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The classes of strongly $V_\lambda^F(A,p)$-summable sequences of fuzzy numbers

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Dedicated to the memories of my father and mother.

Abstract. We introduce the the classes of strongly $V_\lambda^F(A,p)$-summable sequences of fuzzy numbers and give some relations between these classes. We also give a natural relationship between strongly $V_\lambda^F(A,p)$-convergence of sequences of fuzzy numbers and strongly $S_\lambda^F(A)$-statistical convergence of sequences of fuzzy numbers.

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1. Introduction and preliminaries

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [17] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [9] introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties. Later on sequences of fuzzy numbers have been discussed by Diamond and Kloeden [3], Nanda [10], Savaş [14, 15], Esi [4] and many others.

Let $C(\mathbb{R}^n)=\{A\subset\mathbb{R}^n:\ \text{A is compact and convex set}\}$. The space $C(\mathbb{R}^n)$ has a linear structure induced by the operations

$$A+B=\{a+b:\ a\in A,\ b\in B\}\ \text{ and }\ \gamma A=\{\gamma a:\ \text{a}\in A\}$$

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for $A, B \in C(\mathbb{R}^n)$ and $\gamma \in \mathbb{R}$. The Hausdorff distance between A and B in $C(\mathbb{R}^n)$ is defined by

$$\delta_\infty(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}.$$

It is well-known that $(C(\mathbb{R}^n), \delta_\infty)$ is a complete metric space.

A fuzzy number is a function X from $\mathbb{R}^n$ to $[0,1]$ which is normal, fuzzy convex, upper semicontinuous and the closure of $\{x \in \mathbb{R}^n : X(x) > 0\}$ is compact. These properties imply that for each $0 < \alpha \leq 1$, the α-level set $X^\alpha = \{x \in \mathbb{R}^n : X(x) \geq \alpha\}$ is a nonempty compact, convex subset of $\mathbb{R}^n$, with support X^o. If $\mathbb{R}^n$ is replaced by $\mathbb{R}$, then obviously the set $C(\mathbb{R}^n)$ is reduced to the set of all closed bounded intervals $A = \left[\overline{A}, \underline{A}\right]$ on $\mathbb{R}$, and also

$$\delta_\infty(A,B) = \max\left\{|\underline{A} - \underline{B}|, \left|\overline{A} - \overline{B}\right|\right\}.$$

Let $L(\mathbb{R})$ denote the set of all fuzzy numbers. The linear structure of $L(\mathbb{R})$ induces the addition $X+Y$ and the scalar multiplication λX in terms of level α-sets, by

$$[X+Y]^\alpha = [X]^\alpha + [Y]^\alpha \text{ and } [\lambda X]^\alpha = \lambda [X]^\alpha$$

for each $0 \leq \alpha \leq 1$.

The set $\mathbb{R}$ of real numbers can be embedded in $L(\mathbb{R})$ if we define $\overline{r} \in L(\mathbb{R})$ by

$$\overline{r}(t) = \begin{cases} 1, & \text{if } t = r \\ 0, & \text{otherwise.} \end{cases}$$

The additive identity and multiplicative identity of $L(\mathbb{R})$ are denoted by $\overline{0}$ and $\overline{1}$, respectively. Then the arithmetic operations on $L(\mathbb{R})$ are defined as follows:

$$\begin{aligned}
(X \oplus Y)(t) &= \sup\{X(s) \wedge Y(t-s)\}, \ t \in \mathbb{R}, \\
(X \ominus Y)(t) &= \sup\{X(s) \wedge Y(s-t)\}, \ t \in \mathbb{R}, \\
(X \otimes Y)(t) &= \sup\{X(s) \wedge Y(t/s)\}, \ t \in \mathbb{R}, \\
(X/Y)(t) &= \sup\{X(st) \wedge Y(s)\}, \ t \in \mathbb{R}.
\end{aligned}$$

These operations can be defined in terms of α-level sets as follows:

$$\begin{aligned}
[X \oplus Y]^\alpha &= [a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha], \\
[X \ominus Y]^\alpha &= [a_1^\alpha - b_1^\alpha, a_2^\alpha - b_2^\alpha], \\
[X \otimes Y]^\alpha &= \left[\min_{i\in\{1,2\}} a_i^\alpha b_i^\alpha, \max_{i\in\{1,2\}} a_i^\alpha b_i^\alpha\right], \\
\left[X^{-1}\right]^\alpha &= \left[(a_2^\alpha)^{-1}, (a_1^\alpha)^{-1}\right], \ a_i^\alpha > 0, i \in \{1,2\},
\end{aligned}$$

for each $0 < \alpha \leq 1$.

For r in $\mathbb{R}$ and X in $L(\mathbb{R})$, the product rX is defined as follows:

$$rX(t) = \begin{cases} X(r^{-1}t), & \text{if } r \neq 0 \\ 0, & \text{if } r = 0. \end{cases}$$

Define a map by $d : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ by

$$d(X,Y) = \sup_{0 \leq \alpha \leq 1} \delta_\infty(X^\alpha, Y^\alpha).$$

For $X, Y \in L(\mathbb{R})$ define $X \leq Y$ if and only if $X^\alpha \leq Y^\alpha$ for any $\alpha \in [0,1]$. It is known that $(L(\mathbb{R}), d)$ is complete metric space [3].

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of natural numbers into $L(\mathbb{R})$. The fuzzy number X_k denotes the value of the function at $k \in \mathbb{N}$ [9]. We denote by w^F the set of all sequences of fuzzy numbers. A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded [9]. We denote by l_∞^F the set of all bounded sequences $X = (X_k)$ of fuzzy numbers. A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to a fuzzy number X_o if for every $\varepsilon > 0$ there is a positive integer k_o such that $d(X_k, X_o) < \varepsilon$ for $k > k_o$ [9]. We denote by c^F the set of all convergent sequences $X = (X_k)$ of fuzzy numbers.

It is straightforward to see that $c^F \subset l_\infty^F \subset w^F$. Recently, Nanda [10] studied the classes of bounded and convergent sequences of fuzzy numbers and showed that these are complete metric spaces.

The metric d has the following properties:

$$d(cX, cY) = |c|\, d(X,Y), \tag{1}$$

$$d(X+Y, Z+W) \leq d(X,Z) + d(Y,W), \tag{2}$$

for $X, Y, Z, W \in L(\mathbb{R})$ and $c \in \mathbb{R}$.

A metric on $L(\mathbb{R})$ is said to be translation invariant if

$$d(X+Z, Y+Z) = d(X,Y)$$

for all $X, Y, Z \in L(\mathbb{R})$.

The notion of statistical convergence was introduced by Fast [5] and Schoenberg [16], independently. Over the years and under different names statistical convergence has been discussed in the different theories such as the theory of fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [6], Salat [13], Connor [2] and many others. This concept was then applied to sequences of fuzzy numbers by Nuray [12], Kwon and Shim [7], Altin et al. [1], Nuray and Savaş [11] and many others.

A real sequence $x = (x_k)$ is said to be statistically convergent to l if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : \; |x_k - l| \geq \varepsilon\}| = 0,$$

where the vertical bars denote the cardinality of the set which they enclose, in which case we write S-$\lim x = l$.

Let $\Lambda = (\lambda_n)$ be a nondecreasing sequence of positive real numbers tending to infinity with $\lambda_1 = 1$, $\lambda_{n+1} \leq \lambda_n + 1$.

The generalized de la Vallee-Poussin mean is defined by

$$t_r(x) = \frac{1}{\lambda_r} \sum_{k \in I_r} x_k$$

where $I_r = [r - \lambda_r + 1, r]$. A real sequence $x = (x_k)$ is said to be (V, λ)-summable to a number l if $t_r(x) \to l$ as $r \to \infty$ (see for instance Leindler [8]).

2. The classes of strongly summable sequences

Let $A = (a_{ik})$ be an infinite matrix of fuzzy numbers and $p = (p_i)$ be a bounded sequence of positive real numbers, i.e.,

$$0 < h = \inf_i p_i \leq p_i \leq \sup_i p_i = H < \infty,$$

and let $X = (X_k)$ be a sequence of fuzzy numbers. Then we write

$$A_i(X) = \sum_{k=0}^{\infty} a_{ik} X_k$$

if the series converges for each $i \in \mathbb{N}$. We now define

$$V_\lambda^F(A,p)_o = \left\{ X = (X_k) \in w^F : \ \lim_r \frac{1}{\lambda_r} \sum_{i \in I_r} d(A_i(X), \overline{0})^{p_i} = 0 \right\},$$

$$V_\lambda^F(A,p) = \left\{ X = (X_k) \in w^F : \ \lim_r \frac{1}{\lambda_r} \sum_{i \in I_r} d(A_i(X), X_o)^{p_i} = 0 \right\}$$

and

$$V_\lambda^F(A,p)_\infty = \left\{ X = (X_k) \in w^F : \ \sup_r \frac{1}{\lambda_r} \sum_{i \in I_r} d(A_i(X), \overline{0})^{p_i} < \infty \right\}.$$

A sequence $X = (X_k)$ of fuzzy numbers is said to be strongly $V_\lambda^F(A,p)$ convergent to a fuzzy number X_o if there is a fuzzy number such that $X = (X_k) \in V_\lambda^F(A,p)$. In this case we write $X_k \to X_o(V_\lambda^F(A,p))$.

If $p_i = 1$ for all $i \in \mathbb{N}$, then we write the classes $V_\lambda^F(A)_o$, $V_\lambda^F(A)$ and $V_\lambda^F(A)_\infty$ in place of the classes $V_\lambda^F(A,p)_o$, $V_\lambda^F(A,p)$ and $V_\lambda^F(A,p)_\infty$, respectively.

In this section we examine some topological properties of these classes of sequences of fuzzy numbers and investigate some inclusion relations between them.

Theorem 2.1. (a) $V_\lambda^F(A,p)_o \subset V_\lambda^F(A,p) \subset V_\lambda^F(A,p)_\infty$.

(b) $V_\lambda^F(A,p)_o$, $V_\lambda^F(A,p)$ *and* $V_\lambda^F(A,p)_\infty$ *are closed under the operations of addition and scalar multiplication if* d *is a translation invariant metric.*

Proof. (a) It is evident that

$$V_\lambda^F(A,p)_o \subset V_\lambda^F(A,p),$$
$$V_\lambda^F(A,p)_o \subset V_\lambda^F(A,p)_\infty.$$

For $V_\lambda^F(A,p) \subset V_\lambda^F(A,p)$ we use the triangle inequality

$$\begin{aligned} d(A_i(X),\overline{0})^{p_i} &\leq \left[d(A_i(X),X_o)+d(X_o,\overline{0})\right]^{p_i} \\ &\leq K d(A_i(X),X_o)^{p_i} + K\max(1,|X_o|), \end{aligned}$$

where $K=\max(1,\sup_i p_i<\infty)$. So, $X=(X_k)\in V_\lambda^F(A,p)$ implies

$$X=(X_k)\in V_\lambda^F(A,p)_\infty.$$

(b) We consider only $V_\lambda^F(A,p)$. The others can be treated similarly.

Suppose that $X=(X_k), Y=(Y_k)\in V_\lambda^F(A,p)$. By combining Minkowski's inequality with property (2) of the metric d, we derive that

$$d(A_i(X)+A_i(Y),X_o+Y_o)\leq d(A_i(X),X_o)+d(A_i(Y),Y_o).$$

Therefore,

$$d(A_i(X)+A_i(Y),X_o+Y_o)^{p_i}\leq K d(A_i(X),X_o)^{p_i}+K d(A_i(Y),Y_o)^{p_i}$$

where $K=\max(1,\sup_i p_i<\infty)$. This implies that

$$X+Y=(X_k)+(Y_k)\in V_\lambda^F(A,p).$$

Let $X=(X_k)\in V_\lambda^F(A,p)$ and $\alpha\in\mathbb{R}$. Then by taking into account properties (1) and (2) of the metric d,

$$d(A_i(\alpha X),\alpha X_o)^{p_i}\leq|\alpha|^{p_i}\,d(A_i(X),X_o)^{p_i}\leq\max(1,|\alpha|)d(A_i(X),X_o)^{p_i}$$

since $|\alpha|^{p_i}\leq\max(1,|\alpha|)$. Hence $\alpha X\in V_\lambda^F(A,p)$. □

Theorem 2.2. *The space* $V_\lambda^F(A,p)$ *is a complete metric space with the metric*

$$\delta(X,Y)=\sup_{r,i}\left(\frac{1}{\lambda_r}\sum_{i\in I_r}d\big(A_i(X),A_i(Y)\big)^p\right)^{\frac{1}{p}},\quad 1\leq p<\infty.$$

Proof. Let (X^s) be a Cauchy sequence in $V_\lambda^F(A,p)$, where $X^s=(X_k^s)_k=(X_1^s,X_2^s,X_3^s,\dots)\in V_\lambda^F(A,p)$ for each $s\in\mathbb{N}$. Then

$$\delta(X^s,X^t)=\sup_{r,i}\left(\frac{1}{\lambda_r}\sum_{i\in I_r}d\big(A_i(X^s),A_i(X^t)\big)^p\right)^{\frac{1}{p}}\to 0\quad\text{as}\quad s,t\to\infty.$$

Hence

$$d(A_i(X^s), A_i(X^t)) = d\left(\sum_{k=0}^{\infty} a_{ik}X_k^s, \sum_{k=0}^{\infty} a_{ik}X_k^t\right) \\ \to 0 \text{ as } s,t \to \infty, \quad \text{for each } i \in \mathbb{N}.$$

This implies that

$$\sum_{k=0}^{\infty} a_{ik}\left(\lim_{s,t} d(X_k^s, X_k^t)\right) = 0 \quad \text{for each } i \in \mathbb{N}.$$

Hence

$$\lim_{s,t} d(X_k^s, X_k^t) = 0 \quad \text{for each } i \in \mathbb{N}.$$

Therefore $(X^s)_s$ is a Cauchy sequence in $L(\mathbb{R})$. Since $L(\mathbb{R})$ is complete, it is convergent, $\lim_s X_k^s, = X_k$ say, for each $k \in \mathbb{N}$. Since $(X^s)_s$ is a Cauchy sequence, for each $\varepsilon > 0$, there exists $n_o = n_o(\varepsilon)$ such that

$$\delta(X^s, X^t) < \varepsilon \quad \text{for all } s,t \geq n_o.$$

So, we have

$$\lim_{t\to\infty} d(A_i(X^s), A_i(X^t))^p = d(A_i(X^s), A_i(X))^p < \varepsilon^p \quad \text{for all } s \geq n_o.$$

This implies that $\delta(X^s, X^t) < \varepsilon$, for all $s \geq n_o$, that is $X^s \to X$ as $s \to \infty$, where $X = (X_k)$. Since

$$\frac{1}{\lambda_r}\sum_{i\in I_r} d(A_i(X), X_o)^p \leq 2^p \frac{1}{\lambda_r}\sum_{i\in I_r}\{d(A_i(X^{n_o}), X_o)^p + d(A_i(X^{n_o}), A_i(X))^p\} \\ \to 0 \quad \text{as} \quad r \to \infty.$$

So, we obtain $X = (X_k) \in V_\lambda^F(A,p)$. Therefore $V_\lambda^F(A,p)$ is a complete metric space. □

It can be also shown that $V_\lambda^F(A,p)$ is a complete metric space with the metric

$$\delta^1(X,Y) = \sup_{r,i} \frac{1}{\lambda_r}\sum_{i\in I_r} d(A_i(X), A_i(Y))^p, \quad 0 < p < 1.$$

Theorem 2.3. *Let $0 < p_i \leq q_i$ and $(p_i q_i^{-1})$ be bounded. Then*

$$V_\lambda^F(A,q) \subset V_\lambda^F(A,p).$$

Proof. Let $X = (X_k) \in V_\lambda^F(A,q)$ and $w_i = d(A_i(X), X_o)^{q_i}$ and $\gamma_i = p_i q_i^{-1}$ for all $i \in \mathbb{N}$. Then $0 < \gamma_i \leq 1$ for all $i \in \mathbb{N}$. Let $0 < \gamma \leq \gamma_i \leq 1$ for all $i \in \mathbb{N}$. We define the sequences (u_i) and (v_i) as follows: For $w_i \geq 1$, let $u_i = w_i$ and $v_i = 0$ and for $w_i < 1$, let $u_i = 0$ and $v_i = w_i$. Then it is clear that for all $i \in \mathbb{N}$, we have $w_i = u_i + v_i$ and $w_i^{\gamma_i} = u_i^{\gamma_i} + v_i^{\gamma_i}$. Now it follows that $u_i^{\gamma_i} \leq u_i \leq w_i$ and $v_i^{\gamma_i} \leq v_i^{\gamma}$. Therefore

$$\begin{aligned} \frac{1}{\lambda_r} \sum_{i \in I_r} w_i^{\gamma_i} &= \frac{1}{\lambda_r} \sum_{i \in I_r} (u_i + v_i)^{\gamma_i} \\ &\leq \frac{1}{\lambda_r} \sum_{i \in I_r} w_i + \frac{1}{\lambda_r} \sum_{i \in I_r} v_i^{\gamma}. \end{aligned}$$

Now for each r,

$$\begin{aligned} \frac{1}{\lambda_r} \sum_{i \in I_r} v_i^{\gamma} &= \sum_{i \in I_r} (\lambda_r^{-1} v_i^{\gamma})^{\gamma} (\lambda_r^{-1})^{1-\gamma} \\ &\leq \left[\sum_{i \in I_r} ((\lambda_r^{-1} v_i^{\gamma})^{\gamma})^{\frac{1}{\gamma}} \right] \left[\sum_{i \in I_r} ((\lambda_r^{-1}) 1-^{\gamma})^{\frac{1}{1-\gamma}} \right]^{1-\gamma} \\ &= \left(\frac{1}{\lambda_r} \sum_{i \in I_r} v_i \right)^{\gamma} \end{aligned}$$

and so

$$\frac{1}{\lambda_r} \sum_{i \in I_r} w_i^{\gamma_i} \leq \frac{1}{\lambda_r} \sum_{i \in I_r} w_i + \left(\frac{1}{\lambda_r} \sum_{i \in I_r} v_i \right)^{\gamma}.$$

Hence $X = (X_k) \in V_\lambda^F(A,p)$, i.e., $V_\lambda^F(A,q) \subset V_\lambda^F(A,p)$. □

3. Statistical convergence

A sequence $X = (X_k)$ of fuzzy numbers is said to be statistically convergent to a fuzzy number X_o if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left| \{ k \leq n : d(X_k, X_o) \ \geq \varepsilon \} \right| = 0.$$

The set of all statistically convergent sequences of fuzzy numbers is denoted by S^F. We note that if a sequence $X = (X_k)$ of fuzzy numbers converges to a fuzzy number X_o, then it is statistically converges to X_o. But the converse statement is not necessarily valid.

Definition 3.1. A sequence $X = (X_k)$ of fuzzy numbers is said to be $S_\lambda^F(A)$-statistically convergent to a fuzzy number X_o if for every $\varepsilon > 0$,

$$\lim_r \frac{1}{\lambda_r} \left| \{ i \in I_r : d(A_i(X), X_o) \geq \varepsilon \} \right| = 0.$$

The set of all $S_\lambda^F(A)$-statistically convergent sequences of fuzzy numbers is denoted by $S_\lambda^F(A)$. In this case we write $X_k \to X_o$ $(S_\lambda^F(A))$. In the case $A = I$ identity matrix and $\lambda_r = r$, we obtain ordinary statistically convergent sequences of fuzzy numbers S^F, which was defined and studied by Nuray and Savas [11]. In the case $A = I$ identity matrix, we obtain λ-statistically convergent sequences of fuzzy numbers S_λ^F, which was defined and studied by Savaş [14]. In the case $\lambda_r = r$, we obtain $S^F(A)$-statistically convergent sequences of fuzzy numbers. The set of all $S^F(A)$-statistically convergent sequences of fuzzy numbers is denoted by $S^F(A)$.

Now we give the relations between $S^F_\lambda(A)$-statistical convergence and strongly $V^F_\lambda(A,p)$-convergence.

Theorem 3.1. *The following statement are valid:*

(a) $V^F_\lambda(A,p) \subset S^F_\lambda(A)$,
(b) *If* $X = (X_k) \in l^F_\infty \cap S^F_\lambda(A)$, *then* $X = (X_k) \in V^F_\lambda(A,p)$,
(c) $l^F_\infty(A) \cap S^F_\lambda(A) = l^F_\infty(A) \cap V^F_\lambda(A,p)$, *where*

$$l^F_\infty(A) = \left\{ X = (X_k) \in w^F : \sup_i d(A_i(X), X_o) < \infty \right\}.$$

Proof. (a) Let $\varepsilon > 0$ and $X = (X_k) \in V^F_\lambda(A,p)$. Then we have

$$\frac{1}{\lambda_r}\sum_{i\in I_r} d(A_i(X), X_o)^{p_i} \geq \frac{1}{\lambda_r}|\{i \in I_r : d(A_i(X), X_o) \geq \varepsilon\}| \cdot \min(\varepsilon^h, \varepsilon^H).$$

Hence $X = (X_k) \in S^F_\lambda(A)$.

(b) Suppose that $X = (X_k) \in l^F_\infty \cap S^F_\lambda(A)$. Since $X = (X_k) \in l^F_\infty$, we can write $d(A_i(X), X_o) \leq T$ for all $i \in \mathbb{N}$. Given $\varepsilon > 0$, let

$$G_r = \{i \in I_r : d(A_i(X), X_o) \geq \varepsilon\} \quad \text{and} \quad H_r = \{i \in I_r : d(A_i(X), X_o) < \varepsilon\}.$$

Then we have

$$\begin{aligned}\frac{1}{\lambda_r}\sum_{i\in I_r} d(A_i(X), X_o)^{p_i} &= \frac{1}{\lambda_r}\sum_{i\in G_r} d(A_i(X), X_o)^{p_i} + \frac{1}{\lambda_r}\sum_{i\in H_r} d(A_i(X), X_o)^{p_i} \\ &\leq \max(T^h, T^H)\frac{1}{\lambda_r}|G_r| + \max(\varepsilon^h, \varepsilon^H).\end{aligned}$$

Taking the limit as $\varepsilon \to 0$ and $r \to \infty$, it follows that $X = (X_k) \in V^F_\lambda(A,p)$.

(c) Follows from (a) and (b). □

Theorem 3.2. *If* $\liminf_r \lambda_r r^{-1} > 0$, $S^F(A) \subset S^F_\lambda(A)$.

Proof. Let $X = (X_k) \in S^F(A)$.For given $\varepsilon > 0$, we get

$$\{i \leq r : d(A_i(X), X_o) \geq \varepsilon\} \supset G_r$$

where G_r is the same as in Theorem 3.1. Thus,

$$\frac{1}{r}|\{i \leq r : d(A_i(X), X_o) \geq \varepsilon\}| \geq \frac{1}{r}|G_r| = \lambda_r r^{-1}\lambda_r^{-1}|G_r|.$$

Taking limit as $r \to \infty$ and using $\liminf_r \lambda_r r^{-1} > 0$, we get

$$X = (X_k) \in S^F_\lambda(A).$$ □

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