} - \xi Q)_+ \wedge q_k \Big) \cdot \int_{t_k}^{t_{k-1}} \frac{w_{\mathcal{B}_k}(t)\, dt}{t^2 + t + 1},$$

$$\mathbb{G}_{I,Q}^{(1.1.2)}(\xi) = 2\sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_k \in \mathcal{I}_{q,k} \\ q_{k+1} > 2Q}} (2q - \xi Q)_+ \wedge q \int_{t_k}^{t_{k-1}} \frac{2(a + \varepsilon - qt)\, dt}{t^2 + t + 1},$$

$$\mathbb{G}_{I,Q}^{(1.1.3)}(\xi) = 2\sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_k \in \mathcal{I}_{q,k} \\ q_{k+1} > 2Q}} \Big((q_{k+2} - \xi Q)_+ \wedge q + (q_{k+2} - \xi Q)_+ \wedge q_{k+1} \Big) \cdot \int_{t_k}^{t_{k-1}} \frac{w_{\mathcal{C}_k}(t)\, dt}{t^2 + t + 1}.$$

This quantity is estimated employing (5.6), (A.1) and (5.7). The cumulative contribution of the error terms from those formulas to $\sum_I \mathbb{G}_{I,Q}^{(1.1)}(\xi)$ is

$$\ll \sum_I \sum_{\gamma \in \mathcal{F}_I(Q)} \sum_{k=1}^{\infty} \frac{1}{Qqq_{k-1}^2} \leqslant \sum_I \sum_{\gamma \in \mathcal{F}_I(Q)} \frac{1}{Qq} \left(\frac{1}{q'^2} + \frac{1}{qq'} \right) \leqslant \frac{2}{Q} \sum_I \sum_{\gamma \in \mathcal{F}_I(Q)} \frac{1}{qq'} \leqslant \frac{6|I|}{Q},$$

so they can be discarded. Following the outline from the end of Section 5 we find

$$\mathbb{G}_{I,Q}^{(1.1)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_1(\xi), \tag{6.6}$$

with $G_1(\xi)$ as in (1.5).

6.3.2. $q_{k+1} \leqslant 2Q$. In this case $\frac{a+\varepsilon}{q} \leqslant \gamma_{k+1} \leqslant \frac{a_k-\varepsilon}{q_k}$. Furthermore, when $k \geqslant 1$ we have $\frac{a+\varepsilon}{q} \leqslant t_k \leqslant \gamma_{k+1} \leqslant \frac{a_k-\varepsilon}{q_k} \leqslant t_{k-1}$ if $q_{k+1} \leqslant 2Q-q$, and $t_k \leqslant \frac{a+\varepsilon}{q} \leqslant \gamma_{k+1} \leqslant \frac{a_k-\varepsilon}{q_k} \leqslant t_{k-1}$ if $2Q-q \leqslant q_{k+1} \leqslant 2Q$. When $k=0$, the only difference is that $\gamma_1 = t_{-1} \leqslant \frac{a_0-\varepsilon}{q_0} = \frac{a'-\varepsilon}{q'}$. In this case the cumulative contribution of $\mathcal{C}_\leftarrow$ and $\mathcal{C}_\downarrow$ arising from (6.1)–(6.5) when $r=r_k=1$ or $r=r_k=-1$ is

$$\mathbb{G}_{I,Q}^{(1.2)}(\xi) = \mathbb{G}_{I,Q}^{(1.2.1)}(\xi) + \cdots + \mathbb{G}_{I,Q}^{(1.2.5)}(\xi),$$

with

$$\mathbb{G}_{I,Q}^{(1.2.1)}(\xi) = 2\sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_k\in\mathcal{I}_{q,k}\\ q_{k+1}\leqslant 2Q}} \int_{\frac{a_k-\varepsilon}{q_k}}^{t_{k-1}} \frac{W_{\gamma,k}^{(4)}(t)+W_{\gamma,k}^{(8)}(t)}{t^2+t+1}\,dt,$$

$$\mathbb{G}_{I,Q}^{(1.2.2)}(\xi) = 2\sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_k\in\mathcal{I}_{q,k}\\ q_{k+1}\leqslant 2Q}} \int_{\gamma_{k+1}}^{\frac{a_k-\varepsilon}{q_k}} \frac{W_{\gamma,k}^{(3)}(t)+W_{\gamma,k}^{(7)}(t)}{t^2+t+1}\,dt,$$

$$\mathbb{G}_{I,Q}^{(1.2.3)}(\xi) = 2\sum_{k=0}^{\infty} \sideset{}{^*}\sum_{\substack{q_k\in\mathcal{I}_{q,k}\\ q_{k+1}\leqslant 2Q-q}} \int_{t_k}^{\gamma_{k+1}} \frac{W_{\gamma,k}^{(2)}(t)+W_{\gamma,k}^{(6)}(t)}{t^2+t+1}\,dt,$$

$$\mathbb{G}_{I,Q}^{(1.2.4)}(\xi) = 2\sum_{k=0}^{\infty} \sideset{}{^*}\sum_{\substack{q_k\in\mathcal{I}_{q,k}\\ 2Q-q< q_{k+1}\leqslant 2Q}} \int_{t_k}^{\frac{a+\varepsilon}{q}} \frac{W_{\gamma,k}^{(1)}(t)+W_{\gamma,k}^{(5)}(t)}{t^2+t+1}\,dt,$$

$$\mathbb{G}_{I,Q}^{(1.2.5)}(\xi) = 2\sum_{k=0}^{\infty} \sideset{}{^*}\sum_{\substack{q_k\in\mathcal{I}_{q,k}\\ 2Q-q< q_{k+1}\leqslant 2Q}} \int_{\frac{a+\varepsilon}{q}}^{\gamma_{k+1}} \frac{W_{\gamma,k}^{(2)}(t)+W_{\gamma,k}^{(6)}(t)}{t^2+t+1}\,dt.$$

The total contribution of errors from (A.2), (A.3) and (A.4) to $\sum_I \mathbb{G}_{I,Q}^{(1.2.1)}(\xi)$ is

$$\begin{aligned}\ll \sum_I \sum_{\gamma\in\mathcal{F}_I(Q)} \sum_{k=1}^{\infty} \left(\frac{1}{qq_{k-1}^3} + \frac{1}{q^2 q_{k-1} q_k} \right) \\ \leqslant \sum_I \sum_{\gamma\in\mathcal{F}_I(Q)} \left(\frac{1}{qq'^3} + \frac{1}{Qq^2q'} + \frac{1}{(qq')^2} + \frac{1}{q^3q'} \right) \\ \leqslant \frac{2}{Q} \sum_{\gamma\in\mathcal{F}(Q)} \frac{1}{qq'} + \frac{2}{Q} \sum_{q=1}^{Q} \frac{\varphi(q)}{q^2} + \sum_I \frac{8}{|I|Q} \ll Q^{c-1},\end{aligned}$$

showing that they can be discarded in the sequel. The same holds for $\mathbb{G}_{I,Q}^{(1.2.2)}(\xi), \ldots, \mathbb{G}_{I,Q}^{(1.2.5)}(\xi)$, where for $\mathbb{G}_{I,Q}^{(1.2.4)}(\xi)$ and $\mathbb{G}_{I,Q}^{(1.2.5)}(\xi)$ one uses the fact that k takes exactly one value as a result of $q_{k+1} \in (2Q-q, 2Q]$.

To finally estimate $\mathbb{G}_{I,Q}^{(1.2.1)}(\xi), \ldots, \mathbb{G}_{I,Q}^{(1.2.5)}(\xi)$, we proceed as in Section 5. Taking stock on (A.2), (A.3), (A.4), respectively (A.5), (A.6), (A.7), respectively (A.8), (A.9), (A.10), respectively (A.13)–(A.17), respectively (A.17), (A.18), (A.19), we find

$$\mathbb{G}_{I,Q}^{(1.2.j)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{j+1}(\xi), \quad j = 1, \ldots, 5, \tag{6.7}$$

with $G_2(\xi), \ldots, G_6(\xi)$ as in (1.6)–(1.10).

7. Channels with removed slits. The case $r = 0$

In this case $r_k = \pm 1$. The $\mathcal{C}_O$ contributions for $(r, r_k) = (0, 1)$ and respectively $(r, r_k) = (0, -1)$ do coincide. The $\mathcal{C}_{\leftarrow}$ contribution for $(r, r_k) = (0, 1)$ coincides with the $\mathcal{C}_{\downarrow}$ contribution for $(r, r_k) = (0, -1)$. We shall sum as in (5.2), considering

$$\sum\nolimits^* = \sum_{\substack{\gamma\in\mathcal{F}_I(Q) \\ q\equiv a \ (\mathrm{mod}\ 3) \\ q_k \not\equiv a_k \ (\mathrm{mod}\ 3)}}^{*} .$$

It suffices to only analyze the $\mathcal{C}_O$ and $\mathcal{C}_{\leftarrow}$ contributions of $(r, r_k) = (0, 1)$, allowing at the very end β to take both values -1 and 1. The final result will express the $\mathcal{C}_O$ contribution when $(r, r_k) = (0, \pm 1)$, and respectively the sum of the $\mathcal{C}_{\leftarrow}$ contribution for $(r, r_k) = (0, 1)$ and of the $\mathcal{C}_{\downarrow}$ contribution for $(r, r_k) = (0, -1)$.

7.1. The $\mathcal{C}_O$ contribution. The situation is shown in Figure 12. Two cases arise:

7.1.1. $w_{\mathcal{A}_0} \leqslant w_{\mathcal{B}_k}$ **(**$\Longleftrightarrow t \geqslant \gamma_{k+1}$**).** In this case $k \geqslant 1$. Since $\gamma_{k+1} \leqslant t_{k-1}$, we must also have $q_{k+1} \leqslant 2Q$. The channel $\mathcal{A}_0$ is locked by the slit q_{k+1} and $W_{\gamma,k}(t) = w_{\mathcal{A}_0}(t) \cdot (q_{k+1} - \xi Q)_+ \wedge q$, with contribution

$$\mathbb{G}^{(2.1)}_{I,Q}(\xi) = \sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_k \in \mathcal{I}_{q,k} \\ q_{k+1} \leqslant 2Q}} (q_{k+1} - \xi Q)_+ \wedge q \int_{\gamma_{k+1}}^{t_{k-1}} \frac{w_{\mathcal{A}_0}(t)\, dt}{t^2+t+1},$$

estimated in Subsection B.1.1 as

$$\mathbb{G}^{(2.1)}_{I,Q}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_7(\xi), \tag{7.1}$$

where $G_7(\xi)$ is as in (1.11).

7.1.2. $w_{\mathcal{A}_0} > w_{\mathcal{B}_k}$ **(**$\Longleftrightarrow t < \gamma_{k+1}$**) and** $\xi Q < q_{k+1}$**.** In this situation we have

$$W_{\gamma,k}(t) = w_{\mathcal{B}_k}(t)(q_{k+1} - \xi Q) + \big(w_{\mathcal{A}_0}(t) - w_{\mathcal{B}_k}(t)\big) q,$$

with contribution (according to whether $\gamma_{k+1} \leqslant t_{k-1}$ or $t_{k-1} < \gamma_{k+1}$)

$$\begin{aligned} &\mathbb{G}^{(2.2)}_{I,Q}(\xi) \\ &= \sum_{k=0}^{\infty} \sideset{}{^*}\sum_{\substack{q_k \in \mathcal{I}_{q,k} \\ \xi Q < q_{k+1} \leqslant 2Q}} \int_{t_k}^{\gamma_{k+1}} \frac{W_{\gamma,k}(t)}{t^2+t+1}\, dt + \sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_k \in \mathcal{I}_{q,k} \\ q_{k+1} > (\xi \vee 2) Q}} \int_{t_k}^{t_{k-1}} \frac{W_{\gamma,k}(t)}{t^2+t+1}\, dt. \end{aligned}$$

Employing (A.8), (A.9), respectively (5.6), (B.2) and the procedure described at the beginning of Appendix B2, we find, with $G_8(\xi)$ as in (1.12),

$$\mathbb{G}^{(2.2)}_{I,Q}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_8(\xi). \tag{7.2}$$

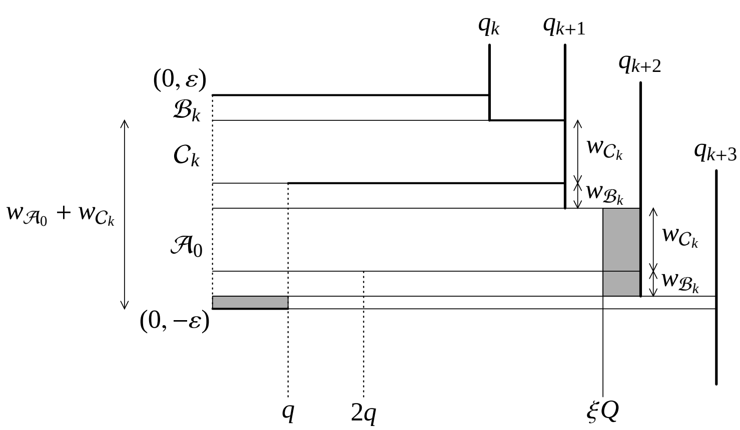

FIGURE 12. The case $t \in I_{\gamma,k}$, $r = 0$, $r_k = \pm 1$, $\mathcal{C}_O$, $w_{\mathcal{B}_k} < w_{\mathcal{A}_0}$, $n_0 = 2$, $N = 1$

7.1.3. $w_{\mathcal{A}_0} > w_{\mathcal{B}_k}$ ($\Longleftrightarrow t < \gamma_{k+1}$) and $\xi Q \geqslant q_{k+1}$. Consider the integer N for which $q_{k+N} \leqslant \xi Q < q_{k+N+1}$, that is $1 \leqslant N := \left\lfloor \frac{\xi Q - q_k}{q} \right\rfloor$. We will keep $N \geqslant 1$ and $q \leqslant Q$ fixed, and sum over

$$y = q_k \in \mathcal{J}_{q,N} := \big(\xi Q - (N+1)q, \xi Q - Nq\big].$$

Let $n_0 := \left\lfloor \frac{w_{\mathcal{A}_0} + w_{\mathcal{C}_k}}{w_{\mathcal{B}_k} + w_{\mathcal{C}_k}} \right\rfloor = \left\lfloor \frac{a_k - q_k t}{qt - a} \right\rfloor \geqslant 1$, so $\gamma_{k+n_0+1} < t \leqslant \gamma_{k+n_0}$ and $B(q_{k+n_0}) = w_{\mathcal{B}_k} + n_0(w_{\mathcal{B}_k} + w_{\mathcal{C}_k}) \leqslant 2\varepsilon < B(q_{k+n_0+1})$. This shows that the channel $\mathcal{A}_0$ is locked by the slits $q_{k+1}, \ldots, q_{k+n_0}, q_{k+n_0+1}$. Suppose that $N \geqslant \xi$. Then $q_k \leqslant \xi Q - Nq \leqslant N(Q-q)$, showing that $q_{k+N} \leqslant NQ$ and $\gamma_{k+N} < t_k$. So $n_0 < N$ and $\xi Q \geqslant q_{k+N} \geqslant q_{k+n_0+1}$, showing that in this case $W_{\gamma,k,N}(t) = 0$, $\forall t \in I_{\gamma,k}$. It remains that $N < \xi$. The following cases arise:

(I) $B(q_{k+N}) \leqslant 2\varepsilon < B(q_{k+N+1})$ ($\Longleftrightarrow \gamma_{k+N+1} < t \leqslant \gamma_{k+N}$). In this case

$$\begin{aligned} W_{\gamma,k,N}(t) = W^{(1)}_{\gamma,k,N}(t) &:= \big(2\varepsilon - B(q_{k+N})\big)(q_{k+N+1} - \xi Q) \\ &= (a_{k+N} - q_{k+N} t)(q_{k+N+1} - \xi Q). \end{aligned} \tag{7.3}$$

(II) $B(q_{k+N+1}) \leqslant 2\varepsilon$ ($\Longleftrightarrow t \leqslant \gamma_{k+N+1}$). In this case

$$\begin{aligned} W_{\gamma,k,N}(t) &= W^{(2)}_{\gamma,k}(t) \\ &:= \big(w_{\mathcal{B}_k}(t) + w_{\mathcal{C}_k}(t)\big)(q_{k+N+1} - \xi Q) + \big(2\varepsilon - B(q_{k+N+1})\big) q \\ &= 1 - \xi Q\big(w_{\mathcal{B}_k}(t) + w_{\mathcal{C}_k}(t)\big). \end{aligned} \tag{7.4}$$

We find

$$\mathbb{G}^{(2.3)}_{I,Q}(\xi) = \mathbb{G}^{(2.3.1)}_{I,Q}(\xi) + \mathbb{G}^{(2.3.2)}_{I,Q}(\xi),$$

with

$$\mathbb{G}^{(2.3.1)}_{I,Q}(\xi) = \sum_{1 \leqslant N < \xi} \sum_{k=0}^{\infty} \sideset{}{^*}\sum_{q_k \in \mathcal{I}_{q,k} \cap \mathcal{J}_{q,N}} \int_{I_{\gamma,k} \cap (\gamma_{k+N+1}, \gamma_{k+N}]} \frac{W^{(1)}_{\gamma,k,N}(t)}{t^2+t+1}\, dt,$$

$$\mathbb{G}^{(2.3.2)}_{I,Q}(\xi) = \sum_{1 \leqslant N < \xi} \sum_{k=0}^{\infty} \sideset{}{^*}\sum_{q_k \in \mathcal{I}_{q,k} \cap \mathcal{J}_{q,N}} \int_{I_{\gamma,k} \cap (\gamma, \gamma_{k+N+1}]} \frac{W^{(2)}_{\gamma,k,N}(t)}{t^2+t+1}\, dt,$$

where the interval $I_{\gamma,k} \cap (\gamma_{k+N+1}, \gamma_{k+N}]$ is given by

$$\begin{cases} \emptyset & \text{if } q_k < N(Q-q) \text{ or } Q + (N+1)(Q-q) \leqslant q_k, \\ (t_k, \gamma_{k+N}] & \text{if } N(Q-q) \leqslant q_k < (N+1)(Q-q), \\ (\gamma_{k+N+1}, \gamma_{k+N}] & \text{if } (N+1)(Q-q) \leqslant q_k < Q + N(Q-q), \\ (\gamma_{k+N+1}, t_{k-1}] & \text{if } Q + N(Q-q) \leqslant q_k < Q + (N+1)(Q-q), \end{cases}$$

and the interval $I_{\gamma,k} \cap (\gamma, \gamma_{k+N+1}]$ is given by

$$\begin{cases} \emptyset & \text{if } q_k < (N+1)(Q-q), \\ (t_k, \gamma_{k+N+1}] & \text{if } (N+1)(Q-q) \leqslant q_k < Q + (N+1)(Q-q), \\ I_{\gamma,k} & \text{if } Q + (N+1)(Q-q) \leqslant q_k. \end{cases}$$

Summing as in (5.2) and employing (B.3), (B.4), (B.5) and Lemmas 3.4 and 3.2, then changing $y + Nq$ to y and making the substitution $(q, y) = (Qu, Qw)$, we find

$$\mathbb{G}^{(2.3.1)}_{I,Q}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_9(\xi), \tag{7.5}$$

with $G_9(\xi)$ as in (1.13).

In a similar way[8] we infer from (B.6) and (B.7), with $G_{10}(\xi)$ as in (1.14),

$$\mathbb{G}^{(2.3.2)}_{I,Q}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{10}(\xi). \tag{7.6}$$

The innermost integrals in (1.13) and the first one in (1.14) can be nonzero only when $N > \xi - 1$, $\xi - 2 < N < \xi$ or $\xi - 2 < N < \xi - 1$.

7.2. The $\mathcal{C}_{\leftarrow}$ contribution. In this case we shall analyze the contribution of the channels $\mathcal{B}_k$ and $\mathcal{C}_k$. All slits q_{k+n}, $n \in \mathbb{Z}$, are removed, while neither q nor any of $2q_k + nq = q_k + q_{k+n}$, $n \in \mathbb{Z}$, is being removed. Since $T(2q_k) = -2w_{\mathcal{A}_0} - 2w_{\mathcal{C}_k} < 0$ and $B(2q_k + nq) = w_{\mathcal{B}_k} - w_{\mathcal{A}_0} - w_{\mathcal{C}_k} + n(w_{\mathcal{B}_k} + w_{\mathcal{C}_k})$, it follows that $\mathcal{B}_k \cup \mathcal{C}_k$ is locked exactly by two of the slits $2q_k + nq$, $n \geqslant 0$. To make this precise, let $n_0 := \left\lfloor \frac{w_{\mathcal{A}_0} + 2w_{\mathcal{C}_k}}{w_{\mathcal{B}_k} + w_{\mathcal{C}_k}} \right\rfloor = \left\lfloor \frac{a_k + a_{k-1} - 2\varepsilon - (q_k + q_{k-1})t}{qt - a} \right\rfloor \geqslant 0$. We have $0 < B(2q_k + n_0 q) = w_{\mathcal{B}_k} - w_{\mathcal{A}_0} - w_{\mathcal{C}_k} + n_0(w_{\mathcal{B}_k} + w_{\mathcal{C}_k}) \leqslant w_{\mathcal{B}_k} + w_{\mathcal{C}_k}$. The situation is described in Figure 13. Consider also

$$\lambda_{k,n} := \frac{a_k + a_{k+n-1} - 2\varepsilon}{q_k + q_{k+n-1}} \searrow \gamma \quad \text{as } n \to \infty, \tag{7.7}$$

so $\lambda_{k,n_0+1} < t \leqslant \lambda_{k,n_0}$. Note that $\lambda_{0,1} \geqslant t_{-1} = \gamma_1 > \lambda_{0,2}$ and $\lambda_{k,0} > t_{k-1}$ when $k \geqslant 1$, showing that for every $k \geqslant 0$ the intervals $(\lambda_{k,n+1}, \lambda_{k,n}]$ cover $I_{\gamma,k}$. The following cases arise:

7.2.1. $0 < B(2q_k + n_0 q) \leqslant w_{\mathcal{B}_k}$ ($\Longleftrightarrow \lambda_{k,n_0+1} < t \leqslant \gamma_{k+n_0}$). In this case $\mathcal{C}_k$ is locked by the slit $2q_k + (n_0 + 1)q$ and $\mathcal{B}_k$ by the slits $2q_k + n_0 q = q_k + q_{k+n_0}$ and $2q_k + (n_0 + 1)q = q_k + q_{k+n_0+1}$. The widths of the three relevant subchannels of $\mathcal{B}_k \cup \mathcal{C}_k$ are (from bottom to top) $w_{\mathcal{C}_k}$, $w_{\mathcal{B}_k} - B(2q_k + n_0 q) = a_{k+n_0} - q_{k+n_0} t$, $B(2q_k + n_0 q) = w_{\mathcal{B}_k} - (a_{k+n_0} - q_{k+n_0} t)$, and so in this situation $W_{\gamma,k}(t)$ is given by

$$\begin{aligned} &W^{(1)}_{\gamma,k,n_0}(t) \\ &= w_{\mathcal{C}_k}(t) \cdot (q_{k+1} + q_{k+n_0} - \xi Q)_+ \wedge q_{k+1} + w_{\mathcal{B}_k}(t) \cdot (q_k + q_{k+n_0} - \xi Q)_+ \wedge q_k \\ &+ (a_{k+n_0} - q_{k+n_0} t)\Big((q_{k+1} + q_{k+n_0} - \xi Q)_+ \wedge q_k - (q_k + q_{k+n_0} - \xi Q)_+ \wedge q_k \Big). \end{aligned}$$

[8]But replacing $y + (N+1)q$ by y

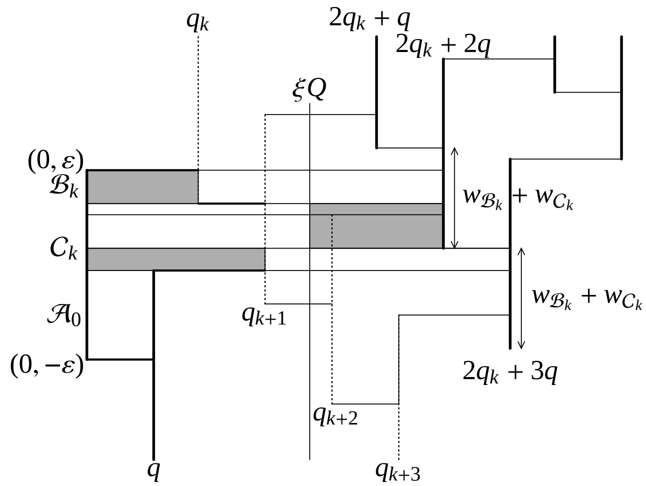

FIGURE 13. The case $t \in I_{\gamma,k}$, $r = 0$, $r_k = 1$, $\mathcal{C}_{\leftarrow}$, $n_0 = 2$, $w_{\mathcal{B}_k} < B(2q_k + n_0 q) \leqslant w_{\mathcal{B}_k} + w_{\mathcal{C}_k}$

7.2.2. $\boldsymbol{w_{\mathcal{B}_k} < B(2q_k + n_0 q) \leqslant w_{\mathcal{B}_k} + w_{\mathcal{C}_k}}$ $\boldsymbol{(\Longleftrightarrow \ \gamma_{k+n_0} < t \leqslant \lambda_{k,n_0})}$**.** Then $\mathcal{B}_k$ is locked by the slit $q_k + q_{k+n_0}$ and $\mathcal{C}_k$ by the slits $q_k + q_{k+n_0}$ and $q_k + q_{k+n_0+1}$. The widths of the three relevant subchannels of $\mathcal{B}_k \cup \mathcal{C}_k$ are $w_{\mathcal{B}_k} + w_{\mathcal{C}_k} - B(2q_k + n_0 q) = w_{\mathcal{C}_k} - (q_{k+n_0} t - a_{k+n_0})$, $B(2q_k + n_0 q) - w_{\mathcal{B}_k} = q_{k+n_0} t - a_{k+n_0}$, $w_{\mathcal{B}_k}$, so $W_{\gamma,k}(t)$ equals

$$\begin{aligned} & W^{(2)}_{\gamma,k,n_0}(t) \\ & = w_{\mathcal{C}_k}(t)(q_{k+1} + q_{k+n_0} - \xi Q)_+ \wedge q_{k+1} + w_{\mathcal{B}_k}(t) \cdot (q_k + q_{k+n_0} - \xi Q)_+ \wedge q_k \\ & \quad - (q_{k+n_0} t - a_{k+n_0}) \\ & \qquad \cdot \Big((q_{k+1} + q_{k+n_0} - \xi Q)_+ \wedge q_{k+1} - (q_k + q_{k+n_0} - \xi Q)_+ \wedge q_{k+1} \Big). \end{aligned}$$

The following five cases arise:

(I) $q_k < (n-1)(Q-q)$. Then $\lambda_{k,n} < t_k$, thus $I_{\gamma,k} \cap (\lambda_{k,n+1}, \lambda_{k,n}] = \emptyset$.

(II) $(n-1)(Q-q) \leqslant q_k < n(Q-q)$. Then $n \geqslant 2$ and $\gamma_{k+n} < t_k \leqslant \lambda_{k,n} < t_{k-1}$ in both cases $k \geqslant 1$ and $k = 0$. The corresponding contribution is

$$\mathbb{G}^{(2.4.1)}_{I,Q}(\xi) = \sum_{n=2}^{\infty} \sum_{k=0}^{\infty} \sideset{}{^*}\sum_{\substack{q_k \in \mathcal{I}_{q,k} \\ (n-1)(Q-q) \leqslant q_k < n(Q-q)}} \int_{t_k}^{\lambda_{k,n}} \frac{W^{(2)}_{\gamma,k,n}(t)}{t^2 + t + 1}\, dt.$$

Employing (B.8)–(B.11), Lemmas 3.8 and 3.3, and the change of variable $(q, y) = (Qu, Qw)$, we find (according to whether $k = 0$ or $k \geqslant 1$), with $G_{11}(\xi)$ as in (1.15),

$$\mathbb{G}^{(2.4.1)}_{I,Q}(\xi) \approx \frac{c_I}{\zeta(2)} G_{11}(\xi). \tag{7.8}$$

(III) $n(Q-q) \leqslant q_k < Q+n(Q-q)$. Upon $\lambda_{0,2} < \gamma_1 = t_{-1} \leqslant \lambda_{0,1} = \frac{a'-\varepsilon}{q'}$, we see that $I_{\gamma,k} \cap (\lambda_{k,n+1}, \lambda_{k,n}] = (\lambda_{k,n+1}, \gamma_{k+n}] \cup (\gamma_{k+n}, \lambda_{k,n}]$ when $k \geqslant 1$ and when $k=0$ and $n \geqslant 2$. When $k=0$ and $n=1$ this interval coincides with $(\lambda_{0,2}, \gamma_1]$. The cumulative contribution is thus

$$\begin{aligned}
&\mathbb{G}_{I,Q}^{(2.4.2)}(\xi) \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_k \in \mathcal{I}_{q,k} \\ n(Q-q) \leqslant q_k < Q+n(Q-q)}} \left(\int_{\lambda_{k,n+1}}^{\gamma_{k+n}} \frac{W_{\gamma,k,n}^{(1)}(t)}{t^2+t+1}\, dt + \int_{\gamma_{k+n}}^{\lambda_{k,n}} \frac{W_{\gamma,k,n}^{(2)}(t)}{t^2+t+1}\, dt \right) \\
&\quad + \sum_{n=2}^{\infty} \sideset{}{^*}\sum_{\substack{q' \in \mathcal{I}_{q,0} \\ q' \geqslant n(Q-q)}} \left(\int_{\lambda_{0,n+1}}^{\gamma_n} \frac{W_{\gamma,0,n}^{(1)}(t)}{t^2+t+1}\, dt + \int_{\gamma_n}^{\lambda_{0,n}} \frac{W_{\gamma,0,n}^{(2)}(t)}{t^2+t+1}\, dt \right) \\
&\quad + \sideset{}{^*}\sum_{q' \in \mathcal{I}_{q,0}} \int_{\lambda_{0,2}}^{\gamma_1} \frac{W_{\gamma,0,1}^{(1)}(t)}{t^2+t+1}\, dt = \mathbb{G}_{I,Q}^{(2.4.2.1)}(\xi) + \mathbb{G}_{I,Q}^{(2.4.2.2)}(\xi) + \mathbb{G}_{I,Q}^{(2.4.2.3)}(\xi).
\end{aligned}$$

For fixed k we have $\frac{q_k-Q}{Q-q} < n \leqslant \frac{q_k}{Q-q}$, so n can take at most $1 + \left\lfloor \frac{Q}{Q-q} \right\rfloor \leqslant \frac{2Q}{Q-q}$ values. Employing (B.12)–(B.15), respectively (B.13)–(B.16), respectively (B.17), (B.18), (B.19), we find

$$\begin{aligned}
\text{(7.9)} \qquad &\mathbb{G}_{I,Q}^{(2.4.2.1)}(\xi) \approx \frac{c_I}{\zeta(2)} G_{12}(\xi), \quad \mathbb{G}_{I,Q}^{(2.4.2.2)}(\xi) \approx \frac{c_I}{\zeta(2)} G_{13}(\xi), \\
&\mathbb{G}_{I,Q}^{(2.4.2.3)}(\xi) \approx \frac{c_I}{\zeta(2)} G_{14}(\xi),
\end{aligned}$$

with $G_{12}(\xi)$ as in (1.16), $G_{13}(\xi)$ as in (1.17), and $G_{14}(\xi)$ as in (1.18).

(IV) $Q+n(Q-q) \leqslant q_k < Q+(n+1)(Q-q)$. Then $k \geqslant 1$ and $(n+2)Q - q_{k+n+1} \leqslant Q-q$. In this case $I_{\gamma,k} \cap (\lambda_{k,n+1}, \lambda_{k,n}] = (\lambda_{k,n+1}, t_{k-1}]$ and $t_{k-1} < \gamma_{k+n}$, so the corresponding contribution is

$$\mathbb{G}_{I,Q}^{(2.4.3)}(\xi) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_k \in \mathcal{I}_{q,k} \\ Q+n(Q-q) \leqslant q_k < Q+(n+1)(Q-q)}} \int_{\lambda_{k,n+1}}^{t_{k-1}} \frac{W_{\gamma,k,n}^{(1)}(t)}{t^2+t+1}\, dt.$$

Note also that n can only take the value $n = \left\lfloor \frac{q_k-Q}{Q-q} \right\rfloor$ for each k. Employing (B.20), (B.21), (B.22) we find, with $G_{15}(\xi)$ as in (1.19),

$$\text{(7.10)} \qquad \mathbb{G}_{I,Q}^{(2.4.3)}(\xi) \approx \frac{c_I}{\zeta(2)} G_{15}(\xi).$$

(V) $q_k \geqslant Q+(n+1)(Q-q)$. Then $k \geqslant 1$, $\lambda_{k,n+1} \geqslant t_{k-1}$ and $I_{\gamma,k} \cap (\lambda_{k,n+1}, \lambda_{k,n}] = \emptyset$.

8. Channels with removed slits. The case $r_k = 0$

The main term and the error term of the $\mathcal{C}_O$ contribution of $(r, r_k) = (1, 0)$ and of $(r, r_k) = (-1, 0)$ coincide, because the corresponding weights are given by the same formulas and one only has to replace the summation condition $q - a \equiv 1 \pmod 3$ by $q - a \equiv -1 \pmod 3$. The same thing holds for the $\mathcal{C}_{\leftarrow}$ contribution of $(r, r_k) = (1, 0)$ and the $\mathcal{C}_{\downarrow}$ contribution of $(r, r_k) = (-1, 0)$ (see Figure 8). So it suffices to take $(r, r_k) = (0, 1)$ in the sequel, doubling the $\mathcal{C}_O$ and the $\mathcal{C}_{\leftarrow}$ contributions. This time we consider

$$\sum\nolimits^{*} = \sum_{\substack{\gamma \in \mathcal{F}_I(Q) \\ q - a \equiv 1 \pmod 3 \\ q_k \equiv a_k \pmod 3}}^{*} .$$

8.1. The channel $\mathcal{C}_O$. The slit q_k is removed, while $q + nq_k$, $n \geqslant 0$, are not, as shown in Figure 14. The following three cases arise:

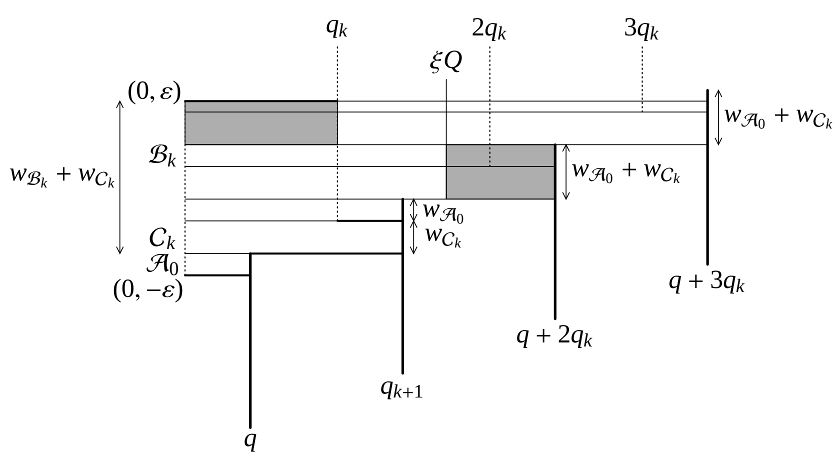

FIGURE 14. The case $t \in I_{\gamma,k}$, $r_k = 0$, $\mathcal{C}_O$, $w_{\mathcal{A}_0} < w_{\mathcal{B}_k}$, $n_0 = 2$, $N = 1$

8.1.1. $w_{\mathcal{A}_0} \geqslant w_{\mathcal{B}_k}$ ($\Longleftrightarrow t \leqslant \gamma_{k+1}$). The channel $\mathcal{B}_k$ is locked by the slit q_{k+1} and $W_{\gamma,k}(t) = w_{\mathcal{B}_k}(t) \cdot (q_{k+1} - \xi Q)_+ \wedge q_k$, with contribution

$$\mathbb{G}_{I,Q}^{(3.1)}(\xi) = \sum_{k=0}^{\infty} \sum_{\substack{q_k \in \mathcal{I}_{q,k} \\ q_{k+1} \leqslant 2Q}}^{*} \int_{t_k}^{\gamma_{k+1}} \frac{W_{\gamma,k}(t)}{t^2 + t + 1}\, dt + \sum_{k=1}^{\infty} \sum_{\substack{q_k \in \mathcal{I}_{q,k} \\ q_{k+1} > 2Q}}^{*} \int_{t_k}^{t_{k-1}} \frac{W_{\gamma,k}(t)}{t^2 + t + 1}\, dt.$$

Employing (A.8) (which also holds for $k = 0$) and (5.6), we find, with $G_{16}(\xi)$ as in (1.20),

$$\mathbb{G}_{I,Q}^{(3.1)}(\xi) \approx \frac{c_I}{\zeta(2)} G_{16}(\xi). \tag{8.1}$$

8.1.2. $w_{\mathcal{A}_0} < w_{\mathcal{B}_k}$ ($\Longleftrightarrow$ $t > \gamma_{k+1}$) and $\xi Q < q_{k+1}$. In this case $k \geqslant 1$, $q_{k+1} \leqslant 2Q$, and

$$W_{\gamma,k}(t) = w_{\mathcal{A}_0}(t) \cdot (q_{k+1} - \xi Q)_+ \wedge q_k + \big(w_{\mathcal{B}_k}(t) - w_{\mathcal{A}_0}(t)\big) q_k,$$

with contribution

$$\mathbb{G}^{(3.2)}_{I,Q}(\xi) = \sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_k \in \mathcal{I}_{q,k} \\ \xi Q < q_{k+1} \leqslant 2Q}} \int_{\gamma_{k+1}}^{t_{k-1}} \frac{W_{\gamma,k}(t)}{t^2+t+1}\, dt.$$

Employing (B.1) and (C.2) we find, with $G_{17}(\xi)$ as in (1.21),

$$\mathbb{G}^{(3.2)}_{I,Q}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{17}(\xi). \tag{8.2}$$

8.1.3. $w_{\mathcal{A}_0} < w_{\mathcal{B}_k}$ ($\Longleftrightarrow t > q_{k+1}$) and $q_{k+1} \leqslant \xi Q$. Again $k \geqslant 1$ and $q_{k+1} \leqslant 2Q$. Consider the integer N for which $q + Nq_k \leqslant \xi Q < q + (N+1)q_k$, that is $1 \leqslant N = \left\lfloor \frac{\xi Q - q}{q_k} \right\rfloor \leqslant \xi$. Consider also $n_0 := \left\lfloor \frac{w_{\mathcal{B}_k} + w_{\mathcal{C}_k}}{w_{\mathcal{A}_0} + w_{\mathcal{C}_k}} \right\rfloor = \left\lfloor \frac{qt-a}{a_k - q_k t} \right\rfloor \geqslant 1$, and let

$$\lambda_{k,n} := \frac{a + na_k}{q + nq_k} \nearrow \gamma_k \quad \text{as } n \to \infty, \tag{8.3}$$

hence $\lambda_{k,n_0} \leqslant t < \lambda_{k,n_0+1}$ and $T(q + (n_0+1)q_k) < 0 \leqslant T(q + n_0 q_k)$. The channel $\mathcal{B}_k$ is locked exactly by the slits

$$q_{k+1} = q + q_k, \ldots, q + n_0 q_k, q + (n_0+1) q_k$$

(see Figure 14) and $W_{\gamma,k}(t)$ is given by

$$\begin{cases} 0 & \text{if } N > n_0, \\ \big(w_{\mathcal{B}_k} + w_{\mathcal{C}_k} - n_0(w_{\mathcal{A}_0} + w_{\mathcal{C}_k})\big)\big(q + (n_0+1)q_k - \xi Q\big) & \text{if } n_0 = N, \\ \big(w_{\mathcal{A}_0} + w_{\mathcal{C}_k}\big)\big(q + (N+1)q_k - \xi Q\big) & \\ \quad + \big(w_{\mathcal{B}_k} + w_{\mathcal{C}_k} - (N+1)(w_{\mathcal{A}_0} + w_{\mathcal{C}_k})\big) q_k & \text{if } N < n_0, \end{cases}$$

or equivalently by

$$\begin{cases} 0 & \text{if } t \in I_{\gamma,k} \cap (t_k, \lambda_{k,N}], \\ W^{(1)}_{\gamma,k,N}(t) & \text{if } t \in I_{\gamma,k} \cap (\lambda_{k,N}, \lambda_{k,N+1}], \\ W^{(2)}_{\gamma,k}(t) & \text{if } t \in I_{\gamma,k} \cap (\lambda_{k,N+1}, t_{k-1}], \end{cases} \tag{8.4}$$

where

$$W^{(1)}_{\gamma,k,N}(t) := (q_{k+1} + Nq_k - \xi Q)\big((q + Nq_k)t - a - Na_k\big),$$
$$W^{(2)}_{\gamma,k}(t) := 1 - \xi Q\big(w_{\mathcal{A}_0}(t) + w_{\mathcal{C}_k}(t)\big).$$

We shall start by keeping $N \geqslant 1$ and $q \leqslant Q$ fixed, and summing over

$$y = q_k \in \mathcal{J}_{q,N} := \left(\frac{\xi Q - q}{N+1}, \frac{\xi Q - q}{N} \right].$$

Since $\lambda_{k,1} = \gamma_{k+1} > t_k$ and $\gamma_k > t_{k-1}$ for all k, nonzero contribution only arises from one of the following two subcases:

(I) $N(q_k-Q)\leqslant Q-q<(N+1)(q_k-Q)$. Then $t_k<\gamma_{k+1}\leqslant\lambda_{k,N}\leqslant t_{k-1}<\lambda_{k,N+1}$ and the contribution is

$$\mathbb{G}_{I,Q}^{(3.3.1)}(\xi)=\sum_{1\leqslant N\leqslant\xi}\sum_{k=1}^{\infty}\sideset{}{^*}\sum_{\substack{q_k\in\mathcal{I}_{q,k}\cap\mathcal{J}_{q,N}\\ Q+\frac{Q-q}{N+1}<q_k\leqslant Q+\frac{Q-q}{N}}}\int_{\lambda_{k,N}}^{t_{k-1}}\frac{W_{\gamma,k,N}^{(1)}(t)}{t^2+t+1}\,dt.$$

In this case N can only take the value $N=\left\lfloor\frac{Q-q}{q_k-Q}\right\rfloor$ for fixed k. Employing (C.3) we find, with $G_{18}(\xi)$ as in (1.22),

$$\mathbb{G}_{I,Q}^{(3.3.1)}(\xi)\approxeq\frac{c_I}{\zeta(2)}G_{18}(\xi).\tag{8.5}$$

(II) $(N+1)(q_k-Q)\leqslant Q-q$. Then $t_k<\gamma_{k+1}\leqslant\lambda_{k,N}<\lambda_{k,N+1}\leqslant t_{k-1}$. The contribution

$$\begin{aligned}&\mathbb{G}_{I,Q}^{(3.3.2)}(\xi)\\&=\sum_{1\leqslant N\leqslant\xi}\sum_{k=1}^{\infty}\sideset{}{^*}\sum_{\substack{q_k\in\mathcal{I}_{q,k}\cap\mathcal{J}_{q,N}\\ q_k\leqslant Q+\frac{Q-q}{N+1}}}\left(\int_{\lambda_{k,N}}^{\lambda_{k,N+1}}\frac{W_{\gamma,k,N}^{(1)}(t)}{t^2+t+1}\,dt+\int_{\lambda_{k,N+1}}^{t_{k-1}}\frac{W_{\gamma,k}^{(2)}(t)}{t^2+t+1}\,dt\right)\end{aligned}$$

is estimated upon (C.4) and (C.7) as

$$\mathbb{G}_{I,Q}^{(3.3.2)}(\xi)\approxeq\frac{c_I}{\zeta(2)}G_{19}(\xi),\tag{8.6}$$

with $G_{19}(\xi)$ as in (1.23). In this case $N\in\{\lfloor\xi\rfloor-1,\lfloor\xi\rfloor\}$ suffices as a result of the inequality $\xi-u<N+2-u$.

8.2. The channel $\mathcal{C}_{\longleftarrow}$. In this case all slits $q+nq_k$ are removed, while slits $2q+nq_k$, $n\geqslant 0$, are not. Exactly two of the latter ones lock $\mathcal{A}_0\cup\mathcal{C}_k$. Letting $n_0:=\left\lfloor\frac{w_{\mathcal{B}_k}+2w_{\mathcal{C}_k}}{w_{\mathcal{A}_0}+w_{\mathcal{C}_k}}\right\rfloor=\left\lfloor\frac{a_{k-2}-2\varepsilon-q_{k-2}t}{a_k-q_kt}\right\rfloor\geqslant 0$, we have $w_{\mathcal{B}_k}\leqslant T(2q+n_0q_k)=2(w_{\mathcal{B}_k}+w_{\mathcal{C}_k})-n_0(w_{\mathcal{A}_0}+w_{\mathcal{C}_k})<2\varepsilon$, so $\mathcal{A}_0\cup\mathcal{C}_k$ is locked exactly by the slits $2q+n_0q_k$ and $2q+(n_0+1)q_k$. The situation is described in Figure 15. Consider also

$$\mu_{k,n}:=\frac{2a+na_k+2\varepsilon}{2q+nq_k}\quad\text{and}\quad\nu_{k,n}:=\frac{a+na_k}{q+nq_k}.\tag{8.7}$$

When $q_k\leqslant 2Q$, $\mu_{k,n}\nearrow\gamma_k>t_{k-1}$ as $n\to\infty$ and $\mu_{k,0}=\frac{a+\varepsilon}{q}\leqslant t_{k-1}$, $\forall k\geqslant 0$. We also have $\mu_{k,n_0-1}\leqslant t<\mu_{k,n_0}$ and the following two situations can arise:

(I) $w_{\mathcal{B}_k}\leqslant T(2q+n_0q_k)<w_{\mathcal{B}_k}+w_{\mathcal{C}_k}$ ($\Longleftrightarrow\mu_{k,n_0-1}\leqslant t<\nu_{k,n_0}$). Then $n_0\geqslant 1$ and the widths of the relevant three subchannels of $\mathcal{A}_0\cup\mathcal{B}_k$ are (from bottom to top) $w_{\mathcal{A}_0}$, $w_{\mathcal{B}_k}+w_{\mathcal{C}_k}-T(2q+n_0q_k)=a+n_0q_k-(q+n_0q_k)t$,

$T(2q+n_0q_k)-w_{\mathcal{B}_k}=w_{\mathcal{C}_k}-\big(a+n_0a_k-(q+n_0q_k)t\big)$. This yields that $W_{\gamma,k}(t)$ is given by

$$\begin{aligned}&W^{(1)}_{\gamma,k,n_0}(t)\\&=w_{\mathcal{A}_0}(t)\cdot(2q+n_0q_k-\xi Q)_+\wedge q+w_{\mathcal{C}_k}(t)\cdot(q_{k+2}+n_0q_k-\xi Q)_+\wedge q_{k+1}\\&\quad-\big(a+n_0a_k-(q+n_0q_k)t)\big)\\&\qquad\cdot\Big((q_{k+2}+n_0q_k-\xi Q)_+\wedge q_{k+1}-(2q+n_0q_k-\xi Q)_+\wedge q_{k+1}\Big).\end{aligned}$$

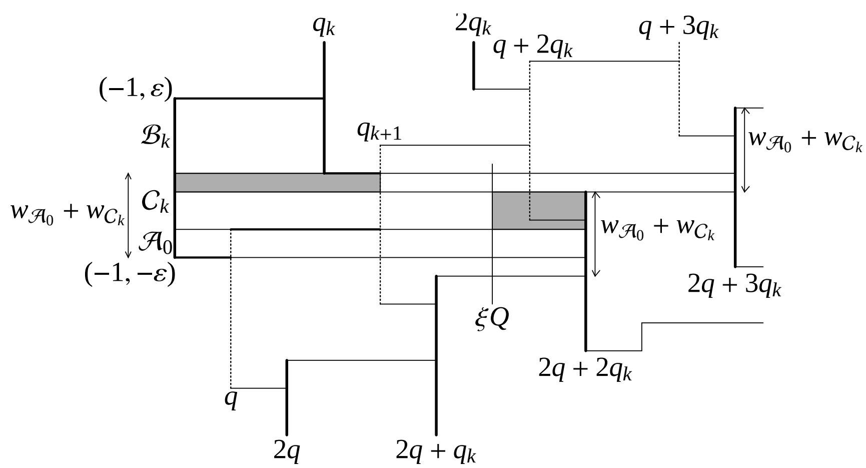

FIGURE 15. The case $t\in I_{\gamma,k}$, $r=1$, $r_k=0$, $r_{k+1}=-1$, $\mathcal{C}_{\leftarrow}$, $n_0=2$

(II) $w_{\mathcal{B}_k}+w_{\mathcal{C}_k}\leqslant T(2q+n_0q_k)<2\varepsilon$ ($\Longleftrightarrow \nu_{k,n_0}\leqslant t<\mu_{k,n_0}$). Then $n_0\geqslant 0$ and the widths of the relevant three subchannels of $\mathcal{A}_0\cup\mathcal{C}_k$ are: $2\varepsilon-T(2q+n_0q_k)=w_{\mathcal{A}_0}-\big((q+n_0q_k)t-a-n_0q_k\big)$, $T(2q+n_0q_k)-(w_{\mathcal{B}_k}+w_{\mathcal{C}_k})=(q+n_0q_k)t-a-n_0a_k$, $w_{\mathcal{C}_k}$. This yields that $W_{\gamma,k}(t)$ is given by

$$\begin{aligned}&W^{(2)}_{\gamma,k,n_0}(t)\\&=w_{\mathcal{A}_0}(t)\cdot(2q+n_0q_k-\xi Q)_+\wedge q+w_{\mathcal{C}_k}(t)\cdot\big(q_{k+2}+n_0q_k-\xi Q\big)_+\wedge q_{k+1}\\&\quad+\big((q+n_0q_k)t-a-n_0a_k\big)\\&\qquad\cdot\Big(\big(q_{k+2}+n_0q_k-\xi Q\big)_+\wedge q-(2q+n_0q_k-\xi Q)_+\wedge q\Big).\end{aligned}$$

The following three cases arise:

8.2.1. $n_0=0$ ($\Longleftrightarrow t<\frac{a+\varepsilon}{q}$). In this case $\mathcal{A}_0$ is locked by the slits $2q$ and q_{k+2}. One sees that $W_{\gamma,k}(t)=W^{(2)}_{\gamma,k,0}(t)$. When $q_{k+2}\leqslant 2Q$ we have $\frac{a+\varepsilon}{q}<t_k$ with zero contribution, so we must take $q_{k+1}>2Q-q$. When

$q_{k+1} > 2Q$ we have $k \geqslant 1$ and $\frac{a+\varepsilon}{q} > t_{k-1}$, with contribution

$$\mathbb{G}_{I,Q}^{(3.4.1)}(\xi) = \sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_k \in \mathcal{I}_{q,k} \\ q_{k+1} > 2Q}} \int_{t_k}^{t_{k-1}} \frac{W_{\gamma,k,0}^{(2)}(t)}{t^2+t+1}\, dt.$$

Employing (5.6), (5.7), (5.9) and $qt - a = w_{\mathcal{B}_k}(t) + w_{\mathcal{C}_k}(t)$ we find, with $G_{20}(\xi)$ as in (1.24),

$$\mathbb{G}_{I,Q}^{(3.4.1)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{20}(\xi). \tag{8.8}$$

When $2Q - q < q_{k+1} \leqslant 2Q$ we have $t_k < \frac{a+\varepsilon}{q} \leqslant t_{k-1}$ and q_k takes exactly one value. The corresponding contribution

$$\mathbb{G}_{I,Q}^{(3.4.2)}(\xi) = \sum_{k=0}^{\infty} \sideset{}{^*}\sum_{\substack{q_k \in \mathcal{I}_{q,k} \\ 2Q-q < q_{k+1} \leqslant 2Q}} \int_{t_k}^{\frac{a+\varepsilon}{q}} \frac{W_{\gamma,k,0}^{(2)}(t)}{t^2+t+1}\, dt$$

is estimated upon (A.11), (A.12), (A.15), (A.16) as

$$\mathbb{G}_{I,Q}^{(3.4.2)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{21}(\xi), \int_0^1 du \int_{2-2u}^{2-u} dw\, F^{(3.4.2)}(\xi; u, w), \tag{8.9}$$

with $G_{21}(\xi)$ as in (1.25).

8.2.2. $n_0 \geqslant 1$ and $k = 0$. Then $\mu_{0,1} = \frac{2a'+a+2\varepsilon}{2q+q'} > \gamma_1 = t_{-1} = \nu_{0,1} > \frac{a+\varepsilon}{q}$, with contribution

$$\mathbb{G}_{I,Q}^{(3.4.3)}(\xi) = \sideset{}{^*}\sum_{\substack{q > Q/2 \\ q' > 2(Q-q)}} \int_{\frac{a+\varepsilon}{q}}^{\gamma_1} \frac{W_{\gamma,0,1}^{(1)}(t)}{t^2+t+1}\, dt.$$

Employing (C.8), (C.9), (C.10) we find, with $G_{22}(\xi)$ as in (1.26),

$$\mathbb{G}_{I,Q}^{(3.4.3)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{22}(\xi). \tag{8.10}$$

8.2.3. $n_0 \geqslant 1$ and $k \geqslant 1$. Note first that $\mu_{k,n} \leqslant t_k$ when $q_k \leqslant \frac{2(Q-q)}{n+1}$ and $t_{k-1} < \mu_{k,n-1}$ when $q_k > Q + \frac{Q-q}{n}$. In both cases $I_{\gamma,k} \cap [\mu_{k,n-1}, \mu_{k,n}]$ has measure zero, so we shall only consider $q_k \in \left[\frac{2(Q-q)}{n+1}, Q + \frac{Q-q}{n}\right]$. The following four subcases arise:

(I) $\frac{2(Q-q)}{n+1} < q_k \leqslant \frac{2(Q-q)}{n} \wedge \left(Q + \frac{Q-q}{n+1}\right)$. Since $q_k > Q$ we have $Q \leqslant \frac{2(Q-q)}{n}$, so $n = 1$ and $q \leqslant \frac{Q}{2}$. Furthermore $\mu_{k,0} < t_k < \nu_{k,1} < \mu_{k,1} < t_{k-1}$ and the contribution is

$$\mathbb{G}_{I,Q}^{(3.4.4)}(\xi) = \sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_k \in \mathcal{I}_{q,k},\, q \leqslant Q/2 \\ q_k \leqslant 2(Q-q) \wedge \frac{3Q-q}{2}}} \left(\int_{t_k}^{\nu_{k,1}} \frac{W_{\gamma,k,1}^{(1)}(t)}{t^2+t+1}\, dt + \int_{\nu_{k,1}}^{\mu_{k,1}} \frac{W_{\gamma,k,1}^{(2)}(t)}{t^2+t+1}\, dt \right).$$

Employing (C.11)–(C.14), we find, with $G_{23}(\xi)$ as in (1.27),

$$\mathbb{G}_{I,Q}^{(3.4.4)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{23}(\xi). \tag{8.11}$$

(II) $\frac{2(Q-q)}{n} < q_k \leqslant Q + \frac{Q-q}{n+1}$. Then $t_k < \mu_{k,n-1} < \nu_{k,n} < \mu_{k,n} \leqslant t_{k-1}$, with contribution

$$\begin{aligned}&\mathbb{G}_{I,Q}^{(3.4.5)}(\xi)\\ &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_k\in\mathcal{I}_{q,k}\\ \frac{2(Q-q)}{n}<q_k\leqslant Q+\frac{Q-q}{n+1}}} \left(\int_{\mu_{k,n-1}}^{\nu_{k,n}} \frac{W_{\gamma,k,n}^{(1)}(t)}{t^2+t+1}\,dt + \int_{\nu_{k,n}}^{\mu_{k,n}} \frac{W_{\gamma,k,n}^{(2)}(t)}{t^2+t+1}\,dt \right).\end{aligned}$$

Employing (C.15)–(C.19), we find (according to whether $n=1$ or $n \geqslant 2$)[9]

$$\mathbb{G}_{I,Q}^{(3.4.5)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{24}(\xi), \tag{8.12}$$

with $G_{24}(\xi)$ as in (1.28).

(III) $Q + \frac{Q-q}{n+1} < q_k \leqslant \frac{2(Q-q)}{n}$. This gives $n=1$, $\mu_{k,0} \leqslant t_k < \nu_{k,1} = \gamma_{k+1} < t_{k-1} < \mu_{k,1}$, and

$$\mathbb{G}_{I,Q}^{(3.4.6)}(\xi) = \sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_k\in\mathcal{I}_{q,k}\\ \frac{3Q-q}{2}<q_k\leqslant 2(Q-q)}} \left(\int_{t_k}^{\nu_{k,1}} \frac{W_{\gamma,k,n}^{(1)}(t)}{t^2+t+1}\,dt + \int_{\nu_{k,1}}^{t_{k-1}} \frac{W_{\gamma,k,n}^{(2)}(t)}{t^2+t+1}\,dt \right).$$

Employing (5.7), (5.9), (A.9) and (C.20) we find, with $G_{25}(\xi)$ as in (1.29),

$$\mathbb{G}_{I,Q}^{(3.4.6)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{25}(\xi). \tag{8.13}$$

(IV) $\frac{2(Q-q)}{n} \vee \left(Q + \frac{Q-q}{n+1}\right) < q_k \leqslant Q + \frac{Q-q}{n}$. For fixed k there is only one value n can take, namely $\left\lfloor \frac{Q-q}{q_k-Q} \right\rfloor$. We have $t_k < \mu_{k,n-1} \leqslant \nu_{k,n} \leqslant t_{k-1} < \mu_{k,n}$ and

$$\begin{aligned}&\mathbb{G}_{I,Q}^{(3.4.7)}(\xi)\\ &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_k\in\mathcal{I}_{q,k}\\ \frac{2(Q-q)}{n}\vee\left(Q+\frac{Q-q}{n+1}\right)<q_k\leqslant Q+\frac{Q-q}{n}}} \left(\int_{\mu_{k,n-1}}^{\nu_{k,n}} \frac{W_{\gamma,k,n}^{(1)}(t)}{t^2+t+1}\,dt + \int_{\nu_{k,n}}^{t_{k-1}} \frac{W_{\gamma,k,n}^{(2)}(t)}{t^2+t+1}\,dt \right).\end{aligned}$$

Employing (C.21)–(C.25) we find, with $G_{26}(\xi)$ as in (1.30),

$$\mathbb{G}_{I,Q}^{(3.4.7)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{26}(\xi). \tag{8.14}$$

[9] Note that $2(Q-q) < q_k \leqslant Q + \frac{Q-q}{2}$ implies $q > \frac{Q}{3}$.

9. Channels with removed slits. The case $r_{k+1} = 0$

In this section we consider

$$\sum^* = \sum_{\substack{\gamma\in\mathcal{F}_I(Q)\\ q\not\equiv a \pmod 3\\ q_{k+1}\equiv a_{k+1} \pmod 3}} .$$

We shall actually sum as in Remark 5.1 over $x = q - a \in q(1-I)$, $x \equiv \pm 1 \pmod 3$, $y = q_{k+1} \in \mathcal{I}_{q,k+1}$, $\beta = 0$, $xy \equiv 1 \pmod{3q}$. The contributions arising from $x \equiv 1 \pmod 3$, respectively $x \equiv -1 \pmod 3$, will have the same main term and error.

9.1. The channel $\mathcal{C}_O$. In the formulas for $\mathcal{C}_O$ and $(r, r_k) = (1, -1)$, respectively $(r, r_k) = (-1, 1)$, the corresponding main term and error coincide. Hence, we only take $(r, r_k, r_{k+1}) = (1, -1, 0)$ and double its $\mathcal{C}_O$ contribution. The slits $q + \ell q_{k+1}$, $l \geqslant 1$, are not removed and lock the channel $\mathcal{C}_k$ (see Figure 16). The following two situations arise:

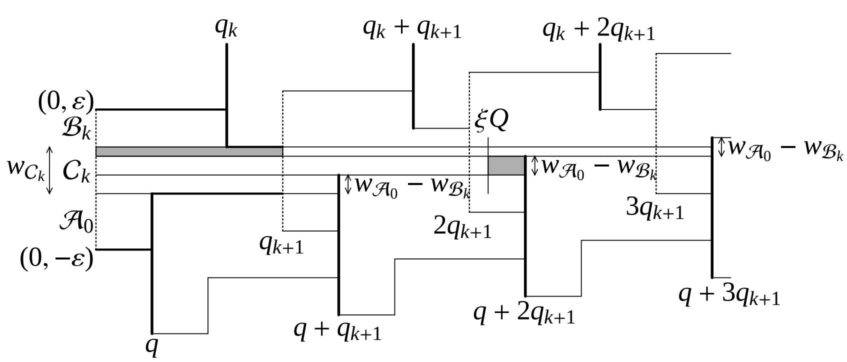

FIGURE 16. The case $t \in I_{\gamma,k}$, $r = \pm 1$, $r_k = \mp 1$, $r_{k+1} = 0$, $\mathcal{C}_O$, $w_{\mathcal{A}_0} > w_{\mathcal{B}_k}$, $N = 1$, $n_0 = 2$

9.1.1. $w_{\mathcal{A}_0} > w_{\mathcal{B}_k}$ ($\Longleftrightarrow t < \gamma_{k+1}$). The analysis is split in two cases:

(I) $\xi Q < q_{k+2}$. Then (since $w_{\mathcal{A}_0} - w_{\mathcal{B}_k} > w_{\mathcal{C}_k} \Longleftrightarrow t < \frac{a+\varepsilon}{q}$) we find

$$W_{\gamma,k}(t) = \begin{cases} W^{(1)}_{\gamma,k}(t) := w_{\mathcal{C}_k}(t)\cdot(q_{k+2} - \xi Q) \wedge q_{k+1} & \text{if } t < \frac{a+\varepsilon}{q},\\ W^{(2)}_{\gamma,k}(t) := \big((w_{\mathcal{A}_0}(t) - w_{\mathcal{B}_k}(t)\big)\cdot(q_{k+2} - \xi Q) \wedge q_{k+1} & \\ \qquad + \big(w_{\mathcal{C}_k}(t) - w_{\mathcal{A}_0}(t) + w_{\mathcal{B}_k}(t)\big) q_{k+1} & \text{if } t \geqslant \frac{a+\varepsilon}{q}. \end{cases}$$

- When $q_{k+1} > 2Q$, we have $k \geqslant 1$ and $t_{k-1} < \gamma_{k+1} < \frac{a+\varepsilon}{q}$, with contribution

$$\mathbb{G}^{(4.1.1.1)}_{I,Q}(\xi) = \sum_{k=1}^{\infty} \sum^*_{\substack{q_{k+1}\in\mathcal{I}_{q,k+1}\\ q_{k+1}>(\xi Q - q)\vee 2Q}} \int_{t_k}^{t_{k-1}} \frac{W^{(1)}_{\gamma,k}(t)}{t^2+t+1}\, dt$$

estimated upon (5.7) as

$$\mathbb{G}_{I,Q}^{(4.1.1.1)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{27}(\xi), \tag{9.1}$$

with $G_{27}(\xi)$ as in (1.31).

• When $q_{k+1} \leqslant 2Q$ we have $\frac{a+\varepsilon}{q} \leqslant \gamma_{k+1} \leqslant t_{k-1}$ and $t_k < \frac{a+\varepsilon}{q} \iff q_{k+2} > 2Q$ for all $k \geqslant 0$. In this case the contribution

$$\begin{aligned}
&\mathbb{G}_{I,Q}^{(4.1.1.2)}(\xi) \\
&= \sum_{k=0}^{\infty} \sideset{}{^*}\sum_{\substack{q_{k+1} \in \mathcal{I}_{q,k+1} \\ 2Q \geqslant q_{k+1} > (\xi \vee 2)Q - q}} \left(\int_{t_k}^{\frac{a+\varepsilon}{q}} \frac{W_{\gamma,k}^{(1)}(t)}{t^2+t+1}\, dt + \int_{\frac{a+\varepsilon}{q}}^{\gamma_{k+1}} \frac{W_{\gamma,k}^{(2)}(t)}{t^2+t+1}\, dt \right) \\
&\quad + \sum_{k=0}^{\infty} \sideset{}{^*}\sum_{\substack{q_{k+1} \in \mathcal{I}_{q,k+1} \\ 2Q - q \geqslant q_{k+1} > \xi Q - q}} \int_{t_k}^{\gamma_{k+1}} \frac{W_{\gamma,k}^{(2)}(t)}{t^2+t+1}\, dt
\end{aligned}$$

is estimated upon (A.15), (A.16), (A.18) (which also holds for $k=0$), (A.9), (A.20), (D.1), (5.4), as

$$\mathbb{G}_{I,Q}^{(4.1.1.2)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{28}(\xi), \tag{9.2}$$

with $G_{28}(\xi)$ as in (1.32).

(II) $\xi Q \geqslant q_{k+2}$. Let N be the unique integer for which $q + N q_{k+1} \leqslant \xi Q < q + (N+1) q_{k+1} = q_{k+2} + N q_{k+1}$, that is $1 \leqslant N := \left\lfloor \frac{\xi Q - q}{q_{k+1}} \right\rfloor \leqslant \xi$. The corresponding range of q_{k+1} is

$$y = q_{k+1} \in \mathcal{J}_{q,N} := \left(\frac{\xi Q - q}{N+1}, \frac{\xi Q - q}{N} \right].$$

We take $n_0 := \left\lfloor \frac{w_{\mathcal{C}_k}}{w_{\mathcal{A}_0} - w_{\mathcal{B}_k}} \right\rfloor = \left\lfloor \frac{a_{k-1} - 2\varepsilon - q_{k-1} t}{a_{k+1} - q_{k+1} t} \right\rfloor \geqslant 1$. Then $t \geqslant \frac{a+\varepsilon}{q}$. We only need $\frac{a+\varepsilon}{q} \leqslant \gamma_{k+1}$, that is $q_{k+1} \leqslant 2Q$. In this case we take

$$\lambda_{k,n} := \frac{n a_{k+1} - a_{k-1} + 2\varepsilon}{n q_{k+1} - q_{k-1}} \nearrow \gamma_{k+1} \leqslant t_{k-1} \quad \text{as } n \to \infty, \tag{9.3}$$

hence $\lambda_{k,n_0} \leqslant t < \lambda_{k,n_0+1}$ and $T(q + n_0 q_{k+1}) = w_{\mathcal{B}_k} + w_{\mathcal{C}_k} - n_0 (w_{\mathcal{A}_0} - w_{\mathcal{B}_k}) \geqslant w_{\mathcal{B}_k} > T\big(q + (n_0+1) q_{k+1}\big)$. Thus $\mathcal{C}_k$ is locked by the slits $q + q_{k+1}, \dots, q + (n_0+1) q_{k+1}$ and $W_{\gamma,k}(t) = W_{\gamma,k,N}(t)$ is given by

$$\begin{cases}
0 & \text{if } n_0 < N, \\
\big(w_{\mathcal{C}_k} - N(w_{\mathcal{A}_0} - w_{\mathcal{B}_k})\big)\big(q + (N+1) q_{k+1} - \xi Q\big) & \text{if } n_0 = N, \\
(w_{\mathcal{A}_0} - w_{\mathcal{B}_k})\big(q + (N+1) q_{k+1} - \xi Q\big) & \\
\qquad + \big(w_{\mathcal{C}_k} - (N+1)(w_{\mathcal{A}_0} - w_{\mathcal{B}_k})\big) q_{k+1} & \text{if } n_0 > N,
\end{cases}$$

$$\begin{cases} 0 & \text{if } t \in I_{\gamma,k} \cap (\gamma, \lambda_{k,N}], \\ W^{(1)}_{\gamma,k,N}(t) & \text{if } t \in I_{\gamma,k} \cap (\lambda_{k,N}, \lambda_{k,N+1}], \\ W^{(2)}_{\gamma,k}(t) & \text{if } t \in I_{\gamma,k} \cap (\lambda_{k,N+1}, \gamma_{k+1}], \end{cases}$$

where

$$\begin{aligned} W^{(1)}_{\gamma,k,N}(t) &:= \big(q + (N+1)q_{k+1} - \xi Q\big) \\ &\qquad \cdot \big((Nq_{k+1} - q_{k-1})t - Na_{k+1} + a_{k-1} - 2\varepsilon\big), \\ W^{(2)}_{\gamma,k}(t) &:= w_{\mathcal{C}_k}(t) q_{k+1} - \big(w_{\mathcal{A}_0}(t) - w_{\mathcal{B}_k}(t)\big)(\xi Q - q). \end{aligned}$$

The following three subcases arise:

(II_1) $q_{k+1} \leqslant Q + \frac{Q-q}{N+1}$. Then $\lambda_{k,N+1} \leqslant t_k$ and $W_{\gamma,k}(t) = W^{(2)}_{\gamma,k}(t)$, $\forall t \in (t_k, \gamma_{k+1}]$, with contribution

$$\mathbb{G}^{(4.1.2)}_{I,Q}(\xi) = \sum_{1 \leqslant N \leqslant \xi} \sum_{k=0}^{\infty} \sideset{}{^*}\sum_{\substack{q \leqslant \xi Q \\ q_{k+1} \in \mathcal{I}_{q,k+1} \cap \mathcal{J}_{q,N} \\ q_{k+1} \leqslant Q + \frac{Q-q}{N+1}}} \int_{t_k}^{\gamma_{k+1}} \frac{W^{(2)}_{\gamma,k}(t)}{t^2+t+1}\, dt.$$

Employing (5.4), (D.1), (A.9) (which also holds for $k=0$), we find

$$\mathbb{G}^{(4.1.2)}_{I,Q}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{29}(\xi), \tag{9.4}$$

with $G_{29}(\xi)$ as in (1.33).

(II_2) $Q + \frac{Q-q}{N+1} < q_{k+1} \leqslant Q + \frac{Q-q}{N}$. In this case $\lambda_{k,N} \leqslant t_k < \lambda_{k,N+1}$. The contribution

$$\begin{aligned} &\mathbb{G}^{(4.1.3)}_{I,Q}(\xi) \\ &= \sum_{1 \leqslant N \leqslant \xi} \sum_{k=0}^{\infty} \sideset{}{^*}\sum_{\substack{q \leqslant \xi Q \\ q_{k+1} \in \mathcal{I}_{q,k+1} \cap \mathcal{J}_{q,N} \\ Q + \frac{Q-q}{N+1} < q_{k+1} \leqslant Q + \frac{Q-q}{N}}} \left(\int_{t_k}^{\lambda_{k,N+1}} \frac{W^{(1)}_{\gamma,k,N}(t)}{t^2+t+1}\, dt + \int_{\lambda_{k,N+1}}^{\gamma_{k+1}} \frac{W^{(2)}_{\gamma,k}(t)}{t^2+t+1}\, dt \right) \end{aligned}$$

is estimated upon (D.4), (D.5), (D.6), (D.9), with $G_{30}(\xi)$ as in (1.34), as

$$\mathbb{G}^{(4.1.3)}_{I,Q}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{30}(\xi). \tag{9.5}$$

(II_3) $Q + \frac{Q-q}{N} < q_{k+1} \leqslant 2Q$. In this case $t_k < \lambda_{k,N}$. The contribution

$$\begin{aligned} &\mathbb{G}^{(4.1.4)}_{I,Q}(\xi) \\ &= \sum_{1 \leqslant N \leqslant \xi} \sum_{k=0}^{\infty} \sideset{}{^*}\sum_{\substack{q \leqslant \xi Q \\ q_{k+1} \in \mathcal{I}_{q,k+1} \cap \mathcal{J}_{q,N} \\ Q + \frac{Q-q}{N} < q_{k+1} \leqslant 2Q}} \left(\int_{\lambda_{k,N}}^{\lambda_{k,N+1}} \frac{W^{(1)}_{\gamma,k,N}(t)}{t^2+t+1}\, dt + \int_{\lambda_{k,N+1}}^{\gamma_{k+1}} \frac{W^{(2)}_{\gamma,k}(t)}{t^2+t+1}\, dt \right) \end{aligned}$$

is estimated upon (D.4)–(D.7), with $G_{31}(\xi)$ as in (1.35), as

(9.6) $$\mathbb{G}^{(4.1.4)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{31}(\xi).$$

9.1.2. $w_{\mathcal{A}_0} < w_{\mathcal{B}_k}$ ($\Longleftrightarrow t > \gamma_{k+1}$). In this situation $k \geqslant 1$ and $\gamma_{k+1} \leqslant t_{k-1}$, so $q_{k+1} \leqslant 2Q$. Two cases arise:

(I) $\xi Q < q_k$. Then $W_{\gamma,k}(t) = w_{\mathcal{C}_k}(t) q_{k+1}$, $\forall t \in (\gamma_{k+1}, t_{k-1}]$, with contribution

$$\mathbb{G}^{(4.2.1)}_{I,Q}(\xi) = \sum_{k=1}^{\infty} \sideset{}{^*}\sum_{q+\xi Q < q_{k+1} \leqslant 2Q} \int_{\gamma_{k+1}}^{t_{k-1}} \frac{W_{\gamma,k}(t)}{t^2+t+1}\, dt.$$

Employing (C.1) we find, with $G_{32}(\xi)$ as in (1.36),

(9.7) $$\mathbb{G}^{(4.2.1)}_{I,Q}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{32}(\xi).$$

(II) $\xi Q \geqslant q_k$. Consider the unique integer $0 \leqslant N = \left\lfloor \frac{\xi Q - q_k}{q_{k+1}} \right\rfloor \leqslant \xi$ for which $q_k + N q_{k+1} < \xi Q \leqslant q_k + (N+1) q_{k+1} = q_{k+2} + N q_{k+1}$, so the range of q_{k+1} is

$$y = q_{k+1} \in \mathcal{J}_{q,N} := \left(\frac{\xi Q + q}{N+2}, \frac{\xi Q + q}{N+1} \right].$$

This time we take $n_0 := \left\lfloor \frac{w_{\mathcal{C}_k}}{w_{\mathcal{B}_k} - w_{\mathcal{A}_0}} \right\rfloor = \left\lfloor \frac{a_{k-1} - 2\varepsilon - q_{k-1} t}{q_{k+1} t - a_{k+1}} \right\rfloor \geqslant 0$ and

(9.8) $$\lambda_{k,n} := \frac{n a_{k+1} + a_{k-1} - 2\varepsilon}{n q_{k+1} + q_{k-1}} \searrow \gamma_{k+1} > t_k \quad \text{as } n \to \infty,$$

hence $\lambda_{k,n_0+1} < t \leqslant \lambda_{k,n_0}$ and $B(q_k + n_0 q_{k+1}) = w_{\mathcal{B}_k} + n_0 (w_{\mathcal{B}_k} - w_{\mathcal{A}_0}) \leqslant w_{\mathcal{B}_k} + w_{\mathcal{C}_k} < B\big(q_k + (n_0+1) q_{k+1}\big)$. Thus $\mathcal{C}_k$ is locked by the slits $q_k + q_{k+1}, \dots, q_k + (n_0+1) q_{k+1}$ and $W_{\gamma,k}(t) = W_{\gamma,k,N}(t)$ is given by

$$\begin{cases} 0 & \text{if } n_0 < N, \\ \big(w_{\mathcal{C}_k} - N(w_{\mathcal{B}_k} - w_{\mathcal{A}_0})\big)\big(q_k + (N+1) q_{k+1} - \xi Q\big) & \text{if } n_0 = N, \\ (w_{\mathcal{B}_k} - w_{\mathcal{A}_0})\big(q_k + (N+1) q_{k+1} - \xi Q\big) & \\ \quad + \big(w_{\mathcal{C}_k} - (N+1)(w_{\mathcal{B}_k} - w_{\mathcal{A}_0})\big) q_{k+1} & \text{if } n_0 > N, \end{cases}$$

or equivalently

$$\begin{cases} 0 & \text{if } t \in (\lambda_{k,N}, t_{k-1}], \\ W^{(3)}_{\gamma,k,N}(t) & \text{if } t \in (\lambda_{k,N+1}, \lambda_{k,N}], \\ W^{(4)}_{\gamma,k}(t) & \text{if } t \in (\gamma_{k+1}, \lambda_{k,N+1}], \end{cases}$$

where

$$\begin{aligned} W^{(3)}_{\gamma,k,N}(t) &:= \big(q_k + (N+1) q_{k+1} - \xi Q\big) \\ &\qquad \cdot \big(N a_{k+1} + a_{k-1} - 2\varepsilon - (N q_{k+1} + q_{k-1}) t\big), \\ W^{(4)}_{\gamma,k}(t) &:= w_{\mathcal{C}_k}(t) q_{k+1} - \big(w_{\mathcal{B}_k}(t) - w_{\mathcal{A}_0}(t)\big)(\xi Q - q_k). \end{aligned}$$

Since $\lambda_{k,0} = t_{k-1}$, the corresponding contribution is given by

$$\mathbb{G}_{I,Q}^{(4.2.2)}(\xi) = \sum_{0\leqslant N\leqslant \xi} \sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_{k+1}\leqslant 2Q \\ q_{k+1}\in \mathcal{I}_{q,k+1}\cap \mathcal{J}_{q,N}}} \left(\int_{\lambda_{k,N+1}}^{\lambda_{k,N}} \frac{W^{(3)}_{\gamma,k,N}(t)}{t^2+t+1}\,dt + \int_{\gamma_{k+1}}^{\lambda_{k,N+1}} \frac{W^{(4)}_{\gamma,k}(t)}{t^2+t+1}\,dt \right).$$

Employing (D.12), (D.13), (D.14) we find, with $G_{33}(\xi)$ as in (1.37),

$$\mathbb{G}_{I,Q}^{(4.2.2)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{33}(\xi). \tag{9.9}$$

9.2. The channel $\mathcal{C}_{\leftarrow}$ when $r = 1$ and $r_k = -1$. The contributions of $(\mathcal{C}_{\leftarrow}, r = 1, r_k = -1)$ and of $(\mathcal{C}_{\downarrow}, r = -1, r_k = 1)$ have the same main and error terms, so we shall consider below the former situation and double the result. The slits $q + nq_{k+1}$ are removed, while $2q + nq_{k+1}$ are not, $n \geqslant 0$ (see Figure 17). Note that $B(2q) = 2(w_{\mathcal{B}_k} + w_{\mathcal{C}_k}) > B(q_{k+1}) = w_{\mathcal{C}_k} + 2w_{\mathcal{B}_k}$. Again, two cases arise:

9.2.1. $w_{\mathcal{A}_0} \leqslant w_{\mathcal{B}_k}$ ($\Longleftrightarrow t \geqslant \gamma_{k+1}$). The slit q_{k+1} locks the channel $\mathcal{A}_0$ and $W_{\gamma,k}(t) = w_{\mathcal{A}_0}(t) \cdot (q_{k+1} - \xi Q)_+ \wedge q$. We must have $\gamma_{k+1} < t_{k-1}$, so $k \geqslant 1$ and $q_{k+1} \leqslant 2Q$. The corresponding contribution is

$$\mathbb{G}_{I,Q}^{(4.3.1)}(\xi) = \sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_{k+1}\in \mathcal{I}_{q,k+1} \\ q_{k+1}\leqslant 2Q}} (q_{k+1} - \xi Q)_+ \wedge q \int_{\gamma_{k+1}}^{t_{k-1}} \frac{w_{\mathcal{A}_0}(t)\,dt}{t^2+t+1}.$$

Employing (D.2) we find, with $G_{34}(\xi)$ as in (1.38),

$$\mathbb{G}_{I,Q}^{(4.3.1)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{34}(\xi). \tag{9.10}$$

9.2.2. $w_{\mathcal{A}_0} > w_{\mathcal{B}_k}$ ($\Longleftrightarrow t < \gamma_{k+1}$). Consider first the subchannel of $\mathcal{A}_0$ of width $w_{\mathcal{B}_k}$ locked by the slit q_{k+1}. Its contribution is

$$\begin{aligned}\mathbb{G}_{I,Q}^{(4.3.2)}(\xi) = {} & \sum_{k=0}^{\infty} \sideset{}{^*}\sum_{\substack{q_{k+1}\in \mathcal{I}_{q,k+1} \\ q_{k+1}\leqslant 2Q}} (q_{k+1} - \xi Q)_+ \wedge q \int_{t_k}^{\gamma_{k+1}} \frac{w_{\mathcal{B}_k}(t)\,dt}{t^2+t+1} \\ & + \sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_{k+1}\in \mathcal{I}_{q,k+1} \\ q_{k+1}> 2Q}} (q_{k+1} - \xi Q)_+ \wedge q \int_{t_k}^{t_{k-1}} \frac{w_{\mathcal{B}_k}(t)\,dt}{t^2+t+1}.\end{aligned}$$

Employing (A.8) and (5.6) we find, with $G_{35}(\xi)$ as in (1.39),

$$\mathbb{G}_{I,Q}^{(4.3.2)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{35}(\xi). \tag{9.11}$$

The remaining part $\tilde{A}_0$ of $\mathcal{A}_0$ (of total width $w_{\mathcal{A}_0} - w_{\mathcal{B}_k}$) is locked by the slits $2q + n_0 q_{k+1}$ and $2q + (n_0 + 1)q_{k+1}$, with n_0 uniquely determined by $2\varepsilon > 2(w_{\mathcal{B}_k} + w_{\mathcal{C}_k}) - n_0(w_{\mathcal{A}_0} - w_{\mathcal{B}_k}) \geqslant 2w_{\mathcal{B}_k} + w_{\mathcal{C}_k}$, or equivalently $n_0 =$

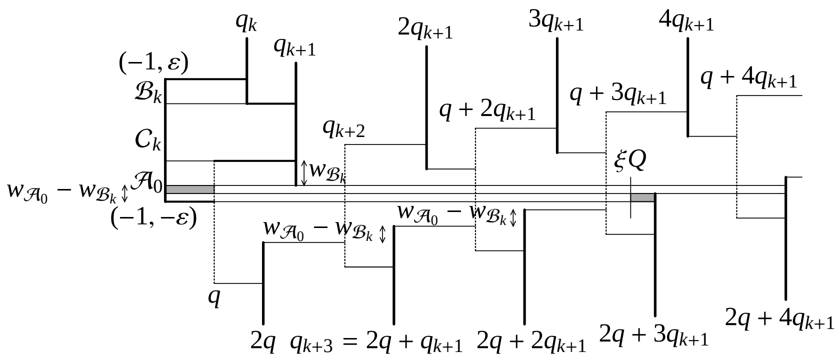

FIGURE 17. The case $t \in I_{\gamma,k}$, $r = 1$, $r_k = -1$, $r_{k+1} = 0$, $\mathcal{C}_{\leftarrow}$, $w_{\mathcal{A}_0} > w_{\mathcal{B}_k}$, $n_0 = 3$

$\left\lfloor \frac{w_{\mathcal{C}_k}}{w_{\mathcal{A}_0} - w_{\mathcal{B}_k}} \right\rfloor = \left\lfloor \frac{a_{k-1} - 2\varepsilon - q_{k-1}t}{a_{k+1} - q_{k+1}t} \right\rfloor \geqslant 0$. The widths of the relevant subchannels (from bottom to top) are

$$2\varepsilon - 2(w_{\mathcal{B}_k} + w_{\mathcal{C}_k}) + n_0(w_{\mathcal{A}_0} - w_{\mathcal{B}_k}) = (n_0 + 1)a_{k+1} - a_{k-1} + 2\varepsilon - \big((n_0 + 1)q_{k+1} - q_{k-1}\big)t$$

and $w_{\mathcal{C}_k} - n_0(w_{\mathcal{A}_0} - w_{\mathcal{B}_k}) = (n_0 q_{k+1} - q_{k-1})t - n_0 a_{k+1} + a_{k-1} - 2\varepsilon$. The following two subcases arise:

(I) $n_0 = 0$ ($\Longleftrightarrow t < \frac{a+\varepsilon}{q}$). From $t_k < \frac{a+\varepsilon}{q}$ we infer $q_{k+2} > 2Q$. In this situation

$$W_{\gamma,k,0}(t) = 2(a + \varepsilon - qt) \cdot (2q - \xi Q)_+ \wedge q + w_{\mathcal{C}_k}(t) \cdot (2q + q_{k+1} - \xi Q)_+ \wedge q,$$

with contribution ($q_{k+1} > 2Q \Longleftrightarrow t_{k-1} < \gamma_{k+1}$)

$$\mathbb{G}_{I,Q}^{(4.3.3)}(\xi) = \sum_{k=0}^{\infty} \sideset{}{^*}\sum_{\substack{q_{k+1} \in \mathcal{I}_{q,k+1} \\ 2Q - q < q_{k+1} \leqslant 2Q}} \int_{t_k}^{\frac{a+\varepsilon}{q}} \frac{W_{\gamma,k,0}(t)}{t^2 + t + 1}\, dt + \sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_{k+1} \in \mathcal{I}_{q,k+1} \\ q_{k+1} > 2Q}} \int_{t_k}^{t_{k-1}} \frac{W_{\gamma,k,0}(t)}{t^2 + t + 1}\, dt.$$

Employing (A.14), (A.15), (A.16), (5.7), (A.1) we find, with $G_{36}(\xi)$ as in (1.40),

$$\mathbb{G}_{I,Q}^{(4.3.3)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{36}(\xi). \tag{9.12}$$

(II) $n_0 \geqslant 1$ ($\Longleftrightarrow t \geqslant \frac{a+\varepsilon}{q}$). From $\frac{a+\varepsilon}{q} \leqslant \gamma_{k+1}$ we infer $q_{k+1} \leqslant 2Q$. Taking $\lambda_{k,n}$ as in (9.3) we obtain, when $t \in [\lambda_{k,n_0}, \lambda_{k,n_0+1})$, that $W_{\gamma,k}(t)$ is given by

$$\begin{aligned}&\big((n_0+1)a_{k+1} - a_{k-1} + 2\varepsilon - \big((n_0+1)q_{k+1} - q_{k-1}\big)t\big)\\ &\qquad \cdot (2q + n_0 q_{k+1} - \xi Q)_+ \wedge q\\ &+ \big((n_0 q_{k+1} - q_{k-1})t - n_0 a_{k+1} + a_{k-1} - 2\varepsilon\big)\\ &\qquad \cdot \big(2q + (n_0+1)q_{k+1} - \xi Q\big)_+ \wedge q.\end{aligned}$$

Ordering λ_{k,n_0}, λ_{k,n_0+1} and t_k, the corresponding contribution takes the form

$$\begin{aligned}\mathbb{G}_{I,Q}^{(4.3.4)}(\xi) = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\Bigg(&\sideset{}{^*}\sum_{\substack{q_{k+1}\in\mathcal{I}_{q,k+1}\\ Q+\frac{Q-q}{n+1}<q_{k+1}\leqslant Q+\frac{Q-q}{n}}} \int_{t_k}^{\lambda_{k,n+1}} \frac{W_{\gamma,k,n}(t)}{t^2+t+1}\,dt\\ &+ \sideset{}{^*}\sum_{\substack{q_{k+1}\in\mathcal{I}_{q,k+1}\\ Q+\frac{Q-q}{n}<q_{k+1}\leqslant 2Q}} \int_{\lambda_{k,n}}^{\lambda_{k,n+1}} \frac{W_{\gamma,k,n}(t)}{t^2+t+1}\,dt\Bigg).\end{aligned}$$

Employing (D.8)–(D.11) we find, with $G_{37}(\xi)$ as in (1.41),

$$\mathbb{G}_{I,Q}^{(4.3.4)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{37}(\xi). \tag{9.13}$$

9.3. The channel $\mathcal{C}_{\leftarrow}$ when $r = -1$ and $r_k = 1$. The $(\mathcal{C}_{\leftarrow}, r = -1, r_k = 1)$ and $(\mathcal{C}_{\downarrow}, r = 1, r_k = -1)$ contributions have the same main and error terms, so we shall consider below the former situation and double the result. This is analogous to Section 9.2, only that this time the slits $q_k + nq_{k+1}$ are removed, while $2q + nq_{k+1}$ are not. Two cases arise:

9.3.1. $w_{\mathcal{B}_k} \leqslant w_{\mathcal{A}_0}$ ($\Longleftrightarrow t \leqslant \gamma_{k+1}$). The slit q_{k+1} locks the channel $\mathcal{B}_k$ and $W_{\gamma,k}(t) = w_{\mathcal{B}_k}(t) \cdot (q_{k+1} - \xi Q)_+ \wedge q_k$. The contribution

$$\begin{aligned}\mathbb{G}_{I,Q}^{(4.4.1)}(\xi) = &\sum_{k=0}^{\infty} \sideset{}{^*}\sum_{\substack{q_{k+1}\in\mathcal{I}_{q,k+1}\\ q_{k+1}\leqslant 2Q}} (q_{k+1} - \xi Q)_+ \wedge q_k \int_{t_k}^{\gamma_{k+1}} \frac{w_{\mathcal{B}_k}(t)\,dt}{t^2+t+1}\\ &+ \sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_{k+1}\in\mathcal{I}_{q,k+1}\\ q_{k+1}>2Q}} (q_{k+1} - \xi Q)_+ \wedge q_k \int_{t_k}^{t_{k-1}} \frac{w_{\mathcal{B}_k}(t)\,dt}{t^2+t+1}\end{aligned}$$

is estimated upon (A.8), (5.3), (5.6), with $G_{38}(\xi)$ as in (1.42), as

$$\mathbb{G}_{I,Q}^{(4.4.1)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{38}(\xi). \tag{9.14}$$

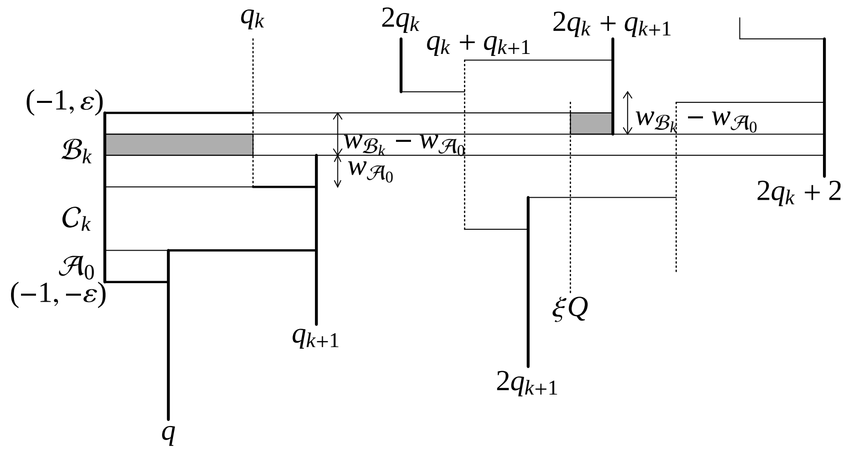

FIGURE 18. The case $t \in I_{\gamma,k}$, $r = -1$, $r_k = 1$, $r_{k+1} = 0$, $\mathcal{C}_{\leftarrow}$, $w_{\mathcal{A}_0} < w_{\mathcal{B}_k}$, $n_0 = 1$

9.3.2. $w_{\mathcal{A}_0} < w_{\mathcal{B}_k}$ ($\iff t > \gamma_{k+1}$). Then $k \geqslant 1$ and $q_{k+1} \leqslant 2Q$. Consider first the subchannel of $\mathcal{B}_k$ of width $w_{\mathcal{A}_0}$ locked by q_{k+1}, with contribution

$$\mathbb{G}_{I,Q}^{(4.4.2)}(\xi) = \sum_{k=1}^{\infty} \sideset{}{^*}\sum_{\substack{q_{k+1} \in \mathcal{I}_{q,k+1} \\ q_{k+1} \leqslant 2Q}} (q_{k+1} - \xi Q)_+ \wedge q_k \int_{\gamma_{k+1}}^{t_{k-1}} \frac{w_{\mathcal{A}_0}(t)\, dt}{t^2+t+1}$$

estimated upon (B.1), with $G_{39}(\xi)$ as in (1.43), as

$$\mathbb{G}_{I,Q}^{(4.4.2)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{39}(\xi). \tag{9.15}$$

The remaining subchannel $\tilde{\mathcal{B}}_k$ of $\mathcal{B}_k$ (of width $w_{\mathcal{B}_k} - w_{\mathcal{A}_0}$) is locked by the slits $2q_k + n_0 q_{k+1}$ and $2q_k + (n_0+1)q_{k+1}$, with n_0 uniquely determined by $0 \leqslant B(2q_k) + n_0(w_{\mathcal{B}_k} - w_{\mathcal{A}_0}) < w_{\mathcal{B}_k} - w_{\mathcal{A}_0} = T(q_{k+1})$, or equivalently $n_0 := \left\lfloor \frac{w_{\mathcal{C}_k}}{w_{\mathcal{B}_k} - w_{\mathcal{A}_0}} \right\rfloor = \left\lfloor \frac{a_{k-1} - 2\varepsilon - q_{k-1}t}{q_{k+1}t - a_{k+1}} \right\rfloor \geqslant 0$. The situation is described in Figure 18. The widths of the relevant two subchannels of $\tilde{\mathcal{B}}_k$ are (from top to bottom) $B(2q_k + n_0 q_{k+1}) = \big((n_0+1)q_{k+1} + q_{k-1}\big)t - (n_0+1)a_{k+1} - a_{k-1} + 2\varepsilon$ and $w_{\mathcal{B}_k} - w_{\mathcal{A}_0} - B(2q_k + n_0 q_{k+1}) = n_0 a_{k+1} + a_{k-1} - 2\varepsilon - (n_0 q_{k+1} + q_{k-1})t$. Taking this time $\lambda_{k,n}$ as in (9.8) we find, for $t \in (\lambda_{k,n_0+1}, \lambda_{k,n_0}]$, that $W_{\gamma,k,n_0}(t)$ is given by

$$\begin{aligned}
&\Big(\big((n_0+1)q_{k+1} + q_{k-1}\big)t - \big((n_0+1)a_{k+1} + a_{k-1} - 2\varepsilon\big)\Big) \\
&\qquad \cdot (2q_k + n_0 q_{k+1} - \xi Q)_+ \wedge q_k \\
&+ \Big(n_0 a_{k+1} + a_{k-1} - 2\varepsilon - (n_0 q_{k+1} + q_{k-1})t\Big) \\
&\qquad \cdot \big(2q_k + (n_0+1)q_{k+1} - \xi Q\big)_+ \wedge q_k.
\end{aligned}$$

Since $\lambda_{k,0} = t_{k-1}$ the corresponding contribution is given by

$$\mathbb{G}_{I,Q}^{(4.4.3)}(\xi) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \sum_{\substack{q_{k+1} \in \mathcal{I}_{q,k+1} \\ q_{k+1} \leqslant 2Q}}^{*} \int_{\lambda_{k,n+1}}^{\lambda_{k,n}} \frac{W_{\gamma,k,n}(t)}{t^2+t+1}\, dt.$$

Employing (D.14) and (D.15) we find, with $G_{40}(\xi)$ as in (1.44),

$$\mathbb{G}_{I,Q}^{(4.4.3)}(\xi) \approxeq \frac{c_I}{\zeta(2)} G_{40}(\xi). \tag{9.16}$$

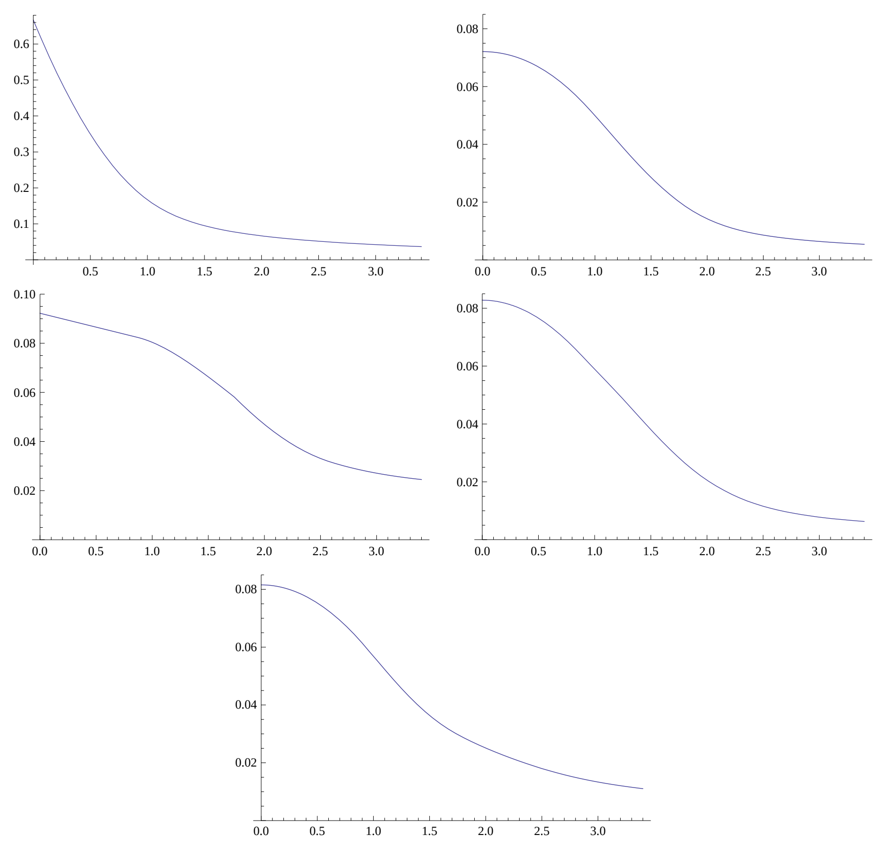

FIGURE 19. The individual contributions of $\mathbb{G}^{(0)}, \dots, \mathbb{G}^{(4)}$ to Φ^{hex}

Appendix A.

$k \geqslant 1$. In this case,

$$\text{(A.1)}\quad \int_{t_k}^{t_{k-1}} \frac{2(a+\varepsilon-qt)\,dt}{t^2+t+1}$$
$$= -2\left(\int_{\gamma}^{t_{k-1}} \frac{qt-a}{t^2+t+1}\,dt - \int_{\gamma}^{t_k} \frac{qt-a}{t^2+t+1}\,dt\right) + \int_{t_k}^{t_{k-1}} \frac{2\varepsilon}{t^2+t+1}\,dt$$
$$= \frac{Q-q}{2Q^2 q_{k-1} q_k(\gamma^2+\gamma+1)}\left(\frac{q_{k+1}-2Q}{q_{k-1}} + \frac{q_{k+2}-2Q}{q_k}\right)$$
$$+ O\left(\frac{1}{Qq^2 q_{k-1}^2}\right).$$

$k \geqslant 1$ and $q_{k+1} \leqslant 2Q$. Here:

$$\text{(A.2)}\quad \int_{\frac{a_k-\varepsilon}{q_k}}^{t_{k-1}} \frac{w_{\mathcal{C}_k}(t)\,dt}{t^2+t+1} = \frac{(2Q-q_{k+1})^2}{8Q^2 q_{k-1} q_k^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{Qqq_k^3}\right).$$

$$\text{(A.3)}\quad \int_{\frac{a_k-\varepsilon}{q_k}}^{t_{k-1}} \frac{2(q_k t - a_k+\varepsilon)\,dt}{t^2+t+1} = \frac{(2Q-q_{k+1})^2}{4Q^2 q_{k-1}^2 q_k(\gamma^2+\gamma+1)} + O\left(\frac{1}{Q^2 qq_k^2}\right).$$

$$\text{(A.4)}\quad \int_{\frac{a_k-\varepsilon}{q_k}}^{t_{k-1}} \frac{w_{\mathcal{A}_0}(t)\,dt}{t^2+t+1}$$
$$= \int_{\frac{a_k-\varepsilon}{q_k}}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt - \int_{t_{k-1}}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt$$
$$= \frac{2Q-q_{k+1}}{4Q^2 q_{k-1} q_k(\gamma^2+\gamma+1)}\left(\frac{q_k+q_{k+1}-2Q}{2q_k} + \frac{q_k-Q}{q_{k-1}}\right)$$
$$+ O\left(\frac{1}{q^2 q_{k-1} q_k^2}\right).$$

$$\text{(A.5)}\quad \int_{\gamma_{k+1}}^{\frac{a_k-\varepsilon}{q_k}} \frac{w_{\mathcal{A}_0}(t)\,dt}{t^2+t+1}$$
$$= \int_{\gamma_{k+1}}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt - \int_{\frac{a_k-\varepsilon}{q_k}}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt$$
$$= \frac{2Q-q_{k+1}}{4Q^2 q_k q_{k+1}(\gamma^2+\gamma+1)}\left(\frac{q_{k+1}-Q}{q_{k+1}} + \frac{q_k+q_{k+1}-2Q}{2q_k}\right)$$
$$+ O\left(\frac{1}{q^2 q_k q_{k+1}^2}\right).$$

$$\int_{\gamma_{k+1}}^{\frac{a_k-\varepsilon}{q_k}} \frac{q_{k+1}t - a_{k+1}}{t^2+t+1}\,dt = \frac{(2Q-q_{k+1})^2}{8Q^2q_k^2q_{k+1}(\gamma^2+\gamma+1)} + O\left(\frac{1}{qq_k^2q_{k+1}^2}\right). \tag{A.6}$$

$$\begin{aligned} \int_{\gamma_{k+1}}^{\frac{a_k-\varepsilon}{q_k}} & \frac{2(a_k-\varepsilon-q_kt)\,dt}{t^2+t+1} \\ & = \frac{(2Q-q_{k+1})^2}{4Q^2q_kq_{k+1}^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{qq_k^2q_{k+1}^2}\right). \end{aligned} \tag{A.7}$$

$$\int_{t_k}^{\gamma_{k+1}} \frac{w_{\mathcal{B}_k}(t)\,dt}{t^2+t+1} = \frac{(q_{k+1}-Q)^2}{2Q^2q_kq_{k+1}^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{qq_k^2q_{k+1}^2}\right). \tag{A.8}$$

$$\begin{aligned} \int_{t_k}^{\gamma_{k+1}} \frac{w_{\mathcal{A}_0}(t)-w_{\mathcal{B}_k}(t)}{t^2+t+1}\,dt & = \int_{t_k}^{\gamma_{k+1}} \frac{a_{k+1}-q_{k+1}t}{t^2+t+1}\,dt \\ & = \frac{(q_{k+1}-Q)^2}{2Q^2q_k^2q_{k+1}(\gamma^2+\gamma+1)} \\ & \quad + O\left(\frac{1}{qq_k^2q_{k+1}^2}\right). \end{aligned} \tag{A.9}$$

$\boldsymbol{k \geqslant 0}$ **and** $\boldsymbol{q_{k+1} \leqslant 2Q-q}$**.** Here:

$$\begin{aligned} & \int_{t_k}^{\gamma_{k+1}} \frac{2(qt-a-\varepsilon)}{t^2+t+1}\,dt \\ & = \int_{\frac{a+\varepsilon}{q}}^{\gamma_{k+1}} \frac{2(qt-a-\varepsilon)}{t^2+t+1}\,dt - \int_{\frac{a+\varepsilon}{q}}^{t_k} \frac{2(qt-a-\varepsilon)}{t^2+t+1}\,dt \\ & = \frac{q_{k+1}-Q}{2Q^2q_kq_{k+1}(\gamma^2+\gamma+1)}\left(\frac{2Q-q_{k+1}}{q_{k+1}} + \frac{2Q-q_{k+2}}{q_k}\right) \\ & \quad + O\left(\frac{1}{q^3q_{k+1}^2}\right). \end{aligned} \tag{A.10}$$

$\boldsymbol{k \geqslant 0}$ **and** $\boldsymbol{2Q-q < q_{k+1} \leqslant 2Q}$**.** It follows that $0 \leqslant q_{k+2}-2Q \leqslant q$, $k = \left\lfloor \frac{2Q-q-q'}{q} \right\rfloor$, and:

$$\begin{aligned} \int_{t_k}^{\frac{a+\varepsilon}{q}} \frac{w_{\mathcal{A}_0}(t)\,dt}{t^2+t+1} & = \int_{t_k}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt - \int_{\frac{a+\varepsilon}{q}}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt \\ & = \frac{(q_{k+2}-2Q)(q_k+2q_{k+1}-2Q)}{8Q^2qq_k^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{q^2q'q_k^2}\right). \end{aligned} \tag{A.11}$$

$$\int_{t_k}^{\frac{a+\varepsilon}{q}} \frac{qt-a}{t^2+t+1}\,dt = \int_{\gamma}^{\frac{a+\varepsilon}{q}} \frac{qt-a}{t^2+t+1}\,dt - \int_{\gamma}^{t_k} \frac{qt-a}{t^2+t+1}\,dt = \frac{(q_{k+2}-2Q)(q_{k-2}+2Q)}{8Q^2qq_k^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{Q^3q^2}\right). \tag{A.12}$$

$$\int_{t_k}^{\frac{a+\varepsilon}{q}} \frac{w_{\mathcal{B}_k}(t)\,dt}{t^2+t+1} = \frac{(q_{k+2}-2Q)^2}{8Q^2q^2q_k(\gamma^2+\gamma+1)} + O\left(\frac{1}{Q^2qq_k^2}\right). \tag{A.13}$$

$$\int_{t_k}^{\frac{a+\varepsilon}{q}} \frac{2(a+\varepsilon-qt)\,dt}{t^2+t+1} = \frac{(q_{k+2}-2Q)^2}{4Q^2qq_k^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{Qqq'q_k^2}\right). \tag{A.14}$$

$$\int_{t_k}^{\frac{a+\varepsilon}{q}} \frac{w_{\mathcal{C}_k}(t)\,dt}{t^2+t+1} = \int_{t_k}^{t_{k-1}} \frac{w_{\mathcal{C}_k}(t)\,dt}{t^2+t+1} - \int_{\frac{a+\varepsilon}{q}}^{t_{k-1}} \frac{w_{\mathcal{C}_k}(t)\,dt}{t^2+t+1} = \frac{q_{k+2}-2Q}{4Q^2qq_k(\gamma^2+\gamma+1)}\left(\frac{Q-q}{q_k}+\frac{2Q-q_{k+1}}{2q}\right) + O\left(\frac{1}{Qqq_k^3}\right) \tag{A.15}$$

if $k \geqslant 1$.

$$\int_{t_0}^{\frac{a+\varepsilon}{q}} \frac{w_{\mathcal{C}_0}(t)\,dt}{t^2+t+1} = \int_{t_0}^{\gamma'} \frac{a'-q't}{t^2+t+1}\,dt - \int_{\frac{a+\varepsilon}{q}}^{\gamma'} \frac{a'-q't}{t^2+t+1}\,dt - \left(\int_{t_0}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt - \int_{\frac{a+\varepsilon}{q}}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt\right) = \frac{q_2-2Q}{4Q^2qq'(\gamma^2+\gamma+1)}\left(\frac{Q-q}{q'}+\frac{2Q-q_1}{2q}\right) + O\left(\frac{1}{Q^2qq'^2}\right). \tag{A.16}$$

$$\begin{aligned}(\text{A.17})\qquad &\int_{\frac{a+\varepsilon}{q}}^{\gamma_{k+1}} \frac{w_{\mathcal{B}_k}(t)\,dt}{t^2+t+1} \\ &= \int_{t_k}^{\gamma_{k+1}} \frac{q_k t - a_k + 2\varepsilon}{t^2+t+1}\,dt - \int_{t_k}^{\frac{a+\varepsilon}{q}} \frac{q_k t - a_k + 2\varepsilon}{t^2+t+1}\,dt \\ &= \frac{2Q-q_{k+1}}{4Q^2 q q_{k+1}(\gamma^2+\gamma+1)} \left(\frac{q_{k+1}-Q}{q_{k+1}} + \frac{q_{k+2}-2Q}{2q} \right) \\ &\quad + O\left(\frac{1}{Q^2 q q_k q_{k+1}} \right).\end{aligned}$$

$$(\text{A.18})\qquad \int_{\frac{a+\varepsilon}{q}}^{\gamma_{k+1}} \frac{a_{k+1}-q_{k+1}t}{t^2+t+1}\,dt = \frac{(2Q-q_{k+1})^2}{8Q^2q^2q_{k+1}(\gamma^2+\gamma+1)} + O\left(\frac{1}{q^3 q_{k+1}^2} \right).$$

$$(\text{A.19})\qquad \int_{\frac{a+\varepsilon}{q}}^{\gamma_{k+1}} \frac{2(qt-a-\varepsilon)\,dt}{t^2+t+1} = \frac{(2Q-q_{k+1})^2}{4Q^2qq_{k+1}^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{Qq^2 q_{k+1}^2} \right).$$

$$\begin{aligned}(\text{A.20})\qquad &\int_{\frac{a+\varepsilon}{q}}^{\gamma_{k+1}} \frac{w_{\mathcal{C}_k}(t)\,dt}{t^2+t+1} \\ &= \frac{(2Q-q_{k+1})^2 q_{k+3}}{8Q^2q^2q_{k+1}^2(\gamma^2+\gamma+1)} + \begin{cases} O\left(\frac{1}{q^3 q_{k-1}^2}\right) & \text{if } k \geqslant 1, \\ O\left(\frac{1}{q^3 q'^2}\right) & \text{if } k = 0. \end{cases}\end{aligned}$$

Appendix B.

B.1. Estimates for the $\mathcal{C}_O$ contribution when $(r, r_k) = (0, 1)$. Here $W^{(1)}_{\gamma,k,n}$ and $W^{(2)}_{\gamma,k}$ are as in (7.3) and (7.4).

B.1.1. For $k \geqslant 1$ and $q_{k+1} \leqslant 2Q$:

$$\begin{aligned}(\text{B.1})\qquad &\int_{\gamma_{k+1}}^{t_{k-1}} \frac{w_{\mathcal{A}_0}(t)\,dt}{t^2+t+1} \\ &= \int_{\gamma_{k+1}}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt - \int_{t_{k-1}}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt \\ &= \frac{2Q-q_{k+1}}{2Q^2q_{k-1}q_{k+1}(\gamma^2+\gamma+1)} \left(\frac{q_{k+1}-Q}{q_{k+1}} + \frac{q_k-Q}{q_{k-1}} \right) \\ &\quad + O\left(\frac{1}{Qq^2q_{k+1}^2} \right).\end{aligned}$$

Summing as in (5.2) and employing Lemma 3.4, we infer

$$\begin{aligned}
&\mathbb{G}_{I,Q}^{(2.1)}(\xi)\\
&= \sum_{\substack{\beta\in\{-1,1\}\\ 1\leqslant q\leqslant Q\\ q\equiv-\beta \pmod 3}} \sum_{k=1}^{\infty} \sum_{\substack{x=3\tilde{x}\in q(1-I),\,(\tilde{x},q)=1\\ y=q_k\in\mathcal{I}_{q,k},\, y\leqslant 2Q-q\\ \tilde{x}y\equiv\frac{\beta q+1}{3} \pmod q}} (y+q-\xi Q)_+\wedge q\int_{\gamma_{k+1}}^{t_{k-1}} \frac{w_{\mathcal{A}_0}(t)}{t^2+t+1}\,dt\\
&\approxeq \frac{c_I C(3)}{3}\sum_{\beta\in\{-1,1\}} \frac{\varphi(q)}{q}\int_Q^{2Q-q} \frac{2Q-q-y}{2Q^2(y-q)(y+q)}\left(\frac{y+q-Q}{y+q}+\frac{y-Q}{y-q}\right)\\
&\qquad\cdot (y+q-\xi Q)_+\wedge q\,dy.
\end{aligned}$$

Estimate (7.1) follows applying Lemma 3.2 and making the change of variable $(q,y)=(Qu,Qw)$.

B.1.2. For $k\geqslant 1$ and $q_{k+1}>2Q$:

$$\begin{aligned}
\text{(B.2)}\qquad &\int_{t_k}^{t_{k-1}} \frac{w_{\mathcal{A}_0}(t)-w_{\mathcal{B}_k}(t)}{t^2+t+1}\,dt\\
&= \int_{t_k}^{\gamma_{k+1}} \frac{a_{k+1}-q_{k+1}t}{t^2+t+1}\,dt - \int_{t_{k-1}}^{\gamma_{k+1}} \frac{a_{k+1}-q_{k+1}t}{t^2+t+1}\,dt\\
&= \frac{Q-q}{2Q^2 q_{k-1}q_k(\gamma^2+\gamma+1)}\left(\frac{q_{k+1}-Q}{q_k}+\frac{q_{k+1}-2Q}{q_{k-1}}\right)\\
&\quad + O\left(\frac{1}{Q^2 q q_k q_{k+1}}\right).
\end{aligned}$$

B.1.3. When $n\geqslant 1$ and $n(Q-q)\leqslant q_k\leqslant (n+1)(Q-q)$:

$$\begin{aligned}
\text{(B.3)}\qquad &\int_{t_k}^{\gamma_{k+n}} \frac{W_{\gamma,k,n}^{(1)}(t)\,dt}{t^2+t+1}\\
&= (q_{k+n+1}-\xi Q)\int_{t_k}^{\gamma_{k+n}} \frac{a_{k+n}-q_{k+n}t}{t^2+t+1}\,dt\\
&= \frac{(q_{k+n}-nQ)^2(q_{k+n+1}-\xi Q)}{2Q^2 q_k^2 q_{k+n}(\gamma^2+\gamma+1)} + O_\xi\left(\frac{1}{q q_k^2 q_{k+n}}\right).
\end{aligned}$$

Here and below $\ll_\xi$ means uniformly for ξ in compact subsets of $[0,\infty)$.

B.1.4. When $n \geqslant 1$ and $(n+1)(Q-q) \leqslant q_k \leqslant Q+n(Q-q)$:

$$\begin{aligned} \text{(B.4)} \qquad & \int_{\gamma_{k+n+1}}^{\gamma_{k+n}} \frac{W^{(1)}_{\gamma,k,n}(t)\,dt}{t^2+t+1} \\ &= (q_{k+n+1}-\xi Q) \int_{\gamma_{k+n+1}}^{\gamma_{k+n}} \frac{a_{k+n}-q_{k+n}t}{t^2+t+1}\,dt \\ &= \frac{q_{k+n+1}-\xi Q}{2q_{k+n}q_{k+n+1}^2(\gamma^2+\gamma+1)} + O_\xi\left(\frac{1}{qq_{k+n}q_{k+n+1}^2}\right). \end{aligned}$$

B.1.5. When $k \geqslant 1$ and $Q+n(Q-q) \leqslant q_k \leqslant Q+(n+1)(Q-q)$:

(B.5)

$$\begin{aligned} & \int_{\gamma_{k+n+1}}^{t_{k-1}} \frac{W^{(1)}_{\gamma,k,n}(t)\,dt}{t^2+t+1} \\ &= (q_{k+n+1}-\xi Q)\left(\int_{\gamma_{k+n+1}}^{\gamma_{k+n}} \frac{a_{k+n}-q_{k+n}t}{t^2+t+1}\,dt - \int_{t_{k-1}}^{\gamma_{k+n}} \frac{a_{k+n}-q_{k+n}t}{t^2+t+1}\,dt\right) \\ &= \frac{\big((n+2)Q-q_{k+n+1}\big)(q_{k+n+1}-\xi Q)}{2Q^2 q_{k-1}q_{k+n+1}(\gamma^2+\gamma+1)} \left(\frac{Q}{q_{k+n+1}} + \frac{q_{k+n}-(n+1)Q}{q_{k-1}}\right) \\ &\quad + O_\xi\left(\frac{1}{qq_{k-1}q_{k+n}q_{k+n+1}}\right). \end{aligned}$$

B.1.6. When $n \geqslant 1$ and $(n+1)(Q-q) \leqslant q_k \leqslant Q+(n+1)(Q-q)$:

$$\int_{t_k}^{\gamma_{k+n+1}} \frac{dt}{t^2+t+1} = \frac{q_k-(n+1)(Q-q)}{Qq_kq_{k+n+1}(\gamma^2+\gamma+1)} + O\left(\frac{1}{Qqq_{k+n+1}^2}\right),$$

$$\begin{aligned} & \int_{t_k}^{\gamma_{n+k+1}} \frac{w_{\mathcal{B}_k}(t)+w_{\mathcal{C}_k}(t)}{t^2+t+1}\,dt \\ &= \int_{\gamma}^{\gamma_{k+n+1}} \frac{qt-a}{t^2+t+1}\,dt - \int_{\gamma}^{t_k} \frac{qt-a}{t^2+t+1}\,dt \\ &= \frac{q_k-(n+1)(Q-q)}{2Q^2q_kq_{k+n+1}(\gamma^2+\gamma+1)}\left(\frac{Q}{q_{n+k+1}} + \frac{Q-q}{q_k}\right) + O\left(\frac{1}{Q^2q_{n+k+1}^3}\right), \end{aligned}$$

yielding

$$\begin{aligned} \text{(B.6)} \qquad & \int_{t_k}^{\gamma_{k+n+1}} \frac{W^{(2)}_{\gamma,k}(t)\,dt}{t^2+t+1} \\ &= \frac{q_{k+n+1}-(n+1)Q}{2Qq_kq_{k+n+1}(\gamma^2+\gamma+1)}\left(\frac{q_{k+n+1}-\xi Q}{q_{k+n+1}} + \frac{q_k-\xi(Q-q)}{q_k}\right) \\ &\quad + O_\xi\left(\frac{1}{Qqq_{k+n+1}^2}\right). \end{aligned}$$

B.1.7. When $q_k \geqslant Q+(n+1)(Q-q)$ formulas (5.8) and (5.9) yield (here $k \geqslant 1$)

$$\begin{aligned}\int_{t_k}^{t_{k-1}} \frac{W^{(2)}_{\gamma,k}(t)\,dt}{t^2+t+1} &= \frac{Q-q}{2Qq_{k-1}q_k(\gamma^2+\gamma+1)}\left(2-\xi\left(\frac{Q-q}{q_{k-1}}+\frac{Q-q}{q_k}\right)\right)\\ &\quad + O_\xi\left(\frac{1}{qq_{k-1}^2q_k}\right).\end{aligned} \tag{B.7}$$

B.2. Estimates for the $\mathcal{C}_{\leftarrow}$ contribution when $(r, r_k) = (0, 1)$. Here $\lambda_{k,n}$ is as in (7.7).

B.2.1. $(n-1)(Q-q) \leqslant q_k \leqslant n(Q-q)$. In this case $n \geqslant 2$. When $k \geqslant 1$ we have

$$\begin{aligned}\int_{t_k}^{\lambda_{k,n}} \frac{w_{\mathcal{C}_k}(t)\,dt}{t^2+t+1} &= \int_{t_k}^{t_{k-1}} \frac{a_{k-1}-2\varepsilon-q_{k-1}t}{t^2+t+1}\,dt - \int_{\lambda_{k,n}}^{t_{k-1}} \frac{a_{k-1}-2\varepsilon-q_{k-1}t}{t^2+t+1}\,dt\\ &= \frac{q_k-(n-1)(Q-q)}{2Q^2q_k(q_k+q_{k+n-1})(\gamma^2+\gamma+1)}\left(\frac{Q-q}{q_k}+\frac{(n+1)Q-q_{k+n}}{q_k+q_{k+n-1}}\right)\\ &\quad + O\left(\frac{1}{qq_{k-1}^2q_k^2}\right).\end{aligned} \tag{B.8}$$

$$\begin{aligned}\int_{t_k}^{\lambda_{k,n}} \frac{w_{\mathcal{B}_k}(t)\,dt}{t^2+t+1} &= \int_{t_k}^{\lambda_{k,n}} \frac{q_kt-a_k+2\varepsilon}{t^2+t+1}\,dt\\ &= \frac{\big(q_k-(n-1)(Q-q)\big)^2}{2Q^2q_k(q_k+q_{k+n-1})^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{qq_k^2q_{k+n}^2}\right).\end{aligned} \tag{B.9}$$

$$\begin{aligned}\int_{t_k}^{\lambda_{k,n}} \frac{q_{k+n}t-a_{k+n}}{t^2+t+1}\,dt &= \int_{\gamma_{k+n}}^{\lambda_{k,n}} \frac{q_{k+n}t-a_{k+n}}{t^2+t+1}\,dt - \int_{\gamma_{k+n}}^{t_k} \frac{q_{k+n}t-a_{k+n}}{t^2+t+1}\,dt\\ &= \frac{q_k-(n-1)(Q-q)}{2Q^2q_k(q_k+q_{k+n-1})(\gamma^2+\gamma+1)}\\ &\quad \cdot\left(\frac{(n+1)Q-q_{k+n}}{q_k+q_{k+n-1}}+\frac{n(Q-q)-q_k}{q_k}\right) + O\left(\frac{1}{qq_k^3q_{k+n}}\right).\end{aligned} \tag{B.10}$$

When $k=0$ formulas (B.9) and (B.10) still hold. The main term in (B.8) remains the same but we need a more careful estimate to control the error term, getting

$$\begin{aligned}
\text{(B.11)}\quad & \int_{t_0}^{\lambda_{0,n}} \frac{w_{\mathcal{C}_0}(t)\,dt}{t^2+t+1} \\
&= \int_{\gamma}^{\lambda_{0,n}} \frac{qt-a}{t^2+t+1}\,dt - \int_{\gamma}^{t_0} \frac{qt-a}{t^2+t+1}\,dt - \int_{t_0}^{\lambda_{0,n}} \frac{q't-a'+2\varepsilon}{t^2+t+1}\,dt \\
&= \frac{q_{n-1}-(n-1)Q}{2Q^2 q'(q'+q_{n-1})(\gamma^2+\gamma+1)} \left(\frac{(n+1)Q-q_n}{q'+q_{n-1}} + \frac{Q-q}{q'} \right) \\
&\quad + O\left(\frac{1}{Qq^2 q_{n-1}^2} \right).
\end{aligned}$$

Since n only takes one value $n = \left\lfloor \frac{Q+q_{k-1}}{Q-q} \right\rfloor$ for fixed k, the error terms in (B.8)–(B.11) do not play a role in the final asymptotic formula.

B.2.2. When $k \geqslant 1$ and $n(Q-q) \leqslant q_k \leqslant Q+n(Q-q)$:

$$\begin{aligned}
\text{(B.12)}\quad & \int_{\lambda_{k,n+1}}^{\lambda_{k,n}} \frac{w_{\mathcal{C}_k}(t)\;dt}{t^2+t+1} \\
&= \int_{\lambda_{k,n+1}}^{t_{k-1}} \frac{a_{k-1}-2\varepsilon-q_{k-1}t}{t^2+t+1}\,dt - \int_{\lambda_{k,n}}^{t_{k-1}} \frac{a_{k-1}-2\varepsilon-q_{k-1}t}{t^2+t+1}\,dt \\
&= \frac{2Q-q}{2Q^2(q_k+q_{k+n-1})(q_k+q_{k+n})(\gamma^2+\gamma+1)} \\
&\quad \cdot \left(\frac{(n+2)Q-q_{k+n+1}}{q_k+q_{k+n}} + \frac{(n+1)Q-q_{k+n}}{q_k+q_{k+n-1}} \right) \\
&\quad + O\left(\frac{1}{(n-1)^3 q^2 q_k^3} \right).
\end{aligned}$$

$$\begin{aligned}
\text{(B.13)}\quad & \int_{\lambda_{k,n+1}}^{\lambda_{k,n}} \frac{w_{\mathcal{B}_k}(t)\,dt}{t^2+t+1} \\
&= \int_{t_k}^{\lambda_{k,n}} \frac{q_k t-a_k+2\varepsilon}{t^2+t+1}\,dt - \int_{t_k}^{\lambda_{k,n+1}} \frac{q_k t-a_k+2\varepsilon}{t^2+t+1}\,dt \\
&= \frac{2Q-q}{2Q^2(q_k+q_{k+n-1})(q_k+q_{k+n})(\gamma^2+\gamma+1)} \\
&\quad \cdot \left(\frac{q_{k+n-1}-(n-1)Q}{q_k+q_{k+n-1}} + \frac{q_{k+n}-nQ}{q_k+q_{k+n}} \right) + O\left(\frac{1}{q q_k q_{k+n}^3} \right).
\end{aligned}$$

$$\int_{\lambda_{k,n+1}}^{\gamma_{k+n}} \frac{a_{k+n}-q_{k+n}t}{t^2+t+1}\,dt = \frac{(q_{k+n}-nQ)^2}{2Q^2 q_{k+n}(q_k+q_{k+n})^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{qq_{k+n}^4}\right). \tag{B.14}$$

$$\int_{\gamma_{k+n}}^{\lambda_{k,n}} \frac{q_{k+n}t-a_{k+n}}{t^2+t+1}\,dt = \frac{\big((n+1)Q-q_{k+n}\big)^2}{2Q^2 q_{k+n}(q_{k-1}+q_{k+n})^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{qq_{k+n}^4}\right). \tag{B.15}$$

B.2.3. When $k=0$ and $n \geqslant 2$ we have $t_0 < \lambda_{0,n+1} < \gamma_n < \lambda_{0,n} \leqslant t_{k-1}$. Analog formulas as (B.12)–(B.15) hold, with same main terms and

$$\begin{aligned}\int_{\lambda_{0,n+1}}^{\lambda_{0,n}} \frac{w_{\mathcal{C}_0}(t)\,dt}{t^2+t+1} &= \int_{\gamma}^{\lambda_{0,n}} \frac{qt-a}{t^2+t+1}\,dt - \int_{\gamma}^{\lambda_{0,n+1}} \frac{qt-a}{t^2+t+1}\,dt \\ &\quad - \left(\int_{t_0}^{\lambda_{0,n}} \frac{q't-a'+2\varepsilon}{t^2+t+1}\,dt - \int_{t_0}^{\lambda_{0,n+1}} \frac{q't-a'+2\varepsilon}{t^2+t+1}\,dt\right) \\ &= \frac{2Q-q}{2Q^2(q'+q_{n-1})(q'+q_n)(\gamma^2+\gamma+1)} \\ &\quad \cdot \left(\frac{(n+2)Q-q_{n+1}}{q'+q_n} + \frac{(n+1)Q-q_n}{q'+q_{n-1}}\right) + O\left(\frac{1}{qq'q_n^3}\right).\end{aligned} \tag{B.16}$$

B.2.4. When $k=0$ and $n=1$ we have $t_0 \leqslant \lambda_{0,2} < \gamma_1 \leqslant \lambda_{0,1}$. We have

$$\begin{aligned}\int_{\lambda_{0,2}}^{\gamma_1} \frac{w_{\mathcal{B}_0}(t)\,dt}{t^2+t+1} &= \int_{t_0}^{\gamma_1} \frac{q't-a'+2\varepsilon}{t^2+t+1}\,dt - \int_{t_0}^{\lambda_{0,2}} \frac{q't-a'+2\varepsilon}{t^2+t+1}\,dt \\ &= \frac{(q_1-Q)^2(q'+2q_1)}{2Q^2q_1^2(q'+q_1)^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{Q^3qq'}\right).\end{aligned} \tag{B.17}$$

$$\int_{\lambda_{0,2}}^{\gamma_1} \frac{a_1-q_1t}{t^2+t+1}\,dt = \frac{(q_1-Q)^2}{2Q^2q_1(q'+q_1)^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{Q^4q}\right) \tag{B.18}$$

$$\begin{aligned}
\text{(B.19)}\qquad & \int_{\lambda_{0,2}}^{\gamma_1} \frac{w_{\mathcal{C}_0}(t)\,dt}{t^2+t+1} \\
&= \int_{\lambda_{0,2}}^{\gamma'} \frac{a'-q't}{t^2+t+1}\,dt - \int_{\gamma_1}^{\gamma'} \frac{a'-q't}{t^2+t+1}\,dt \\
&\quad - \int_{\lambda_{0,2}}^{\frac{a+2\varepsilon}{q}} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt + \int_{\gamma_1}^{\frac{a+2\varepsilon}{q}} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt \\
&= \frac{q_1-Q}{2Q^2q_1(q'+q_1)(\gamma^2+\gamma+1)}\left(\frac{3Q-q_2}{q'+q_1}+\frac{2Q-q_1}{q_1}\right) \\
&\quad + O\left(\frac{1}{Q^3qq'}\right).
\end{aligned}$$

B.2.5. When $Q+n(Q-q) \leqslant q_k \leqslant Q+(n+1)(Q-q)$:

$$\begin{aligned}
\text{(B.20)}\qquad & \int_{\lambda_{k,n+1}}^{t_{k-1}} \frac{w_{\mathcal{C}_k}(t)\,dt}{t^2+t+1} \\
&= \frac{\big((n+2)Q-q_{k+n+1}\big)^2}{2Q^2q_{k-1}(q_k+q_{k+n})^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{Qqq_{k-1}q_{k+n}^2}\right).
\end{aligned}$$

$$\begin{aligned}
\text{(B.21)}\qquad & \int_{\lambda_{k,n+1}}^{t_{k-1}} \frac{w_{\mathcal{B}_k}(t)\,dt}{t^2+t+1} \\
&= \int_{t_k}^{t_{k-1}} \frac{q_kt-a_k+2\varepsilon}{t^2+t+1}\,dt - \int_{t_k}^{\lambda_{k,n+1}} \frac{q_kt-a_k+2\varepsilon}{t^2+t+1}\,dt \\
&= \frac{(n+2)Q-q_{k+n+1}}{2Q^2q_{k-1}(q_k+q_{k+n})(\gamma^2+\gamma+1)} \\
&\quad \cdot \left(\frac{Q-q}{q_{k-1}}+\frac{q_{k+n}-nQ}{q_k+q_{k+n}}\right) + O\left(\frac{1}{Q^2qq_k^2}\right).
\end{aligned}$$

$$\begin{aligned}
\text{(B.22)}\qquad & \int_{\lambda_{k,n+1}}^{t_{k-1}} \frac{a_{k+n}-q_{k+n}t}{t^2+t+1}\,dt \\
&= \int_{\lambda_{k,n+1}}^{\gamma_{k+n}} \frac{a_{k+n}-q_{k+n}t}{t^2+t+1}\,dt - \int_{t_{k-1}}^{\gamma_{k+n}} \frac{a_{k+n}-q_{k+n}t}{t^2+t+1}\,dt \\
&= \frac{(n+2)Q-q_{k+n+1}}{2Q^2q_{k-1}(q_k+q_{k+n})(\gamma^2+\gamma+1)} \\
&\quad \cdot \left(\frac{q_{k+n}-nQ}{q_k+q_{k+n}}+\frac{q_{k+n}-(n+1)Q}{q_{k-1}}\right) + O\left(\frac{1}{Q^2qq_{k+n}^2}\right).
\end{aligned}$$

Appendix C.

C.1. Estimates for the $\mathcal{C}_O$ contribution when $(r, r_k) = (1, 0)$. Here $\lambda_{k,n}$ is as in (8.3) and $W^{(1)}_{\gamma,k,n}$ and $W^{(2)}_{\gamma,k}$ as in (8.4).

C.1.1. For $k \geqslant 1$ and $q_{k+1} \leqslant 2Q$:

$$\int_{\gamma_{k+1}}^{t_{k-1}} \frac{w_{\mathcal{C}_k}(t)\,dt}{t^2+t+1} = \frac{(2Q-q_{k+1})^2}{2Q^2 q_{k-1} q_{k+1}^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{q q_{k-1}^2 q_k^2}\right). \tag{C.1}$$

$$\begin{aligned}\int_{\gamma_{k+1}}^{t_{k-1}} \frac{w_{\mathcal{B}_k}(t) - w_{\mathcal{A}_0}(t)}{t^2+t+1}\,dt &= \int_{\gamma_{k+1}}^{t_{k-1}} \frac{q_{k+1}t - a_{k+1}}{t^2+t+1}\,dt \\ &= \frac{(2Q-q_{k+1})^2}{2Q^2 q_{k-1}^2 q_{k+1}(\gamma^2+\gamma+1)} \\ &\quad + O\left(\frac{1}{q q_{k-1}^3 q_{k+1}}\right).\end{aligned} \tag{C.2}$$

C.1.2. For $k \geqslant 1$ and $n(q_k - Q) \leqslant Q - q \leqslant (N+1)(q_k - Q)$:

$$\begin{aligned}\int_{\lambda_{k,n}}^{t_{k-1}} \frac{(q+nq_k)t - (a+na_k)}{t^2+t+1}\,dt &= \frac{\big((n+1)Q - (q+nq_k)\big)^2}{2Q^2 q_{k-1}^2 (q+nq_k)(\gamma^2+\gamma+1)} \\ &\quad + O\left(\frac{1}{q q_{k-1}^3 (q+nq_k)}\right)\end{aligned} \tag{C.3}$$

C.1.3. For $k \geqslant 1$ and $(n+1)(q_k - Q) \leqslant Q - q$:

$$\begin{aligned}\int_{\lambda_{k,n}}^{\lambda_{k,n+1}} \frac{W^{(1)}_{\gamma,k,n}(t)\ dt}{t^2+t+1} &= \frac{q_{k+1} + nq_k - \xi Q}{2(q+nq_k)(q_{k+1}+nq_k)^2(\gamma^2+\gamma+1)} \\ &\quad + O_\xi\left(\frac{1}{q q_k (q+nq_k)(q_{k+1}+nq_k)}\right).\end{aligned} \tag{C.4}$$

$$\begin{aligned}&\int_{\lambda_{k,n+1}}^{t_{k-1}} \frac{dt}{t^2+t+1} \\ &\quad = \frac{(n+2)Q - q_{k+1} - nq_k}{Q q_{k-1}(q_{k+1}+nq_k)(\gamma^2+\gamma+1)} + O\left(\frac{1}{q q_{k-1}^2 q_k}\right).\end{aligned} \tag{C.5}$$

$$\begin{aligned}
(\text{C.6})\qquad & \int_{\lambda_{k,n+1}}^{t_{k-1}} \frac{w_{\mathcal{A}_0}(t)+w_{\mathcal{C}_k}(t)}{t^2+t+1}\,dt \\
&= \int_{\lambda_{k,n+1}}^{\gamma_k} \frac{a_k-q_kt}{t^2+t+1}\,dt - \int_{t_{k-1}}^{\gamma_k} \frac{a_k-q_kt}{t^2+t+1}\,dt \\
&= \frac{(n+2)Q-q_{k+1}-nq_k}{2Q^2q_{k-1}(q_{k+1}+nq_k)(\gamma^2+\gamma+1)} \left(\frac{Q}{q_{k+1}+nq_k} + \frac{q_k-Q}{q_{k-1}} \right) \\
&\quad + O\left(\frac{1}{q_{k-1}^3 q_{k+1}^2} \right).
\end{aligned}$$

$$\begin{aligned}
(\text{C.7})\qquad \int_{\lambda_{k,n+1}}^{t_{k-1}} \frac{W_{\gamma,k}^{(2)}(t)}{t^2+t+1}\,dt &= \frac{(n+2)Q-q_{k+1}-nq_k}{2Qq_{k-1}(q_{k+1}+nq_k)(\gamma^2+\gamma+1)} \\
&\quad \cdot \left(\frac{q_{k+1}+nq_k-\xi Q}{q_{k+1}+nq_k} + \frac{q_{k-1}-\xi(q_k-Q)}{q_{k-1}} \right) \\
&\quad + O_\xi\left(\frac{1}{qq_{k-1}^3} \right).
\end{aligned}$$

C.2. Estimates for the $\mathcal{C}_{\leftarrow}$ contribution when $(r,r_k)=(1,0)$. Here $\mu_{k,n}$ and $\nu_{k,n}$ are as in (8.7).

C.2.1. For $q_2>2Q$:

$$\begin{aligned}
(\text{C.8})\quad \int_{\frac{a+\varepsilon}{q}}^{\gamma_1} \frac{w_{\mathcal{C}_0}(t)\,dt}{t^2+t+1} &= -\left(\int_{t_0}^{\gamma_1} \frac{q't-a'+2\varepsilon}{t^2+t+1}\,dt - \int_{t_0}^{\frac{a+\varepsilon}{q}} \frac{q't-a'+2\varepsilon}{t^2+t+1}\,dt \right) \\
&\quad + \left(\int_{\gamma}^{\gamma_1} \frac{qt-a}{t^2+t+1}\,dt - \int_{\gamma}^{\frac{a+\varepsilon}{q}} \frac{qt-a}{t^2+t+1}\,dt \right) \\
&= \frac{(2Q-q_1)^2q_3}{8Q^2q^2q_1^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{Q^2q^2q'} \right).
\end{aligned}$$

$$\begin{aligned}
(\text{C.9})\qquad \int_{\frac{a+\varepsilon}{q}}^{\gamma_1} \frac{w_{\mathcal{A}_0}(t)\,dt}{t^2+t+1} &= \int_{\frac{a+\varepsilon}{q}}^{u_0} \frac{w_{\mathcal{A}_0}(t)\,dt}{t^2+t+1}\,dt - \int_{\gamma}^{u_0} \frac{w_{\mathcal{A}_0}(t)\,dt}{t^2+t+1} \\
&= \frac{(2Q-q_1)(3q_1-2Q)}{8Q^2qq_1^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{Q^3q^2} \right).
\end{aligned}$$

$$(\text{C.10})\qquad \int_{\frac{a+\varepsilon}{q}}^{\gamma_1} \frac{a_1-q_1t}{t^2+t+1}\,dt = \frac{(2Q-q_1)^2}{8Q^2q^2q_1(\gamma^2+\gamma+1)} + O\left(\frac{1}{q^3Q^2} \right).$$

C.2.2. When $n=1$ and $\frac{2(Q-q)}{n+1} \leqslant q_k \leqslant \frac{2(Q-q)}{n} \wedge \left(Q + \frac{Q-q}{n+1}\right)$:

$$\begin{aligned}\int_{t_k}^{\mu_{k,n}} \frac{w_{\mathcal{A}_0}(t)\,dt}{t^2+t+1} &= \int_{t_k}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt - \int_{\mu_{k,n}}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt \\ &= \frac{(n+1)q_k-2(Q-q)}{2Q^2q_k(2q+nq_k)(\gamma^2+\gamma+1)} \\ &\quad\cdot\left(\frac{q_{k+1}-Q}{q_k} + \frac{q+nq_k-nQ}{2q+nq_k}\right) + O\left(\frac{1}{Qq^2q_k^2}\right).\end{aligned} \tag{C.11}$$

$$\begin{aligned}&\int_{t_k}^{\mu_{k,n}} \frac{w_{\mathcal{C}_k}(t)\,dt}{t^2+t+1} \\ &= \int_{t_k}^{t_{k-1}} \frac{a_{k-1}-2\varepsilon-q_{k-1}t}{t^2+t+1}\,dt - \int_{\mu_{k,n}}^{t_{k-1}} \frac{a_{k-1}-2\varepsilon-q_{k-1}t}{t^2+t+1}\,dt \\ &= \frac{(n+1)q_k-2(Q-q)}{2Q^2q_k(2q+nq_k)(\gamma^2+\gamma+1)} \\ &\quad\cdot\left(\frac{Q-q}{q_k} + \frac{(n+2)Q-q-(n+1)q_k}{2q+nq_k}\right) + O\left(\frac{1}{Qqq_{k-1}q_k^2}\right).\end{aligned} \tag{C.12}$$

$$\begin{aligned}&\int_{t_k}^{\nu_{k,n}} \frac{a+na_k-(q+nq_k)t}{t^2+t+1}\,dt \\ &\qquad = \frac{(q+nq_k-Q)^2}{2Q^2q_k^2(q+nq_k)(\gamma^2+\gamma+1)} + O\left(\frac{1}{qq_k^4}\right).\end{aligned} \tag{C.13}$$

$$\begin{aligned}&\int_{\nu_{k,n}}^{\mu_{k,n}} \frac{(q+nq_k)t-(a+na_k)}{t^2+t+1}\,dt \\ &\qquad = \frac{(q+nq_k-Q)^2}{2Q^2(q+nq_k)(2q+nq_k)^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{qq_k^4}\right).\end{aligned} \tag{C.14}$$

C.2.3. When $\frac{2(Q-q)}{n} \leqslant q_k \leqslant Q + \frac{Q-q}{n+1}$, $n, k \geqslant 1$:
For $n \geqslant 2$ we have

$$\begin{aligned}\int_{\mu_{k,n-1}}^{\mu_{k,n}} \frac{w_{\mathcal{A}_0}(t)\,dt}{t^2+t+1} &= \int_{\mu_{k,n-1}}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt - \int_{\mu_{k,n}}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt \\ &= \frac{2Q-q_k}{2Q^2(2q+nq_k)\big(2q+(n-1)q_k\big)(\gamma^2+\gamma+1)} \\ &\quad\cdot\left(\frac{q+(n-1)(q_k-Q)}{2q+(n-1)q_k} + \frac{q+n(q_k-Q)}{2q+nq_k}\right) \\ &\quad + O\left(\frac{1}{(n-1)^3q^2q_k^3}\right).\end{aligned} \tag{C.15}$$

For $n=1$, the error can be improved (since $2Q-q_k \leqslant 2q$ and $\mu_{k,1}-\gamma \leqslant \frac{1}{qq_k}$) as follows:

$$\begin{aligned}\text{(C.16)}\quad & \int_{\mu_{k,0}}^{\mu_{k,1}} \frac{w_{\mathcal{A}_0}(t)\,dt}{t^2+t+1} \\ &= -\int_{\mu_{k,0}}^{\mu_{k,1}} \frac{qt-a-\varepsilon}{t^2+t+1}\,dt + \int_{\mu_{k,0}}^{\mu_{k,1}} \frac{\varepsilon\,dt}{t^2+t+1} \\ &= \frac{2Q-q_k}{2Q^2(2q+q_k)(2q)(\gamma^2+\gamma+1)} \left(\frac{q}{2q} + \frac{q+q_k-Q}{2q+q_k} \right) \\ &\quad + O\left(\frac{1}{Q^2 q q_k^2} \right).\end{aligned}$$

Since $0 \leqslant (n+1)Q-q-nq_k \leqslant Q-q < q'$ and $0 \leqslant nq_k-(n+1)Q \leqslant 2Q$, we find:

$$\begin{aligned}\text{(C.17)}\quad & \int_{\mu_{k,n-1}}^{\mu_{k,n}} \frac{w_{\mathcal{C}_k}(t)\;dt}{t^2+t+1} \\ &= \int_{\mu_{k,n-1}}^{t_{k-1}} \frac{a_{k-1}-2\varepsilon-q_{k-1}t}{t^2+t+1}\,dt - \int_{\mu_{k,n}}^{t_{k-1}} \frac{a_{k-1}-2\varepsilon-q_{k-1}t}{t^2+t+1}\,dt \\ &= \frac{2Q-q_k}{2Q^2(2q+nq_k)\big(2q+(n-1)q_k\big)(\gamma^2+\gamma+1)} \\ &\quad \cdot \left(\frac{Q-q-n(q_k-Q)}{2q+(n-1)q_k} + \frac{Q-q-(n+1)(q_k-Q)}{2q+nq_k} \right) \\ &\quad + O\left(\frac{1}{qq_{k-1}^2\big(q+(n-1)q_k\big)^2} \right).\end{aligned}$$

$$\begin{aligned}\text{(C.18)}\quad & \int_{\mu_{k,n-1}}^{\nu_{k,n}} \frac{a+na_k-(q+nq_k)t}{t^2+t+1}\,dt \\ &= \frac{\big(Q-q-n(q_k-Q)\big)^2}{2Q^2(q+nq_k)\big(2q+(n-1)q_k\big)^2(\gamma^2+\gamma+1)} \\ &\quad + O\left(\frac{1}{nqq_k^2\big(q+(n-1)q_k\big)^2} \right).\end{aligned}$$

$$\begin{aligned}\text{(C.19)}\quad & \int_{\nu_{k,n}}^{\mu_{k,n}} \frac{(q+nq_k)t-(a+na_k)}{t^2+t+1}\,dt \\ &= \frac{\big(q+n(q_k-Q)\big)^2}{2Q^2(q+nq_k)(2q+nq_k)^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{n^3qq_k^4} \right).\end{aligned}$$

We check that the contribution of error terms is negligible. Note that when $n=1$ we must have $2q \geqslant 2Q-q_k \geqslant 2Q-\frac{3Q-q}{2} = \frac{Q+q}{2}$, so $q \geqslant \frac{Q}{3}$. The errors in (C.15)–(C.19) add up to

$$\ll \sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\sum_{\gamma\in\mathcal{F}(Q)} \frac{q}{(n-1)^3q^2q_k^3} + \sum_{k=1}^{\infty}\sum_{\gamma\in\mathcal{F}(Q)} \frac{1}{Q^2q_k^2} + \sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\sum_{\gamma\in\mathcal{F}(Q)} \frac{q_{k+1}}{qq_{k-1}^2q_k^2}$$

$$+ \sum_{k=1}^{\infty}\sum_{\gamma\in\mathcal{F}(Q)} \frac{q_{k+1}}{q^3q_{k-1}^2} + \sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\sum_{\gamma\in\mathcal{F}(Q)} \frac{q_{k+1}}{n(n-1)^2qq_k^4} + \sum_{k=1}^{\infty}\sum_{\gamma\in\mathcal{F}(Q)} \frac{q_{k+1}}{q^3q_k^2}$$

$$+ \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{n^3q_k^4} + \frac{1}{|I|}\cdot\frac{1}{Q} \ll \sum_{\gamma\in\mathcal{F}(Q)} \frac{Q}{q^3q'^2} + \sum_{\frac{Q}{3}\leqslant q\leqslant Q} \frac{\varphi(q)}{q^3} + Q^{c-1} \ll Q^{c-1}.$$

C.2.4. For $k \geqslant 1$ and $q_{k+2} \leqslant 2Q$:

$$\int_{\gamma_{k+1}}^{t_{k-1}} \frac{q_{k+1}t - a_{k+1}}{t^2+t+1}\,dt = \frac{(2Q-q_{k+1})^2}{2Q^2q_{k-1}^2q_{k+1}(\gamma^2+\gamma+1)} + O\left(\frac{1}{qq_{k-1}^3q_{k+1}}\right). \tag{C.20}$$

C.2.5. For $n, k \geqslant 1$ and $\frac{2(Q-q)}{n} \vee \left(Q + \frac{Q-q}{n+1}\right) \leqslant q_k \leqslant Q + \frac{Q-q}{n}$:

(C.21)

$$\int_{\mu_{k,n-1}}^{t_{k-1}} \frac{w_{\mathcal{A}_0}(t)\,dt}{t^2+t+1}$$
$$= \int_{\mu_{k,n-1}}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt - \int_{t_{k-1}}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt$$
$$= \frac{(n+1)Q-q-nq_k}{2Q^2q_{k-1}\big(2q+(n-1)q_k\big)(\gamma^2+\gamma+1)}\left(\frac{q+(n-1)(q_k-Q)}{2q+(n-1)q_k} + \frac{q_k-Q}{q_{k-1}}\right)$$
$$+ O\left(\frac{1}{q\big(2q+(n-1)q_k\big)^2}\left(\frac{Q-q}{Q^2q_{k-1}} + \frac{1}{q\big(2q+(n-1)q_k\big)}\right)\right).$$

When $n \geqslant 2$, the error is $\leqslant \frac{2}{q^2q_{k-1}q_k^2}$. When $n = 1$, we can improve on the error term:

$$\int_{\mu_{k,0}}^{t_{k-1}} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt = -\int_{\frac{a+\varepsilon}{q}}^{t_{k-1}} \frac{a+\varepsilon-qt}{t^2+t+1}\,dt + \int_{\frac{a+\varepsilon}{q}}^{t_{k-1}} \frac{\varepsilon\,dt}{t^2+t+1}$$
$$= \frac{2Q-q_{k+1}}{4Q^2qq_{k-1}(\gamma^2+\gamma+1)}\left(1 - \frac{2Q-q_{k+1}}{2q_{k-1}}\right) + O\left(\frac{1}{Qq^2q_{k-1}^2} + \frac{1}{q^2q_{k-1}^3} + \frac{1}{Q^2q^2q_{k-1}}\right). \tag{C.22}$$

$$\int_{\mu_{k,n-1}}^{t_{k-1}} \frac{w_{\mathcal{C}_k}(t)\,dt}{t^2+t+1} = \frac{\big((n+1)Q-q-nq_k\big)^2}{2Q^2q_{k-1}\big(2q+(n-1)q_k\big)^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{qq_{k-1}^2\big(2q+(n-1)q_k\big)^2}\right). \tag{C.23}$$

$$\int_{\mu_{k,n-1}}^{\nu_{k,n}} \frac{a+na_k-(q+nq_k)t}{t^2+t+1}\,dt = \frac{\big((n+1)Q-q-nq_k\big)^2}{2Q^2(q+nq_k)\big(2q+(n-1)q_k\big)^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{q^3q_k^2}\right). \tag{C.24}$$

$$\int_{\nu_{k,n}}^{t_{k-1}} \frac{(q+nq_k)t-(a+na_k)}{t^2+t+1}\,dt = \frac{\big((n+1)Q-q-nq_k\big)^2}{2Q^2q_{k-1}^2(q+nq_k)(\gamma^2+\gamma+1)} + O\left(\frac{1}{Q^2qq_k^2}\right). \tag{C.25}$$

Appendix D.

When $k \geqslant 1$ and $q_{k+1} \leqslant 2Q$:

$$\begin{aligned}\int_{t_k}^{\gamma_{k+1}} \frac{w_{\mathcal{C}_k}(t)\,dt}{t^2+t+1} &= \int_{t_k}^{t_{k-1}} \frac{a_{k-1}-2\varepsilon-q_{k-1}t}{t^2+t+1}\,dt - \int_{\gamma_{k+1}}^{t_{k-1}} \frac{a_{k-1}-2\varepsilon-q_{k-1}t}{t^2+t+1}\,dt \\ &= \frac{q_{k+1}-Q}{2Q^2q_kq_{k+1}(\gamma^2+\gamma+1)}\left(\frac{Q-q}{q_k}+\frac{2Q-q_{k+1}}{q_{k+1}}\right) + O\left(\frac{1}{qq_{k-1}q_k^3}\right).\end{aligned} \tag{D.1}$$

$$\begin{aligned}\int_{\gamma_{k+1}}^{t_{k-1}} \frac{w_{\mathcal{A}_0}(t)\,dt}{t^2+t+1} &= \int_{\gamma_{k+1}}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt - \int_{t_{k-1}}^{u_0} \frac{a+2\varepsilon-qt}{t^2+t+1}\,dt \\ &= \frac{2Q-q_{k+1}}{2Q^2q_{k-1}q_{k+1}}\left(\frac{q_{k+1}-Q}{q_{k+1}}+\frac{q_k-Q}{q_{k-1}}\right) + O\left(\frac{1}{q^3q_{k+1}^2}\right).\end{aligned} \tag{D.2}$$

When $q_{k+1} > 2Q$:

(D.3)
$$\begin{aligned}&\int_{t_k}^{\gamma_{k+1}} \frac{2(a+\varepsilon-qt)\,dt}{t^2+t+1}\\&= -\left(\int_{\gamma}^{\gamma_{k+1}} \frac{qt-a}{t^2+t+1}\,dt - \int_{\gamma}^{t_k} \frac{qt-a}{t^2+t+1}\,dt\right) + \int_{t_k}^{\gamma_{k+1}} \frac{2\varepsilon\,dt}{t^2+t+1}\\&= \frac{q_{k+1}-Q}{2Q^2 q_k q_{k+1}(\gamma^2+\gamma+1)}\left(\frac{q_{k+2}-2Q}{q_k} + \frac{q_{k+1}-2Q}{q_{k+1}}\right) + O\left(\frac{1}{q^3 q_k^2}\right).\end{aligned}$$

D.1. Estimates for the $\mathcal{C}_O$ and the $\mathcal{C}_{\leftarrow}$ contributions when $(r, r_k) = (1, -1)$ and $w_{\mathcal{A}_0} > w_{\mathcal{B}_k}$. Here $\lambda_{k,n}$ is as in (9.3).

D.1.1. If $Q + \frac{Q-q}{n+1} \leqslant q_{k+1} \leqslant 2Q$ and $k \geqslant 1$, then

(D.4)
$$\begin{aligned}&\int_{\lambda_{k,n+1}}^{\gamma_{k+1}} \frac{w_{\mathcal{C}_k}(t)\,dt}{t^2+t+1}\\&= \int_{\lambda_{k,n+1}}^{t_{k-1}} \frac{a_{k-1}-2\varepsilon-q_{k-1}t}{t^2+t+1}\,dt - \int_{\gamma_{k+1}}^{t_{k-1}} \frac{a_{k-1}-2\varepsilon-q_{k-1}t}{t^2+t+1}\,dt\\&= \frac{2Q-q_{k+1}}{2Q^2 q_{k+1}(nq_{k+1}+2q)(\gamma^2+\gamma+1)}\left(\frac{(n+1)(2Q-q_{k+1})}{nq_{k+1}+2q} + \frac{2Q-q_{k+1}}{q_{k+1}}\right)\\&\quad + O\left(\frac{1}{q q_{k-1}^2 q_{k+1}^2}\right).\end{aligned}$$

When $k = 0$, one can improve on the error as follows:

$$\begin{aligned}\text{(D.5)}\quad &\int_{\lambda_{0,n+1}}^{\gamma_1} \frac{w_{\mathcal{C}_0}(t)\,dt}{t^2+t+1}\\&= -\left(\int_{t_0}^{\gamma_1} \frac{q't-a'+2\varepsilon}{t^2+t+1}\,dt - \int_{t_0}^{\lambda_{0,n+1}} \frac{q't-a'+2\varepsilon}{t^2+t+1}\,dt\right)\\&\quad + \left(\int_{\gamma}^{\gamma_1} \frac{qt-a}{t^2+t+1}\,dt - \int_{\gamma}^{\lambda_{0,n+1}} \frac{qt-a}{t^2+t+1}\,dt\right)\\&= \frac{2Q-q_1}{2Q^2 q_1(nq_1+2q)(\gamma^2+\gamma+1)}\left(\frac{(n+1)(2Q-q_1)}{nq_1+2q} + \frac{2Q-q_1}{q_1}\right)\\&\quad + O\left(\frac{1}{Q^2 q^2 q'}\right).\end{aligned}$$

D.1.2. When $n \geqslant 1$ and $Q + \frac{Q-q}{n+1} \leqslant q_{k+1} \leqslant 2Q$:

$$\int_{\lambda_{k,n+1}}^{\gamma_{k+1}} \frac{w_{\mathcal{A}_0}(t) - w_{\mathcal{B}_k}(t)}{t^2+t+1}\,dt = \frac{(2Q-q_{k+1})^2}{2Q^2 q_{k+1}(nq_{k+1}+2q)^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{qq_{k+1}^4}\right). \tag{D.6}$$

$$\int_{\lambda_{k,n}}^{\lambda_{k,n+1}} \frac{(nq_{k+1}-q_{k-1})t - na_{k+1} + a_{k-1} - 2\varepsilon}{t^2+t+1}\,dt = \frac{(2Q-q_{k+1})^2}{2Q^2(nq_{k+1}+2q)^2\big((n-1)q_{k+1}+2q\big)(\gamma^2+\gamma+1)} + O\left(\frac{1}{Qq^2q_{k+1}^2}\right). \tag{D.7}$$

D.1.3. When $n \geqslant 1$ and $Q + \frac{Q-q}{n+1} \leqslant q_{k+1} \leqslant Q + \frac{Q-q}{n}$, we have $0 \leqslant (n+1)(q_{k+1}-Q)+q-Q \leqslant \frac{Q-q}{n}$ and

$$\int_{t_k}^{\lambda_{k,n+1}} \frac{(n+1)a_{k+1} - a_{k-1} + 2\varepsilon - \big((n+1)q_{k+1}-q_{k-1}\big)t}{t^2+t+1}\,dt = \frac{\big((n+1)(q_{k+1}-Q)+q-Q\big)^2}{2Q^2q_k^2(nq_{k+1}+2q)(\gamma^2+\gamma+1)} + O\left(\frac{1}{n^4qq_k^2q_{k+1}^2}\right). \tag{D.8}$$

(D.9)

$$\int_{t_k}^{\lambda_{k,n+1}} \frac{(nq_{k+1}-q_{k-1})t - na_{k+1} + a_{k-1} - 2\varepsilon}{t^2+t+1}\,dt = \int_{\lambda_{k,n}}^{\lambda_{k,n+1}} \cdots - \int_{\lambda_{k,n}}^{t_k} \cdots$$
$$= \frac{(n+1)(q_{k+1}-Q)+q-Q}{2Q^2q_k(nq_{k+1}+2q)(\gamma^2+\gamma+1)}\left(\frac{2Q-q_{k+1}}{nq_{k+1}+2q} + \frac{Q-q-n(q_{k+1}-Q)}{q_k}\right) + O\left(\frac{1}{n^2Qqq_kq_{k+1}^2}\right).$$

D.1.4. When $n \geqslant 1$ and $Q + \frac{Q-q}{n} \leqslant q_{k+1} \leqslant 2Q$:

$$\int_{\lambda_{k,n}}^{\lambda_{k,n+1}} \frac{(n+1)a_{k+1} - a_{k-1} + 2\varepsilon - \big((n+1)q_{k+1}-q_{k-1}\big)t}{t^2+t+1}\,dt = \frac{(2Q-q_{k+1})^2}{2Q^2(nq_{k+1}+2q)\big((n-1)q_{k+1}+2q\big)^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{(n^2-n+1)q^3q_{k+1}^2}\right). \tag{D.10}$$

$$\text{(D.11)} \quad \int_{\lambda_{k,n}}^{\lambda_{k,n+1}} \frac{(nq_{k+1}-q_{k-1})t - na_{k+1} + a_{k-1} - 2\varepsilon}{t^2+t+1}\,dt$$
$$= \frac{(2Q-q_{k+1})^2}{2Q^2(nq_{k+1}+2q)^2\big((n-1)q_{k+1}+2q\big)(\gamma^2+\gamma+1)} + O\left(\frac{1}{n^2Qq^2q_{k+1}^2}\right).$$

D.2. Estimates for the $\mathcal{C}_O$ and the $\mathcal{C}_\downarrow$ contributions when $(r, r_k) = (1, -1)$ and $w_{\mathcal{A}_0} < w_{\mathcal{B}_k}$. Here we take $\lambda_{k,n}$ as in (9.8), $k \geqslant 1$, and $q_{k+1} \leqslant 2Q$.

$$\text{(D.12)} \quad \int_{\gamma_{k+1}}^{\lambda_{k,n+1}} \frac{w_{\mathcal{C}_k}(t)\,dt}{t^2+t+1}$$
$$= \int_{\gamma_{k+1}}^{t_{k-1}} \frac{a_{k-1}-2\varepsilon-q_{k-1}t}{t^2+t+1}\,dt - \int_{\lambda_{k,n+1}}^{t_{k-1}} \frac{a_{k-1}-2\varepsilon-q_{k-1}t}{t^2+t+1}\,dt$$
$$= \frac{(2Q-q_{k+1})^2\big((2n+2)q_{k+1}+q_{k-1}\big)}{2Q^2q_{k+1}^2\big((n+1)q_{k+1}+q_{k-1}\big)^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{qq_{k-1}^2q_{k+1}^2}\right).$$

$$\text{(D.13)} \quad \int_{\gamma_{k+1}}^{\lambda_{k,n+1}} \frac{w_{\mathcal{B}_k}(t) - w_{\mathcal{A}_0}(t)}{t^2+t+1}\,dt$$
$$= \int_{\gamma_{k+1}}^{\lambda_{k,n+1}} \frac{q_{k+1}t - a_{k+1}}{t^2+t+1}\,dt$$
$$= \frac{(2Q-q_{k+1})^2}{2Q^2q_{k+1}\big((n+1)q_{k+1}+q_{k-1}\big)^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{qq_{k+1}^4}\right).$$

$$\text{(D.14)} \quad \int_{\lambda_{k,n+1}}^{\lambda_{k,n}} \frac{na_{k+1}+a_{k-1}-2\varepsilon-(nq_{k+1}+q_{k-1})t}{t^2+t+1}\,dt$$
$$= \frac{(2Q-q_{k+1})^2}{2Q^2(nq_{k+1}+q_{k-1})\big((n+1)q_{k+1}+q_{k-1}\big)^2(\gamma^2+\gamma+1)} + O\left(\frac{1}{(n^3+1)qq_{k-1}^2q_{k+1}^2}\right).$$

$$\begin{aligned}(D.15)\qquad &\int_{\lambda_{k,n+1}}^{\lambda_{k,n}} \frac{\big((n+1)q_{k+1}+q_{k-1}\big)t-(n+1)a_{k+1}-a_{k-1}+2\varepsilon}{t^2+t+1}\,dt \\ &= \frac{(2Q-q_{k+1})^2}{2Q^2(nq_{k+1}+q_{k-1})^2\big((n+1)q_{k+1}+q_{k-1}\big)(\gamma^2+\gamma+1)} \\ &\quad + O\left(\frac{1}{(n^3+1)qq_{k-1}^2q_{k+1}^2}\right).\end{aligned}$$

References

[1] Blank, S. J.; Krikorian, Nishan. Thom's problem on irrational flows. *Internat. J. Math.* **4** (1993), 721–726. MR1245349 (95b:58131), Zbl 0813.58048.

[2] Boca, Florin P.; Cobeli, Cristian; Zaharescu, Alexandru. Distribution of lattice points visible from the origin. *Comm. Math. Phys.* **213** (2000), 433–470. MR1785463 (2001j:11094), Zbl 0989.11049.

[3] Boca, Florin P.; Gologan, Radu N.; Zaharescu, Alexandru. The average length of a trajectory in a certain billiard in a flat two-torus. *New York J. Math.* **9** (2003), 303–330. MR2028172 (2004m:37065), Zbl 1066.37021.

[4] Boca, Florin P.; Gologan, Radu N.; Zaharescu, Alexandru. The statistics of the trajectory of a billiard in a flat two-torus. *Comm. Math. Phys.* **240** (2003), 53–73. MR2004979 (2004f:37007), Zbl 1078.37006.

[5] Boca, Florin P.; Gologan, Radu N. On the distribution of the free path length of the linear flow in a honeycomb. *Ann. Inst. Fourier* **59** (2009), 1043–1075. MR2543662, Zbl 1173.37036.

[6] Boca, Florin P.; Zaharescu, Alexandru. On the correlations of directions in the Euclidean plane. *Trans. Amer. Math. Soc.* **358** (2006), 1797–1825. MR2186997 (2007e:11077), Zbl 1154.11022.

[7] Boca, Florin P.; Zaharescu, Alexandru. The distribution of the free path lengths in the periodic two-dimensional Lorentz gas in the small-scatterer limit. *Comm. Math. Phys.* **269** (2007), 425–471. MR2274553 (2008f:37013), Zbl 1143.37002.

[8] Bykovskiĭ, V. A.; Ustinov, A. V. The statistics of particle trajectories in the nonhomogeneous Sinai problem for a two-dimensional lattice. *Izv. Math.* **73** (2009), no.4, 669-688. MR2583964, Zbl 1177.82023.

[9] Bourgain, Jean; Golse, François; Wennberg, Bernt. On the distribution of free path lengths for the periodic Lorentz gas. *Comm. Math. Phys.* **190** (1998), 491–508. MR1600299 (98k:82146), Zbl 0910.60082.

[10] Caglioti, Emanuele; Golse, François. On the distribution of free path lengths for the periodic Lorentz gas. III. *Comm. Math. Phys.* **236** (2003), 199–221. MR1981109 (2004h:82060), Zbl 1041.82016.

[11] Caglioti, Emanuele; Golse, François. The Boltzmann–Grad limit of the periodic Lorentz gas in two space dimensions. *C. R. Math. Acad. Sci. Paris* **346** (2008), 477–482. MR2417573 (2009d:82126), Zbl 1145.82019.

[12] Dahlqvist, Per. The Lyapunov exponent in the Sinai billiard in the small scatterer limit. *Nonlinearity* **10** (1997), 159–173. MR1430746 (98a:58102), Zbl 0907.58038.

[13] Golse, François. The periodic Lorentz gas in the Boltzmann–Grad limit. *Proceedings of the International Congress of Mathematicians* (Madrid, 2006). *Vol. 3: Invited Lectures. Eur. Math. Soc., Zürich* 2006, pp. 183–201. MR2275676 (2007k:82112), Zbl 1109.82020.

[14] Golse, François. Recent results on the periodic Lorentz gas. arXiv:0906.0191.

[15] Lorentz, A. Le mouvement des électrons dans les métaux. *Arch. Néerl.* **10** (1905) 337–371. JFM 36.0922.02 Reprinted in *Collected Papers*, Vol. 3, *Martinus Nijhoff, The Hague*, 1936.

[16] Marklof, Jens. Kinetic transport in crystals. *Proceedings of the XVIth International Congress of Mathematical Physics* (Prague, 2009). To appear. arXiv:0909.3463.

[17] Marklof, Jens; Strömbergsson, Andreas. The distribution of free path lengths in the periodic Lorentz gas and related lattice point problems. *Ann. of Math.* (2) **172** (2010) 1949–2033.

[18] Marklof, Jens; Strömbergsson Andreas. The Boltzmann–Grad limit of the periodic Lorentz gas. To appear in *Ann. of Math.* arXiv:0801.0612.

[19] Marklof, Jens; Strömbergsson, Andreas. Kinetic transport in the two-dimensional periodic Lorentz gas. *Nonlinearity* **21** (2008), 1413–1422. MR2425325 (2009j:82029), Zbl 1153.82020.

[20] Pólya, G. Zahlentheoretisches und wahrscheinlichkeitstheoretisches über die sichtweite im walde. *Arch. Math. Phys.* **27** (1918), 135–142. JFM 46.0284.01.

[21] Smillie, John; Ulcigrai, Corinna. Symbolic coding for linear trajectories in the regular octogon. arXiv:0905.0871.

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