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C^*-algebra of the $\mathbb{Z}^n$-tree

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Abstract. Let $\Lambda = \mathbb{Z}^n$ with lexicographic ordering. Λ is a totally ordered group. Let $X = \Lambda^+ * \Lambda^+$. Then X is a Λ-tree. Analogous to the construction of graph C^*-algebras, we form a groupoid whose unit space is the space of ends of the tree. The C^*-algebra of the Λ-tree is defined as the C^*-algebra of this groupoid. We prove some properties of this C^*-algebra.

Contents

1. Introduction

Since the introduction of C^*-algebras of groupoids, in the late 1970's, several classes of C^*-algebras have been given groupoid models. One such class is the class of graph C^*-algebras.

In their paper [10], Kumjian, Pask, Raeburn and Renault associated to each locally finite directed graph E a locally compact groupoid $\mathcal{G}$, and showed that its groupoid C^*-algebra $C^*(\mathcal{G})$ is the universal C^*-algebra generated by families of partial isometries satisfying the Cuntz–Krieger relations determined by E. In [16], Spielberg constructed a locally compact groupoid $\mathcal{G}$ associated to a general graph E and generalized the result to a general directed graph.

We refer to [13] for the detailed theory of topological groupoids and their C^*-algebras.

A directed graph $E = (E^0, E^1, o, t)$ consists of a countable set E^0 of vertices and E^1 of edges, and maps $o, t : E^1 \to E^0$ identifying the origin

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(source) and the terminus (range) of each edge. For the purposes of this discussion it is sufficient to consider row-finite graphs with no sinks.

For the moment, let T be a bundle of of row-finite directed trees with no sinks, that is a disjoint union of trees that have no sinks or infinite emitters, i.e., no singular vertices. We denote the set of finite paths of T by T^* and the set of infinite paths by ∂T.

For each $p \in T^*$, define

$$V(p) := \{px : x \in \partial T,\ t(p) = o(x)\}.$$

For $p, q \in T^*$, we see that:

$$V(p) \cap V(q) = \begin{cases} V(p) & \text{if } p = qr \text{ for some } r \in T^* \\ V(q) & \text{if } q = pr \text{ for some } r \in T^* \\ \emptyset & \text{otherwise.} \end{cases}$$

It is fairly easy to see that:

Lemma 1.1. *The cylinder sets $\{V(p) : p \in T^*\}$ form a base of compact open sets for a locally compact, totally disconnected, Hausdorff topology of ∂T.*

We want to define a groupoid that has ∂T as a unit space. For $x = x_1x_2\ldots$, and $y = y_1y_2\ldots \in \partial T$, we say *$x$ is shift equivalent to y with lag $k \in \mathbb{Z}$* and write $x \sim_k y$, if there exists $n \in \mathbb{N}$ such that $x_i = y_{k+i}$ for each $i \geq n$. It is not difficult to see that shift equivalence is an equivalence relation.

Definition 1.2. Let $\mathcal{G} := \{(x, k, y) \in \partial T \times \mathbb{Z} \times \partial T : x \sim_k y\}$. For pairs in $\mathcal{G}^2 := \{((x, k, y), (y, m, z)) : (x, k, y), (y, m, z) \in \mathcal{G}\}$, we define

$$(x, k, y) \cdot (y, m, z) = (x, k + m, z). \tag{1.1}$$

For arbitrary $(x, k, y) \in \mathcal{G}$, we define

$$(x, k, y)^{-1} = (y, -k, x). \tag{1.2}$$

With the operations (1.1) and (1.2), and source and range maps $s, r : \mathcal{G} \longrightarrow \partial T$ given by $s(x, k, y) = y,\ r(x, k, y) = x,\ \mathcal{G}$ is a groupoid with unit space ∂T.

For $p, q \in T^*$, with $t(p) = t(q)$, define $U(p, q) := \{px, l(p) - l(q), qx) : x \in \partial T,\ t(p) = o(x)\}$, where $l(p)$ denotes the length of the path p. The sets $\{U(p, q) : p, q \in T^*, t(p) = t(q)\}$ make $\mathcal{G}$ a locally compact r-discrete groupoid with (topological) unit space equal to ∂T.

Now let E be a directed graph. We form a graph whose vertices are the paths of E and edges are (ordered) pairs of paths as follows:

Definition 1.3. Let $\widetilde{E}$ denote the following graph:

$$\widetilde{E}^0 = E^*$$
$$\widetilde{E}^1 = \{(p, q) \in E^* \times E^* : q = pe \quad \text{for some} \quad e \in E^1\}$$
$$o(p, q) = p,\ t(p, q) = q.$$

The following lemma, due to Spielberg [16], is straightforward.

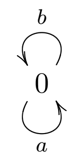
Lemma 1.4. [16, Lemma 2.4] *$\widetilde{E}$ is a bundle of trees.*

Notice that if E is a row-finite graph with no sinks, then $\widetilde{E}$ is a bundle of row-finite trees with no sinks.

If $\mathcal{G}(E)$ is the groupoid obtained as in Definition 1.2, where $\widetilde{E}$ plays the role of T then, in [16] Spielberg showed, in its full generality, that the graph C^*-algebra of E is equal to the C^*-algebra of the groupoid $\mathcal{G}(E)$. We refer to [16], for readers interested in the general construction and the proof.

We now examine the C^*-algebra $\mathcal{O}_2$, which is the C^*-algebra of the graph

$$E \qquad =$$



Denoting the vertex of E by 0 and the edges of E by a and b, as shown in the graph, the vertices of $\widetilde{E}$ are $0, a, b, aa, ab, ba, bb$, etc. And the graph $\widetilde{E}$ is the binary tree.

Take a typical path p of E, say $p = aaabbbbbabbaaaa$. Writing aaa as 3 and $bbbbb$ as $5'$, etc. we can write p as $35'12'4$ which is an element of $\mathbb{Z}^+ * \mathbb{Z}^+$ (the free product of two copies of $\mathbb{Z}^+$). In other words, the set of vertices of $\widetilde{E}$ is $G_1^+ * G_2^+$, where $G_1^+ = \mathbb{Z}^+ = G_2^+$, and the vertex 0 is the empty word. The elements of $\partial\widetilde{E}$ are the infinite sequence of n's and m's, where $n \in G_1^+$ and $m \in G_2^+$.

Motivated by this construction, we wish to explore the C^*-algebra of the case when Λ is an ordered abelian group, and X is the free product of two copies of Λ^+. In this paper we study the special case when $\Lambda = \mathbb{Z}^n$ endowed with the lexicographic ordering, where $n \in \{2,\ 3,\ \ldots\}$.

The paper is organized as follows. In Section 2 we develop the topology of the $\mathbb{Z}^n$-tree. In Section 3 we build the C^*-algebra of the $\mathbb{Z}^n$-tree by first building the groupoid $\mathcal{G}$ in a fashion similar to that of the graph groupoid. In Section 4, by explicitly exploring the partial isometries generating the C^*-algebra, we give a detailed description of the C^*-algebra. In Section 5, we look at the crossed product of the C^*-algebra by the gauge action and study the fixed-point algebra. Finally in Section 6 we provide classification of the C^*-algebra. We prove that the C^*-algebra is simple, purely infinite, nuclear and classifiable.

I am deeply indebted to Jack Spielberg without whom none of this would have been possible. I also wish to thank Mark Tomforde for many helpful discussions and for providing material when I could not find them otherwise.

2. The $\mathbb{Z}^n$-tree and its boundary

Let $n \in \{2, 3, \ldots\}$ and let $\Lambda = \mathbb{Z}^n$ together with lexicographic ordering, that is, $(k_1, k_2, \ldots, k_n) < (m_1, m_2, \ldots, m_n)$ if either $k_1 < m_1$, or $k_1 = m_1$,

..., $k_{d-1} = m_{d-1}$, and $k_d < m_d$. We set

$$\partial\Lambda^+ = \{(k_1, k_2, \ldots, k_{n-1}, \infty) : k_i \in \mathbb{N} \cup \{\infty\},\ k_i = \infty \Rightarrow k_{i+1} = \infty\}.$$

Let $G_i = \Lambda^+ = \{a \in \Lambda : a > 0\}$ for $i = 1,\ 2$, and let $\partial G_i = \partial\Lambda^+$ for $i = 1,\ 2$. That is, we take two copies of Λ^+ and label them as G_1 and G_2, and two copies of $\partial\Lambda^+$ and label them as ∂G_1 and ∂G_2. Now consider the set $X = G_1 * G_2$. We denote the empty word by 0. Thus, $X = \bigcup_{d=1}^{\infty}\{a_1a_2\ldots a_d : a_i \in G_k \Rightarrow a_{i+1} \in G_{k\pm1} \text{ for } 1 \leq i < d\} \bigcup \{0\}$. We note that X is a Λ-tree, as studied in [4].

Let $\partial X = \{a_1a_2\ldots a_d : a_i \in G_k \Rightarrow a_{i+1} \in G_{k\pm1}, \text{ for } 1 \leq i < d-1 \text{ and } a_{d-1} \in G_k \Rightarrow a_d \in \partial G_{k\pm1}\} \bigcup \{a_1a_2\ldots : a_i \in G_k \Rightarrow a_{i+1} \in G_{k\pm1} \text{ for each } i\}$. In words, ∂X contains either a finite sequence of elements of Λ from sets with alternating indices, where the last element is from $\partial\Lambda^+$, or an infinite sequence of elements of Λ from sets with alternating indices. For $a \in \Lambda^+$ and $b \in \partial\Lambda^+$, define $a + b \in \partial\Lambda^+$ by componentwise addition.

For $p = a_1a_2\ldots a_k \in X$ and $q = b_1b_2\ldots b_m \in X \cup \partial X$, i.e., $m \in \mathbb{N} \cup \{\infty\}$, define pq as follows:

(i) If $a_k, b_1 \in G_i \cup \partial G_i$ (i.e., they belong to sets with the same index), then $pq := a_1a_2\ldots a_{k-1}(a_k + b_1)b_2\ldots b_m$. Observe that since $a_k \in \Lambda$, the sum $a_k + b_1$ is defined and is in the same set as b_1.

(ii) If a_k and b_1 belong to sets with different indices, then
$$pq := a_1a_2\ldots a_kb_1b_2\ldots b_m.$$

In other words, we concatenate p and q in the most natural way (using the group law in $\Lambda * \Lambda$).

For $p \in X$ and $q \in X \cup \partial X$, we write $p \preceq q$ to mean q extends p, i.e., there exists $r \in X \cup \partial X$ such that $q = pr$.

For $p \in \partial X$ and $q \in X \cup \partial X$, we write $p \preceq q$ to mean q extends p, i.e., for each $r \in X$, $r \preceq p$ implies that $r \preceq q$.

We now define two length functions. Define $l : X \cup \partial X \longrightarrow (\mathbb{N} \cup \{\infty\})^n$ by $l(a_1a_2\ldots a_k) := \sum_{i=1}^{k} a_i$.

And define $l_i : X \cup \partial X \longrightarrow \mathbb{N} \cup \{\infty\}$ to be the i^{th} component of l, i.e., $l_i(p)$ is the i^{th} component of $l(p)$. It is easy to see that both l and l_i are additive.

Next, we define basic open sets of ∂X. For $p, q \in X$, we define

$$V(p) := \{px : x \in \partial X\} \quad \text{and} \quad V(p; q) := V(p) \setminus V(q).$$

Notice that

$$V(p) \cap V(q) = \begin{cases} \emptyset & \text{if } p \npreceq q \text{ and } q \npreceq p \\ V(p) & \text{if } q \preceq p \\ V(q) & \text{if } p \preceq q. \end{cases} \tag{2.1}$$

Hence

$$V(p) \setminus V(q) = \begin{cases} V(p) & \text{if } p \npreceq q \text{ and } q \npreceq p \\ \emptyset & \text{if } q \preceq p. \end{cases}$$

Therefore, we will assume that $p \preceq q$ whenever we write $V(p;q)$.
Let $\mathcal{E} := \{V(p) : p \in X\} \bigcup \{V(p;q) : p, q \in X\}$.

Lemma 2.1. *$\mathcal{E}$ separates points of ∂X, that is, if $x, y \in \partial X$ and $x \neq y$ then there exist two sets $A, B \in \mathcal{E}$ such that $x \in A$, $y \in B$, and $A \cap B = \emptyset$.*

Proof. Suppose $x, y \in \partial X$ and $x \neq y$. Let $x = a_1a_2 \dots a_s$, $y = b_1b_2 \dots b_m$. Assume, without loss of generality, that $s \leq m$. We consider two cases:

Case I. There exists $k < s$ such that $a_k \neq b_k$ (or they belong to different G_i's).

Then $x \in V(a_1a_2 \dots a_k)$, $y \in V(b_1b_2 \dots b_k)$ and

$$V(a_1a_2 \dots a_k) \cap V(b_1b_2 \dots b_k) = \emptyset.$$

Case II. $a_i = b_i$ for each $i < s$.

Notice that if $s = \infty$, that is, if both x and y are infinite sequences then there should be a $k \in \mathbb{N}$ such that $a_k \neq b_k$ which was considered in Case I. Hence $s < \infty$. Again, we distinguish two subcases:

(a) $s = m$. Therefore $x = a_1a_2 \dots a_s$ and $y = a_1a_2 \dots b_s$, and $a_s, b_s \in \partial G_i$, with $a_s \neq b_s$. Assuming, without loss of generality, that $a_s < b_s$, let $a_s = (k_1, k_2, \dots, k_{n-1}, \infty)$, and $b_s = (r_1, r_2, \dots, r_{n-1}, \infty)$ where $(k_1, k_2, \dots, k_{n-1}) < (r_1, r_2, \dots, r_{n-1})$. Therefore there must be an index i such that $k_i < r_i$; let j be the largest such. Hence $a_s + e_j \leq b_s$, where e_j is the n-tuple with 1 at the j^{th} spot and 0 elsewhere. Letting $c = a_s + e_j$, we see that $x \in A = V(a_1a_2 \dots a_{s-1}; c)$, $y \in B = V(c)$, and $A \cap B = \emptyset$.

(b) $s < m$. Then $y = a_1a_2 \dots a_{s-1}b_sb_{s+1} \dots b_m$ $(m \geq s+1)$.
Since $b_{s+1} \in (G_i \cup \partial G_i) \setminus \{0\}$ for $i = 1, 2$, choose $c = e_n \in G_i$ (same index as b_{s+1} is in). Then $x \in A = V(a_1a_2 \dots a_{s-1}; a_1a_2 \dots a_{s-1}b_sc)$, $y \in B = V(a_1a_2 \dots a_{s-1}b_sc)$, and $A \cap B = \emptyset$.

This completes the proof. □

Lemma 2.2. *$\mathcal{E}$ forms a base of compact open sets for a locally compact Hausdorff topology on ∂X.*

Proof. First we prove that $\mathcal{E}$ forms a base. Let $A = V(p_1; p_2)$ and $B = V(q_1; q_2)$. Notice that if $p_1 \npreceq q_1$ and $q_1 \npreceq p_1$ then $A \cap B = \emptyset$. Suppose, without loss of generality, that $p_1 \preceq q_1$ and let $x \in A \cap B$. Then by construction, $p_1 \preceq q_1 \preceq x$ and $p_2 \npreceq x$ and $q_2 \npreceq x$. Since $p_2 \npreceq x$ and $q_2 \npreceq x$, we can choose $r \in X$ such that $q_1 \preceq r$, $p_2 \npreceq r$, $q_2 \npreceq r$, and $x = ra$ for some $a \in \partial X$. If $x \npreceq p_2$ and $x \npreceq q_2$ then r can be chosen so that $r \npreceq p_2$ and $r \npreceq q_2$, hence $x \in V(r) \subseteq A \cap B$.

Suppose now that $x \preceq p_2$. Then $x = ra$, for some $r \in X$ and $a \in \partial X$. By extending r if necessary, we may assume that $a \in \partial \Lambda^+$. Then we may write $p_2 = rby$ for some $b \in \Lambda^+$, and $y \in \partial X$ with $a < b$. Let $b' = b - (0, \dots, 0, 1)$, and $s_1 = rb'$. Notice that $x \preceq s_1 \preceq p_2$ and $s_1 \neq p_2$. If $x \npreceq q_2$ then we can choose r so that $r \not\preceq q_2$. Therefore $x \in V(r; s_1) \subseteq A \cap B$. If $x \preceq q_2$, construct s_2 the way as s_1 was constructed, where q_2 takes the place of p_2. Then either $s_1 \preceq s_2$ or $s_2 \preceq s_1$. Set

$$s = \begin{cases} s_1 & \text{if } s_1 \preceq s_2 \\ s_2 & \text{if } s_2 \preceq s_1. \end{cases}$$

Then $x \in V(r; s) \subseteq A \cap B$. The cases when A or B is of the form $V(p)$ are similar, in fact easier.

That the topology is Hausdorff follows from the fact that $\mathcal{E}$ separates points.

Next we prove local compactness. Given $p, q \in X$ we need to prove that $V(p; q)$ is compact. Since $V(p; q) = V(p) \setminus V(q)$ is a (relatively) closed subset of $V(p)$, it suffices to show that $V(p)$ is compact. Let $A_0 = V(p)$ be covered by an open cover $\mathcal{U}$ and suppose that A_0 does not admit a finite subcover. Choose $p_1 \in X$ such that $l_i(p_1) \geq 1$ and $V(pp_1)$ does not admit a finite subcover, for some $i \in \{1, \ldots, n-1\}$. We consider two cases:

Case I. Suppose no such p_1 exists.

Let $a = e_n \in G_1$, $b = e_n \in G_2$. Then $V(p) = V(pa) \cup V(pb)$. Hence either $V(pa)$ or $V(pb)$ is not finitely covered, say $V(pa)$, then let $x_1 = a$. After choosing x_s, since $V(px_1 \ldots x_s) = V(px_1 \ldots x_s a) \cup V(px_1 \ldots x_s b)$, either $V(px_1 \ldots x_s a)$ or $V(px_1 \ldots x_s b)$ is not finitely covered. And we let $x_{s+1} = a$ or b accordingly. Now let $A_j = V(px_1 \ldots x_j)$ for $j \geq 1$ and let $x = px_1x_2 \ldots \in \partial X$. Notice that $A_0 \supseteq A_1 \supseteq A_2 \ldots$, and $x \in \bigcap_{j=0}^{\infty} A_j$. Choose $A' \in \mathcal{U}$, $q, r \in X$, such that $x \in V(q; r) \subseteq A'$. Clearly $q \preceq x$ and $r \not\preceq x$. Once again, we distinguish two subcases:

(a) $x \not\preceq r$. Then, for a large enough k we get $q \preceq px_1x_2 \ldots x_k$ and $px_1x_2 \ldots x_k \not\preceq r$. Therefore $A_k = V(px_1x_2 \ldots x_k) \subseteq A'$, which contradicts to that A_k is not finitely covered.

(b) $x \preceq r$. Notice $l_1(x) = l_1(p)$ and since $x = px_1x_2 \ldots \preceq r$, we have $l_1(x) = l_1(p) < l_1(r)$. Therefore $V(r)$ is finitely covered, say by $B_1, B_2, \ldots, B_s \in \mathcal{U}$. For large enough k, $q \preceq px_1x_2 \ldots x_k$. Therefore $A_k = V(px_1x_2 \ldots x_k) \subseteq V(q) = V(q; r) \cup V(r) \subseteq A' \cup \bigcup_{j=1}^{n} B_j$, which is a finite union. This is a contradiction.

Case II. Let $p_1 \in X$ such that $l_i(p_1) \geq 1$ and $V(pp_1)$ is not finitely covered, for some $i \in \{1, \ldots, n-1\}$.

Having chosen $p_1, \ldots, p_s$ let p_{s+1} with $l_i(p_{s+1}) \geq 1$ and $V(pp_1 \ldots p_{s+1})$ not finitely covered, for some $i \in \{1, \ldots, n-1\}$. If no such p_{s+1} exists then we are back in to Case I with $V(pp_1p_2 \ldots p_s)$ playing the role of $V(p)$. Now let $x = pp_1p_2 \ldots \in \partial X$ and let $A_j = V(pp_1 \ldots p_j)$. We get $A_0 \supseteq A_1 \supseteq \ldots$, and $x = pp_1p_2 \ldots \in \bigcap_{j=0}^{\infty} A_j$. Choose $A' \in \mathcal{U}$ such that $x \in V(q; r) \subseteq A'$. Notice that $q \preceq x$ and $n-1$ is finite, hence there exists $i_0 \in \{1, \ldots, n-1\}$ such that $l_{i_0}(x) = \infty$. Since $l_{i_0}(r) < \infty$, we have $x \not\preceq r$. Therefore, for large enough k, $q \preceq pp_1 \ldots p_k \not\preceq r$, implying $A_k \subseteq A'$, a contradiction.

Therefore $V(p)$ is compact. □

3. The groupoid and C^*-algebra of the $\mathbb{Z}^n$-tree

We are now ready to form the groupoid which will eventually be used to construct the C^*-algebra of the Λ-tree.

For $x, y \in \partial X$ and $k \in \Lambda$, we write $x \sim_k y$ if there exist $p, q \in X$ and $z \in \partial X$ such that $k = l(p) - l(q)$ and $x = pz, y = qz$.

Notice that:

(a) If $x \sim_k y$ then $y \sim_{-k} x$.

(b) $x \sim_0 x$.

(c) If $x \sim_k y$ and $y \sim_m z$ then $x = \mu t,\ y = \nu t,\ y = \eta s,\ z = \beta s$ for some $\mu, \nu, \eta, \beta \in X$ $t, s \in \partial X$ and $k = l(\mu) - l(\nu),\ m = l(\eta) - l(\beta)$.

If $l(\eta) \leq l(\nu)$ then $\nu = \eta\delta$ for some $\delta \in X$. Therefore $y = \eta\delta t$, implying $s = \delta t$, hence $z = \beta\delta t$. Therefore $x \sim_r z$, where $r = l(\mu) - l(\beta\delta) = l(\mu) - l(\beta) - l(\delta) = l(\mu) - l(\beta) - (l(\nu) - l(\eta)) = [l(\mu) - l(\nu)] + [l(\eta) - l(\beta)] = k + m$.

Similarly, if $l(\eta) \geq l(\nu)$ we get $x \sim_r z$, where $r = k + m$.

Definition 3.1. Let $\mathcal{G} := \{(x, k, y) \in \partial X \times \Lambda \times \partial X : x \sim_k y\}$.

For pairs in $\mathcal{G}^2 := \{((x, k, y), (y, m, z)) : (x, k, y), (y, m, z) \in \mathcal{G}\}$, we define

$$(x, k, y) \cdot (y, m, z) = (x, k + m, z). \tag{3.1}$$

For arbitrary $(x, k, y) \in \mathcal{G}$, we define

$$(x, k, y)^{-1} = (y, -k, x). \tag{3.2}$$

With the operations (3.1) and (3.2), and source and range maps $s, r : \mathcal{G} \longrightarrow \partial X$ given by $s(x, k, y) = y,\ r(x, k, y) = x,\ \mathcal{G}$ is a groupoid with unit space ∂X.

We want to make $\mathcal{G}$ a locally compact r-discrete groupoid with (topological) unit space ∂X.

For $p, q \in X$ and $A \in \mathcal{E}$, define $[p, q]_A = \{(px, l(p) - l(q), qx) : x \in A\}$.

Lemma 3.2. *For $p, q, r, s \in X$ and $A, B \in \mathcal{E}$,*

$$[p, q]_A \cap [r, s]_B = \begin{cases} [p, q]_{A \cap \mu B} & \text{if there exists } \mu \in X \text{ such that } r = p\mu,\ s = q\mu \\ [r, s]_{(\mu A) \cap B} & \text{if there exists } \mu \in X \text{ such that } p = r\mu,\ q = s\mu \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. Let $t \in [p, q]_A \cap [r, s]_B$. Then $t = (px, k, qx) = (ry, m, sy)$ for some $x \in A, y \in B$. Clearly $k = m$. Furthermore, $px = ry$ and $qx = sy$. Suppose that $l(p) \leq l(r)$. Then $r = p\mu$ for some $\mu \in X$, hence $px = p\mu y$, implying $x = \mu y$. Hence $qx = q\mu y = sy$, implying $q\mu = s$. Therefore $t = (px, k, qx) = (p\mu y, k, q\mu y)$, that is, $t = (px, k, qx)$ for some $x \in A \cap \mu B$.

The case when $l(r) \leq l(p)$ follows by symmetry. The reverse containment is clear. $\square$

Proposition 3.3. *Let $\mathcal{G}$ have the relative topology inherited from $\partial X \times \Lambda \times \partial X$. Then $\mathcal{G}$ is a locally compact Housdorff groupoid, with base $\mathcal{D} = \{[a,b]_A : a,b \in X, A \in \mathcal{E}\}$ consisting of compact open subsets.*

Proof. That $\mathcal{D}$ is a base follows from Lemma 3.2. $[a,b]_A$ is a closed subset of $aA \times \{l(a)-l(b)\} \times bA$, which is a compact open subset of $\partial X \times \Lambda \times \partial X$. Hence $[a,b]_A$ is compact open in $\mathcal{G}$.

To prove that inversion is continuous, let $\phi : \mathcal{G} \longrightarrow \mathcal{G}$ be the inversion function. Then $\phi^{-1}([a,b]_A) = [b,a]_A$. Therefore ϕ is continuous. In fact ϕ is a homeomorphism.

For the product function, let $\psi : \mathcal{G}^2 \longrightarrow \mathcal{G}$ be the product function. Then $\psi^{-1}([a,b]_A) = \bigcup_{c \in X}(([a,c]_A \times [c,b]_A) \cap \mathcal{G}^2)$ which is open (is a union of open sets). $\square$

Remark 3.4. We remark the following points:

(a) Since the set $\mathcal{D}$ is countable, the topology is second countable.

(b) We can identify the unit space, ∂X, of $\mathcal{G}$ with the subset $\{(x,0,x) : x \in \partial X\}$ of $\mathcal{G}$ via $x \mapsto (x,0,x)$. The topology on ∂X agrees with the topology it inherits by viewing it as the subset $\{(x,0,x) : x \in \partial X\}$ of $\mathcal{G}$.

Proposition 3.5. *For each $A \in \mathcal{E}$ and each $a,b \in X$, $[a,b]_A$ is a $\mathcal{G}$-set. $\mathcal{G}$ is r-discrete.*

Proof.

$$
\begin{aligned}
[a,b]_A &= \{(ax, l(a)-l(b), bx) : x \in A\} \\
\Rightarrow ([a,b]_A)^{-1} &= \{(bx, l(b)-l(a), ax) : x \in A\}.
\end{aligned}
$$

Hence, $((ax,l(a)-l(b),bx)(by,l(b)-l(a),ay)) \in [a,b]_A \times ([a,b]_A)^{-1} \bigcap \mathcal{G}^2$ if and only if $x = y$. And in that case, $(ax,l(a)-l(b),bx)\cdot(bx,l(b)-l(a),ax) = (ax,0,ax) \in \partial X$, via the identification stated in Remark 3.4(b). This gives $[a,b]_A \cdot ([a,b]_A)^{-1} \subseteq \partial X$. Similarly, $([a,b]_A)^{-1} \cdot [a,b]_A \subseteq \partial X$. Therefore $\mathcal{G}$ has a base of compact open $\mathcal{G}$-sets, implying $\mathcal{G}$ is r-discrete. $\square$

Define $C^*(\Lambda)$ to be the C^* algebra of the groupoid $\mathcal{G}$. Thus $C^*(\Lambda) = \overline{\operatorname{span}}\{\chi_S : S \in \mathcal{D}\}$.

For $A = V(p) \in \mathcal{E}$,

$$
\begin{aligned}
[a,b]_A = [a,b]_{V(p)} &= \{(ax, l(a)-l(b), bx) : x \in V(p)\} \\
&= \{(ax, l(a)-l(b), bx) : x = pt,\ t \in \partial X\} \\
&= \{(apt, l(a)-l(b), bpt) : t \in \partial X\} \\
&= [ap, bp]_{\partial X}.
\end{aligned}
$$

And for $A = V(p;q) = V(p) \setminus V(q) \in \mathcal{E}$,

$$\begin{aligned}
[a,b]_A &= \{(ax, l(a)-l(b), bx) : x \in V(p) \setminus V(q)\} \\
&= \{(ax, l(a)-l(b), bx) : x \in V(p)\} \setminus \{(ax, l(a)-l(b), bx) : x \in V(q)\} \\
&= [ap, bp]_{\partial X} \setminus [aq, bq]_{\partial X}.
\end{aligned}$$

Denoting $[a,b]_{\partial X}$ by $U(a,b)$ we get:

$$\mathcal{D} = \{U(a,b) : a, b \in X\} \bigcup \{U(a,b) \setminus U(c,d) : a,b,c,d \in X, a \preceq c, b \preceq d\}.$$

Moreover $\chi_{U(a,b)\setminus U(c,d)} = \chi_{U(a,b)} - \chi_{U(c,d)}$, whenever $a \preceq c, b \preceq d$. This gives us:

$$C^*(\Lambda) = \overline{\text{span}}\{\chi_{U(a,b)} : a, b \in X\}.$$

4. Generators and relations

For $p \in X$, let $s_p = \chi_{U(p,0)}$, where 0 is the empty word. Then:

$$\begin{aligned}
s_p^*(x,k,y) &= \overline{\chi_{U(p,0)}((x,k,y)^{-1})} \\
&= \chi_{U(p,0)}(y, -k, x) \\
&= \chi_{U(0,p)}(x,k,y).
\end{aligned}$$

Hence $s_p^* = \chi_{U(0,p)}$.

And for $p, q \in X$,

$$\begin{aligned}
s_p s_q(x,k,y) &= \sum_{y \sim_m z} \chi_{U(p,0)}((x,k,y)(y,m,z))\ \chi_{U(q,0)}((y,m,z)^{-1}) \\
&= \sum_{y \sim_m z} \chi_{U(p,0)}(x, k+m, z)\ \chi_{U(q,0)}(z, -m, y).
\end{aligned}$$

Each term in this sum is zero except when $x = pz$, with $k + m = l(p)$, and $z = qy$, with $l(q) = -m$. Hence, $k = l(p) - m = l(p) + l(q)$, and $x = pz = pqy$. Therefore $s_p s_q(x,k,y) = \chi_{U(pq,0)}(x,k,y)$; that is, $s_p s_q = \chi_{U(pq,0)} = s_{pq}$.

Moreover,

$$\begin{aligned}
s_p s_q^*(x,k,y) &= \sum_{y \sim_m z} \chi_{U(p,0)}((x,k,y)(y,m,z))\ \chi_{U(0,q)}((y,m,z)^{-1}) \\
&= \sum_{y \sim_m z} \chi_{U(p,0)}(x, k+m, z)\ \chi_{U(0,q)}(z, -m, y) \\
&= \sum_{y \sim_m z} \chi_{U(p,0)}(x, k+m, z)\ \chi_{U(q,0)}(y, m, z).
\end{aligned}$$

Each term in this sum is zero except when $x = pz,\ k + m = l(p),\ y = qz$, and $l(q) = m$. That is, $k = l(p) - l(q)$, and $x = pz,\ y = qz$. Therefore $s_p s_q^*(x,k,y) = \chi_{U(p,q)}(x,k,y)$; that is, $s_p s_q^* = \chi_{U(p,q)}$.

Notice also that

$$s_p^* s_q(x,k,y) = \sum_{y \sim_m z} \chi_{U(0,p)}((x,k,y)(y,m,z))\ \chi_{U(q,0)}((y,m,z)^{-1})$$

$$= \sum_{y\sim_m z} \chi_{U(0,p)}(x, k+m, z)\ \chi_{U(q,0)}(z, -m, y).$$

is nonzero exactly when $z = px,\ l(p) = -(k+m),\ z = qy$, and $l(q) = -m$, which implies that $px = qy,\ l(p) = -k - m = -k + l(q)$. This implies that $s_p^* s_q$ is nonzero only if either $p \preceq q$ or $q \preceq p$.

If $p \preceq q$ then there exists $r \in X$ such that $q = pr$. But $-k = l(p) - l(q) \Rightarrow k = l(q) - l(p) = l(r)$. And $qy = pry \Rightarrow x = ry$. Therefore $s_p^* s_q = s_r$. And if $q \preceq p$ then there exists $r \in X$ such that $p = qr$. Then $(s_p^* s_q)^* = s_q^* s_p = s_r$. Hence $s_p^* s_q = s_r^*$. In short,

$$s_p^* s_q = \begin{cases} s_r & \text{if } q = pr \\ s_r^* & \text{if } p = qr \\ 0 & \text{otherwise.} \end{cases}$$

We have established that

$$C^*(\Lambda) = \overline{\operatorname{span}}\{s_p s_q^* : p, q \in X\}. \tag{4.1}$$

Let $\mathcal{G}_0 := \{(x, 0, y) \in \mathcal{G} : x, y \in \partial X\}$. Then $\mathcal{G}_0$, with the relative topology, has the basic open sets $[a, b]_A$, where $A \in \mathcal{E}, a, b \in X$ and $l(a) = l(b)$. Clearly $\mathcal{G}_0$ is a subgroupoid of $\mathcal{G}$. And

$$\begin{aligned} C^*(\mathcal{G}_0) &= \overline{\operatorname{span}}\{\chi_{U(p,q)} : p, q \in X, l(p) = l(q)\} \\ &\subseteq \overline{\operatorname{span}}\{\chi_{[p,q]_A} : p, q \in X, l(p) = l(q), A \subseteq \partial X \text{ is compact open}\} \\ &\subseteq C^*(\mathcal{G}_0). \end{aligned}$$

The second inclusion is due to the fact that $[p, q]_A$ is compact open whenever $A \subseteq \partial X$ is, hence $\chi_{[a,b]_A} \in C_c(\mathcal{G}_0) \subseteq C^*(\mathcal{G}_0)$.

We wish to prove that the C^*-algebra $C^*(\mathcal{G}_0)$ is an AF algebra. But first notice that for any $\mu \in X$, $V(\mu) = V(\mu e_n') \cup V(\mu e_n'')$, where $e_n' = e_n = (0, \dots, 0, 1) \in G_1$ and $e_n'' = e_n \in G_2$.

Take a basic open set $A = V(\mu) \setminus \left(\bigcup_{k=1}^{m_1} V(\nu_k)\right)$. It is possible to rewrite A as $V(p) \setminus \left(\bigcup_{k=1}^{m_2} V(r_k)\right)$ with $\mu \neq p$. Here is a relatively simple example (pointed out to the author by Spielberg): $V(\mu) \setminus V(\mu e_n') = V(\mu e_n'')$, where $e_n' = e_n \in G_1$ and $e_n'' = e_n \in G_2$.

Lemma 4.1. *Suppose $A = V(\mu) \setminus (\bigcup_{k=1}^{s} V(\mu\nu_k)) \neq \emptyset$. Then we can write A as $A = V(p) \setminus \left(\bigcup_{k=1}^{m_1} V(pr_k)\right)$ where $l(p)$ is the largest possible, that is, if $A = V(q) \setminus \left(\bigcup_{j=1}^{m_2} V(qs_j)\right)$ then $l(q) \leq l(p)$.*

Proof. We take two cases:

Case I. For each $k = 1, \dots, s$, there exists $i \in \{1, \dots, n-1\}$ with $l_i(\nu_k) \geq 1$.

Choose $p = \mu,\ r_k = \nu_k$ for each k (i.e., leave A the way it is). Suppose now that $A = V(q) \setminus \left(\bigcup_{j=1}^{m_2} V(qs_j)\right)$ with $l(p) \leq l(q)$. We will prove that $l(p) = l(q)$. Assuming the contrary, suppose $l(p) < l(q)$. Let $x \in A \Rightarrow x = qy$ for some $y \in \partial X$. Since $qy \in V(p) \setminus (\bigcup_{k=1}^{s} V(pr_k))$, $p \preceq qy$. But $l(p) < l(q) \Rightarrow p \preceq q$. Let $q = pr$, since $p \neq q$, $r \neq 0$. Let $r = a_1 a_2 \dots a_d$.

Either $a_1 \in G_1 \backslash \{0\}$ or $a_1 \in G_2 \backslash \{0\}$. Suppose, for definiteness, $a_1 \in G_1 \backslash \{0\}$. Take $t = (0, \ldots, 0, \infty) \in \partial G_2$. Since $l(r_k) > l(t)$ for each $k = 1, \ldots, s$, we get $pr_k \not\preceq pt$ for each $k = 1, \ldots, s$, moreover $pt \in V(p)$. Hence $pt \in A$. But $pr \not\preceq pt \Rightarrow q \not\preceq pt \Rightarrow pt \notin V(q) \Rightarrow pt \notin V(q) \setminus \left(\bigcup_{j=1}^{m_2} V(qs_j)\right)$ which is a contradiction to $A = V(q) \setminus \left(\bigcup_{j=1}^{m_2} V(qs_j)\right)$. Therefore $l(p) = l(q)$. In fact, $p = q$.

Case II. There exists $k \in \{1, \ldots, s\}$ with $l_i(\nu_k) = 0$, for each $i = 1, \ldots, n-1$.

After rearranging, suppose that $l_i(\nu_k) = 0$ for each $k = 1, \ldots, \alpha$ and each $i = 1, \ldots, n-1$; and that for each $k = \alpha + 1, \ldots, s$, $l_i(\nu_k) \geq 1$ for some $i \leq n-1$. We can also assume that $l(\nu_1)$ is the largest of $l(\nu_k)$'s for $k \leq \alpha$. Then

$$\begin{aligned} A &= V(\mu) \setminus \left(\bigcup_{k=1}^{s} V(\mu\nu_k)\right) \\ &= \left[V(\mu) \setminus \left(\bigcup_{k=1}^{\alpha} V(\mu\nu_k)\right)\right] \bigcap \left[V(\mu) \setminus \left(\bigcup_{k=\alpha+1}^{s} V(\mu\nu_k)\right)\right]. \end{aligned}$$

Let $me_n = l(\nu_1)$ which is non zero. We will prove that if we can rewrite A as $V(q) \setminus \left(\bigcup_{k=1}^{m_2} V(qs_k)\right)$ with $l(\mu) \leq l(q)$ then $q = \mu r$ with $0 \leq l(r) \leq m(e_n)$.

Clearly if $\mu \not\preceq q$, then $A \cap V(q) = \emptyset$. So, if $A \cap V(q) \setminus \left(\bigcup_{k=1}^{m_2} V(qs_k)\right) \neq \emptyset$ then $\mu \preceq q$. Now let $q = \mu r$, and let $\nu_1 = a_1 a_2 \ldots a_d$. Observe that since for each j, $a_j \in \Lambda^+$ and that $l_k(\nu_1) = 0$ for each $k \leq n-1$, we have $l_k(a_j) = 0$ for all $k \leq n-1$. Also, by assumption, $l(\nu_1) > 0$, therefore either $a_d \in G_1 \setminus \{0\}$ or $a_d \in G_2 \setminus \{0\}$. Suppose, for definiteness, that $a_d \in G_1 \backslash \{0\}$. Let $a'_d = a_d - e_n$ and let $\nu' = a_1 a_2 \ldots a'_d$ (or just $a_1 a_2 \ldots a_{d-1}$, if $a'_d = 0$). If $V(\mu\nu') \cap A = \emptyset$ then we can replace ν_1 by ν' in the expression of A and and (after rearranging the $\nu'_i s$) choose a new ν_1. Since $A \neq \emptyset$ this process of replacement must stop with $V(\mu\nu') \cap A \neq \emptyset$. Letting $e'_n = e_n \in G_1$ and $e''_n = e_n \in G_2$, then $V(\mu\nu') = V(\mu\nu' e'_n) \cup V(\mu\nu' e''_n) = V(\mu\nu_1) \cup V(\mu\nu' e''_n)$. Since $V(\mu\nu_1) \cap A = \emptyset$, $A \cap V(\mu\nu' e''_n) \neq \emptyset$ hence $\nu' e''_n \notin \{\nu_1, \ldots, \nu_\alpha\}$. Take $t' = (0, \ldots, 0, \infty) \in \partial G_1$ and $t'' = (0, \ldots, 0, \infty) \in \partial G_2$. Then $\mu\nu' e''_n t',\ \mu\nu' e''_n t'' \in V(\mu) \setminus (\bigcup_{k=1}^{\alpha} V(\mu\nu_k))$. Moreover, for each $k = \alpha+1, \ldots, s$, we have $l(\nu' e''_n t'),\ l(\nu' e''_n t'') < l(\nu_k)$, implying $\mu\nu' e''_n t',\ \mu\nu' e''_n t'' \in V(\mu) \setminus \left(\bigcup_{k=\alpha+1}^{s} V(\mu\nu_k)\right)$. Hence $\mu\nu' e''_n t',\ \mu\nu' e''_n t'' \in V(q) \setminus \left(\bigcup_{k=1}^{m_2} V(qs_k)\right)$. Therefore $q \preceq \mu\nu' e''_n \Rightarrow \mu r \preceq \mu\nu' e''_n \Rightarrow 0 \leq l(r) \leq l(\nu' e''_n) = l(\nu') + e_n = me_n$. Therefore there is only a finite possible r's we can choose form. [In fact, since $r \preceq \nu' e''_n$, there are at most m of them to choose from.] $\square$

To prove that $C^*(\mathcal{G}_0)$ is an AF algebra, we start with a finite subset $\mathcal{U}$ of the generating set $\{\chi_{U(p,q)} : p, q \in X, l(p) = l(q)\}$ and show that there is a finite dimensional C^*-subalgebra of $C^*(\mathcal{G}_0)$ that contains the set $\mathcal{U}$.

Theorem 4.2. *$C^*(\mathcal{G}_0)$ is an AF algebra.*

Proof. Suppose that $\mathcal{U} = \{\chi_{U(p_1,q_1)}, \chi_{U(p_2,q_2)}, \dots \chi_{U(p_s,q_s)}\}$ is a (finite) subset of the generating set of $C^*(\mathcal{G}_0)$. Let

$$\mathcal{S} := \{V(p_1), V(q_1), V(p_2), V(q_2), \dots, V(p_s), V(q_s)\}.$$

We "disjointize" the set $\mathcal{S}$ as follows. For a subset $\mathbf{F}$ of $\mathcal{S}$, write

$$A_{\mathbf{F}} := \bigcap_{A \in \mathbf{F}} A \setminus \bigcup_{A \notin \mathbf{F}} A.$$

Define

$$\mathcal{C} := \{A_{\mathbf{F}} : \mathbf{F} \subseteq \mathcal{S}\}.$$

Clearly, the set $\mathcal{C}$ is a finite collection of pairwise disjoint sets. A routine computation reveals that for any $E \in \mathcal{S}$, $E = \bigcup\{C \in \mathcal{C} : C \subseteq E\}$. It follows from (2.1) that for any $\mathbf{F} \subseteq \mathcal{S}$, $\bigcap_{A \in \mathbf{F}} A = V(p)$, for some $p \in X$, if it is not empty. Hence,

$$A_{\mathbf{F}} = V(p) \setminus \bigcup_{i=1}^{k} V(pr_i)$$

for some $p \in X$ and some $r_i \in X$. Let $p_{\mathbf{F}} \in X$ be such that $A_{\mathbf{F}} = V(p) \setminus \bigcup_{i=1}^{k} V(pr_i)$ and $l(p_{\mathbf{F}})$ is maximum (as in Lemma 4.1). Then

$$\begin{aligned} A_{\mathbf{F}} &= p_{\mathbf{F}} \left(\partial X \setminus \left(\bigcup_{i=1}^{k} V(r_i) \right) \right) \\ &= p_{\mathbf{F}} C_{\mathbf{F}}, \end{aligned}$$

where $C_{\mathbf{F}} = \partial X \setminus \left(\bigcup_{i=1}^{k} V(r_i) \right)$. Now $V(p_\alpha) = p_{F_1} C_{F_1} \cup p_{F_2} C_{F_2} \cup \ldots \cup p_{F_k} C_{F_k}$ where $\{F_1, F_2, \dots, F_k\} = \{F \subseteq \mathcal{S} : V(p_\alpha) \in F\}$. Notice that $p_{F_i} C_{F_i} \subseteq V(p_\alpha)$ for each i, hence $p_\alpha \preceq p_{F_i}$. Hence $p_{F_i} C_{F_i} = p_\alpha t_i C_{F_i}$, for some $t_i \in X$. Therefore $V(p_\alpha) = p_\alpha U_1 \cup p_\alpha U_2 \cup \ldots \cup p_\alpha U_k$ where $U_i = t_i C_{F_i}$. Similarly $V(q_\alpha) = q_\alpha V_1 \cup q_\alpha V_2 \cup \ldots \cup q_\alpha V_m$, where each $q_\alpha V_i \in \mathcal{C}$ is subset of $V(q_\alpha)$. Consider the set

$$\mathcal{B} := \{[p, q]_{C \cap D} : pC, qD \in \mathcal{C} \text{ and } p = p_\alpha, q = q_\alpha, 1 \leq \alpha \leq s\}.$$

Since $\mathcal{C}$ is a finite collection, this collection is finite too. We will prove that $\mathcal{B}$ is pairwise disjoint.

Suppose $[p, q]_{C \cap D} \bigcap [p', q']_{C' \cap D'}$ is non-empty. Clearly $p(C \cap D) \bigcap p'(C' \cap D') \neq \emptyset$, and $q(C \cap D) \bigcap q'(C' \cap D') \neq \emptyset$. Therefore, among other things, $pC \cap p'C' \neq \emptyset$ and $qD \cap q'D' \neq \emptyset$, but by construction, $\{pC, qD, p'C', q'D'\}$ is pairwise disjoint. Hence $pC = p'C'$ and $qD = q'D'$. Suppose, without loss of generality, that $l(p) \leq l(p')$. Then $p' = pr$ and $q' = qs$ for some $r, s \in X$, hence $[p', q']_{C' \cap D'} = [pr, qs]_{C' \cap D'}$. Let $(px, 0, qx) \in [p, q]_{C \cap D} \bigcap [pr, qs]_{C' \cap D'}$. Then $px = prt$ and $qx = qst$, for some $t \in C' \cap D'$, hence $x = rt = st$. Therefore $r = s$ (since $l(r) = l(p') - l(p) = l(q') - l(q) = l(s)$). Hence $pC = p'C' = prC'$, and $qD = q'D' = qrD'$, implying $C = rC'$ and $D = rD'$. This gives us $C \cap D = rC' \cap rD' = r(C' \cap D')$. Hence $[p', q']_{C' \cap D'} =$

$[pr, qr]_{C' \cap D'} = [p, q]_{r(C' \cap D')} = [p, q]_{C \cap D}$. Therefore $\mathcal{B}$ is a pairwise disjoint collection.

For each $[p, q]_{C \cap D} \in \mathcal{B}$, since $C \cap D$ is of the form $V(\mu) \setminus \bigcup_{i=1}^{k} V(\mu\nu_i)$, we can rewrite $C \cap D$ as μW, where $W = \partial X \setminus \bigcup_{i=1}^{k} V(\nu_i)$ and $l(\mu)$ is maximal (by Lemma 4.1). Then $[p, q]_{C \cap D} = [p, q]_{\mu W} = [p\mu, q\mu]_W$. Hence each $[p, q]_{C \cap D} \in \mathcal{B}$ can be written as $[p, q]_W$ where $l(p) = l(q)$ is maximal and $W = \partial X \setminus \bigcup_{i=1}^{k} V(\nu_i)$.

Consider the collection $\mathcal{D} := \{\chi_{[p,q]_W} : [p, q]_W \in \mathcal{B}\}$. We will show that, for each $1 \leq \alpha \leq s$, $\chi_{U(p_\alpha, q_\alpha)}$ is a sum of elements of $\mathcal{D}$ and that $\mathcal{D}$ is a self-adjoint system of matrix units. For the first, let $V(p_\alpha) = p_\alpha U_1 \cup p_\alpha U_2 \cup \ldots \cup p_\alpha U_k$ and $V(q_\alpha) = q_\alpha V_1 \cup q_\alpha V_2 \cup \ldots \cup q_\alpha V_m$. One more routine computation gives us:

$$\begin{aligned} U(p_\alpha, q_\alpha) = [p_\alpha, q_\alpha]_{\partial X} &= \bigcup_{i,j=1}^{k,m} \left([p_\alpha, p_\alpha]_{U_i} \cdot [p_\alpha, q_\alpha]_{\partial X} \cdot [q_\alpha, q_\alpha]_{V_j} \right) \\ &= \bigcup_{i,j=1}^{k,m} [p_\alpha, q_\alpha]_{U_i \cap V_j}. \end{aligned}$$

Since the union is disjoint,

$$\chi_{U(p_\alpha, q_\alpha)} = \sum_{i,j=1}^{k,m} \chi_{[p_\alpha, q_\alpha]_{U_i \cap V_j}}.$$

And each $\chi_{[p_\alpha, q_\alpha]_{U_i \cap V_j}}$ is in the collection $\mathcal{D}$. Therefore $\mathcal{U} \subseteq \operatorname{span}(\mathcal{D})$.

To show that $\mathcal{D}$ is a self-adjoint system of matrix units, let $\chi_{[p,q]_W}, \chi_{[r,s]_V} \in \mathcal{D}$. Then

$$\begin{aligned} \chi_{[p,q]_W} \cdot \chi_{[r,s]_V}(x_1, 0, x_2) &= \sum_{y_1, y_2} \chi_{[p,q]_W}\big((x_1, 0, x_2)(y_1, 0, y_2)\big) \cdot \chi_{[r,s]_V}(y_2, 0, y_1) \\ &= \sum_{y_2} \chi_{[p,q]_W}(x_1, 0, y_2) \cdot \chi_{[r,s]_V}(y_2, 0, x_2), \end{aligned}$$

where the last sum is taken over all y_2 such that $x_1 \sim_0 y_2 \sim_0 x_2$. Clearly the above sum is zero if $x_1 \notin pW$ or $x_2 \notin sV$. Also, recalling that qW and rV are either equal or disjoint, we see that the above sum is zero if they are disjoint. For the preselected x_1, if $x_1 = pz$ then $y_2 = qz$ (is uniquely chosen). Therefore the above sum is just the single term $\chi_{[p,q]_W}(x_1, 0, y_2) \cdot \chi_{[r,s]_V}(y_2, 0, x_2)$. Suppose that $qW = rV$. We will show that $l(q) = l(r)$, which implies that $q = r$ and $W = V$.

Given this,

$$\begin{aligned} \chi_{[p,q]_W} \cdot \chi_{[r,s]_V}(x_1, 0, x_2) &= \chi_{[p,q]_W}(x_1, 0, y_2) \cdot \chi_{[r,s]_V}(y_2, 0, x_2) \\ &= \chi_{[p,q]_W}(x_1, 0, y_2) \cdot \chi_{[q,s]_W}(y_2, 0, x_2) \\ &= \chi_{[p,s]_W}(x_1, 0, x_2). \end{aligned}$$

To show that $l(q) = l(r)$, assuming the contrary, suppose $l(q) < l(r)$ then $r = qc$ for some non-zero $c \in X$, implying $V = cW$. Hence $[r,s]_V = [r,s]_{cW} = [rc, sc]_W$, which contradicts the maximality of $l(r) = l(s)$. By symmetry $l(r) < l(q)$ is also impossible. Hence $l(q) = l(r)$ and $W = V$. This concludes the proof. $\square$

5. Crossed product by the gauge action

Let $\hat{\Lambda}$ denote the dual of Λ, i.e., the abelian group of continuous homomorphisms of Λ into the circle group $\mathbb{T}$ with pointwise multiplication: for $t, s \in \hat{\Lambda}$, $\langle \lambda, ts\rangle = \langle \lambda, t\rangle\langle \lambda, s\rangle$ for each $\lambda \in \Lambda$, where $\langle \lambda, t\rangle$ denotes the value of $t \in \hat{\Lambda}$ at $\lambda \in \Lambda$.

Define an action called the **gauge action**: $\alpha : \hat{\Lambda} \longrightarrow \mathrm{Aut}(C^*(\mathcal{G}))$ as follows. For $t \in \hat{\Lambda}$, first define $\alpha_t : C_c(\mathcal{G}) \longrightarrow C_c(\mathcal{G})$ by $\alpha_t(f)(x, \lambda, y) = \langle \lambda, t\rangle f(x, \lambda, y)$ then extend $\alpha_t : C^*(\mathcal{G}) \longrightarrow C^*(\mathcal{G})$ continuously. Notice that $(A, \hat{\Lambda}, \alpha)$ is a C^*- dynamical system.

Consider the linear map Φ of $C^*(\mathcal{G})$ onto the fixed-point algebra $C^*(\mathcal{G})^\alpha$ given by

$$\Phi(a) = \int_{\hat{\Lambda}} \alpha_t(a)\, dt, \text{ for } a \in C^*(\mathcal{G}).$$

where dt denotes a normalized Haar measure on $\widehat{\Lambda}$.

Lemma 5.1. *Let Φ be defined as above.*

(a) *The map Φ is a faithful conditional expectation; in the sense that $\Phi(a^*a) = 0$ implies $a = 0$.*
(b) *$C^*(\mathcal{G}_0) = C^*(\mathcal{G})^\alpha$.*

Proof. Since the action α is continuous, we see that Φ is a conditional expectation from $C^*(\mathcal{G})$ onto $C^*(\mathcal{G})^\alpha$, and that the expectation is faithful. For $p, q \in X$, $\alpha_t(s_p s_q^*)(x, l(p) - l(q), y) = \langle l(p) - l(q), t\rangle s_p s_q^*(x, l(p) - l(q), y)$. Hence if $l(p) = l(q)$ then $\alpha_t(s_p s_q^*) = s_p s_q^*$ for each $t \in \hat{\Lambda}$. Therefore α fixes $C^*(\mathcal{G}_0)$. Hence $C^*(\mathcal{G}_0) \subseteq C^*(\mathcal{G})^\alpha$. By continuity of Φ it suffices to show that $\Phi(s_p s_q^*) \in C^*(\mathcal{G}_0)$ for all $p, q \in X$.

$$\int_{\hat{\Lambda}} \alpha_t(s_p s_q^*)\, dt = \int_{\hat{\Lambda}} \langle l(p) - l(q), t\rangle s_p s_q^*\, dt = 0, \text{ when } l(p) \neq l(q).$$

It follows from (4.1) that $C^*(\mathcal{G})^\alpha \subseteq C^*(\mathcal{G}_0)$. Therefore $C^*(\mathcal{G})^\alpha = C^*(\mathcal{G}_0)$. $\square$

We study the crossed product $C^*(\mathcal{G}) \times_\alpha \widehat{\Lambda}$. Recall that $C_c(\hat{\Lambda}, A)$, which is equal to $C(\hat{\Lambda}, A)$, since $\hat{\Lambda}$ is compact, is a dense *-subalgebra of $A \times_\alpha \hat{\Lambda}$. Recall also that multiplication (convolution) and involution on $C(\hat{\Lambda}, A)$ are, respectively, defined by:

$$(f \cdot g)(s) = \int_{\hat{\Lambda}} f(t)\alpha_t(g(t^{-1}s))\, dt$$

and

$$f^*(s) = \alpha(f(s^{-1})^*).$$

The functions of the form $f(t) = \langle \lambda, t\rangle s_p s_q^*$ from $\hat{\Lambda}$ into A form a generating set for $A \times_\alpha \hat{\Lambda}$. Moreover the fixed-point algebra $C^*(\mathcal{G}_0)$ can be imbedded into $A \times_\alpha \hat{\Lambda}$ as follows: for each $b \in C^*(\mathcal{G}_0)$, define the function $b : \hat{\Lambda} \longrightarrow A$ as $b(t) = b$ (the constant function). Thus $C^*(\mathcal{G}_0)$ is a subalgebra of $A \times_\alpha \hat{\Lambda}$.

Proposition 5.2. *The C^*-algebra $B := C^*(\mathcal{G}_0)$ is a hereditary C^*-subalgebra of $A \times_\alpha \hat{\Lambda}$.*

Proof. To prove the theorem, we prove that $B \cdot A \times_\alpha \hat{\Lambda} \cdot B \subseteq B$. Since $A \times_\alpha \hat{\Lambda}$ is generated by functions of the form $f(t) = \langle \lambda, t\rangle s_p s_q^*$, it suffices to show that $b_1 \cdot f \cdot b_2 \in B$ whenever $b_1, b_2 \in B$ and $f(t) = \langle \lambda, t\rangle s_p s_q^*$ for $\lambda \in \Lambda, p, q \in X$.

$$\begin{aligned}
(b_1 \cdot f \cdot b_2)(z) &= \int_{\hat{\Lambda}} b_1(t)\alpha_t((f \cdot b_2)(t^{-1}z))\, dt \\
&= \int_{\hat{\Lambda}} b_1\alpha_t \left(\int_{\hat{\Lambda}} f(w)\alpha_w(b_2(w^{-1}t^{-1}z))\, dw \right) dt \\
&= \int_{\hat{\Lambda}} \int_{\hat{\Lambda}} b_1\alpha_t(f(w)\alpha_w(b_2))\, dw\, dt \\
&= \int_{\hat{\Lambda}} \int_{\hat{\Lambda}} b_1\alpha_t(\langle \lambda, w\rangle s_p s_q^*) b_2\, dw\, dt, \text{ since } \alpha_w(b_2) = \alpha_t(b_2) = b_2 \\
&= \int_{\hat{\Lambda}} \int_{\hat{\Lambda}} b_1 \langle \lambda, w\rangle \alpha_t(s_p s_q^*) b_2\, dw\, dt \\
&= \int_{\hat{\Lambda}} \int_{\hat{\Lambda}} b_1 \langle \lambda, w\rangle \langle l(p) - l(q), t\rangle s_p s_q^* b_2\, dw\, dt \\
&= \int_{\hat{\Lambda}} \langle \lambda, w\rangle\, dw \int_{\hat{\Lambda}} \langle l(p) - l(q), t\rangle\, dt\ b_1 s_p s_q^* b_2 \\
&= 0 \text{ unless } \lambda = 0 \text{ and } l(p) - l(q) = 0.
\end{aligned}$$

And in that case (in the case when $\lambda = 0$ and $l(p) - l(q) = 0$) we get $(b_1 \cdot f \cdot b_2)(z) = b_1 s_p s_q^* b_2 \in B$ (since $l(p) = l(q)$). Therefore B is hereditary. □

Let I_B denote the ideal in $A \times_\alpha \hat{\Lambda}$ generated by B. The following corollary follows from Theorem 4.2 and Proposition 5.2.

Corollary 5.3. *I_B is an AF algebra.*

We want to prove that $A \times_\alpha \hat{\Lambda}$ is an AF algebra, and to do this we consider the dual system. Define $\hat{\alpha} : \hat{\hat{\Lambda}} = \Lambda \longrightarrow \mathrm{Aut}(A \times_\alpha \hat{\Lambda})$ as follows: For $\lambda \in \Lambda$ and $f \in C(\hat{\Lambda}, A)$, we define $\hat{\alpha}_\lambda(f) \in C(\hat{\Lambda}, A)$ by: $\hat{\alpha}_\lambda(f)(t) = \langle \lambda, t\rangle f(t)$. Extend $\hat{\alpha}_\lambda$ continuously.

As before we use $\cdot$ to represent multiplication in $A \times_\alpha \hat{\Lambda}$.

Lemma 5.4. $\hat{\alpha}_\lambda(I_B) \subseteq I_B$ *for each* $\lambda \geq 0$.

Proof. Since the functions of the form $f(t) = \langle \lambda, t\rangle s_p s_q^*$ make a generating set for $A \times_\alpha \hat{\Lambda}$, it suffices to show that if $\lambda > 0$ then $\hat{\alpha}_\lambda(f \cdot b \cdot g) \in I_B$ for $f(t) = \langle \lambda_1, t\rangle s_{p_1} s_{q_1}^*$, $g(t) = \langle \lambda_2, t\rangle s_{p_2} s_{q_2}^*$, and $b = s_{p_0} s_{q_0}^*$, with $l(p_0) = l(q_0)$. First

$$
\begin{aligned}
&(f \cdot b \cdot g)(z) \\
&\qquad = \int_{\hat{\Lambda}} f(t)\alpha_t((b \cdot g)(t^{-1}z))\, dt \\
&\qquad = \int_{\hat{\Lambda}} f(t)\alpha_t \left[\int_{\hat{\Lambda}} b(w)\alpha_w(g(w^{-1}t^{-1}z))\, dw\right] dt \\
&\qquad = \int_{\hat{\Lambda}} f(t) \left[\int_{\hat{\Lambda}} b\alpha_{tw}(g(w^{-1}t^{-1}z)\, dw)\right] dt \\
&\qquad = \int_{\hat{\Lambda}} f(t) \left[\int_{\hat{\Lambda}} b\alpha_{w}(g(w^{-1}z)\, dw)\right] dt \\
&\qquad = \int_{\hat{\Lambda}} \int_{\hat{\Lambda}} f(t) b\alpha_w(g(w^{-1}z))\, dw\, dt \\
&\qquad = \int_{\hat{\Lambda}} \int_{\hat{\Lambda}} \langle \lambda_1, t\rangle s_{p_1} s_{q_1}^* s_{p_0} s_{q_0}^* \langle \lambda_2, w^{-1}z\rangle \alpha_w(s_{p_2} s_{q_2}^*)\, dw\, dt \\
&\qquad = \int_{\hat{\Lambda}} \int_{\hat{\Lambda}} \langle \lambda_1, t\rangle s_{p_1} s_{q_1}^* s_{p_0} s_{q_0}^* \langle \lambda_2, w^{-1}z\rangle \langle l(p_2) - l(q_2), w\rangle s_{p_2} s_{q_2}^*\, dw\, dt.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\hat{\alpha}_\lambda(f \cdot b \cdot g)(z) \\
&= \langle \lambda, z\rangle \int_{\hat{\Lambda}} \int_{\hat{\Lambda}} \langle \lambda_1, t\rangle s_{p_1} s_{q_1}^* s_{p_0} s_{q_0}^* \langle \lambda_2, w^{-1}z\rangle \langle l(p_2) - l(q_2), w\rangle s_{p_2} s_{q_2}^*\, dw\, dt \\
&= \int_{\hat{\Lambda}} \int_{\hat{\Lambda}} \langle \lambda_1, t\rangle s_{p_1} s_{q_1}^* s_{p_0} s_{q_0}^* \langle \lambda, w^{-1}z\rangle \langle \lambda, w\rangle \langle \lambda_2, w^{-1}z\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad \langle l(p_2) - l(q_2), w\rangle s_{p_2} s_{q_2}^*\, dw\, dt \\
&= \int_{\hat{\Lambda}} \int_{\hat{\Lambda}} \langle \lambda_1, t\rangle s_{p_1} s_{q_1}^* s_{p_0} s_{q_0}^* \langle \lambda + \lambda_2, w^{-1}z\rangle \langle \lambda + l(p_2) - l(q_2), w\rangle s_{p_2} s_{q_2}^*\, dw\, dt;
\end{aligned}
$$

letting $\lambda' = \lambda \in G_1$, then this last integral gives us

$$
\begin{aligned}
&= \int_{\hat{\Lambda}} \int_{\hat{\Lambda}} \langle \lambda_1, t\rangle s_{p_1} s_{q_1}^* s_{\lambda'}^* s_{\lambda'} s_{p_0} s_{q_0}^* s_{\lambda'}^* s_{\lambda'} \langle \lambda + \lambda_2, w^{-1}z\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad \langle \lambda + l(p_2) - l(q_2), w\rangle s_{p_2} s_{q_2}^*\, dw\, dt \\
&= \int_{\hat{\Lambda}} \int_{\hat{\Lambda}} \langle \lambda_1, t\rangle s_{p_1} s_{\lambda' q_1}^* s_{\lambda' p_0} s_{\lambda' q_0}^* \langle \lambda + \lambda_2, w^{-1}z\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad \langle \lambda + l(p_2) - l(q_2), w\rangle s_{\lambda' p_2} s_{q_2}^*\, dw\, dt \\
&= (f' \cdot b' \cdot g')(z),
\end{aligned}
$$

where $f'(t) = \langle \lambda_1, t\rangle s_{p_1} s^*_{\lambda' q_1}$, $g'(t) = \langle \lambda + \lambda_2, t\rangle s_{\lambda' p_2} s^*_{q_2}$, and $b' = s_{\lambda' p_0} s^*_{\lambda' q_0}$. Therefore $\hat{\alpha}_\lambda(f \cdot b \cdot g) \in I_B$. □

For each $\lambda \in \Lambda$ define $I_\lambda := \hat{\alpha}_\lambda(I_B)$. Clearly each I_λ is an ideal of $A \times_\alpha \hat{\Lambda}$ and is an AF algebra. Let $\lambda_1 < \lambda_2$ then $\lambda_2 - \lambda_1 > 0 \Rightarrow I_{\lambda_2 - \lambda_1} = \hat{\alpha}_{\lambda_2 - \lambda_1}(I_B) \subseteq I_B$. Therefore $I_{\lambda_2} = \hat{\alpha}_{\lambda_2}(I_B) = \hat{\alpha}_{\lambda_1}(\hat{\alpha}_{\lambda_2 - \lambda_1}(I_B)) \subseteq I_{\lambda_1}$. That is, $I_{\lambda_1} \supseteq I_{\lambda_2}$ whenever $\lambda_1 < \lambda_2$. In particular $I_B = I_0 \supseteq I_\lambda$ for each $\lambda \geq 0$. Furthermore, if $f \in C(\hat{\Lambda}, A)$ is given by $f(t) = \langle \lambda, t\rangle s_p s_q^*$, for $\lambda \in \Lambda$ and $p, q \in X$ then $\hat{\alpha}_\beta(f)(t) = \langle \beta, t\rangle f(t) = \langle \beta, t\rangle\langle \lambda, t\rangle s_p s_q^* = \langle \beta + \lambda, t\rangle s_p s_q^*$.

For $f \in C(\mathcal{G})$ given by $f(t) = \langle \lambda, t\rangle s_p s_q^*$, let us compute f^*, $f \cdot f^*$, and $f^* \cdot f$ so we can use them in the next lemma.

$$\begin{aligned}
f^*(t) &= \alpha_t(f(t^{-1})^*) \\
&= \alpha_t\left(\left(\langle \lambda, t^{-1}\rangle s_p s_q^*\right)^*\right) \\
&= \overline{\langle \lambda, t^{-1}\rangle}\alpha_t\left(s_q s_p^*\right) \\
&= \langle \lambda, t\rangle\langle l(q) - l(p), t\rangle s_q s_p^* \\
&= \langle \lambda + l(q) - l(p), t\rangle s_q s_p^*,
\end{aligned}$$

$$\begin{aligned}
(f \cdot f^*)(z) &= \int_{\hat{\Lambda}} f(t)\alpha_t(f^*(t^{-1}z))\,dt \\
&= \int_{\hat{\Lambda}} \langle \lambda, t\rangle s_p s_q^* \alpha_t\left(\langle \lambda + l(q) - l(p), t^{-1}z\rangle s_q s_p^*\right)\,dt \\
&= \int_{\hat{\Lambda}} \langle \lambda, t\rangle s_p s_q^* \langle \lambda + l(q) - l(p), t^{-1}z\rangle\langle l(q) - l(p), t\rangle s_q s_p^*\,dt \\
&= \int_{\hat{\Lambda}} \langle \lambda + l(q) - l(p), t\rangle s_p s_q^* \langle \lambda + l(q) - l(p), t^{-1}z\rangle s_q s_p^*\,dt \\
&= \int_{\hat{\Lambda}} \langle \lambda + l(q) - l(p), z\rangle s_p s_q^* s_q s_p^*\,dt \\
&= \langle \lambda + l(q) - l(p), z\rangle s_p s_q^* s_q s_p^* \\
&= \langle \lambda + l(q) - l(p), z\rangle s_p s_p^*,
\end{aligned}$$

and

$$\begin{aligned}
(f^* \cdot f)(z) &= \langle \left(\lambda + l(q) - l(p)\right) + l(p) - l(q), z\rangle s_q s_q^* \\
&= \langle \lambda, z\rangle s_q s_q^*.
\end{aligned}$$

Lemma 5.5. *Let $\lambda \in \Lambda$, $p, q \in X$, $f(t) = \langle \lambda, t\rangle s_q s_q^*$, and let $g(t) = \langle \lambda, t\rangle s_p s_q^*$.*

(a) *If $\lambda \geq 0$ then $f \in I_B$.*
(b) *If $\lambda + l(q) \geq l(p)$ then $g \in I_B$.*

Proof. To prove (a), $s_q s_q^* \in C^*(\mathcal{G}_0) \subseteq I_B$. Then $f \in I_B$, since $\lambda \geq 0$, by Lemma 5.4. To prove (b), $(g \cdot g^*)(z) = \langle \lambda + l(q) - l(p), z\rangle s_p s_p^*$. By (a), $g \cdot g^* \in I_B$, implying $g \in I_B$. ☐

Theorem 5.6. *$A \times_\alpha \hat{\Lambda}$ is an AF algebra.*

Proof. Let $f(t) = \langle \lambda, t\rangle s_p s_q^*$. Choose $\beta \in \Lambda$ large enough such that $\beta + \lambda + l(q) \geq l(p)$. Then

$$\begin{aligned}\hat{\alpha}_\beta(f)(z) &= \langle \beta, z\rangle\langle \lambda, z\rangle s_p s_q^* \\ &= \langle \beta + \lambda, z\rangle s_p s_q^*.\end{aligned}$$

Applying Lemma 5.5(b), $\hat{\alpha}_\beta(f) \in I_B$. Thus $\hat{\alpha}_{-\beta}(\hat{\alpha}_\beta(f)) \in I_{-\beta}$, implying $f \in I_{-\beta}$. Therefore $A \times_\alpha \hat{\Lambda} = \overline{\bigcup_{\lambda \leq 0} I_\lambda}$. Since each I_λ is an AF algebra, so is $A \times_\alpha \hat{\Lambda}$. ☐

6. Final results

Let us recall that an r-discrete groupoid G is *locally contractive* if for every nonempty open subset U of the unit space there is an open G-set Z with $s(\bar{Z}) \subseteq U$ and $r(\bar{Z}) \subsetneqq s(Z)$. A subset E of the unit space of a groupoid G is said to be invariant if its saturation $[E] = r(s^{-1}(E))$ is equal to E.

An r-discrete groupoid G is *essentially free* if the set of all x's in the unit space G^0 with $r^{-1}(x) \cap s^{-1}(x) = \{x\}$ is dense in the unit space. When the only open invariant subsets of G^0 are the empty set and G^0 itself, then we say that G is *minimal*.

Lemma 6.1. *$\mathcal{G}$ is locally contractive, essentially free and minimal.*

Proof. To prove that $\mathcal{G}$ is locally contractive, let $U \subseteq \mathcal{G}^0$ be nonempty open. Let $V(p;q) \subseteq U$. Choose $\mu \in X$ such that $p \preceq \mu$, $q \npreceq \mu$ and $\mu \npreceq q$. Then $V(\mu) \subseteq V(p;q) \subseteq U$. Let $Z = [\mu, 0]_{V(\mu)}$. Then $Z = \bar{Z}$, $s(Z) = V(\mu) \subseteq U$, $r(Z) = \mu V(\mu) \subsetneqq V(\mu) \subseteq U$. Therefore $\mathcal{G}$ is locally contractive.

To prove that $\mathcal{G}$ is essentially free, let $x \in \partial X$. Then $r^{-1}(x) = \{(x, k, y) : x \sim_k y\}$ and $s^{-1}(x) = \{(z, m, x) : z \sim_m x\}$. Hence $r^{-1}(x) \cap s^{-1}(x) = \{(x, k, x) : x \sim_k x\}$. Notice that $r^{-1}(x) \cap s^{-1}(x) = \{x\}$ exactly when $x \sim_k x$ which implies $k = 0$. If $k \neq 0$ then $x = pt = qt$, for some $p, q \in X$, $t \in \partial X$ such that $l(p) - l(q) = k$. If $k > 0$ then $l(p) > l(q)$ and we get $q \preceq p$. Hence $p = qb$, for some $b \in X \setminus \{0\}$. Therefore $x = qbt = qt$, implying $bt = t$. Hence $x = qbbb\ldots$. Similarly, if $k < 0$ then $x = pbbb\ldots$, for some $b \in X$, with $l(b) > 0$. Therefore, to prove that $\mathcal{G}$ is essentially free, we need to prove that if U is an open set containing an element of the form $pbbb\ldots$, with $l(b) > 0$, then it contains an element that cannot be written in the form of $qddd\ldots$, with $l(d) > 0$. Suppose $pbbb\ldots \in U$, where U is open in $\mathcal{G}^0$. then $U \supseteq V(\mu;\nu)$ for some $\mu, \nu \in X$. Choose $\eta \in X$ such that $\mu \preceq \eta$, $\nu \npreceq \eta$, and $\eta \npreceq \nu$. Then $V(\eta) \subseteq V(\mu;\nu) \subseteq U$. Now take $a_1 = (1, 0, \ldots, 0) \in G_1$, $a_2 = (2, 0, \ldots, 0) \in G_2$, $a_3 = (3, 0, \ldots, 0) \in G_1$, $a_4 = (4, 0, \ldots, 0) \in G_2$, etc. Now $t = \eta a_1 a_2 a_3 \ldots \in V(\eta) \subseteq U$, but t cannot be written in the form of $qddd\ldots$.

To prove that $\mathcal{G}$ is minimal, let $E \subseteq \mathcal{G}^0$ be nonempty open and invariant, i.e., $E = r(s^{-1}(E))$. We want to show that $E = \mathcal{G}^0$. Since E is open, there exist $p, q \in X$ such that $V(p;q) \subseteq E$. But

$$s^{-1}(V(p;q)) = \{(\mu x, l(\mu) - l(p\nu), p\nu x) : q \not\preceq p\mu x\}.$$

Let $x \in \mathcal{G}^0$. Choose $\nu \in X$ such that $p\nu \not\preceq q$ and $q \not\preceq p\nu$. Then $(x, -l(p\nu), p\nu x) \in s^{-1}(V(p;q)) \subseteq s^{-1}(E)$ and $r(x, -l(p\nu), p\nu x) = x$. That is, $x \in r(s^{-1}(V(p;q)))$, hence $E = \mathcal{G}^0$. Therefore $\mathcal{G}$ is minimal. $\square$

Proposition 6.2 ([1, Proposition 2.4])**.** *Let G be an r-discrete groupoid, essentially free and locally contractive. Then every non-zero hereditary C^*-subalgebra of $C^*_r(G)$ contains an infinite projection.*

Corollary 6.3. *$C^*_r(\mathcal{G})$ is simple and purely infinite.*

Proof. This follows from Lemma 6.1 and Proposition 6.2. $\square$

Theorem 6.4. *$C^*(\mathcal{G})$ is simple, purely infinite, nuclear and classifiable.*

Proof. It follows from the Takesaki–Takai Duality Theorem that $C^*(\mathcal{G})$ is stably isomorphic to $C^*(\mathcal{G}) \times_\alpha \hat{\Lambda} \times_{\hat{\alpha}} \Lambda$. Since $C^*(\mathcal{G}) \times_\alpha \hat{\Lambda}$ is an AF algebra and that $\Lambda = \mathbb{Z}^2$ is amenable, $C^*(\mathcal{G})$ is nuclear and classifiable. We prove that $C^*(\mathcal{G}) = C^*_r(\mathcal{G})$. From Theorem 4.2 we get that the fixed-point algebra $C^*(\mathcal{G}_0)$ is an AF algebra. The inclusion $C_c(\mathcal{G}_0) \subseteq C_c(\mathcal{G}) \subseteq C^*(\mathcal{G})$ extends to an injective $*$-homomorphism $C^*(\mathcal{G}_0) \subseteq C^*(\mathcal{G})$ (injectivity follows since $C^*(\mathcal{G}_0)$ is an AF algebra). Since $C^*(\mathcal{G}_0) = C^*_r(\mathcal{G}_0)$, it follows that $C^*(\mathcal{G}_0) \subseteq C^*_r(\mathcal{G})$. Let E be the conditional expectation of $C^*(\mathcal{G})$ onto $C^*(\mathcal{G}_0)$ and λ be the canonical map of $C^*(\mathcal{G})$ onto $C^*_r(\mathcal{G})$. IF E^r is the conditional expectation of $C^*_r(\mathcal{G})$ onto $C^*(\mathcal{G}_0)$, then $E^r \circ \lambda = E$. It then follows that λ is injective. Therefore $C^*(\mathcal{G}) = C^*_r(\mathcal{G})$. Simplicity and pure infiniteness follow from Corollary 6.3. $\square$

Remark 6.5. Kirchberg–Phillips classification theorem states that simple, unital, purely infinite, and nuclear C^*-algebras are classified by their K-theory [12]. In the continuation of this project, we wish to compute the K-theory of $C^*(\mathbb{Z}^n)$.

Another interest is to generalize the study and/or the result to a more general ordered group or even a “larger” group, such as $\mathbb{R}^n$

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