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The matrix Stieltjes moment problem: a description of all solutions

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Abstract. We describe all solutions of the matrix Stieltjes moment problem in the general case (no conditions besides solvability are assumed). We use Krein's formula for the generalized Π-resolvents of positive Hermitian operators in the form of V. A. Derkach and M. M. Malamud.

Contents

1. Introduction

The matrix Stieltjes moment problem consists of finding a left-continuous nondecreasing matrix function $M(x) = (m_{k,l}(x))_{k,l=0}^{N-1}$ on $\mathbb{R}_+ = [0, +\infty)$, $M(0) = 0$, such that

$$\int_{\mathbb{R}_+} x^n dM(x) = S_n, \qquad n \in \mathbb{Z}_+, \tag{1}$$

where $\{S_n\}_{n=0}^{\infty}$ is a given sequence of Hermitian $(N \times N)$ complex matrices, $N \in \mathbb{N}$. This problem is said to be determinate if there exists a unique solution and indeterminate in the opposite case.

In the scalar ($N = 1$) indeterminate case the Stieltjes moment problem was solved by M. G. Krein (see [8], [9]), while in the scalar degenerate case the problem was solved by F. R. Gantmacher in [7, Chapter XVI].

The operator (and, in particular, the matrix) Stieltjes moment problem was introduced by M. G. Krein and M. A. Krasnoselskiy in [10]. They obtained the necessary and sufficient conditions of solvability for this problem.

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Let us introduce the following matrices

$$\Gamma_n = (S_{i+j})_{i,j=0}^n = \begin{pmatrix} S_0 & S_1 & \dots & S_n \\ S_1 & S_2 & \dots & S_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ S_n & S_{n+1} & \dots & S_{2n} \end{pmatrix}, \tag{2}$$

$$\widetilde{\Gamma}_n = (S_{i+j+1})_{i,j=0}^n = \begin{pmatrix} S_1 & S_2 & \dots & S_{n+1} \\ S_2 & S_3 & \dots & S_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n+1} & S_{n+2} & \dots & S_{2n+1} \end{pmatrix}, \quad n \in \mathbb{Z}_+. \tag{3}$$

The moment problem (1) has a solution if and only if

$$\Gamma_n \geq 0, \quad \widetilde{\Gamma}_n \geq 0, \quad n \in \mathbb{Z}_+. \tag{4}$$

In 2004, Yu. M. Dyukarev performed a deep investigation of the moment problem (1) in the case when

$$\Gamma_n > 0, \quad \widetilde{\Gamma}_n > 0, \quad n \in \mathbb{Z}_+, \tag{5}$$

and some limit matrix intervals (which he called the limit Weyl intervals) are nondegenerate, see [6]. He obtained a parameterization of all solutions of the moment problem in this case.

Our aim here is to obtain a description of all solutions of the moment problem (1) in the general case. No conditions besides the solvability (i.e., conditions (4)) will be assumed. We shall apply an operator approach which was used in [16] and Krein's formula for the generalized Π-resolvents of nonnegative Hermitian operators [14], [11]. We shall use Krein's formula in the form which was proposed by V. A. Derkach and M. M. Malamud in [4]. We should also notice that these authors presented a detailed proof of Krein's formula.

Notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$ the sets of real numbers, complex numbers, positive integers, integers and nonnegative integers, respectively; $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{C}_+ = \{z \in \mathbb{C} : \ \operatorname{Im} z > 0\}$. The space of n-dimensional complex vectors $a = (a_0, a_1, \dots, a_{n-1})$, will be denoted by $\mathbb{C}^n$, $n \in \mathbb{N}$. If $a \in \mathbb{C}^n$ then a^* means the complex conjugate vector. By $\mathbb{P}$ we denote the set of all complex polynomials.

Let $M(x)$ be a left-continuous nondecreasing matrix function $M(x) = (m_{k,l}(x))_{k,l=0}^{N-1}$ on $\mathbb{R}_+$, $M(0) = 0$, and $\tau_M(x) := \sum_{k=0}^{N-1} m_{k,k}(x)$; $\Psi(x) = (dm_{k,l}/d\tau_M)_{k,l=0}^{N-1}$ (the Radon–Nikodym derivative). We denote by $L^2(M)$ a set (of classes of equivalence) of vector functions $f : \mathbb{R} \to \mathbb{C}^N$, $f = (f_0, f_1, \dots, f_{N-1})$, such that (see, e.g., [15])

$$\|f\|^2_{L^2(M)} := \int_{\mathbb{R}} f(x)\Psi(x)f^*(x)d\tau_M(x) < \infty.$$

The space $L^2(M)$ is a Hilbert space with the scalar product

$$(f,g)_{L^2(M)} := \int_{\mathbb{R}} f(x)\Psi(x)g^*(x)d\tau_M(x), \quad f,g \in L^2(M).$$

For a separable Hilbert space H we denote by $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ the scalar product and the norm in H, respectively. The indices may be omitted in obvious cases. By E_H we denote the identity operator in H, i.e., $E_H x = x$, $x \in H$.

For a linear operator A in H we denote by $D(A)$ its domain, by $R(A)$ its range, and by $\ker A$ its kernel. By A^* we denote its adjoint if it exists. By $\rho(A)$ we denote the resolvent set of A; $N_z = \ker(A^* - zE_H)$. If A is bounded, then $\|A\|$ stands for its operator norm. For a set of elements $\{x_n\}_{n\in T}$ in H, we denote by $\mathrm{Lin}\{x_n\}_{n\in T}$ and $\mathrm{span}\{x_n\}_{n\in T}$ the linear span and the closed linear span (in the norm of H), respectively. Here T is an arbitrary set of indices. For a set $M \subseteq H$ we denote by $\bar{M}$ the closure of M with respect to the norm of H.

If H_1 is a subspace of H, by $P_{H_1} = P_{H_1}^H$ we denote the operator of the orthogonal projection on H_1 in H. If $\mathcal{H}$ is another Hilbert space, by $[H,\mathcal{H}]$ we denote the space of all bounded operators from H into $\mathcal{H}$; $[H] := [H,H]$. $\mathfrak{C}(H)$ is the set of closed linear operators A such that $\overline{D(A)} = H$.

2. The matrix Stieltjes moment problem: solvability

Consider the matrix Stieltjes moment problem (1). Let us check that conditions (4) are necessary for the solvability of the problem (1). In fact, suppose that the moment problem has a solution $M(x)$. Choose an arbitrary function $a(x) = (a_0(x), a_1(x), \ldots, a_{N-1}(x))$, where

$$a_j(x) = \sum_{k=0}^{n} \alpha_{j,k} x^k, \quad \alpha_{j,k} \in \mathbb{C},\ n \in \mathbb{Z}_+.$$

This function belongs to $L^2(M)$ and

$$0 \le \int_{\mathbb{R}_+} a(x)dM(x)a^*(x) = \sum_{k,l=0}^{n} \int_{\mathbb{R}_+} (\alpha_{0,k}, \alpha_{1,k}, \ldots, \alpha_{N-1,k}) x^{k+l} dM(x)$$

$$*(\alpha_{0,l}, \alpha_{1,l}, \ldots, \alpha_{N-1,l})^* = \sum_{k,l=0}^{n} (\alpha_{0,k}, \alpha_{1,k}, \ldots, \alpha_{N-1,k}) S_{k+l}$$

$$*(\alpha_{0,l}, \alpha_{1,l}, \ldots, \alpha_{N-1,l})^* = A\Gamma_n A^*,$$

where

$$A = (\alpha_{0,0}, \alpha_{1,0}, \ldots, \alpha_{N-1,0}, \alpha_{0,1}, \alpha_{1,1}, \ldots, \alpha_{N-1,1}, \ldots, \alpha_{0,n}, \alpha_{1,n}, \ldots, \alpha_{N-1,n}),$$

and we have used the rules for the multiplication of block matrices. In a similar manner we get

$$0 \leq \int_{\mathbb{R}_+} a(x) x dM(x) a^*(x) = A\widetilde{\Gamma}_n A^*,$$

and therefore conditions (4) hold.

On the other hand, let the moment problem (1) be given and suppose that conditions (4) are true. For the prescribed moments

$$S_j = (s_{j;k,l})_{k,l=0}^{N-1}, \quad s_{j;k,l} \in \mathbb{C}, \quad j \in \mathbb{Z}_+,$$

we consider the following block matrices

$$\Gamma = (S_{i+j})_{i,j=0}^{\infty} = \begin{pmatrix} S_0 & S_1 & S_2 & \dots \\ S_1 & S_2 & S_3 & \dots \\ S_2 & S_3 & S_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{6}$$

$$\widetilde{\Gamma} = (S_{i+j+1})_{i,j=0}^{\infty} = \begin{pmatrix} S_1 & S_2 & S_3 & \dots \\ S_2 & S_3 & S_4 & \dots \\ S_3 & S_4 & S_5 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{7}$$

The matrix Γ can be viewed as a scalar semi-infinite matrix

$$\Gamma = (\gamma_{n,m})_{n,m=0}^{\infty}, \quad \gamma_{n,m} \in \mathbb{C}. \tag{8}$$

Notice that

$$\gamma_{rN+j,tN+n} = s_{r+t;j,n}, \quad r,t \in \mathbb{Z}_+,\ 0 \leq j,n \leq N-1. \tag{9}$$

The matrix $\widetilde{\Gamma}$ can be also viewed as a scalar semi-infinite matrix

$$\widetilde{\Gamma} = (\widetilde{\gamma}_{n,m})_{n,m=0}^{\infty} = (\gamma_{n+N,m})_{n,m=0}^{\infty}. \tag{10}$$

The conditions in (4) imply that

$$(\gamma_{k,l})_{k,l=0}^{r} \geq 0, \quad r \in \mathbb{Z}_+; \tag{11}$$

$$(\gamma_{k+N,l})_{k,l=0}^{r} \geq 0, \quad r \in \mathbb{Z}_+. \tag{12}$$

We shall use the following important fact (e.g., [2, Supplement 1]):

Theorem 1. *Let $\Gamma = (\gamma_{n,m})_{n,m=0}^{\infty}$, $\gamma_{n,m} \in \mathbb{C}$, be a semi-infinite complex matrix such that condition (11) holds. Then there exist a separable Hilbert space H with a scalar product $(\cdot,\cdot)_H$ and a sequence $\{x_n\}_{n=0}^{\infty}$ in H, such that*

$$\gamma_{n,m} = (x_n, x_m)_H, \quad n,m \in \mathbb{Z}_+, \tag{13}$$

and $\operatorname{span}\{x_n\}_{n=0}^{\infty} = H$.

Proof. Consider an arbitrary infinite-dimensional linear vector space V, e.g., the space of all complex sequences $(u_n)_{n\in\mathbb{Z}_+}$, $u_n\in\mathbb{C}$. Let $X=\{x_n\}_{n=0}^\infty$ be an arbitrary infinite sequence of linear independent elements in V. Let $L=\mathrm{Lin}\{x_n\}_{n\in\mathbb{Z}_+}$ be the linear span of elements of X. Introduce the following functional:

$$[x,y]=\sum_{n,m=0}^\infty \gamma_{n,m}a_n\overline{b_m}, \tag{14}$$

for $x,y\in L$,

$$x=\sum_{n=0}^\infty a_n x_n,\quad y=\sum_{m=0}^\infty b_m x_m,\quad a_n,b_m\in\mathbb{C}.$$

Here and in what follows we assume that for elements of linear spans all but a finite number of coefficients are zero. The space V with $[\cdot,\cdot]$ will be a quasi-Hilbert space. Factorizing and making the completion we obtain the required space H (see [3]). □

From (9) it follows that

$$\gamma_{a+N,b}=\gamma_{a,b+N},\quad a,b\in\mathbb{Z}_+. \tag{15}$$

In fact, if $a=rN+j$, $b=tN+n$, $0\le j,n\le N-1$, $r,t\in\mathbb{Z}_+$, we can write

$$\gamma_{a+N,b}=\gamma_{(r+1)N+j,tN+n}=s_{r+t+1;j,n}=\gamma_{rN+j,(t+1)N+n}=\gamma_{a,b+N}.$$

By Theorem 1 there exist a Hilbert space H and a sequence $\{x_n\}_{n=0}^\infty$ in H, such that $\mathrm{span}\{x_n\}_{n=0}^\infty=H$, and

$$(x_n,x_m)_H=\gamma_{n,m},\quad n,m\in\mathbb{Z}_+. \tag{16}$$

Set $L:=\mathrm{Lin}\{x_n\}_{n=0}^\infty$. Notice that elements $\{x_n\}$ are *not necessarily linearly independent*. Thus, for an arbitrary $x\in L$ there can exist different representations:

$$x=\sum_{k=0}^\infty \alpha_k x_k,\quad \alpha_k\in\mathbb{C}, \tag{17}$$

$$x=\sum_{k=0}^\infty \beta_k x_k,\quad \beta_k\in\mathbb{C}. \tag{18}$$

(Here all but a finite number of coefficients α_k, β_k are zero). Using (15), (16) we can write

$$\begin{aligned}\left(\sum_{k=0}^{\infty}\alpha_k x_{k+N}, x_l\right) &= \sum_{k=0}^{\infty}\alpha_k (x_{k+N}, x_l) = \sum_{k=0}^{\infty}\alpha_k \gamma_{k+N,l} = \sum_{k=0}^{\infty}\alpha_k \gamma_{k,l+N} \\ &= \sum_{k=0}^{\infty}\alpha_k (x_k, x_{l+N}) = \left(\sum_{k=0}^{\infty}\alpha_k x_k, x_{l+N}\right) \\ &= (x, x_{l+N}), \quad l \in \mathbb{Z}_+.\end{aligned}$$

In a similar manner we obtain that

$$\left(\sum_{k=0}^{\infty}\beta_k x_{k+N}, x_l\right) = (x, x_{l+N}), \quad l \in \mathbb{Z}_+,$$

and therefore

$$\left(\sum_{k=0}^{\infty}\alpha_k x_{k+N}, x_l\right) = \left(\sum_{k=0}^{\infty}\beta_k x_{k+N}, x_l\right), \quad l \in \mathbb{Z}_+.$$

Since $\bar{L} = H$, we obtain that

$$\sum_{k=0}^{\infty}\alpha_k x_{k+N} = \sum_{k=0}^{\infty}\beta_k x_{k+N}. \tag{19}$$

Let us introduce the following operator:

$$Ax = \sum_{k=0}^{\infty}\alpha_k x_{k+N}, \quad x \in L,\ x = \sum_{k=0}^{\infty}\alpha_k x_k. \tag{20}$$

Relations (17), (18) and (19) show that this definition does not depend on the choice of a representation for $x \in L$. Thus, this definition is correct. In particular, we have

$$Ax_k = x_{k+N}, \quad k \in \mathbb{Z}_+. \tag{21}$$

Choose arbitrary $x, y \in L$, $x = \sum_{k=0}^{\infty}\alpha_k x_k$, $y = \sum_{n=0}^{\infty}\gamma_n x_n$, and write

$$\begin{aligned}(Ax, y) &= \left(\sum_{k=0}^{\infty}\alpha_k x_{k+N}, \sum_{n=0}^{\infty}\gamma_n x_n\right) = \sum_{k,n=0}^{\infty}\alpha_k\overline{\gamma_n}(x_{k+N}, x_n) \\ &= \sum_{k,n=0}^{\infty}\alpha_k\overline{\gamma_n}(x_k, x_{n+N}) = \left(\sum_{k=0}^{\infty}\alpha_k x_k, \sum_{n=0}^{\infty}\gamma_n x_{n+N}\right) \\ &= (x, Ay).\end{aligned}$$

By relation (12) we get

$$(Ax,x) = \left(\sum_{k=0}^{\infty} \alpha_k x_{k+N}, \sum_{n=0}^{\infty} \alpha_n x_n \right) = \sum_{k,n=0}^{\infty} \alpha_k \overline{\alpha_n} (x_{k+N}, x_n)$$
$$= \sum_{k,n=0}^{\infty} \alpha_k \overline{\alpha_n} \gamma_{k+N,n} \geq 0.$$

Thus, the operator A is a linear nonnegative Hermitian operator in H with the domain $D(A) = L$. Such an operator has a nonnegative self-adjoint extension [13, Theorem 7, p.450]. Let $\widetilde{A} \supseteq A$ be an arbitrary nonnegative self-adjoint extension of A in a Hilbert space $\widetilde{H} \supseteq H$, and $\{\widetilde{E}_\lambda\}_{\lambda\in\mathbb{R}_+}$ be its left-continuous orthogonal resolution of unity. Choose an arbitrary $a \in \mathbb{Z}_+$, $a = rN + j$, $r \in \mathbb{Z}_+$, $0 \leq j \leq N-1$. Notice that

$$x_a = x_{rN+j} = Ax_{(r-1)N+j} = \cdots = A^r x_j.$$

Using (9), (16) we can write

$$s_{r+t;j,n} = \gamma_{rN+j,tN+n} = (x_{rN+j}, x_{tN+n})_H = (A^r x_j, A^t x_n)_H$$
$$= (\widetilde{A}^r x_j, \widetilde{A}^t x_n)_{\widetilde{H}} = \left(\int_{\mathbb{R}_+} \lambda^r d\widetilde{E}_\lambda x_j, \int_{\mathbb{R}_+} \lambda^t d\widetilde{E}_\lambda x_n \right)_{\widetilde{H}}$$
$$= \int_{\mathbb{R}_+} \lambda^{r+t} d(\widetilde{E}_\lambda x_j, x_n)_{\widetilde{H}} = \int_{\mathbb{R}_+} \lambda^{r+t} d\left(P^{\widetilde{H}}_H \widetilde{E}_\lambda x_j, x_n \right)_H.$$

Let us write the last relation in a matrix form:

$$S_{r+t} = \int_{\mathbb{R}_+} \lambda^{r+t} d\widetilde{M}(\lambda), \quad r,t \in \mathbb{Z}_+, \tag{22}$$

where

$$\widetilde{M}(\lambda) := \left(\left(P^{\widetilde{H}}_H \widetilde{E}_\lambda x_j, x_n \right)_H \right)_{j,n=0}^{N-1}. \tag{23}$$

If we set $t = 0$ in relation (22), we obtain that the matrix function $\widetilde{M}(\lambda)$ is a solution of the matrix Stieltjes moment problem (1). In fact, from the properties of the orthogonal resolution of unity it easily follows that $\widetilde{M}(\lambda)$ is left-continuous nondecreasing and $\widetilde{M}(0) = 0$.

Thus, we obtained another proof of the solvability criterion for the matrix Stieltjes moment problem (1):

Theorem 2. *Let a matrix Stieltjes moment problem (1) be given. This problem has a solution if and only if conditions (4) hold true.*

3. A description of solutions

Let B be an arbitrary nonnegative Hermitian operator in a Hilbert space $\mathcal{H}$. Choose an arbitrary nonnegative self-adjoint extension $\widehat{B}$ of B in a Hilbert space $\widehat{\mathcal{H}} \supseteq \mathcal{H}$. Let $R_z(\widehat{B})$ be the resolvent of $\widehat{B}$ and $\{\widehat{E}_\lambda\}_{\lambda\in\mathbb{R}_+}$

be the orthogonal left-continuous resolution of unity of $\widehat{B}$. Recall that the operator-valued function $\mathbf{R}_z = P^{\widehat{\mathcal{H}}}_{\mathcal{H}} R_z(\widehat{B})$ is called *a generalized* Π-*resolvent of* B, $z \in \mathbb{C}\backslash\mathbb{R}$ [11]. If $\widehat{\mathcal{H}} = \mathcal{H}$ then $R_z(\widehat{B})$ is called *a canonical* Π-*resolvent*. The function $\mathbf{E}_\lambda = P^{\widehat{\mathcal{H}}}_{\mathcal{H}} \widehat{E}_\lambda$, $\lambda \in \mathbb{R}$, we call a Π-*spectral function* of a non-negative Hermitian operator B. There exists a one-to-one correspondence between generalized Π-resolvents and Π-spectral functions established by the following relation ([2]):

$$(\mathbf{R}_z f, g)_{\mathcal{H}} = \int_{\mathbb{R}_+} \frac{1}{\lambda - z} d(\mathbf{E}_\lambda f, g)_{\mathcal{H}}, \quad f, g \in \mathcal{H},\ z \in \mathbb{C}\backslash\mathbb{R}. \tag{24}$$

Denote the set of all generalized Π-resolvents of B by

$$\Omega^0(-\infty, 0) = \Omega^0(-\infty, 0)(B).$$

Let a moment problem (1) be given and conditions (4) hold. Consider the operator A defined as in (20). Formula (23) shows that Π-spectral functions of the operator A produce solutions of the matrix Stieltjes moment problem (1). Let us show that an arbitrary solution of (1) can be produced in this way.

Choose an arbitrary solution $\widehat{M}(x) = (\widehat{m}_{k,l}(x))_{k,l=0}^{N-1}$ of the matrix Stieltjes moment problem (1). Consider the space $L^2(\widehat{M})$ and let Q be the operator of multiplication by an independent variable in $L^2(\widehat{M})$. The operator Q is self-adjoint and its resolution of unity is given by (see [15])

$$E_b - E_a = E([a,b)) : h(x) \to \chi_{[a,b)}(x) h(x), \tag{25}$$

where $\chi_{[a,b)}(x)$ is the characteristic function of an interval $[a,b)$, $0 \le a < b \le +\infty$. Set

$$\vec{e}_k = (e_{k,0}, e_{k,1}, \ldots, e_{k,N-1}), \quad e_{k,j} = \delta_{k,j}, \quad 0 \le j \le N-1,$$

where $k = 0, 1, \ldots N-1$. A set of (equivalence classes of) functions $f \in L^2(\widehat{M})$ such that (the corresponding class includes) $f = (f_0, f_1, \ldots, f_{N-1})$, $f \in \mathbb{P}$, we denote by $\mathbb{P}^2(\widehat{M})$. It is said to be a set of vector polynomials in $L^2(\widehat{M})$. Set $L^2_0(\widehat{M}) := \overline{\mathbb{P}^2(\widehat{M})}$.

For an arbitrary (representative) $f \in \mathbb{P}^2(\widehat{M})$ there exists a unique representation of the following form:

$$f(x) = \sum_{k=0}^{N-1} \sum_{j=0}^{\infty} \alpha_{k,j} x^j \vec{e}_k, \quad \alpha_{k,j} \in \mathbb{C}. \tag{26}$$

Here the sum is assumed to be finite. Let $g \in \mathbb{P}^2(\widehat{M})$ have a representation

$$g(x) = \sum_{l=0}^{N-1} \sum_{r=0}^{\infty} \beta_{l,r} x^r \vec{e}_l, \quad \beta_{l,r} \in \mathbb{C}. \tag{27}$$

Then we can write

$$(f,g)_{L^2(\widehat{M})} = \sum_{k,l=0}^{N-1} \sum_{j,r=0}^{\infty} \alpha_{k,j}\overline{\beta_{l,r}} \int_{\mathbb{R}} x^{j+r} \vec{e}_k d\widehat{M}(x) \vec{e}_l^{\,*} \tag{28}$$
$$= \sum_{k,l=0}^{N-1} \sum_{j,r=0}^{\infty} \alpha_{k,j}\overline{\beta_{l,r}} \int_{\mathbb{R}} x^{j+r} d\widehat{m}_{k,l}(x)$$
$$= \sum_{k,l=0}^{N-1} \sum_{j,r=0}^{\infty} \alpha_{k,j}\overline{\beta_{l,r}} s_{j+r;k,l}.$$

On the other hand, we can write

$$\left(\sum_{j=0}^{\infty} \sum_{k=0}^{N-1} \alpha_{k,j} x_{jN+k}, \sum_{r=0}^{\infty} \sum_{l=0}^{N-1} \beta_{l,r} x_{rN+l} \right)_H \tag{29}$$
$$= \sum_{k,l=0}^{N-1} \sum_{j,r=0}^{\infty} \alpha_{k,j}\overline{\beta_{l,r}} (x_{jN+k}, x_{rN+l})_H$$
$$= \sum_{k,l=0}^{N-1} \sum_{j,r=0}^{\infty} \alpha_{k,j}\overline{\beta_{l,r}} \gamma_{jN+k,rN+l}$$
$$= \sum_{k,l=0}^{N-1} \sum_{j,r=0}^{\infty} \alpha_{k,j}\overline{\beta_{l,r}} s_{j+r;k,l}.$$

From relations (28), (29) it follows that

$$(f,g)_{L^2(\widehat{M})} = \left(\sum_{j=0}^{\infty} \sum_{k=0}^{N-1} \alpha_{k,j} x_{jN+k}, \sum_{r=0}^{\infty} \sum_{l=0}^{N-1} \beta_{l,r} x_{rN+l} \right)_H. \tag{30}$$

Let us introduce the following operator:

$$Vf = \sum_{j=0}^{\infty} \sum_{k=0}^{N-1} \alpha_{k,j} x_{jN+k}, \tag{31}$$

for $f(x) \in \mathbb{P}^2(\widehat{M})$, $f(x) = \sum_{k=0}^{N-1} \sum_{j=0}^{\infty} \alpha_{k,j} x^j \vec{e}_k$, $\alpha_{k,j} \in \mathbb{C}$. Let us show that this definition is correct. In fact, if vector polynomials f, g have representations (26), (27), and $\|f-g\|_{L^2(\widehat{M})} = 0$, then from (30) it follows that $V(f-g) = 0$. Thus, V is a correctly defined operator from $\mathbb{P}^2(\widehat{M})$ into H.

Relation (30) shows that V is an isometric transformation from $\mathbb{P}^2(\widehat{M})$ onto L. By continuity we extend it to an isometric transformation from $L^2_0(\widehat{M})$ onto H. In particular, we note that

$$Vx^j \vec{e}_k = x_{jN+k}, \quad j \in \mathbb{Z}_+; \quad 0 \le k \le N-1. \tag{32}$$

Set $L^2_1(\widehat{M}) := L^2(\widehat{M}) \ominus L^2_0(\widehat{M})$, and $U := V \oplus E_{L^2_1(\widehat{M})}$. The operator U is an isometric transformation from $L^2(\widehat{M})$ onto $H \oplus L^2_1(\widehat{M}) =: \widehat{H}$. Set

$$\widehat{A} := UQU^{-1}.$$

The operator $\widehat{A}$ is a nonnegative self-adjoint operator in $\widehat{H}$. Let $\{\widehat{E}_\lambda\}_{\lambda\in\mathbb{R}_+}$ be its left-continuous orthogonal resolution of unity. Notice that

$$\begin{aligned} UQU^{-1}x_{jN+k} &= VQV^{-1}x_{jN+k} = VQx^j\vec{e}_k = Vx^{j+1}\vec{e}_k = x_{(j+1)N+k} \\ &= x_{jN+k+N} = Ax_{jN+k}, \quad j \in \mathbb{Z}_+; \quad 0 \le k \le N-1. \end{aligned}$$

By linearity we get

$$UQU^{-1}x = Ax, \quad x \in L = D(A),$$

and therefore $\widehat{A} \supseteq A$. Choose an arbitrary $z \in \mathbb{C}\backslash\mathbb{R}$ and write

$$\begin{aligned} (33) \qquad \int_{\mathbb{R}_+} \frac{1}{\lambda - z} d(\widehat{E}_\lambda x_k, x_j)_{\widehat{H}} &= \left(\int_{\mathbb{R}_+} \frac{1}{\lambda - z} d\widehat{E}_\lambda x_k, x_j \right)_{\widehat{H}} \\ &= \left(U^{-1} \int_{\mathbb{R}_+} \frac{1}{\lambda - z} d\widehat{E}_\lambda x_k, U^{-1}x_j \right)_{L^2(\widehat{M})} \\ &= \left(\int_{\mathbb{R}_+} \frac{1}{\lambda - z} dU^{-1}\widehat{E}_\lambda U\vec{e}_k, \vec{e}_j \right)_{L^2(\widehat{M})} \\ &= \left(\int_{\mathbb{R}_+} \frac{1}{\lambda - z} dE_\lambda\vec{e}_k, \vec{e}_j \right)_{L^2(\widehat{M})} \\ &= \int_{\mathbb{R}_+} \frac{1}{\lambda - z} d(E_\lambda\vec{e}_k, \vec{e}_j)_{L^2(\widehat{M})}, \end{aligned}$$

$0 \le k, j \le N-1$. Using (25) we can write

$$(E_\lambda\vec{e}_k, \vec{e}_j)_{L^2(\widehat{M})} = \widehat{m}_{k,j}(\lambda),$$

and therefore

$$(34) \quad \int_{\mathbb{R}_+} \frac{1}{\lambda - z} d(P^{\widehat{H}}_H \widehat{E}_\lambda x_k, x_j)_H = \int_{\mathbb{R}_+} \frac{1}{\lambda - z} d\widehat{m}_{k,j}(\lambda), \quad 0 \le k, j \le N-1.$$

By the Stieltjes–Perron inversion formula (see, e.g., [1]) we conclude that

$$(35) \qquad \widehat{m}_{k,j}(\lambda) = (P^{\widehat{H}}_H \widehat{E}_\lambda x_k, x_j)_H.$$

Proposition 1. *Let the matrix Stieltjes moment problem (1) be given and conditions (4) hold. Let A be a nonnegative Hermitian operator which is defined by (20). The deficiency index of A is equal to (n,n), $0 \le n \le N$.*

Proof. Choose an arbitrary $u \in L$, $u = \sum_{k=0}^\infty c_k x_k$, $c_k \in \mathbb{C}$. Suppose that $c_k = 0$, $k \ge N + R + 1$, for some $R \in \mathbb{Z}_+$. Consider the following system of linear equations:

$$-zd_k = c_k, \quad k = 0, 1, \ldots, N-1; \tag{36}$$

$$d_{k-N} - zd_k = c_k, \quad k = N, N+1, N+2, \ldots; \tag{37}$$

where $\{d_k\}_{k\in\mathbb{Z}_+}$ are unknown complex numbers, $z \in \mathbb{C}\backslash\mathbb{R}$ is a fixed parameter. Set

$$\begin{aligned} d_k &= 0, && k \geq R+1, \\ d_j &= c_{N+j} + zd_{N+j}, && j = R, R-1, R-2, \ldots, 0. \end{aligned} \tag{38}$$

For such defined numbers $\{d_k\}_{k\in\mathbb{Z}_+}$, all equations in (37) are satisfied. But equations (36) are not necessarily satisfied. Set

$$v = \sum_{k=0}^{\infty} d_k x_k, \quad v \in L.$$

Notice that

$$(A - zE_H)v = \sum_{k=0}^{\infty} (d_{k-N} - zd_k)x_k,$$

where $d_{-1} = d_{-2} = \cdots = d_{-N} = 0$. By the construction of d_k we have

$$(A - zE_H)v - u = \sum_{k=0}^{\infty} (d_{k-N} - zd_k - c_k)x_k = \sum_{k=0}^{N-1} (-zd_k - c_k)x_k, \tag{39}$$

$$u = (A - zE_H)v + \sum_{k=0}^{N-1} (zd_k + c_k)x_k, \quad u \in L.$$

Set

$$\begin{aligned} H_z &:= \overline{(A - zE_H)L} = (\overline{A} - zE_H)D(\overline{A}), \\ y_k &:= x_k - P^H_{H_z} x_k, \quad k = 0, 1, \ldots, N-1. \end{aligned} \tag{40}$$

Set

$$H_0 := \operatorname{span}\{y_k\}_{k=0}^{N-1}.$$

Notice that the dimension of H_0 is less or equal to N, and $H_0 \perp H_z$. From (39) it follows that $u \in L$ can be represented in the following form:

$$u = u_1 + u_2, \quad u_1 \in H_z, \quad u_2 \in H_0. \tag{41}$$

Therefore we get $L \subseteq H_z \oplus H_0$; $H \subseteq H_z \oplus H_0$, and finally $H = H_z \oplus H_0$. Thus, H_0 is the corresponding defect subspace. So, the defect numbers of A are less or equal to N. Since the operator A is nonnegative, they are equal. □

Theorem 3. *Let a matrix Stieltjes moment problem (1) be given and conditions (4) hold. Let an operator A be constructed for the moment problem as in (20). All solutions of the moment problem have the following form*

$$M(\lambda) = (m_{k,j}(\lambda))_{k,j=0}^{N-1}, \quad m_{k,j}(\lambda) = (\mathbf{E}_\lambda x_k, x_j)_H, \tag{42}$$

where $\mathbf{E}_\lambda$ *is a* Π*-spectral function of the operator* A*. Moreover, the correspondence between all* Π*-spectral functions of* A *and all solutions of the moment problem is one-to-one.*

Proof. It remains to prove that different Π-spectral functions of the operator A produce different solutions of the moment problem (1). Suppose to the contrary that two different Π-spectral functions produce the same solution of the moment problem. That means that there exist two nonnegative self-adjoint extensions $A_j \supseteq A$, in Hilbert spaces $H_j \supseteq H$, such that

$$P_H^{H_1} E_{1,\lambda} \neq P_H^{H_2} E_{2,\lambda}, \tag{43}$$

$$(P_H^{H_1} E_{1,\lambda} x_k, x_j)_H = (P_H^{H_2} E_{2,\lambda} x_k, x_j)_H, \quad 0 \leq k,j \leq N-1, \quad \lambda \in \mathbb{R}_+, \tag{44}$$

where $\{E_{n,\lambda}\}_{\lambda\in\mathbb{R}_+}$ are orthogonal left-continuous resolutions of unity of operators A_n, $n = 1,2$. Set $L_N := \mathrm{Lin}\{x_k\}_{k=0,N-1}$. By linearity we get

$$(P_H^{H_1} E_{1,\lambda} x, y)_H = (P_H^{H_2} E_{2,\lambda} x, y)_H, \quad x,y \in L_N, \quad \lambda \in \mathbb{R}_+. \tag{45}$$

Denote by $R_{n,\lambda}$ the resolvent of A_n, and set $\mathbf{R}_{n,\lambda} := P_H^{H_n} R_{n,\lambda}$, $n = 1,2$. From (45), (24) it follows that

$$(\mathbf{R}_{1,z} x, y)_H = (\mathbf{R}_{2,z} x, y)_H, \quad x,y \in L_N, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{46}$$

Choose an arbitrary $z \in \mathbb{C}\backslash\mathbb{R}$ and consider the space H_z defined as above. Since

$$R_{j,z}(A - zE_H)x = (A_j - zE_{H_j})^{-1}(A_j - zE_{H_j})x = x, \quad x \in L = D(A),$$

we get

$$R_{1,z} u = R_{2,z} u \in H, \quad u \in H_z; \tag{47}$$

$$\mathbf{R}_{1,z} u = \mathbf{R}_{2,z} u, \quad u \in H_z,\ z \in \mathbb{C}\backslash\mathbb{R}. \tag{48}$$

We can write

$$\begin{aligned}(\mathbf{R}_{n,z} x, u)_H = (R_{n,z} x, u)_{H_n} = (x, R_{n,\bar{z}} u)_{H_n} = (x, \mathbf{R}_{n,\bar{z}} u)_H,\\ x \in L_N,\ u \in H_{\bar{z}},\ n = 1,2,\end{aligned} \tag{49}$$

and therefore we get

$$(\mathbf{R}_{1,z} x, u)_H = (\mathbf{R}_{2,z} x, u)_H, \quad x \in L_N,\ u \in H_{\bar{z}}. \tag{50}$$

By (39) an arbitrary element $y \in L$ can be represented as $y = y_{\bar{z}} + y'$, $y_{\bar{z}} \in H_{\bar{z}}$, $y' \in L_N$. Using (46) and (48) we get

$$\begin{aligned}(\mathbf{R}_{1,z} x, y)_H = (\mathbf{R}_{1,z} x, y_{\bar{z}} + y')_H = (\mathbf{R}_{2,z} x, y_{\bar{z}} + y')_H \\ = (\mathbf{R}_{2,z} x, y)_H, \quad x \in L_N,\ y \in L.\end{aligned}$$

Since $\bar{L} = H$, we obtain

$$\mathbf{R}_{1,z} x = \mathbf{R}_{2,z} x, \quad x \in L_N,\ z \in \mathbb{C}\backslash\mathbb{R}. \tag{51}$$

For an arbitrary $x \in L$, $x = x_z + x'$, $x_z \in H_z$, $x' \in L_N$, using relations (48), (51) we obtain

$$\mathbf{R}_{1,z}x = \mathbf{R}_{1,z}(x_z + x') = \mathbf{R}_{2,z}(x_z + x') = \mathbf{R}_{2,z}x, \quad x \in L,\ z \in \mathbb{C}\backslash\mathbb{R}, \tag{52}$$

and

$$\mathbf{R}_{1,z}x = \mathbf{R}_{2,z}x, \quad x \in H,\ z \in \mathbb{C}\backslash\mathbb{R}. \tag{53}$$

By (24) that means that the Π-spectral functions coincide and we obtain a contradiction. □

We shall recall some basic definitions and facts from [4]. Let A be a closed Hermitian operator in a Hilbert space H, $\overline{D(A)} = H$.

Definition 1. A collection $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$ in which $\mathcal{H}$ is a Hilbert space and $\Gamma_1, \Gamma_2 \in [D(A^*), \mathcal{H}]$, is called *a space of boundary values* (SBV) for A^*, if:

(1) $(A^*f, g)_H - (f, A^*g)_H = (\Gamma_1 f, \Gamma_2 g)_{\mathcal{H}} - (\Gamma_2 f, \Gamma_1 g)_{\mathcal{H}}$, $\forall f, g \in D(A^*)$.
(2) The mapping $\Gamma : f \to \{\Gamma_1 f, \Gamma_2 f\}$ from $D(A^*)$ to $\mathcal{H} \oplus \mathcal{H}$ is surjective.

Naturally associated with each SBV are self-adjoint operators $\widetilde{A}_1, \widetilde{A}_2$ $(\subset A^*)$ with

$$D(\widetilde{A}_1) = \ker \Gamma_1,\ D(\widetilde{A}_2) = \ker \Gamma_2.$$

The operator Γ_2 restricted to the defect subspace $N_z = \ker(A^* - zE_H)$, $z \in \rho(\widetilde{A}_2)$, is fully invertible. For $\forall z \in \rho(\widetilde{A}_2)$ set

$$\gamma(z) = (\Gamma_2|_{N_z})^{-1} \in [\mathcal{H}, N_z]. \tag{54}$$

Definition 2. The operator-valued function $M(z)$ defined for $z \in \rho(\widetilde{A}_2)$ by

$$M(z)\Gamma_2 f_z = \Gamma_1 f_z, \quad f_z \in N_z, \tag{55}$$

is called a *Weyl function* of A, corresponding to the SBV $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$.

The Weyl function can be also obtained from the equality:

$$M(z) = \Gamma_1 \gamma(z), \quad z \in \rho(\widetilde{A}_2). \tag{56}$$

For an arbitrary operator $\widetilde{A} = \widetilde{A}^* \subset A^*$ there exists an SBV with ([5])

$$D(\widetilde{A}_2) = \ker \Gamma_2 = D(\widetilde{A}). \tag{57}$$

(There even exists a family of such SBV).

An extension $\widehat{A}$ of A is called *proper* if $A \subset \widehat{A} \subset A^*$ and $(\widehat{A}^*)^* = \widehat{A}$. Two proper extensions $\widehat{A}_1$ and $\widehat{A}_2$ are *disjoint* if $D(\widehat{A}_1) \cap D(\widehat{A}_2) = D(A)$ and *transversals* if they are disjoint and $D(\widehat{A}_1) + D(\widehat{A}_2) = D(A^*)$.

Suppose that the operator A is nonnegative, $A \geq 0$. In this case there exist two nonnegative self-adjoint extensions of A in H, Friedrich's extension A_μ and Krein's extension A_M, such that for an arbitrary nonnegative self-adjoint extension $\widehat{A}$ of A in H it holds:

$$(A_\mu + xE_H)^{-1} \leq (\widehat{A} + xE_H)^{-1} \leq (A_M + xE_H)^{-1}, \quad x \in \mathbb{R}_+. \tag{58}$$

Recall some definitions and facts from [11], [13]. For the nonnegative operator A we put into correspondence the following operator:

$$T = (E_H - A)(E_H + A)^{-1} = -E_H + 2(E_H + A)^{-1},$$
$$D(T) = (A + E_H)D(A). \tag{59}$$

The operator T is a Hermitian contraction (i.e., $\|T\| \leq 1$). Its domain is not dense in H if A is not self-adjoint. The defect subspace $H \ominus D(T) = N_{-1}$ and its dimension is equal to the defect number $n(A)$ of A. The inverse transformation to (59) is given by

$$A = (E_H - T)(E_H + T)^{-1} = -E_H + 2(E_H + T)^{-1},$$
$$D(A) = (T + E_H)D(T). \tag{60}$$

Relations (59), (60) (with $\widehat{T}, \widehat{A}$ instead of T, A) also establish a bijective correspondence between self-adjoint contractive extensions $\widehat{T} \supseteq T$ in H and self-adjoint nonnegative extensions $\widehat{A} \supseteq A$ in H ([13, p. 451]).

Consider an arbitrary Hilbert space $\widehat{H} \supseteq H$. It is not hard to see that relations (59), (60) (with $\widehat{T}, \widehat{A}$ instead of T, A) establish a bijective correspondence between self-adjoint contractive extensions $\widehat{T} \supseteq T$ in $\widehat{H}$ and self-adjoint nonnegative extensions $\widehat{A} \supseteq A$ in $\widehat{H}$, as well.

There exist extremal self-adjoint contractive extensions of T in H such that for an arbitrary self-adjoint contractive extension $\widetilde{T} \supseteq T$ in H,

$$T_\mu \leq \widetilde{T} \leq T_M. \tag{61}$$

Notice that

$$A_\mu = -E_H + 2(E_H + T_\mu)^{-1}, \quad A_M = -E_H + 2(E_H + T_M)^{-1}. \tag{62}$$

Set

$$C = T_M - T_\mu. \tag{63}$$

Consider the following subspace:

$$\Upsilon = \ker\left(C|_{N_{-1}}\right). \tag{64}$$

Definition 3. Let a closed nonnegative Hermitian operator A be given. For the operator A we are in *a completely indeterminate case* if $\Upsilon = \{0\}$.

By Theorem 1.4 in [12], on the set $\{x \in H : T_\mu x = T_M x\} = \ker C$, all self-adjoint contractive extensions in a Hilbert space $\widetilde{H} \supseteq H$ coincide. Thus, all such extensions are extensions of the operator T_{ext}:

$$T_{\mathrm{ext}} x = \begin{cases} Tx, & x \in D(T) \\ T_\mu x = T_M x, & x \in \ker C. \end{cases} \tag{65}$$

Introduce the following operator:

$$A_{\mathrm{ext}} = -E_H + 2(E_H + T_{\mathrm{ext}})^{-1} \supseteq A. \tag{66}$$

Thus, *the set of all nonnegative self-adjoint extensions of A coincides with the set of all nonnegative self-adjoint extensions of A_{ext}.* Since $T_{\mathrm{ext},\mu} = T_\mu$ and $T_{\mathrm{ext},M} = T_M$, A_{ext} is in the completely indeterminate case.

Proposition 2. *Let A be a closed nonnegative Hermitian operator with finite defect numbers, such that A is in the completely indeterminate case. Then extensions A_μ and A_M given by (62) are transversal.*

Proof. Notice that

$$D(A_M) \cap D(A_\mu) = D(A). \tag{67}$$

In fact, suppose that there exists $y \in D(A_M) \cap D(A_\mu)$, $y \notin D(A)$. Since $A_M \subset A^*$ and $A_\mu \subset A^*$ we have $A_M y = A_\mu y$. Set

$$g := (A_M + E_H)y = (A_\mu + E_H)y.$$

Then

$$T_M g = -g + 2(E_H + A_M)^{-1} g = -g + 2y,$$
$$T_\mu g = -g + 2(E_H + A_\mu)^{-1} g = -g + 2y,$$

and therefore $Cg = (T_M - T_\mu)g = 0$. Since $y \notin D(A)$, then $g \in N_{-1}$. We obtain a contradiction, since A is in the completely indeterminate case.

Introduce the following sets:

$$D_M := (A_M + E_H)^{-1} N_{-1}, \quad D_\mu := (A_\mu + E_H)^{-1} N_{-1}. \tag{68}$$

Since $D(A_M) = (A_M + E_H)^{-1} D(T_M)$, $D(A_\mu) = (A_\mu + E_H)^{-1} D(T_\mu)$, we have

$$D_M \subset D(A_M),\ D_\mu \subset D(A_\mu), \tag{69}$$

and

$$D_M \cap D(A) = \{0\}, \quad D_\mu \cap D(A) = \{0\}, \tag{70}$$

By (67), (69) and (70) we obtain that

$$D_M \cap D_\mu = \{0\}. \tag{71}$$

Set

$$D := D_M \dot{+} D_\mu. \tag{72}$$

By (68) we obtain that the sets D_M and D_μ have the linear dimension $n(A)$. Elementary arguments show that D has the linear dimension $2n(A)$. Since $D(A_\mu) \subset D(A^*)$, $D(A_M) \subset D(A^*)$, we can write

$$D(A) \dot{+} D_M \dot{+} D_\mu \subseteq D(A^*) = D(A) \dot{+} N_z \dot{+} N_{\bar{z}}, \tag{73}$$

where $z \in \mathbb{C} \backslash \mathbb{R}$.

Let

$$g_1, g_2, \ldots, g_{2n(A)},$$

be $2n(A)$ linearly independent elements from D. Let

$$g_j = g_{A,j} + g_{z,j} + g_{\bar{z},j}, \quad 1 \le j \le 2n(A), \tag{74}$$

where $g_{A,j} \in D(A)$, $g_{z,j} \in N_z$, $g_{\bar{z},j} \in N_{\bar{z}}$. Set

$$\widehat{g}_j := g_j - g_{A,j}, \quad 1 \leq j \leq 2n(A). \tag{75}$$

If for some $\alpha_j \in \mathbb{C}$, $1 \leq j \leq 2n(A)$, we have

$$0 = \sum_{j=1}^{2n(A)} \alpha_j \widehat{g}_j = \sum_{j=1}^{2n(A)} \alpha_j g_j - \sum_{j=1}^{2n(A)} \alpha_j g_{A,j},$$

then

$$\sum_{j=1}^{2n(A)} \alpha_j g_j = 0,$$

and $\alpha_j = 0$, $1 \leq j \leq 2n(A)$. Therefore elements $\widehat{g}_j$, $1 \leq j \leq 2n(A)$ are linearly independent. Thus, they form a linear basis in a finite-dimensional subspace $N_z \dot{+} N_{\bar{z}}$. Then

$$N_z \dot{+} N_{\bar{z}} \subseteq D, \tag{76}$$

$$D(A^*) = D(A) \dot{+} N_z \dot{+} N_{\bar{z}} \subseteq D(A) \dot{+} D = D_L. \tag{77}$$

So, we get the equality

$$D(A) \dot{+} D_M \dot{+} D_\mu = D(A^*). \tag{78}$$

Since $D(A) + D_M \subseteq D(A_M)$, $D_\mu \subseteq D(A_\mu)$, we get

$$D(A^*) = D(A) + D_M + D_\mu \subseteq D(A_M) + D(A_\mu).$$

Since $D(A_M) + D(A_\mu) \subseteq D(A^*)$, we get

$$D(A^*) = D(A_M) + D(A_\mu). \tag{79}$$

From (67), (79) the result follows. □

We shall use the following classes of functions [4]. Let $\mathcal{H}$ be a Hilbert space. Denote by $R_{\mathcal{H}}$ the class of operator-valued functions $F(z) = F^*(\bar{z})$ holomorphic in $\mathbb{C}\backslash\mathbb{R}$ with values (for $z \in \mathbb{C}_+$) in the set of maximal dissipative operators in $\mathfrak{C}(\mathcal{H})$. Completing the class $R_{\mathcal{H}}$ by ideal elements we get the class $\widetilde{R}_{\mathcal{H}}$. Thus, $\widetilde{R}_{\mathcal{H}}$ is a collection of functions holomorphic in $\mathbb{C}\backslash\mathbb{R}$ with values (for $z \in \mathbb{C}_+$) in the set of maximal dissipative linear relations $\theta(z) = \theta^*(\bar{z})$ in $\mathcal{H}$. The indeterminate part of the relation $\theta(z)$ does not depend on z and the relation $\theta(z)$ admits the representation

$$\theta(z) = \{\langle h_1, F_1(z)h_1 + h_2\rangle : \ h_1 \in D(F_1(z)), \ h_2 \in \mathcal{H}_2\}, \tag{80}$$

where $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $F_1(z) \in R_{\mathcal{H}_1}$.

Definition 4 ([4]). An operator-valued function $F(z) \in R_{\mathcal{H}}$ belongs to the class $S_{\mathcal{H}}^{-0}(-\infty, 0)$ if $\forall n \in \mathbb{N}$, $\forall z_j \in \mathbb{C}_+$, $h_j \in D(F(z_j))$, $\xi_j \in \mathbb{C}$, we have

$$\sum_{i,j=1}^{n} \frac{(z_i^{-1}F(z_i)h_i, h_j) - (h_i, z_j^{-1}F(z_j)h_j)}{z_i - \overline{z_j}} \xi_i \overline{\xi_j} \geq 0. \tag{81}$$

Completing the class $S_{\mathcal{H}}^{-0}(-\infty, 0)$ with ideal elements (80) we obtain the class $\widetilde{S}_{\mathcal{H}}^{-0}(-\infty, 0)$.

From Theorem 9 in [4, p.46] taking into account Proposition 2 we have the following conclusion (see also Remark 17 in [4, p.49]):

Theorem 4. *Let A be a closed nonnegative Hermitian operator in a Hilbert space H in the completely indeterminate case. Let $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$ be an arbitrary SBV for A such that $\widetilde{A}_2 = A_\mu$ and $M(z)$ be the corresponding Weyl function. Then the formula*

$$\mathbf{R}_z = (A_\mu - zE_H)^{-1} - \gamma(z)(\tau(z) + M(z) - M(0))^{-1}\gamma^*(\bar{z}), \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{82}$$

establishes a bijective correspondence between $\mathbf{R}_z \in \Omega^0(-\infty, 0)(A)$ and $\tau \in \widetilde{S}_{\mathcal{H}}^{-0}(-\infty, 0)$. The function $\tau(z) \equiv \tau = \tau^*$ in (82) corresponds to the canonical Π-resolvents and only to them.

Now we can state our main result.

Theorem 5. *Let a matrix Stieltjes moment problem (1) be given and conditions (4) hold. Let an operator A be the closure of the operator constructed for the moment problem in (20). Then the following statements are true:*

(1) *The moment problem (1) is determinate if and only if Friedrich's extension A_μ and Krein's extension A_M coincide: $A_\mu = A_M$. In this case the unique solution of the moment problem is generated by the orthogonal spectral function $\mathbf{E}_\lambda$ of A_μ by formula (42).*
(2) *If $A_\mu \neq A_M$, define the extended operator A_{ext} for A as in (66). Let $\{\mathcal{H}, \Gamma_1, \Gamma_2\}$ be an arbitrary SBV for A_{ext} such that $\widetilde{A}_2 = (A_{\mathrm{ext}})\mu$ and $M(z)$ be the corresponding Weyl function. All solutions of the moment problem (1) have the following form:*

$$M(\lambda) = (m_{k,j}(\lambda))_{k,j=0}^{N-1}, \tag{83}$$

where

$$\int_{\mathbb{R}_+} \frac{dm_{k,j}(\lambda)}{\lambda - z} = \left((A_\mu - zE_H)^{-1}x_k, x_j\right)_H - \left(\gamma(z)(\tau(z) + M(z) - M(0))^{-1}\gamma^*(\bar{z})x_k, x_j\right)_H, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{84}$$

where $\tau \in \widetilde{S}_{\mathcal{H}}^{-0}(-\infty, 0)$. Moreover, the correspondence between all $\tau \in \widetilde{S}_{\mathcal{H}}^{-0}(-\infty, 0)$ and all solutions of the moment problem (1) is one-to-one.

Proof. This follows directly from Theorems 3 and 4. □

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