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Typicality of normal numbers with respect to the Cantor series expansion

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Abstract. Fix a sequence of integers $Q = \{q_n\}_{n=1}^{\infty}$ such that q_n is greater than or equal to 2 for all n. In this paper, we improve upon results by J. Galambos and F. Schweiger showing that almost every (in the sense of Lebesgue measure) real number in $[0, 1)$ is Q-normal with respect to the Q-Cantor series expansion for sequences Q that satisfy a certain condition. We also provide asymptotics describing the number of occurrences of blocks of digits in the Q-Cantor series expansion of a typical number. The notion of strong Q-normality, that satisfies a similar typicality result, is introduced. Both of these notions are equivalent for the b-ary expansion, but strong normality is stronger than normality for the Cantor series expansion. In order to show this, we provide an explicit construction of a sequence Q and a real number that is Q-normal, but not strongly Q-normal. We use the results in this paper to show that under a mild condition on the sequence Q, a set satisfying a weaker notion of normality, studied by A. Rényi, 1956, will be dense in $[0, 1)$.

Contents

1. Introduction

Definition 1.1. Let b and k be positive integers. A *block of length k in base b* is an ordered k-tuple of integers in $\{0, 1, \ldots, b-1\}$. A *block of length*

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k is a block of length k in some base b. A *block* is a block of length k in base b for some integers k and b.

Definition 1.2. Given an integer $b \geq 2$, the *b-ary expansion* of a real x in $[0,1)$ is the (unique) expansion of the form

$$x = \sum_{n=1}^{\infty} \frac{E_n}{b^n} = 0.E_1E_2E_3\ldots$$

such that E_n is in $\{0, 1, \ldots, b-1\}$ for all n with $E_n \neq b-1$ infinitely often.

Denote by $N_n^b(B,x)$ the number of times a block B occurs with its starting position no greater than n in the b-ary expansion of x.

Definition 1.3. A real number x in $[0,1)$ is *normal in base b* if for all k and blocks B in base b of length k, one has

$$\lim_{n\to\infty} \frac{N_n^b(B,x)}{n} = b^{-k}. \tag{1}$$

A number x is *simply normal in base b* if (1) holds for $k = 1$.

Borel introduced normal numbers in 1909 and proved that almost all (in the sense of Lebesgue measure) real numbers in $[0,1)$ are normal in all bases. The best known example of a number that is normal in base 10 is due to Champernowne [3]. The number

$$H_{10} = 0.1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ldots,$$

formed by concatenating the digits of every natural number written in increasing order in base 10, is normal in base 10. Any H_b, formed similarly to H_{10} but in base b, is known to be normal in base b. Since then, many examples have been given of numbers that are normal in at least one base. One can find a more thorough literature review in [4] and [5].

The Q-Cantor series expansion, first studied by Georg Cantor in [9], is a natural generalization of the b-ary expansion.

Definition 1.4. $Q = \{q_n\}_{n=1}^{\infty}$ is a *basic sequence* if each q_n is an integer greater than or equal to 2.

Definition 1.5. Given a basic sequence Q, the *Q-Cantor series expansion* of a real x in $[0,1)$ is the (unique) expansion of the form

$$x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n} \tag{2}$$

such that E_n is in $\{0, 1, \ldots, q_n - 1\}$ for all n with $E_n \neq q_n - 1$ infinitely often. We abbreviate (2) with the notation $x = 0.E_1E_2E_3\ldots$ with respect to Q.

Clearly, the b-ary expansion is a special case of (2) where $q_n = b$ for all n. If one thinks of a b-ary expansion as representing an outcome of repeatedly rolling a fair b-sided die, then a Q-Cantor series expansion may be thought of as representing an outcome of rolling a fair q_1 sided die, followed by a fair q_2 sided die and so on. For example, if $q_n = n+1$ for all n, then the Q-Cantor series expansion of $e-2$ is

$$e - 2 = \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots .$$

If $q_n = 10$ for all n, then the Q-Cantor series expansion for $1/4$ is

$$\frac{1}{4} = \frac{2}{10} + \frac{5}{10^2} + \frac{0}{10^3} + \frac{0}{10^4} + \cdots .$$

For a given basic sequence Q, let $N_n^Q(B,x)$ denote the number of times a block B occurs starting at a position no greater than n in the Q-Cantor series expansion of x. Additionally, define

$$Q_n^{(k)} = \sum_{j=1}^{n} \frac{1}{q_j q_{j+1} \cdots q_{j+k-1}}.$$

A. Rényi [7] defined a real number x to be normal with respect to Q if for all blocks B of length 1,

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n^{(1)}} = 1. \tag{3}$$

If $q_n = b$ for all n, then (3) is equivalent to *simple normality in base* b, but not equivalent to *normality in base* b. Thus, we want to generalize normality in a way that is equivalent to normality in base b when all $q_n = b$.

Definition 1.6. A real number x is *Q-normal of order k* if for all blocks B of length k,

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n^{(k)}} = 1.$$

We say that x is *Q-normal* if it is Q-normal of order k for all k. A real number x is *Q-ratio normal of order k* if for all blocks B and B' of length k, we have

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{N_n^Q(B',x)} = 1.$$

x is *Q-ratio normal* if it is Q-ratio normal of order k for all positive integers k.

We make the following definitions:

Definition 1.7. A basic sequence Q is *k-divergent* if $\lim_{n\to\infty} Q_n^{(k)} = \infty$. Q is *fully divergent* if Q is k-divergent for all k. Q is *k-convergent* if it is not k-divergent.

Definition 1.8. A basic sequence Q is *infinite in limit* if $q_n \to \infty$.

For Q that are infinite in limit, it has been shown that the set of all x in $[0,1)$ that are Q-normal of order k has full Lebesgue measure if and only if Q is k-divergent [7]. Therefore, if Q is infinite in limit, then the set of all x in $[0,1)$ that are Q-normal has full Lebesgue measure if and only if Q is fully divergent. Suppose that Q is 1-divergent. Given an arbitrary nonnegative integer a, F. Schweiger [8] proved that for almost every x with $\epsilon > 0$, one has

$$N_n((a),x) = Q_n^{(1)} + O\left(\sqrt{Q_n^{(1)}} \cdot \log^{3/2+\epsilon} Q_n^{(1)}\right).$$

J. Galambos proved an even stronger result in [10]. He showed that for almost every x in $[0,1)$ and for all nonnegative integers a,

$$N_n^Q((a),x) = Q_n^{(1)} + O\left(\sqrt{Q_n^{(1)}} \left(\log\log Q_n^{(1)}\right)^{1/2}\right).$$

We provide the following main results:

1. (1) A notion of strong Q-normality is provided and we construct an explicit example of a basic sequence Q and a real number that is Q-normal, but not strongly Q-normal (Theorem 2.15).
2. (2) (Theorem 4.9) If Q is a basic sequence that is infinite in limit and B is a block of length k, then for almost every real number x in $[0,1)$, we have
$$N_n^Q(B,x) = Q_n^{(k)} + O\left(\sqrt{Q_n^{(k)}} \left(\log\log Q_n^{(k)}\right)^{1/2}\right).$$
3. (3) If Q is infinite in limit, then almost every real number is Q-normal of order k if and only if Q is k-divergent (Theorem 4.11).
4. (4) If Q is k-convergent for some k, then the set of numbers that are Q-normal is empty (Proposition 5.1). If Q is infinite in limit, then the set of Q-ratio normal numbers is dense in $[0,1)$ (Corollary 5.3).

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2. Strongly normal numbers

2.1. Basic definitions and results. In this section, we will introduce a notion of normality that is stronger than Q-normality. This notion of normality will arise naturally later in this paper and will be useful for studying the typicality of Q-normal numbers. We will first need to make definitions similar to those of $N_n^Q(B,x)$ and $Q_n^{(k)}$.

Given a real number $x \in [0,1)$, a basic sequence Q, a block B of length k, a positive integer $p \in [1,k]$, and a positive integer n, we will denote by $N_{n,p}^Q(B,x)$ the number of times that the block B occurs in the Q-Cantor series expansion of x with starting position of the form $j \cdot k + p$ for $0 \le j < \frac{n}{k}$.

If n and k are positive integers, define

$$\rho(n,k) = \lceil n/k \rceil - 1 = \max\left\{ i \in \mathbb{Z} : i < \frac{n}{k} \right\}.$$

Suppose that Q is a basic sequence and that n, p, and k are positive integers with $p \in [1,k]$. We will write

$$Q_{n,p}^{(k)} = \sum_{j=0}^{\rho(n,k)} \frac{1}{q_{jk+p}q_{jk+p+1}\cdots q_{jk+p+k-1}}.$$

Definition 2.1. Let k be a positive integer. Then a basic sequence Q is *strongly k-divergent*[1] if for all positive integers p with $p \in [1,k]$, we have $\lim_{n\to\infty} Q_{n,p}^{(k)} = \infty$. A basic sequence Q is *strongly fully divergent* if it is strongly k-divergent for all k.

Given a real number $x \in [0,1)$, a basic sequence Q, a block B of length k, a positive integer $p \in [1,k]$, and a positive integer n, we will denote by $N_{n,p}^{Q}(B,x)$ the number of times the block B occurs in the Q-Cantor series expansion of x with positions of the form $j \cdot k + p$ for $0 \leq j < \frac{n}{k}$.

Definition 2.2. Suppose that Q is a basic sequence. A real number x in $[0,1)$ is *strongly Q-normal of order k* if for all blocks B of length $m \leq k$ and all $p \in [1,m]$, we have

$$\lim_{n\to\infty} \frac{N_{n,p}^{Q}(B,x)}{Q_{n,p}^{(m)}} = 1.$$

A real number x is *strongly Q-normal* if it is strongly Q normal of order k for all k.

We will use the following lemmas frequently and without mention:

Lemma 2.3. *Given a real number $x \in [0,1)$, a basic sequence Q, a block B of length k, a positive integer $p \in [1,k]$, and a positive integer n, we have*

$$N_{n,1}^{Q}(B,x) + N_{n,2}^{Q}(B,x) + \cdots + N_{n,k}^{Q}(B,x) = N_{n}^{Q}(B,x) + O(1) \text{ and}$$

$$Q_{n,1}^{(k)} + Q_{n,2}^{(k)} + \cdots + Q_{n,k}^{(k)} = Q_{n}^{(k)} + O(1).$$

Proof. This follows directly from the definitions of $N_n^Q(B,x)$ and $Q_n^{(k)}$. □

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[1]It is not true that k-divergent basic sequences must be strongly k-divergent. The following example of a 2-divergent basic sequence that is not strongly 2-divergent was suggested by C. Altomare (verbal communication): let the basic sequence $Q = \{q_n\}$ be given by

$$q_n = \begin{cases} \max(2, \lfloor n^{1/4} \rfloor) & \text{if } n \equiv 0 \pmod 4 \\ \max(2, \lfloor n^{1/4} \cdot \log^2 n \rfloor) & \text{if } n \equiv 1 \pmod 4 \\ \max(2, \lfloor n^{3/4} \rfloor) & \text{if } n \equiv 2 \pmod 4 \\ \max(2, \lfloor n^{3/4} \cdot \log^2 n \rfloor) & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Lemma 2.4. *If $g_1, g_2, \dots, g_n$ are nonnegative functions on the natural numbers, then*

$$o(g_1) + o(g_2) + \cdots + o(g_n) = o(g_1 + g_2 + \cdots + g_n).$$

Theorem 2.5. *If Q is a basic sequence and x is strongly Q-normal of order k, then x is Q-normal of order k.*

Proof. Let $m \leq k$ be a positive integer and let B be a block of length k. Since x is strongly Q-normal of k, we know that for all $p \in [1, m]$, $N_{n,p}^Q(B, x) = Q_{n,p}^{(k)} + o\left(Q_{n,p}^{(k)}\right)$. Thus, we see that

$$N_n^Q(B, x) = \sum_{p=1}^{m} N_{n,p}^Q(B, x) = \sum_{p=1}^{m} \left(Q_{n,p}^{(k)} + o\left(Q_{n,p}^{(k)}\right)\right)$$
$$= \sum_{p=1}^{m} Q_{n,p}^{(k)} + o\left(\sum_{p=1}^{m} Q_{n,p}^{(k)}\right) = Q_n^{(k)} + o\left(Q_n^{(k)}\right),$$

so $\lim_{n\to\infty} \frac{N_n^Q(B,x)}{Q_n^{(k)}} = 1$. Therefore, x is Q-normal of order k. □

Corollary 2.6. *Suppose that Q is a basic sequence. If x is strongly Q-normal, then x is Q-normal.*

2.2. Construction of a number that is Q-normal, but not strongly Q-normal of order 2. In this subsection, we will work towards giving an example of a basic sequence Q and a real number x that is Q-normal, but not strongly Q-normal of order 2. We will use the conventions found in [6].

Given a block B, $|B|$ will represent the length of B. Given nonnegative integers $l_1, l_2, \dots, l_n$, at least one of which is positive, and blocks $B_1, B_2, \dots, B_n$, the block $B = l_1 B_1 l_2 B_2 \dots l_n B_n$ will be the block of length $l_1|B_1| + \cdots + l_n|B_n|$ formed by concatenating l_1 copies of B_1, l_2 copies of B_2, through l_n copies of B_n. For example, if $B_1 = (2, 3, 5)$ and $B_2 = (0, 8)$, then $2B_1 1B_2 0B_2 = (2, 3, 5, 2, 3, 5, 0, 8)$. We will need the following definitions:

Definition 2.7. A *weighting* μ is a collection of functions $\mu^{(1)}, \mu^{(2)}, \mu^{(3)}, \dots$ with $\sum_{j=0}^{\infty} \mu^{(1)}(j) = 1$ such that for all k, $\mu^{(k)} : \{0, 1, 2, \dots\}^k \to [0, 1]$ and $\mu^{(k)}(b_1, b_2, \dots, b_k) = \sum_{j=0}^{\infty} \mu^{(k+1)}(b_1, b_2, \dots, b_k, j)$.

Definition 2.8. The *uniform weighting in base b* is the collection λ_b of functions $\lambda_b^{(1)}, \lambda_b^{(2)}, \lambda_b^{(3)}, \dots$ such that for all k and blocks B of length k in base b

$$\lambda_b^{(k)}(B) = b^{-k}. \tag{4}$$

Definition 2.9. Let p and b be positive integers such that $1 \leq p \leq b$. A weighting μ is (p, b)-*uniform* if for all k and blocks B of length k in base p, we have

$$\mu^{(k)}(B) = \lambda_b^{(k)}(B) = b^{-k}. \tag{5}$$

Given blocks B and y, let $N(B,y)$ be the number of occurrences of the block B in the block y.

Definition 2.10. Let ϵ be a real number such that $0<\epsilon<1$ and let k be a positive integer. Assume that μ is a weighting. A block of digits y is (ϵ,k,μ)-*normal*[2] if for all blocks B of length $m\le k$, we have

$$\mu^{(m)}(B)|y|(1-\epsilon)\le N(B,y)\le \mu^{(m)}(B)|y|(1+\epsilon). \tag{6}$$

For the rest of this subsection, we use the following conventions. Given sequences of nonnegative integers $\{l_i\}_{i=1}^\infty$ and $\{b_i\}_{i=1}^\infty$ with each $b_i\ge 2$ and a sequence of blocks $\{x_i\}_{i=1}^\infty$, we set

$$L_i=|l_1x_1\dots l_ix_i|=\sum_{j=1}^{i} l_j|x_j|, \tag{7}$$

$$q_n=b_i \text{ for } L_{i-1}<n\le L_i, \tag{8}$$

and

$$Q=\{q_n\}_{n=1}^\infty. \tag{9}$$

Moreover, if $(E_1,E_2,\dots)=l_1x_1l_2x_2\dots$, we set

$$x=\sum_{n=1}^\infty \frac{E_n}{q_1q_2\cdots q_n}. \tag{10}$$

Given $\{q_n\}_{n=1}^\infty$ and $\{l_i\}_{i=1}^\infty$, it is assumed that x and Q are given by the formulas above.

Definition 2.11. A *block friendly family* is a 6-tuple

$$W=\{(l_i,b_i,p_i,\epsilon_i,k_i,\mu_i)\}_{i=1}^\infty$$

with nondecreasing sequences $\{l_i\}_{i=1}^\infty$, $\{b_i\}_{i=1}^\infty$, $\{p_i\}_{i=1}^\infty$ and $\{k_i\}_{i=1}^\infty$ of nonnegative integers for which $b_i\ge 2$, $b_i\to\infty$ and $p_i\to\infty$, such that $\{\mu_i\}_{i=1}^\infty$ is a sequence of (p_i,b_i)-uniform weightings and $\{\epsilon_i\}_{i=1}^\infty$ strictly decreases to 0.

Definition 2.12. Let $W=\{(l_i,b_i,p_i,\epsilon_i,k_i,\mu_i)\}_{i=1}^\infty$ be a block friendly family. If $\lim k_i=K<\infty$, then let $R(W)=\{0,1,2,\dots,K\}$. Otherwise, let $R(W)=\{0,1,2,\dots\}$. A sequence $\{x_i\}_{i=1}^\infty$ of (ϵ_i,k_i,μ_i)-normal blocks of nondecreasing length is said to be W-*good* if for all k in R, the following three conditions hold:

$$\frac{b_i^k}{\epsilon_{i-1}-\epsilon_i}=o(|x_i|); \tag{11}$$

$$\frac{l_{i-1}}{l_i}\cdot\frac{|x_{i-1}|}{|x_i|}=o(i^{-1}b_i^{-k}); \tag{12}$$

[2] Definition 2.10 is a generalization of the concept of (ϵ,k)-normality, originally due to Besicovitch [2].

$$\frac{1}{l_i} \cdot \frac{|x_{i+1}|}{|x_i|} = o(b_i^{-k}). \tag{13}$$

We now state a key theorem of [6].

Theorem 2.13. *Let W be a block friendly family and $\{x_i\}_{i=1}^{\infty}$ a W-good sequence. If $k \in R(W)$, then x is Q-normal of order k. If $k_i \to \infty$, then x is Q-normal.*

If b and w are positive integers where b is greater than or equal to 2 and $w \geq 3$ is odd, then we let $C_{b,w}$ be one of the blocks formed by concatenating all the blocks of length w in base b in such a way that there are at least twice as many copies of the block (0) at odd positions as the block (1). For example, we could pick

$$\begin{aligned} C_{2,3} &= 1(0,0,0)1(1,0,1)1(0,1,0)1(0,0,1)1(0,1,1)1(1,0,0)1(1,1,0)1(1,1,1) \\ &= (0,0,0,1,0,1,0,1,0,0,0,1,0,1,1,1,0,0,1,1,0,1,1,1), \end{aligned}$$

which has 9 copies of (0) at the odd positions and 3 copies of (1) at the odd positions. Note that $|C_{b,w}| = wb^w$. The next lemma is proven identically to Lemma 4.2 in [6]:

Lemma 2.14. *If $K < w$ and $\epsilon = \frac{K}{w}$, then $C_{b,w}$ is (ϵ, K, λ_b)-normal.*

Theorem 2.15. [3] *There exists a basic sequence Q and a real number x such that x is Q-normal, but not strongly Q-normal of order 2.*

Proof. Let $x_1 = (0,1)$, $b_1 = 2$, and $l_1 = 0$. For $i \geq 2$, let $x_i = C_{2i,(2i+1)^2}$, $b_i = 2i$, and $l_i = (2i)^{9i+8}$. Set $\epsilon_1 = 1/2$, $k_1 = 1$, $p_1 = 2$ and $\mu_1 = \lambda_2$. For $i \geq 2$, put $\epsilon_i = 1/(2i+1)$, $k_i = 2i+1$, $p_i = b_i$, $\mu_i = \lambda_{2i}$, and

$$W = \{(l_i, b_i, p_i, \epsilon_i, k_i, \mu_i)\}_{i=1}^{\infty}.$$

Thus, since $x_i = C_{b,w}$ where $b = 2i$ and $w = (2i+1)^2$, x_i is $(\epsilon_i, k_i, \lambda_{b_i})$-normal by Lemma 2.14.

In order to show that $\{x_i\}$ is a W-good sequence we need to verify (11), (12), and (13). Since $k_i \to \infty$, we let k be an arbitrary positive integer. We will make repeated use of the fact that $|x_i| = (2i+1)^2 \cdot (2i)^{(2i+1)^2}$. We first verify (11):

$$\lim_{i\to\infty} |x_i| \bigg/ \left(\frac{(2i)^k}{\frac{1}{2(i-1)+1} - \frac{1}{2i+1}} \right) = \lim_{i\to\infty} \frac{2(2i+1)^2 \cdot (2i)^{(2i+1)^2}}{(2i)^k \cdot (4i^2-1)} = \infty.$$

[3]Theorem 2.13 may be used to construct other explicit examples of Q-normal numbers that satisfy some unusual conditions. Given a basic sequence Q, we say that x is *Q-distribution normal* if the sequence $\{q_1 q_2 \cdots q_n x\}_n$ is uniformly distributed mod 1. [1] uses Theorem 2.13 to give an example of a basic sequence Q and a real number x such that x is Q-normal, but $q_1 q_2 \cdots q_n x \pmod 1 \to 0$, so x is not Q-distribution normal.

We next verify (12). Since $l_{i-1}/l_i < 1$, $(2i-1)^2/(2i+1)^2 < 1$ and

$$\left(1-\frac{1}{i}\right)^{(2i+1)^2} < e^{-2(2i+1)},$$

we have

$$\begin{aligned}\lim_{i\to\infty}\frac{\frac{l_{i-1}}{l_i}\cdot\frac{x_{i-1}}{x_i}}{i^{-1}(2i)^{-k}} &\leq \lim_{i\to\infty} i\cdot(2i)^k\cdot 1\cdot\frac{(2i-1)^2}{(2i+1)^2}\cdot\frac{(2i-2)^{(2i-1)^2}}{(2i)^{(2i+1)^2}}\\ &\leq \lim_{i\to\infty} i(2i)^k\cdot 1\cdot\left(1-\frac{1}{i}\right)^{(2i+1)^2}\cdot(2i-2)^{-8i}\\ &\leq \lim_{i\to\infty} i(2i+1)^k e^{-2(2i+1)}(2i-2)^{-8i} = 0.\end{aligned}$$

Lastly, we verify (13). Since $(2i+3)^2/(2i+1)^2 \leq 2$, $(1+2/(2i+1))^{8i} < e^8$, and

$$\left(1+\frac{2}{2i+1}\right)^{(2i+1)^2} < 2e^{2(2i+1)},$$

we have

$$\begin{aligned}\lim_{i\to\infty}\frac{\frac{1}{l_i}\cdot\frac{|x_{i+1}|}{|x_i|}}{(2i)^{-k}} &= \lim_{i\to\infty}(2i)^{-9i-8+k}\cdot\frac{(2i+3)^2}{(2i+1)^2}\cdot\frac{(2i+2)^{(2i+3)^2}}{(2i)^{(2i+1)^2}}\\ &\leq \lim_{i\to\infty}(2i)^{-9i-8+k}\cdot 2\cdot\left(1+\frac{1}{i}\right)^{(2i+1)^2}\cdot(2i+2)^{(8i+8)}\\ &\leq \lim_{i\to\infty} 4e^{2(2i+1)}\left(1+\frac{1}{i}\right)^{8i+8}(2i)^{-i+k}\\ &\leq \lim_{i\to\infty} 4e^{2(2i+1)+8}\cdot(2i)^{-i+k} = 0.\end{aligned}$$

Since λ_{b_i} is (p_i, b_i)-uniform, $\{x_i\}$ is a W-good sequence and by Theorem 2.13, x is Q-normal.

Since the length of each block x_i is even, so there will always be at least twice as many copies of the block (0) as the block (1) in any initial segment of digits of x, so x is not strongly Q-normal of order 2. □

3. Random variables associated with normality

For this section, we must recall a few basic notions from probability theory. Given a random variable X, we will denote the expected value of X as $\mathrm{E}[X]$. We will denote the variance of X as $\mathrm{Var}[X]$. Lastly, $\mathrm{P}(X=j)$ will represent the probability that $X=j$.

We consider x as a random variable which has uniform distribution on the interval $[0,1)$. If $x = 0.E_1(x)E_2(x)E_3(x)\ldots$ with respect to Q, then we consider $E_1(x), E_2(x), E_3(x), \ldots$ to be random variables. So for all n, we have

$$\mathrm{P}\left(E_n(x)=j\right)=\begin{cases}\frac{1}{q_n} & \text{if } 0\le j\le q_n-1\\ 0 & \text{if } j\ge q_n.\end{cases}$$

Lemma 3.1. *If Q is a basic sequence, then the random variables $E_1(x)$, $E_2(x)$, $E_3(x),\dots$ are independent.*

Proof. Suppose n_1 and n_2 are distinct positive integers and $0\le F_j<q_j-1$ for all j. Then

$$\begin{aligned}&\mathrm{P}\left(E_{n_1}(x)=F_{n_1},E_{n_2}(x)=F_{n_2}\right)\\ &\quad=\lambda\left(\{x\in[0,1):E_{n_1}(x)=F_{n_1}\text{ and }E_{n_2}(x)=F_{n_2}\}\right)\\ &\quad=\frac{1}{q_{n_1}q_{n_2}}=\frac{1}{q_{n_1}}\cdot\frac{1}{q_{n_2}}=\mathrm{P}\left(E_{n_1}(x)=F_{n_1}\right)\cdot\mathrm{P}\left(E_{n_2}(x)=F_{n_2}\right).\end{aligned}$$ □

Suppose that Q is a basic sequence, b is a natural number, B is a block of length k, and $m=ik+p$ is an integer with $p\in[0,k-1]$. We set

$$\zeta_{b,n}^{Q}(x)=\begin{cases}1 & \text{if } E_n(x)=b\\ 0 & \text{if } E_n(x)\ne n,\end{cases}$$

$$\zeta_{B,i,p}^{Q}(x)=\begin{cases}1 & \text{if } E_{ik+p,k}(x)=B\\ 0 & \text{if } E_{ik+p,k}(x)\ne B,\end{cases}$$

$$F_m^{(k)}=\mathrm{E}\left[\zeta_{B,i,p}^{Q}(x)\right],\quad V_m^{(k)}=\mathrm{Var}\left[\zeta_{B,i,p}^{Q}(x)\right],\quad t_{n,p}^{(k)}=\sum_{i=0}^{\rho(n,k)}V_{ik+p}^{(k)}.$$

Lemma 3.2. *For all non-negative integers b, the random variables $\zeta_{b,1}^{Q}(x)$, $\zeta_{b,2}^{Q}(x)$, $\zeta_{b,3}^{Q}(x),\dots$ are independent.*

Proof. This follows directly from Lemma 3.1 as the random variables $E_1(x)$, $E_2(x)$, $E_3(x),\dots$ are independent. □

Lemma 3.3. *If $B=(b_1,b_2,\dots,b_k)$ is a block of length k, then*

$$\zeta_{B,i,p}^{Q}(x)=\zeta_{b_1,ik+p}^{Q}(x)\cdot\zeta_{b_2,ik+p+1}^{Q}(x)\cdots\zeta_{b_k,ik+p+k-1}^{Q}(x).$$

Proof. By definition,

$$\zeta_{B,i,p}^{Q}(x)=\begin{cases}1 & \text{if } E_{ik+p,k}=B\\ 0 & \text{if } E_{ik+p,k}\ne B.\end{cases}$$

In other words, $\zeta_{B,i,p}^{Q}(x)=1$ if

$$\zeta_{b_1,ik+p}^{Q}(x)=\zeta_{b_2,ik+p+1}^{Q}(x)=\cdots=\zeta_{b_k,ik+p+k-1}^{Q}(x)=1$$

and $\zeta_{B,i,p}^{Q}(x)=0$ otherwise. □

Corollary 3.4. *For all blocks $B=(b_1,b_2,\dots,b_k)$ of length k and nonnegative integers $p_1,p_2\in[1,k]$, i_1, and i_2 with $(i_1,p_1)\neq(i_2,p_2)$, the random variables $\zeta^Q_{B,i_1,p_1}(x)$ and $\zeta^Q_{B,i_2,p_2}(x)$ are independent.*

Proof. Using Lemma 3.2 and Lemma 3.3, we see that

$$\begin{aligned}
&\mathrm{E}\left[\zeta^Q_{B,i_1,p_1}(x)\cdot\zeta^Q_{B,i_2,p_2}(x)\right]\\
&=\mathrm{E}\left[\left(\Pi_{j=0}^{k-1}\zeta^Q_{b_j,i_1k+p_1+j}(x)\right)\cdot\left(\Pi_{j=0}^{k-1}\zeta^Q_{b_j,i_2k+p_2+j}(x)\right)\right]\\
&=\left(\Pi_{j=0}^{k-1}\mathrm{E}\left[\zeta^Q_{b_j,i_1k+p_1+j}(x)\right]\right)\cdot\left(\Pi_{j=0}^{k-1}\mathrm{E}\left[\zeta^Q_{b_j,i_2k+p_2+j}(x)\right]\right)\\
&=\mathrm{E}\left[\Pi_{j=0}^{k-1}\zeta^Q_{b_j,i_1k+p_1+j}(x)\right]\cdot\mathrm{E}\left[\Pi_{j=0}^{k-1}\zeta^Q_{b_j,i_2k+p_2+j}(x)\right]\\
&=\mathrm{E}\left[\zeta^Q_{B,i_1,p_1}(x)\right]\cdot\mathrm{E}\left[\zeta^Q_{B,i_2,p_2}(x)\right]. \qquad \square
\end{aligned}$$

Lemma 3.5. *If $B=(b_1,b_2,\dots,b_k)$ is a block of length k, then*

$$F_m^{(k)}=\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}}\ \textit{and}$$

$$V_m^{(k)}=\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}}-\left(\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}}\right)^2.$$

Proof. We first compute the expected value of $\zeta^Q_{B,i,p}(x)$. By Lemma 3.2 and Lemma 3.3, we see that

$$\begin{aligned}
\mathrm{E}\left[\zeta^Q_{B,i,p}(x)\right]&=\mathrm{E}\left[\zeta^Q_{b_1,ik+p}(x)\cdot\zeta^Q_{b_2,ik+p+1}(x)\cdots\zeta^Q_{b_k,ik+p+k-1}(x)\right]\\
&=\mathrm{E}\left[\zeta^Q_{b_1,ik+p}(x)\right]\cdot\mathrm{E}\left[\zeta^Q_{b_2,ik+p+1}(x)\right]\cdots\mathrm{E}\left[\zeta^Q_{b_k,ik+p+k-1}(x)\right]\\
&=\frac{1}{q_{ik+p}}\cdot\frac{1}{q_{ik+p+1}}\cdots\frac{1}{q_{ik+p+k-1}}\\
&=\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}}.
\end{aligned}$$

Next, we recall that $\mathrm{Var}\left[\zeta^Q_{B,i,p}(x)\right]=\mathrm{E}\left[\zeta^Q_{B,i,p}(x)^2\right]-\mathrm{E}\left[\zeta^Q_{B,i,p}(x)\right]^2$. Since $\zeta^Q_{B,i,p}(x)$ may only be 0 or 1, we see that $\left(\zeta^Q_{B,i,p}(x)\right)^2=\zeta^Q_{B,i,p}(x)$, so

$$\begin{aligned}
&\mathrm{Var}\left[\zeta^Q_{B,i,p}(x)\right]\\
&=\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}}-\left(\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}}\right)^2. \qquad \square
\end{aligned}$$

Lastly, we remark that $Q_{n,p}^{(k)}=\sum_{i=0}^{\rho(n,k)}F_{ik+p}^{(k)}$ by Lemma 3.5 and will use this fact frequently and without mention.

4. Typicality of normal numbers

We will need the following:

Theorem 4.1. [4] *Let $X_1, X_2, \ldots, X_n$ be independent random variables. Assume that there exists a constant $c > 0$ such that $|X_j| < c$ for all j. Let $G_j = E[X_j], U_j = Var[X_j]$, and $t_n = \sum_{j=1}^n U_j$. If $t_n \to \infty$, then, with probability one,*

$$\limsup_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n - G_1 - G_2 - \ldots G_n}{\sqrt{2t_n \log\log t_n}} = 1.$$

Corollary 4.2. *Under the same assumptions of Theorem 4.1, with probability one,*

$$X_1 + X_2 + \cdots + X_n = G_1 + G_2 + \cdots + G_n + O\left(t_n^{1/2}(\log\log t_n)^{1/2}\right).$$

We will also need the Borel–Cantelli Lemma:

Theorem 4.3 (The Borel–Cantelli Lemma)**.** *If $\sum_{n=1}^\infty P(A_n) < \infty$, then $P(A_n\ i.o.) = 0$.*

Given a basic sequence Q, we will define $t_{n,p}^{(k)} = \sum_{i=0}^{\rho(n,k)} V_{jk+p}^{(k)}$.

Lemma 4.4. *If Q is a basic sequence and n, k, and p are positive integers with $p \in [1, k]$, then*

$$\frac{1}{2}Q_{n,p}^{(k)} \le t_{n,p}^{(k)} < Q_{n,p}^{(k)}.$$

Proof.

$$t_{n,p}^{(k)} = \sum_{i=0}^{\rho(n,k)} \left(\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} - \left(\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}}\right)^2\right)$$
$$< \sum_{i=0}^{\rho(n,k)} \frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} = \sum_{i=0}^{\rho(n,k)} F_{ik+p}^{(k)} = Q_{n,p}^{(k)}.$$

To show the other direction of the inequality, we recall that since Q is a basic sequence, $q_m \ge 2$ for all m, so for all i

$$\sum_{i=0}^{\rho(n,k)} \left(\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} - \left(\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}}\right)^2\right)$$
$$\ge \sum_{i=0}^{\rho(n,k)} \left(\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} - \frac{1}{2^k}\left(\frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}}\right)\right)$$
$$\ge \sum_{i=0}^{\rho(n,k)} \frac{1}{2} \cdot \frac{1}{q_{ik+p}q_{ik+p+1}\cdots q_{ik+p+k-1}} = \frac{1}{2}Q_{n,p}^{(k)}.$$

□

[4] See, for example, [11].

Lemma 4.5. *If Q is infinite in limit and B is a block of length k, then for almost every real number x in $[0,1)$, we have*

$$N_{n,p}^Q(B,x) = Q_{n,p}^{(k)} + O\left(\sqrt{Q_{n,p}^{(k)}}\left(\log\log Q_{n,p}^{(k)}\right)^{1/2}\right). \tag{14}$$

Proof. We consider two cases. The first case is when $\lim_{n\to\infty} Q_{n,p}^{(k)} < \infty$. We see that

$$\lim_{n\to\infty} Q_{n,p}^{(k)} = \lim_{n\to\infty} \sum_{i=0}^{\rho(n,k)} \mathrm{P}\left(\zeta_{B,i,p}^Q = 1\right) < \infty,$$

so by Theorem 4.3, we have $\mathrm{P}\left(\zeta_{B,i,p}^Q = 1 \text{ i.o. }\right) = 0$. Thus, for almost every $x \in [0,1)$, $\lim_{n\to\infty} N_{n,p}^Q(B,x) < \infty$ and (14) holds.

Second, we consider the case where $\lim_{n\to\infty} Q_{n,p}^{(k)} = \infty$. By Lemma 4.4, we have $\lim_{n\to\infty} t_{n,p}^{(k)} \geq \lim_{n\to\infty} Q_{n,p}^{(k)} = \infty$. Note that

$$N_{n,p}^Q(B,x) = \sum_{i=0}^{\rho(n,k)} \zeta_{B,i,p}(x).$$

By Corollary 4.2,

$$N_{n,p}^Q(B,x) = \sum_{i=0}^{\rho(n,k)} F_{ik+p}^{(k)} + O\left(\sqrt{t_{n,p}^{(k)}}\left(\log\log t_{n,p}^{(k)}\right)^{1/2}\right)$$

for almost every $x \in [0,1)$. By Lemma 4.4, $t_{n,p}^{(k)} < Q_{n,p}^{(k)}$, so the lemma follows. □

Lemma 4.5 allows us to prove the following results on strongly normal numbers:

Theorem 4.6. *Suppose that Q is strongly k-divergent and infinite in limit. Then almost every $x \in [0,1)$ is strongly Q-normal of order k.*

Proof. Let B be a block of length $m \leq k$ and $p \in [1,m]$. Then by Lemma 4.5, for almost every $x \in [0,1)$, we have that

$$N_{n,p}^Q(B,x) = Q_{n,p}^{(m)} + O\left(\sqrt{Q_{n,p}^{(m)}}\left(\log\log Q_{n,p}^{(m)}\right)^{1/2}\right),$$

so

$$\frac{N_{n,p}^Q(B,x)}{Q_{n,p}^{(m)}} = 1 + O\left(\frac{\sqrt{Q_{n,p}^{(m)}}\left(\log\log Q_{n,p}^{(m)}\right)^{1/2}}{Q_{n,p}^{(m)}}\right).$$

However, Q is strongly k-divergent, so $Q_{n,p}^{(m)} \to \infty$ and

$$\lim_{n\to\infty} \frac{N_{n,p}^Q(B,x)}{Q_{n,p}^{(m)}} = \lim_{n\to\infty} \left(1 + O\left(\frac{\sqrt{Q_{n,p}^{(m)}}\left(\log\log Q_{n,p}^{(m)}\right)^{1/2}}{Q_{n,p}^{(m)}}\right)\right) = 1.$$

Since there are finitely many choices of m and p and only countably many choices of B, the result follows. □

Corollary 4.7. *If Q is strongly fully divergent and infinite in limit, then almost every real $x \in [0,1)$ is strongly Q-normal.*

We now work towards proving a result much stronger than Corollary 4.7 on the typicality of Q-normal numbers. We will need the following lemma in addition to Lemma 4.5:

Lemma 4.8. *If Q is a basic sequence and k and p are positive integers with $p \in [1,k]$, then*

$$\sum_{p=1}^{k} \left(Q_{n,p}^{(k)} + O\left(\sqrt{Q_{n,p}^{(k)}}\left(\log\log Q_{n,p}^{(k)}\right)^{1/2}\right)\right) = Q_n^{(k)} + O\left(\sqrt{Q_n^{(k)}}\left(\log\log Q_n^{(k)}\right)^{1/2}\right).$$

Proof. We first note that

$$\sum_{p=1}^{k} Q_{n,p}^{(k)} \le Q_n^{(k)} + \left(Q_n^{(k)} - Q_{n-k}^{(k)}\right).$$

Since $Q_n^{(k)} - Q_{n-k}^{(k)} \le (k+1)2^{-k} \to 0$, we see that

$$\sum_{p=1}^{k} Q_{n,p}^{(k)} = Q_n^{(k)} + o(1). \tag{15}$$

Next, note that

$$\sum_{p=1}^{k} \sqrt{Q_{n,p}^{(k)}}\left(\log\log Q_{n,p}^{(k)}\right)^{1/2} \le k\sqrt{\sum_{p=1}^{k} Q_{n,p}^{(k)}}\left(\log\log \sum_{p=1}^{k} Q_{n,p}^{(k)}\right)^{1/2}. \tag{16}$$

By (15) and (16),

$$\sum_{p=1}^{k} O\left(\sqrt{Q_{n,p}^{(k)}}\left(\log\log Q_{n,p}^{(k)}\right)^{1/2}\right) = O\left(\sqrt{Q_n^{(k)}}\left(\log\log Q_n^{(k)}\right)^{1/2}\right). \tag{17}$$

Thus, the lemma follows by combining (15) and (17). □

Theorem 4.9. *If Q is a basic sequence that is infinite in limit and B is a block of length k, then for almost every real number x in $[0,1)$, we have*

$$N_n^Q(B,x) = Q_n^{(k)} + O\left(\sqrt{Q_n^{(k)}}\left(\log\log Q_n^{(k)}\right)^{1/2}\right).$$

Proof. We first note that

$$N_n^Q(B,x) = \sum_{p=1}^{k} N_{n,p}(B,x) + O(1). \tag{18}$$

Thus, by (18) and Lemma 4.5, for almost every $x \in [0,1)$, we have

$$N_n^Q(B,x) = \sum_{p=1}^{k}\left(Q_{n,p}^{(k)} + O\left(\sqrt{Q_{n,p}^{(k)}}\left(\log\log Q_{n,p}^{(k)}\right)^{1/2}\right)\right) + O(1). \tag{19}$$

Thus, the theorem follows by applying Lemma 4.8 to (19). □

We recall the following standard result on infinite products:

Lemma 4.10. *If $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that $0 \le a_n < 1$ for all n, then the infinite product $\prod_{n=1}^{\infty}(1-a_n)$ converges if and only if the sum $\sum_{n=1}^{\infty} a_n$ is convergent.*

Theorem 4.11. *Suppose that Q is a basic sequence that is infinite in limit. Then almost every real number in $[0,1)$ is Q-normal of order k if and only if Q is k-divergent.*

Proof. First, we suppose that Q is k-divergent. Then by Theorem 4.9, for almost every $x \in [0,1)$, we have

$$\lim_{n\to\infty}\frac{N_n^Q(B,x)}{Q_n^{(k)}} = \lim_{n\to\infty}\frac{Q_n^{(k)} + O\left(\sqrt{Q_n^{(k)}}\left(\log\log Q_n^{(k)}\right)^{1/2}\right)}{Q_n^{(k)}} = 1.$$

We now suppose that Q is k-convergent.We will now use similar reasoning to that found in [7]. Set $B = (0,0,\ldots,0)$ (k zeros). We will show that the set of real numbers in $[0,1)$ whose Q-Cantor series expansion does not contain the block B has positive measure. Call this set V. We see that

$$\lambda(V) = \prod_{n=1}^{\infty}\left(1 - \frac{1}{q_n q_{n+1}\cdots q_{n+k-1}}\right).$$

Set $a_n = q_n q_{n+1}\cdots q_{n+k-1}$. Since Q is k-convergent, we have $\sum a_n < \infty$. Thus, $\lambda(V) > 0$ by Lemma 4.10. □

Corollary 4.12. *Suppose that Q is a basic sequence that is infinite in limit. Then almost every real number in $[0,1)$ is Q-normal if and only if Q is fully divergent.*

5. Ratio normal numbers

We are now in a position to compare the prevelance of Q-normal numbers to Q-ratio normal numbers, depending on properties of the basic sequence Q. In particular, we will show that if Q is infinite in limit, then the set of Q-ratio normal numbers is dense in $[0,1)$ even though the set of Q-normal numbers may be empty. Suppose that Q is a k-convergent basic sequence and define

$$Q_\infty^{(k)} = \lim_{n\to\infty} Q_n^{(k)} < \infty. \tag{20}$$

Proposition 5.1. *If Q is a basic sequence that is k-convergent for some k, then the set of Q-normal numbers is empty.*

Proof. We make the observation that since $q_n \geq 2$ for all n, $Q_\infty^{(k)} \leq \frac{1}{2} Q_\infty^{(k-1)}$ for all k. Thus, there exists a $K > 0$ such that for all $k > K$, we have $Q_\infty^{(k)} < 1$. Thus, no blocks of length $k > K$ can occur in any Q-normal number and the set of Q-normal numbers is empty. □

If $B = (b_1, b_2, \cdots, b_k)$ is a block of length k, we write

$$\max(B) = \max(b_1, b_2, \cdots, b_k).$$

If $E = (E_1, E_2, \cdots)$, then set $E_{n,k} = (E_n, E_{n+1}, \cdots, E_{n+k-1})$.

Proposition 5.2. *If $Q = \{q_n\}_{n=1}^\infty$ is infinite in limit, then there exists a real number that is Q-ratio normal.*

Proof. Let $Q' = \{q'_n\}_{n=1}^\infty$ be any fully divergent basic sequence that is infinite in limit. Then we know that there exists a Q'-normal number by Corollary 4.12. Let $x = 0.E'_1 E'_2 E'_3 \ldots$ with respect to Q' be Q'-normal and let $E' = (E'_1, E'_2, \ldots)$. Set

$$M_k = \min\{m : q_n > k \ \forall n \geq m\},$$

$E_n = \min(E'_n, q_n - 1)$, and $E = (E_1, E_2, \ldots)$. Suppose that B and B' are two blocks of length k and let $l = \max(\max(B), \max(B')) + 2$.

Thus, if $n > M_l$, then $E'_{n,k} = B$ is equivalent to $E_{n,k} = B$ and $E'_{n,k} = B'$ is equivalent to $E_{n,k} = B'$. Since x is Q'-normal, there are infinitely many occurences of every block. Additionally, $E_n \leq q_n - 1$ for all n, so $\sum_{n=1}^\infty \frac{E_n}{q_1 q_2 \ldots q_n}$ is Q-ratio normal. □

Corollary 5.3. *If Q is infinite in limit, then the set of numbers that are Q-ratio normal is dense in $[0,1)$.*

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