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Why is Helfenstein's claim about equichordal points false?

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Abstract. This article explains why a paper by Heinz G. Helfenstein entitled *Ovals with equichordal points*, J. London Math. Soc. **31** (1956), 54–57, is incorrect. We point out a computational error which renders his conclusions invalid. More importantly, we explain that the method presented there cannot be used to solve the equichordal point problem. Today, there is a solution to the problem: Marek R. Rychlik, *A complete solution to the equichordal point problem of Fujiwara, Blaschke, Rothe and Weizenböck*, Inventiones Mathematicae **129** (1997), 141–212. However, some mathematicians still point to Helfenstein's paper as a plausible path to a simpler solution. We show that Helfenstein's method cannot be salvaged. The fact that Helfenstein's argument is not correct was known to Wirsing, but he did not explicitly point out the error. This article points out the error and the reasons for the failure of Helfenstein's approach in an accessible, and hopefully enjoyable way.

Contents

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1. The equichordal point problem

The equichordal point problem enjoyed significant popularity since its original formulation by Fujiwara in 1916 and Blaschke, Röthe and Weizenböck in 1917, because it can be formulated in easy to understand terms of elementary geometry, and it is hard to solve. The starting point is the definition of an equichordal point:

Definition 1. Let C be a Jordan curve and let O be a point inside it. This point is called *equichordal* if every chord of C through this point has the same length.

Then the equichordal point problem may be formulated as follows:

Question 1. *Is there a curve with two equichordal points?*

Why two? Because circles and a lot of other shapes have one equichordal point, and because Fujiwara pointed out that it is impossible for a shape to have three equichordal points.

The full solution to the problem appears in the article [5]. The paper is considered (and it is!) rather hard to read and its length is 72 pages. Thus, although it is a great resource for anyone studying this and related problems, it is not always easy to extract the information one needs.

In this set of notes we use the information provided in [5] to construct a counterexample to a published article by Helfenstein [1] in 1956. Helfenstein made a claim which would lead to a simple solution of the equichordal point problem (under 10 pages, perhaps) if it is augmented with a few relatively easy facts to prove. It has been the hope of many that such a simple proof exists. However, as we will see, Helfenstein's paper is incorrect, and thus there is no hope for a simple proof, at least along the lines of Helfenstein's argument.

In the convex geometry community, the equichordal point problem, and our solution of it, are quite well known. The community has had an especially hard time coming to grips with the hard analytical methods used in our article (and also prior articles of Wirsing [8] and Schäfke and Volkmer [6]). We found on several occasions that the argument of Helfenstein continues to have some legitimacy because no one has explicitly shown where it is incorrect [3]. At the end of this paper we cite Grünbaums's argument from [3] which is representative of this opinion, although the experts on the problem (including Wirsing) have clearly dismissed Helfenstein's paper. Therefore, it will be beneficial to analyze Helfenstein's argument from today's perspective, and explain why it is incorrect. We hope that the reading is entertaining and allows one to understand some of the trappings of the problem, and perhaps even appreciate the length and complexity of our solution.

Helfenstein's paper contains an incorrect statement which must have resulted from an error in a mundane calculation, involving only elementary calculus. This will be clear from what follows. With the aid of a Computer Algebra System (CAS), we reconstructed and corrected the intermediate calculations, and arrived at the opposite conclusion, which clearly shows an error in Helfenstein's argument in an elementary way.

More importantly, the main idea of Helfenstein's paper is also incorrect, and it cannot be salvaged by simply correcting the error in calculation he made. We show this in the strongest possible way: we construct a counterexample by referring to the relevant portions of [5]. However, we will make the argument as simple and as self-contained as possible.

2. A summary of Helfenstein's paper

In 1954 Heinz Helfenstein submitted an article [1], in which he claims that there is no oval with 2 equichordal points that is 6 times differentiable. He calls a curve with 2 equichordal points a *2e-curve*. We will use this abbreviated term below, as synonymous with "curve with 2 equichordal points".

We proceed to summarize the terminology, technique and results of Helfenstein's paper. For simplicity, we will assume that the curve C under consideration is a convex oval and it is symmetric in various ways (cf. Figure 1):

(1) It is symmetric with respect to a point O inside C.
(2) Let R and S be the two hypothetical equichordal points. We may assume that O bisects the interval RS.
(3) C is symmetric with respect to the straight line passing through R and S.
(4) Let A and B be the two points collinear with R and S which belong to C. We may assume that the distance $|AB| = 1$. Moreover, due to prior symmetries, O bisects AB.
(5) The distance BS is called c. By the symmetries assumed, we have $AR = c$.
(6) The construction is valid when $0 < c < 1$. The order of the points A, R, O, S and B on the straight line on which all these points lie is:
   (a) A, R, O, S, B if $0 < c < 1/2$;
   (b) A, S, O, R, B if $1/2 < c < 1$.
   For $c = 1/2$, $R = O = S$.

It should be stated that the above picture of a hypothetical 2e-curve is correct, based on many independent analyses. In particular, the symmetries are well established.

The next assumption used in the paper [1] is that locally near A and B the curve C may be represented by a graph of a function. Helfenstein uses two orthogonal coordinate systems, one centered at A and one centered at

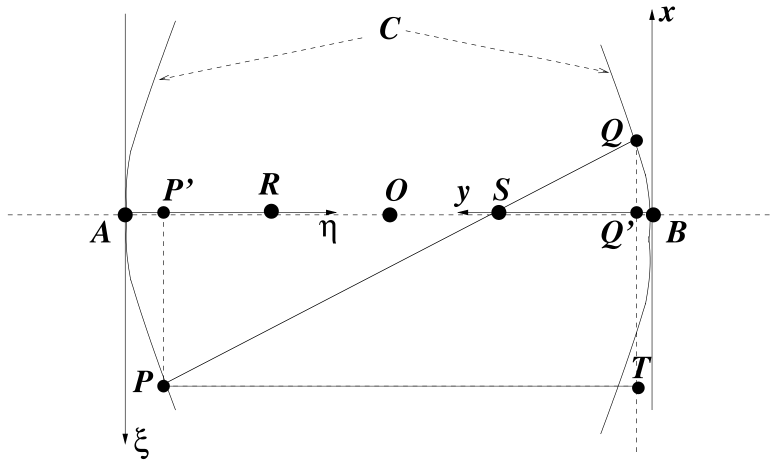

FIGURE 1. Illustration of Helfenstein's notation.

B. The coordinates of the system centered at A are called ξ and η, and the positive direction of the η axis is AB. The coordinates of the system centered at B are called x and y, and the positive direction of the y axis is BA, so that the y and η axes point in the opposite directions. We assume that C near the points B and A may be represented by the equations: $y = f(x)$ and $\eta = f(\xi)$, respectively. There is an agreement of results supporting the claim that C is represented by a smooth function $f(x)$ near the points A and B. It should be emphasized that $f(x)$ is *locally defined*, i.e., its domain is some interval $(-\epsilon, \epsilon)$, where $\epsilon > 0$. There is a proof that ϵ may be as large as $1/2$, but this will not be material in these notes. One can also prove that $f(x)$ is *real-analytic*, i.e., it may be represented by a power series convergent on the interval $|x| < \epsilon$. Again, the analyticity is not material in these notes, but Helfenstein assumes that the function has six derivatives. It should be noted that Helfenstein's assumption in regard to differentiablility is faulty. The fact that he assumes six-fold differentiability is a result of a computational error, as will be demonstrated below.

The next important construction in Helfenstein's paper is that of a functional equation satisfied by $f(x)$. The derivation presented in the Helfenstein's paper is correct, and is consistent with the construction used in our solution of the equichordal point problem [5]. It nicely illustrates the transition from geometry to analysis, which is a hallmark feature of the equichordal point problem. We will repeat Helfenstein's construction here.

Let $P(\xi, \eta)$ be a point near A and let $Q(x, y)$ be a point near B, both on the curve C and both represented in the respective coordinate systems. Let P' and Q' be the orthogonal projections of P and Q onto the line AB. Let T be the projection of P onto the line QQ', perpendicular to AB. Helfenstein observes that the triangles $QQ'S$ and QTP are similar. From this observation, the following equations result:

$$\frac{x}{\sqrt{x^2+(c-y)^2}} = \frac{x+\xi}{1},$$
$$\frac{c-y}{\sqrt{x^2+(c-y)^2}} = \frac{1-y-\eta}{1}.$$

By solving with respect to ξ and η we obtain:

$$\xi = \frac{x}{\sqrt{x^2+(c-y)^2}} - x \tag{1}$$

$$\eta = \frac{-(c-y)}{\sqrt{x^2+(c-y)^2}} + (1-y). \tag{2}$$

By construction, $y = f(x)$ and $\eta = f(\xi)$. Therefore, we obtain this functional equation:

$$f\left(\frac{x}{\sqrt{x^2+(c-f(x))^2}} - x\right) = \frac{-(c-f(x))}{\sqrt{x^2+(c-f(x))^2}} + (1-f(x)). \tag{3}$$

The manner in which Helfenstein uses this equation is also well established: we repeatedly differentiate both sides at $x = 0$ to obtain the consecutive derivatives of $f(x)$ at $x = 0$. Thus, we try to solve the equation by finding the $f(x)$'s Taylor series at $x = 0$. Helfenstein uses the following notation, defining coefficients of the Taylor expansion at $x = 0$: for $n = 0, 1, 2, \ldots$:

$$\left.\frac{d^n f(x)}{dx}\right|_{x=0} = n!\, a_n. \tag{4}$$

Often we will write $a_n(c)$ instead of a_n when it is necessary to consider the dependence of a_n upon the parameter c.

The consensus of several methods is that these equations can be used to determine a_n by a recurrence relation, and thus determine the Taylor expansion of $f(x)$ at $x = 0$ up to an arbitrary order. Moreover, the symmetries imply that $f(x)$ is an even function:

$$f(-x) = f(x).$$

This implies that $a_n = 0$ for odd n. Moreover, $f(0) = 0$ follows from the assumptions made, that C passes through A and B.

We come to a point where Helfenstein makes a calculation error in calculating the third nontrivial coefficient, a_6. Only simple calculus (chain rule) is involved. Calculating a_2 and a_4 by hand would test anyone's patience, but today it is conveniently done with the aid of a Computer Algebra System (CAS). Helfenstein calculated a_2 and a_4 correctly. Calculation of a_6 must have been very challenging without a CAS, and indeed it resulted in an important error which affects the entire argument.

We wrote a simple CAS program which determines the coefficients a_n for n up to 10. In theory, the program can find a_n for arbitrarily large n, but even CAS consumes an amount of time that probably grows exponentially with n. The CAS we used is the open source, free system Maxima [4], although any other CAS can solve this problem. We included our program as Appendix A.

The results are presented below for even n only. Moreover, we list numbers

$$b_n = a_n\left(\frac{1}{2}+\frac{\sqrt{z}}{2}\right), \tag{5}$$

which are analytic in z iff a_n are invariant under the substitution $c \mapsto 1-c$. Thus, it is much easier to read off the invariance by looking at b_n.

The program generated the coefficients in standard TEX format. Additionally, the program factored a_n as rational functions of c, for easy comparison of a_2 and a_4 with Helfenstein's paper. Here is the result:

$$a_2 = \frac{1}{2\,(2\,c^2-2\,c+1)}$$

$$a_4 = -\frac{12\,c^4-24\,c^3+12\,c^2-1}{8\,(2\,c^2-2\,c+1)^2\,(2\,c^4-4\,c^3+6\,c^2-4\,c+1)}$$

$$a_6 = \frac{80c^{10}-400c^9+680c^8-320c^7-500c^6+940c^5-712c^4+284c^3-52c^2+1}{16\,(2\,c^2-2\,c+1)^4\,(c^4-2\,c^3+5\,c^2-4\,c+1)\,(2\,c^4-4\,c^3+6\,c^2-4\,c+1)}$$

$$b_2 = \frac{1}{z+1}$$

$$b_4 = -\frac{3\,z^2-6\,z-1}{z^4+8\,z^3+14\,z^2+8\,z+1}$$

$$b_6 = \frac{10\,z^5-110\,z^4-20\,z^3+204\,z^2+42\,z+2}{z^8+24\,z^7+172\,z^6+488\,z^5+678\,z^4+488\,z^3+172\,z^2+24\,z+1}.$$

On the web we listed the coefficients up to a_{10} without folding or breaking them up. The corresponding b_n are clearly analytic, i.e., do not contain half-integer powers of z. This means that a_n is invariant under the substitution $c \mapsto 1-c$ for n up to 10. We can push this calculation further, up to, say, $n = 20$, always with the same result: it is invariant under this substitution.

Helfenstein's paper contains correct expressions for a_2 and a_4. However, he did not include the expression for a_6. Since he derived a false conclusion about a_6, as we will see below the only possible explanation is that he made a computational error in the intermediate calculations. Helfenstein's argument is founded on an unproven claim that if a 2e-curve exists for some value c then a_n must be invariant under the substitution $c \mapsto 1-c$. More precisely, if we consider $a_n = a_n(c)$ (i.e., as a function of c) then the condition $a_n(c) = a_n(1-c)$ is necessary (according to Helfenstein) for a 2e-curve to exist for a particular value of c.

Finally, we are ready to explain Helfenstein's main argument, and how he arrived at the erroneous six-fold differentiability condition. He correctly noted that the expressions for a_2 and a_4 are invariant under the substitution $c \mapsto 1-c$. He then considers a_6 as a candidate for a coefficient which is not invariant under this substitution. In contradiction with our findings, he claims that it is not invariant under the substitution $c \mapsto 1-c$. Helfenstein writes:

> A sixth differentiation finally yields an expression for $a_6(c)$ which is not identical to $a_6(1-c)$.

The form of a_6 is omitted in Helfenstein's paper and the intermediate calculations of it are missing. He then proceeds to determine specific values of c which solve the equation:

$$a_6(c) = a_6(1-c).$$

He claims that the above equation is equivalent to a certain polynomial equation of the 9-th degree:

$$144\,c^9 - 648\,c^8 + 1176\,c^7 - 1092\,c^6 + 168\,c^5 + 798\,c^4 - 846\,c^3 + 357\,c^2 - 59\,c + 1 = 0.$$

Subsequently, he demonstrates that equation does not have roots in the range

$$\frac{2-\sqrt{3}}{4} < c < \frac{2+\sqrt{3}}{4}$$

which is known to be the region of c, outside of which there is no 2e-curve, based on elementary arguments which preceded Helfenstein's paper. Clearly, the 9-th degree polynomial and the subsequent conclusions are a result of a calculation error.

Helfenstein's claim is that there are no 2e-curves which are six-fold differentiable. The reason is that he needs this much differentiability to calculate a_6. As we have shown, the six-fold differentiability of the function $f(x)$ at $x = 0$ is not sufficient to disprove the existence of a 2e-curve. Moreover, we verified with a CAS that Helfenstein's argument fails for curves which are ten-fold differentiable, on the basis of our calculations of a_n and b_n for n up to 10.

Of course, now it is time to pause, and suggest that the argument fails for all n. This shows that Helfenstein's approach cannot succeed, even if we had unlimited computing power and find as many coefficients a_n as necessary.

3. More differentiability does not help

One could hope that by finding more derivatives of $f(x)$ we will eventually find a coefficient $a_n(c)$ for which $a_n(c)$ and $a_n(1-c)$ do not coincide. In this section we will show that $a_n(c) = a_n(1-c)$ for all n, given that all derivatives up to n exist. This, of course, demonstrates that Helfenstein's method cannot disprove the existence of a 2e-curve.

The key point is to understand the *local* existence and uniqueness of solutions of Helfenstein's functional equation. It should be noted that his paper is an attempt to disprove local existence. What he missed is the fact that the *locally defined* solution to his functional equation does exist! Furthermore, he missed that the local existence does not imply that a 2e curve exists.

The existence and uniqueness results are contained in the *Inventiones* article [5], but we will formulate those results here in a manner more suitable for these notes.

The graph of a solution to the Helfenstein's functional equation gives rise to two Jordan arcs, C_A and C_B, contained in neighborhoods of A and B respectively. The arcs C_A and C_B are defined by the equations

$$\eta = f(\xi),$$
$$y = f(x)$$

in the respective coordinate system. A picture is worth a thousand words. Thus, if the reader still cannot imagine how the two arcs C_A and C_B may connect together when they are maximally extended, without forming a 2e-curve, a plausible scenario can be visualized by a schematic drawing in Figure 2. The alternative would be for

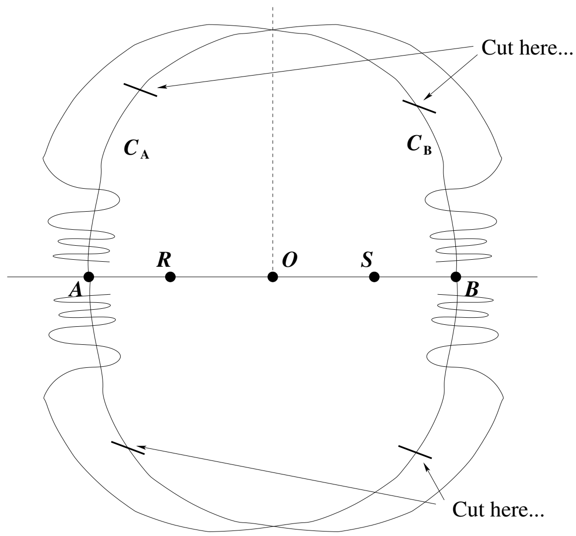

FIGURE 2. A schematic figure of local curves connecting.

the two arcs to meet exactly and form one smooth curve. The fact that they do not meet in this manner is the thrust of the *Inventiones* article [5]. It should be noted that the figure has perfect reflectional symmetry, both with respect to reflections in the line AB and in the center point O. Because Helfenstein did not look away from the line AB (i.e., outside the "cuts", which mark the ends of Jordan arcs C_A and C_B) he failed to notice that his assumptions may be satisfied by a *local equichordal configuration*. And indeed, this is what really happens. While the above figure is only a schematic, we and others performed numerical computations which are in perfect agreement with the above picture in regard to its topology. It should be noted that the oscillations by the curves near A and B continue up to the line AB. The size of the oscillations settles down to a fixed, positive amplitude.

The extremely important idea in understanding the equichordal point problem has been that the problem should be phrased as a problem about iterations of mappings, i.e., should be framed as a problem of *dynamical systems theory*. To remain faithful to Helfenstein's notation, we define a map $G_c : U_c \to \mathbb{R}^2$, where $U_c \subset \mathbb{R}^2$ is an open, punctured unit disk centered at $S(0, c)$:

$$U_c = \left\{(x, y) \in \mathbb{R}^2 \, : \, 0 < x^2 + (y - c)^2 < 1\right\}.$$

The map G_c is defined by the formula $G_c(x, y) = (\xi, \eta)$ where ξ and η are given by Equations (1)–(2). For better understanding, one should consider c to be a parameter, and think of the mapping $G : U \to \mathbb{R}^2$ defined by $G(\xi, \eta, c) = G_c(x, y)$, where $U \subset \mathbb{R}^3$:

$$U = \left\{(x, y, c) \in \mathbb{R}^3 \, : \, 0 < c < 1, \; 0 < x^2 + (y - c)^2 < 1\right\}.$$

Occasionally, there is a technical advantage to including c in the set of variables, for instance, when stating joint continuity, differentiability, etc., which includes the parameter.

Geometrically, G_c acts on a point $Q(x,y)$ in an almost obvious way. The preliminary idea is to map it to the point $Q(x,y)$ to the point $P(\xi,\eta)$, where ξ and η are computed from Equations (1)–(2). However, this point is subsequently identified with a point $Q_1(x',y')$ where *numerically* $x'=\xi$ and $y'=\eta$. Thus, the action of G_c on a point Q is described as a composition of two maps ("walks"):

$$Q \mapsto P \mapsto Q_1$$

where the two walks may be described as follows:

(1) We start at Q, and walk towards S and pass it, until we have walked a total distance of 1;
(2) We walk from P towards O and pass it, until we reach Q_1, and satisfy the distance condition $|QO| = |OQ_1|$.

In short:

$$G_c(Q) = Q_1.$$

For every $c \in (0,1)$ the domain of G_c is the open disk about S of radius 1, without its center (a punctured disk). The reader should note that according to our conventions A and S lie on the same side of O when $c < 1/2$ and on the opposite sides when $c > 1/2$. Although it is possible to enlarge the domain with some additional conventions, we will not do so, and adhere to the "natural" domain described above.

The role of the substitution $c \mapsto 1-c$ is explained in the following

Proposition 1. *For an arbitrary $c \in (0,1)$, Let Q be in the domain of G_c and let $Q_1 = G_c(Q)$. Then G_{1-c} is well-defined at Q_1 and*

$$G_{1-c}(Q_1) = Q.$$

In particular, the mappings G_c and G_{1-c} are inverses of each other and $G_c(U_c) = U_{1-c}$. In addition the mapping $G_c : U_c \to U_{1-c}$ is a diffeomorphism.

Proof. Let us consider point $P(\xi,\eta)$ and another point, $P_1(\tilde{\xi},\tilde{\eta})$ defined as the reflection of $Q(x,y)$ through the point O. We claim that:

(1) P_1, R and Q_1 are collinear.
(2) $|P_1Q_1| = 1$.

Then the equation $G_{1-c}(Q_1) = Q$ follows from the above claims and the definition of G in terms of walks. Indeed, the claims imply that the two walks defining $G_{1-c}(Q_1)$ are:

$$Q_1 \mapsto P_1 \mapsto Q.$$

Both properties follow from the observation that the quadrilateral QPP_1Q_1 is a parallelogram, Indeed, its two diagonals are bisected by O. In particular, the side P_1Q_1 is parallel to PQ which has length 1. Therefore, both sides have length 1. Point S lies on the side PQ by definition. Thus, R lies on the opposite side P_1Q_1 because O bisects RS by definition. □

This above statement and proof carefully avoid complicated algebraic equations. An attempt to prove the above proposition by brute force calculations is likely to fail. For a reader wanting to understand an algebraic approach to the above lemma, we included a CAS program Appendix C which arithmetically verifies the claims in the above proof. Another way[1] which uses the polar coordinates $x = r\cos\theta$, $y = c + r\sin\theta$, is left to the reader as an exercise.

The significance of Proposition 1 in the context of Helfenstein's paper: the invariant sets of G_c and $G_c^{-1} = G_{1-c}$ are identical. It should also be noted that the following two conditions are equivalent for a fixed c:

(1) A function $y = f(x)$ satisfies the functional equation (3).
(2) The graph $C_B = \{(x, f_c(x))\}$ is an invariant set of G_c.

The invariance should be understood locally in the neighborhood of B or near $x = 0$. Local invariance of a Jordan arc C_B means that C_B is contained in the domain of G_c. There is a neighborhood K of B such that $C_B \cap K = G_c(C_B) \cap K$.

The uniqueness of the solutions easily implies that Helfenstein's method cannot work. Let us denote by $f_c(x)$ any solution of the functional equation (3), defined in some neighborhood of $x = 0$. If we know that the solution is unique then $y = f_c(x)$ is a solution of the functional equation for both c and $c' = 1-c$. Uniqueness implies $f_c(x) = f_{1-c}(x)$ and equality $a_n(c) = a_n(1-c)$ follows for all n.

4. Local existence and uniqueness

It turns out that a properly formulated existence and uniqueness theorem eliminates Helfenstein's approach as viable, but it also eliminates the possibility that a continuous 2e-curve exists. The key to such a strong result is a consideration of curves closely related to the solutions to the functional equation (3) which do not satisfy $f(0) = 0$, but instead $f(0) = y_0$, where y_0 varies in the range $|y_0| < \min(c, 1-c)$. Such a family provides a good coordinate system near B. It should be noted that for any solution of Equation (3) we have $f(0) = 0$. Thus, a function satisfying $f(0) = y_0$ cannot be a solution. However, it can be a solution of the *second iteration* of the associated iterative method by which solutions to Equation (3) may be found. Also, we will formulate a new functional equation (6) whose solutions determine these new functions f. As y_0 varies, the resulting family of curves is an *invariant family of curves*, i.e., one curve of the family is mapped to another member of the family under the iterative process used to find the solutions of the original functional equation (3).

Figure 3 schematically depicts the idea of the resulting coordinate system formed by the invariant family of curves. In its caption, the figure states several claims reflecting the behavior of G_c near the line RS. Instead of $y = f(x)$ it will be appropriate to write $y = F(x, y_0, c)$, where the dependence on y_0 and c is made explicit. Thus, for fixed y_0 and c, we simply set $f(x) = F(x, y_0, c)$. Equation (6) is precisely the functional equation that the function F (with *three* independent variables!) must satisfy in order to justify all claims. Geometrically, $y = F(x, y_0, c)$ describes a 2-parameter family of curves in $\mathbb{R}^3$.

This section is mainly expository, as the proof can be extracted from our *Inventiones* article [5]. The technique is covered in the monograph of Hirsch, Pugh and Shub [2]. Therefore, we walk the reader through the constructions and provide some motivations leading up to the theorem, which we formulate at the end.

[1] We are indebted to the reviewer for pointing out this method.

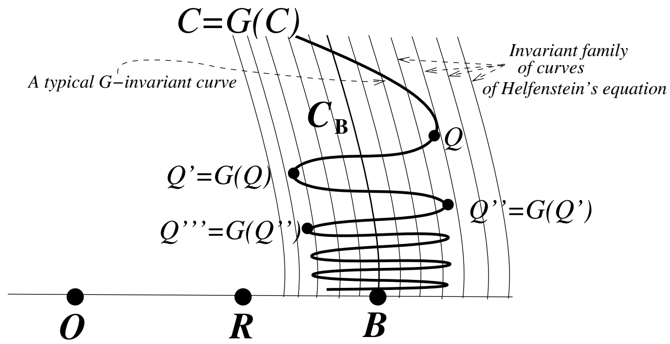

FIGURE 3. The invariant family of curves $y = F(x, y_0, c)$ near B for the map $G = G_c$. We assume that $1/2 < c < 1$, which implies that R is between O and B. Only a half of each curve is drawn in the upper halfplane above the line SB. A typical invariant curve C such that $C = G_c(C)$ is depicted, as it enters the neighborhood of B foliated by curves of the family. A sample trajectory $Q^{(i)} = Q, Q', Q'', Q''', \ldots$, where $i = 0, 1, 2, \ldots$, under iteration of G, is depicted. Unless even and odd subsequences $Q^{(2i)}$ and $Q^{(2i+1)}$, which must converge, both converge to the same point B, the invariant curve C must oscillate between two points lying on the straight line RS.

Throughout this section we use the following set

$$V = \left\{ (y_0, c) \in \mathbb{R}^2 \, : \, 0 < \left| c - \frac{1}{2} \right| < \frac{1}{2}, \; |y_0| < \min(c, 1-c) \right\}.$$

We shall consider a family of curves in $\mathbb{R}^3$, $\{\Gamma(y_0, c)\}$, locally represented as a graph $y = F(x, y_0, c)$ and passing through point $(0, y_0)$, i.e.,

$$y_0 = F_c(x, y_0).$$

When c is fixed, we use the notation $F_c(x, y_0)$ instead of $F(x, y_0, c)$ and $\Gamma_c(y_0)$ for the curve in $\mathbb{R}^2$ given by the equation $y = F_c(x, y_0)$. This emphasizes c's role as a parameter.

We allow $(y_0, c) \in V$; equivalently, when c is fixed, we require that $|y_0| < \min(c, 1-c)$. Moreover, we require the local invariance condition: the curve $\Gamma_c(y_0)$ is mapped by G_c to another curve of the family. Which one? It is easy to verify that

$$G_c(0, y) = (0, -y).$$

Thus, we know that the point $(0, -y_0)$ lies in the image of $\Gamma_c(c)$ and thus

$$G_c\left(\Gamma_c(y_0)\right) = \Gamma_c(-y_0).$$

locally in a neighborhood of $x = 0$.

It can be seen that this invariance condition is equivalent to a functional equation that the function $F(x, y_0, c)$ must satisfy at least in a neighborhood of the line $x = 0$:

$$F\left(\frac{x}{\sqrt{x^2+(c-F(x,y_0,c))^2}} - x, -y_0, c\right) = \frac{-(c-F(x,y_0,c))}{\sqrt{x^2+(c-F(x,y_0,c))^2}} + (1-F(x,y_0,c)). \tag{6}$$

Conceptually, both y_0 and c are parameters in F and once we fix them, we also consider the function $F_{y_0,c} : J \to \mathbb{R}$, $J \subseteq \mathbb{R}$, given by:

$$F_{y_0,c}(x) = F(x, y_0, c).$$

Also, conceptually, F is a 2-parameter family of ordinary 1-variable functions. We will use the resulting three notations for F interchangeably. The domain J of $F_{y_0,c}$ shall depend on the parameters. We will assume that J is an open interval, symmetric about $x = 0$, and that its length varies continuously:

$$J = J_\varphi(y_0, c) = \{x \in \mathbb{R} : -\varphi(y_0, c) < x < \varphi(y_0, c)\}$$

where $\varphi : V \to \mathbb{R}$ is a certain positive, continuous function. Thus, equivalently $F : \mathcal{J}_\varphi \to \mathbb{R}$ where $\mathcal{J}_\varphi \subset \mathbb{R}^3$ is given by:

$$\mathcal{J}_\varphi = \left\{(x, y_0, c) \in \mathbb{R}^3 : 0 < \left|c - \frac{1}{2}\right| < \frac{1}{2},\ |y_0| < \min(c, 1-c),\ |x| < \varphi(y_0, c)\right\}.$$

For fixed c, we will also consider the section of $\mathcal{J}_\varphi$:

$$\mathcal{J}_\varphi(c) = \left\{(x, y_0) \in \mathbb{R}^2 : |y_0| < \min(c, 1-c),\ |x| < \varphi(y_0, c)\right\}.$$

Let

$$H_c = \begin{cases} G_c & \text{if } c > 1/2, \\ G_c^{-1} & \text{if } c < 1/2. \end{cases}$$

We recall that $G_c^{-1} = G_{1-c}$. The reason for this definition is that the theorem below is applicable either to G_c or G_c^{-1} depending on whether $c > 1/2$ or $c < 1/2$. To maintain symmetry of these two cases, we formulate the theorem for H_c instead of G_c. We recall that $G_c^{-1} = G_{1-c}$ which means that it would be sufficient to consider only the range $c > 1/2$.

Theorem 1. *There exists a continuous, positive function $\varphi : V \to \mathbb{R}$ such that:*

(1) *The set $\mathcal{J}_\varphi(c)$ is contained in the domain of H_c.*
(2) *We have $H_c(\mathcal{J}_\varphi(c)) \subseteq \mathcal{J}_\varphi(c)$.*
(3) *For every $(x, y) \in \mathcal{J}_\varphi(c)$ the limit $\lim_{n\to\infty} H_c^n(x, y)$ exists and it lies in the set*

$$V_c = \{(x, y) : x = 0, |y| < \min(c, 1-c)\}.$$

(4) *The mapping $\pi_c : \mathcal{J}_\phi(c) \to V_c$ defined by*

$$\pi_c(x, y) = \lim_{n\to\infty} H_c^n(x, y)$$

is real-analytic and it is a fiber bundle. *In particular $\pi_c(0, y) = (0, y)$; equivalently $\pi_c|V_c = \mathrm{id}_{V_c}$.*

(5) *The sets $\Gamma_c(y_0)$ defined for all y_0 s.t. $|y_0| < \min(c, 1-c)$ by the formula:*

$$\Gamma_c(y_0) = \pi_c^{-1}(0, y_0) = \left\{(x,y) \in \mathcal{J}_\varphi(c) \,:\, \lim_{n\to\infty} H_c^n(x,y) = (0,y_0)\right\}$$

may also be equivalently defined by the equation

$$y = F(x, y_0, c)$$

where $F : \mathcal{J}_\varphi \to \mathbb{R}$ is a unique, real-analytic function.

(6) *For all y_0 s.t. $|y_0| < \min(c, 1-c)$ we have*

$$H_c\left(\Gamma_c(y_0)\right) \subset \Gamma_c(-y_0).$$

In particular,

$$H_c^2\left(\Gamma_c(y_0)\right) \subset \Gamma_c(y_0)$$

i.e., the curve $\Gamma_c(y_0)$ is invariant under the action of H_c^2. The curve $\Gamma_c(0)$ is invariant under H_c.

(7) *We have the following decomposition:*

$$\mathcal{J}_\varphi(c) = \bigcup_{y_0\,:\,|y_0|<\min(c,1-c)} \Gamma_c(y_0).$$

(8) *The convergence in the definition of $\Gamma_c(y_0)$ is exponentially fast. More precisely, there exist continuous functions $\mu, D : V \to \mathbb{R}$ such that $0 < \mu < 1$ and $0 < D < \infty$ and such that for all $(x,y) \in \mathcal{J}_\varphi(c)$ and all nonnegative integers n:*

$$\|H_c^n(x,y) - (0,y_0)\| \le D(y_0,c)\,\mu(y_0,c)^n.$$

(9) *Each curve $\Gamma_c(y_0)$ is tangent to the x-axis, i.e., it is normal to the line RS. Alternatively, for every $(y_0, c) \in V$:*

$$\left.\frac{\partial F(x,y_0,c)}{\partial x}\right|_{x=0} = 0.$$

An outline of the proof is a subject of another section. Let us note that the curve $\Gamma(0,c)$ and the corresponding equation $y = F(x,0,c)$ solve Helfenstein's functional equation (3).

5. Consequences of local existence and uniqueness

We shall start with perhaps the least understood consequence of Theorem 1: the nonexistence of nondifferentiable, continuous 2e-curves. We will show only *local differentiability* here, near points A and B.

Proposition 2. *Let $c \in (0, 1/2) \cup (1/2, 1)$. Let $K \subseteq \mathbb{R}^2$ be a closed subset contained in U_c, the domain of G_c, such that $G_c(K) = K$. Moreover, let $S \notin K$. Let φ, $\mathcal{J}_\varphi(c)$ and π_c be given by Theorem 1. Then*

$$\mathcal{J}_\varphi(c) \cap K \subseteq \bigcup_{E\in V\cap K} \pi_c^{-1}(E).$$

Proof. The assumption $S \notin K$ implies that K is a compact subset of U_c. Indeed, otherwise it would either have S in it, or contain a sequence $Q_n \in K$ converging to the circular portion of the boundary of U_c. Invariance implies that $G(Q_n)$ converges to S, so $S \in K$, which contradicts our assumptions.

To prove the main claim, let us consider a point $Q \in \mathcal{J}_\varphi(c) \cap K$. Let us consider the limit $E = \lim_{n\to\infty} H_c^n(Q)$ which exists by Theorem 1. The set K is closed and invariant under H_c, and thus $E \in K$. Hence, $Q \in \pi_c^{-1}(E)$. □

Proposition 2 can be applied to a hypothetical 2e-curve C of G_c. Since C must intersect the line RS at exactly two points A and B then K must be contained in the union of two analytic Jordan arcs $\pi^{-1}(\{A, B\})$. So, C is automatically *locally analytic* near points A and B.

Proposition 2 may also be used to deduce an interesting property of the real (not-hypothetical!) curves depicted schematically in Figure 2. It depicts two G_c-invariant curves which are reflections of each other and widely oscillate near one of the points A or B. Let C be one of these curves. The curve is an *immersed* copy of $\mathbb{R}$, but it is not a submanifold of $\mathbb{R}^2$ and is not closed. Nevertheless, we may apply Proposition 2 to the closure of C. We obtain the following result:

Proposition 3. *Let C be an invariant set of G_c which is a submanifold disjoint from the set $x = 0$, and let $\overline{C}$ be its closure. Then either $\overline{C}$ is an analytic curve or contains an interval which is a subset of the line $x = 0$.*

The interpretation of this result is that either the curve can be completed to an analytic submanifold of the plane by adding to it the points of the line $x = 0$ which lie in its closure, or it must widely oscillate near the line $x = 0$. This fact is interesting because there have been many attempts at constructing a 2e-invariant curve by starting with a point Q and its image $Q' = G(Q)$, and then connecting them somehow by a path. One can obtain an invariant curve C by iterating this path. Authors of these attempts often make a claim of continuity of their curve as it approaches the line RS. The above proposition shows that such attempts must fail to construct a continuous solution, unless the constructed curve is analytic near A and B. As we will point out, the *Inventiones* article [5] proves that if C is a 2e-curve which is locally analytic near the line RS then it is globally analytic. The article disproves the existence of an analytic 2e-curve, and thus a locally analytic one also, and furthermore by the above proposition, a continuous 2e-curve.

To prove that local differentiability of C near A and B implies global differentiability at all points, a global proof is required. Such a proof is given in Theorems 3 and 4 in the *Inventiones* article [5]. We refer the reader to the proofs there. Let us just comment that to prove global differentiability of C we need two facts. The first fact that the map $G_c^{\pm 1}$ is defined on and differentiable on the entire hypothetical 2e-curve C. The second fact is that for every point $Q \in C$ the limit $\lim_{n\to\infty} G^{\pm n}(Q)$ exists and is either A or B. Then the local differentiable structure near A or B may be "transplanted" by a suitable iteration of a local diffeomorphism G to any point of C.

Theorem 1 and Proposition 2 imply that for every $c \in (0, 1)$, $c \neq 1/2$, the function $f(x) = F(x, 0, c)$ is the unique *continuous* solution of Helfenstein's functional equation (3), and this solution is automatically analytic. A simple consequence of this uniqueness is that every solution is automatically even: $f(x) = f(-x)$. Indeed, if $f(x)$ solves the functional equation then so does $\tilde{f}(x) = f(-x)$, by inspection. Hence, in view of uniqueness, $\tilde{f}(x) \equiv f(x)$.

To gain a little bit more clarity, we make the following definition:

Definition 2. We will call a pair of continuous Jordan arcs (C_A, C_B) *a local equichordal configuration with respect to the points* R *and* S iff:

(1) The intersection of any straight line parallel to RS with C_A or C_B consists of at most one point.
(2) The intersection of C_A and C_B with the straight line RS consists of exactly one point, denoted A and B respectively.
(3) For every pair of points P and Q such that:
    (a) P is in C_A and Q is in C_B;
    (b) the points P and Q and one of the points R or S, are collinear;
    the distance $|PQ| = 1$.
(4) The arcs C_A and C_B are symmetric with respect to the reflection in the point O, the center of the segment RS, and with respect to the axis RS.

Less formally, if we only look at the chords of C whose ends are close to A and B, points R and S apppear equichordal for the curve C. We request that a local equichordal configuration have the symmetry properties, which would follow if a 2e-curve existed. However, we do not require that a local equichordal configuration be a part of a 2e-curve.

It is clear to a reader of Helfenstein's paper that he attempts to prove nonexistence of a local equichordal configuration and deduce from it that a 2e-curve does not exist. The twist is that there exists a local equichordal configuration, but still no 2e-curve exists. To Helfenstein's credit, in 1956 figures like Figure 2 were uncommon, used primarily as counterexamples in topology. The classical example is the curve

$$y = \sin\frac{1}{x}$$

which, together with the y axis, joined somehow together to make the figure connected, form an example of a set that is not locally connected. Later on, the science of chaos made a discovery that curves like this are common in studying differential equations describing real mechanical systems. A search on the terms *homoclinic connection* and *heteroclinic connection* yield many references to such systems. It should be noted that first examples of this kind were known already to Poincaré and thus available to Helfenstein.

The reader of Helfenstein's paper can easily verify that Helfenstein tries to prove the nonexistence of $f(x)$ described by Theorem 1. Since Theorem 1 shows that the localized version of the equichordal point problem has a solution for all c in $0, 1$, $c \neq 1/2$, this means that any local nonexistence argument focused on a small neighborhood of the line RS must fail. This is a deeper reason why the problem had remained open for 80 years until our article [5].

Corollary 1. *For every* c *in the range* $0 < c < 1$ *Helfenstein's functional equation has a unique solution* $f(x)$ *defined in a neighborhood of* $x = 0$*, and which is infinitely differentiable and for all* n*. Moreover, the solution for* c *and* $1-c$ *is the same, which implies that for all* c*:*

$$a_n(c) \equiv a_n(1-c).$$

This corollary is the consequence of the symmetry with respect to the straight line passing through O and perpendicular to the line AB. This symmetry exchanges the role of c and $1-c$. Thus, if a function $f(x)$ solves the problem for c then it also solves it for c replaced with $1 - c$. Helfenstein was clearly aware of this corollary, but made an incorrect assumption about the existence and uniqueness.

6. Equichordal point problem solved, after all

Although $a_n(c) \equiv a_n(1-c)$ for all n, the absence of 2e-curves is proved by different means in the *Inventiones* article [5]. Thus, the line of reasoning used by Helfenstein proved to be one of the numerous traps that await anyone studing the problem. The Helfenstein's invariance condition appears to have no significance in the *Inventiones* solution.

As we have shown, the function $f(x)$ with all the symmetry properties, which is also a solution to Helfenstein's functional equation, does exist *locally* in a neighborhood of $x = 0$. It is the locality of the argument that is the main error in Helfenstein's paper. Only a *global argument*, which takes into account the behavior of $f(x)$ to the point of breakdown of its properties as a *single-valued* function, can eliminate the possibility of a solution to the equichordal point problem.

Helfenstein's argument is purely local, i.e., it does not refer to the behavior of the function outside a small neighborhood of zero. It is clear that the entire oval cannot be represented as a graph of a single-valued function $f(x)$. For instance, the standard unit circle is often represented as the graph of the equation $y = \pm\sqrt{1-x^2}$, where $|x| < 1$. However, the right-hand side has *two* possible values. Obviously, any convex oval can be represented by a two-valued function with two "branch points", when the two values coalesce to form a closed curve.

Thus, it is clear that the breakdown of the representation of C as a graph of an equation $y = f(x)$ must occur at some point, something that Helfenstein does not discuss. Other papers deal with this issue. Wirsing in his 1958 article [8], and Shäfke and Volkmer in their 1992 article [6], represent the curve in polar coordinates and use the equation $r = g(\theta)$ to represent the curve. This equation does not suffer from the limit on the range, and is capable of capturing the solution to the Helfenstein's equation near A and B *simultaneously*. Moreover, $g(\theta) \equiv 1/2$ represents the unit circle, which naturally plays a special role in the asymptotic considerations as $c \to 1/2$. The Wirsing and Shäfke and Volkmer research reveals the nature of the obstruction to the existence of an oval with two equichordal points: if a solution is well-behaved near A, it must lose continuity near B and vice versa. Hence, there is no *globally defined* solution in polar coordinates, either.

7. An outline of the proof of Theorem 1

The first step of the proof is to interpret the solution of the functional equation (6) together with its side condition:

$$\begin{aligned} &F\left(\frac{x}{\sqrt{x^2+(c-F(x,y_0,c))^2}} - x, -y_0, c\right) \\ &\qquad = \frac{-(c-F(x,y_0,c))}{\sqrt{x^2+(c-F(x,y_0,c))^2}} + (1-F(x,y_0,c)), \\ &y_0 = F(x,y_0,c). \end{aligned}$$

as a question about invariant manifolds. We define a map $\mathcal{G} : U \to \mathbb{R}^3$, where $U \subset \mathbb{R}^3$ has already been defined before:

$$U = \left\{(x, y, c) \in \mathbb{R}^3 \,:\, 0 < c < 1,\ 0 < x^2 + (y - c)^2 < 1\right\}$$

defined by the formula $\mathcal{G}(x, y, c) = (\xi, \eta, c)$ where

$$\xi = \frac{x}{\sqrt{x^2 + (c - y)^2}} - x$$
$$\eta = \frac{-(c - y)}{\sqrt{x^2 + (c - y)^2}} + (1 - y).$$

Also see Equations (1)–(2). It is also true that $\mathcal{G}(x, y, c) = (G_c(x, y), c)$, using our prior notation, i.e., it simply extends G_c to three dimensions by adding a trivial action on the parameter c.

It can be seen that the surface W given by the equation $y = F(x, y_0, c)$ is locally invariant under the mapping $\mathcal{G}$ iff $F(x, y_0, c)$ solves its functional equation, i.e., $\mathcal{G}(W \cap U) \subseteq W$. Because $|y| < \min(c, 1 - c)$, we do not have to worry about issues such as nondifferentiability of $\mathcal{G}(x, y, c)$ or $\mathcal{G}^{-1}(x, y, c)$ when $x^2 + (c - y)^2 = 0$, i.e., at the point $(0, c, c)$; it is outside of V. Moreover, every point of V is a periodic point of $\mathcal{G}$ of period 2, because it is easy to see that:

$$\mathcal{G}(0, y, c) = (0, -y, c).$$

Thus, the point $(0, 0, c)$ (corresponding to the point A in Helfenstein's paper) is a fixed point of $\mathcal{G}$.

In particular, V is an *invariant manifold* of dimension 2. It is a union of two open segments of a straight line. The next step in the proof is to notice that the manifold V is *normally hyperbolic* in the sense of Hirsch, Pugh and Shub [2]. Without repeating lengthy definitions, we will explain what this means. We start with linearizing $\mathcal{G}$ at all points of V:

$$\mathcal{G}(u, y + v, c + w) = \mathcal{G}(0, y, c) + D\mathcal{G}(0, y, c) \cdot (u, v, w) + O(\|(u, v, w)\|^2)$$

where $D\mathcal{G}(0, y, c)$ is the (Fréchet) derivative of $\mathcal{G}$, i.e., a 2×2 matrix. With a little bit of work, or using a Maxima program presented in Appendix B, we find:

$$D\mathcal{G}(0, y, c) = \begin{bmatrix} -\frac{1}{y-c} - 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It happens that $D\mathcal{G}(0, y, c)$ is diagonal, and thus it has three eigenvectors: $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e_2} = (0, 1, 0)$ and $\mathbf{e_3} = (0, 0, 1)$, with eigenvalues:

$$\lambda_1 = \frac{1}{c - y} - 1,$$
$$\lambda_2 = -1,$$
$$\lambda_3 = 1,$$

respectively.

What is important is that the tangent space to V, spanned by the set $\{\mathbf{e}_2, \mathbf{e}_3\}$, is invariant and $|\lambda_2| = |\lambda_3| = 1$. This means that under iteration of $\mathcal{G}$ the manifold V does not expand or contract. It is also important that in the normal direction to V we have linear contraction or expansion. This can be verified by performing two iterations of $\mathcal{G}$, starting from the point $(0, y, c)$. In two iterations, the direction $\mathbf{e}_1$ scales by the product of the first eigenvalues for $D\mathcal{G}(0, y, c)$ and $D\mathcal{G}(0, -y, c)$, i.e., the multiplier

$$\mu = \left(\frac{1}{c-y} - 1\right) \cdot \left(\frac{1}{c+y} - 1\right) = \frac{(1-c)^2 - y^2}{c^2 - y^2}.$$

In particular, for $|y| < \sqrt{\min(c, 1-c)}$ the multiplier μ is positive. If $0 < c < 1/2$, the numerator is larger than the denominator and $\mu < 1$. If $1/2 < c < 1$ then the numerator is smaller than the denominator, and $\mu > 1$. Of course, when $c = 1/2$ then $\mu = 1$.

Let us consider the partition $V = V^+ \cup V^-$, where

$$V^+ = V \cap \left\{(x, y, c) \,:\, c > \frac{1}{2}\right\},$$
$$V^- = V \cap \left\{(x, y, c) \,:\, c < \frac{1}{2}\right\}.$$

The manifolds V_+ and V_- are both invariant and satisfy the definition of *normal hyperbolicity*, i.e., under the linearization the normal direction (which in our case is $\mathbf{e}_1$) is scaled by a multiplier $\mu \neq 1$. The multiplier μ may depend on the point of $V_\pm$ (and it does!), but it may not approach 1 along the trajectory of a point, i.e., along the set

$$\{(0, y, c),\, (\mathcal{G}(0, y, c),\, \mathcal{G}(\mathcal{G}(0, y, c)),\, \mathcal{G}(\mathcal{G}(\mathcal{G}(0, y, c))),\, \ldots\}.$$

Since in our case this set consists of two points:

$$\{(0, y, c),\, (0, -y, c)\}.$$

Thus, in our case of normal hyperbolicity is easy to verify.

The theory of normally hyperbolic invariant manifolds immediately implies Theorem 1. In passing, we verified all technical assumptions necessary to apply it. Stating the detailed results would involve a large amount of notations and definitions, so we shall not repeat it here, and refer an interested reader to the classical presentation [2].

We note that our setup satisfied the definition of "immediate absolute r-normal hypebolicity" and thus it satisfies the strongest assumptions used in the classical monograph of Hirsch, Pugh and Shub [2]. Thus, the strongest conclusions also hold. It takes very superficial understanding to conclude the existence of a continuous $F(x, c)$, infinitely differentiable over x, with derivatives continuous in c. This suffices to prove $a_n(c) \equiv a_n(1-c)$. With better understanding, one can show joint analyticity in x and c.

It should also be noted that our presentation reflects the modern view point, but Wirsing [8] proved a theorem in *Abschnitt 4* similar to Theorem 1. He covers only the analytic case, but the proof is very concise and to the point, and even today it has some appeal despite the lack of generality.

8. Conclusions

By repeating the calculations in Helfenstein's paper [1] we identified the mistake in the paper. The existence of the mistake was asserted by Wirsing [8], but he did not directly identify the place in which it occurred, which led to half a century of confusion in regard to the validity of Helfenstein's paper. We resolved this issue to our satisfaction. In addition, we showed that Helfenstein's method cannot be a basis of a new, simpler solution of the equichordal point problem than the one currently known [5].

9. Notes and references

9.1. Theorem 1. Theorem 1 corresponds to Theorem 2 of the *Inventiones* article [5]. There is a difference in notation, because in these notes we adopted Helfenstein's notation. The symmetry properties are introduced in the text of the *Inventiones* article in the parts leading up to Theorem 2.

It should be noted that Theorem 1 is in the "easy" part of the article, and therefore any determined reader is capable of understanding it with minimal effort, given some familiarity with invariant manifold theory. Today, invariant manifold theory is quite well understood, with many excellent expositions. We still prefer the work of Hirsch, Pugh and Shub [2].

The analyticity of $F(x, c)$ follows from an argument of [5] used to prove Theorem 7 there. However, the argument is not easy to separate from a more complex situation it is addressing. Finally, the analytic version of Theorem 1 follows from *Abschnitt 4* of Wirsing's 1958 article [8]. Wirsing was not the first one to invent the method of proof. It goes back to the works of Hadamard and Perron, and this fact is well known today. The method is used by Hirsch, Pugh and Shub [2], and the references to Hadamard and Perron can be found there.

9.2. Comments by B. Grünbaum. The following comments are found in the notes by B. Grünbaum dated 2010 included as part of not yet published revision of the book by V. Klee [3][2]:

> But more unexpected has been the reaction to the work of Helfenstein [Hel56]. His rather infelicitously formulated claim is: "We shall show in this paper the nonexistence of real and, in one special point, at least six times differentiable 2e-curves." What his argumentation shows (assuming there are no errors, and nobody pointed out any specific errors in the paper) is that a contradiction is reached from the assumption of existence of a 2e-curve together with the assumption that the curve assumed to exist is six times differentiable at a specific point. This would seem a reasonable result, and the nonexistence would be established provided the differentiability assumption could be proved for all 2e-curves assumed to exist. Hence one would think (and this was explicitly stated by Süss [Süs]) that the non-existence of 2e-curves would follow from Helfenstein's result together with the following result of Wirsing [Wir58]: If a 2e-curve exists, it must be analytic, that is, infinitely differentiable at all points. Indeed, to belabor the completely obvious, if a curve is infinitely differentiable at all points, then it is six times differentiable at a specific point — and the contradiction reached by Helfenstein proves the nonexistence. It is mystery to

[2]According to an e-mail communication with B. Grünbaum, there is no current plan to publish the revised version of Klee's book, or his comments, or revise the comments.

> me why Wirsing thought that Helfenstein's work must be in error since "it is contradicted by the present investigation" ("durch die vorliegende Untersuchung widerlegt wird"). But an even greater mystery is why Klee, in all his discussions of equichordality, accepted Wirsing's statement as valid, and Helfenstein's result as invalid. Wirsing committed a logical error, and Klee and others uncritically accepted it a valid. The reviews of Helfenstein's paper in the Math. Reviews and the Zentralblatt fail to claim any errors in it. Not all details of Helfenstein's proof are given in [Hel56] — but no error has been found either.

Of course, [Hel56] refers to the earlier cited article [1]. The Wirsing's work cited as [Wir58] is [8]. The last citation in the above comment, [Süss], is [7].

Clearly, this inaccurate account of the state of the equichordal point problem still lingers in the public domain. Grünbaum writes in the introduction ([3], p. i):

> About mid-May 2010 it occurred to me that it might be appropriate to have the fifty-years old collection made available to participants at the "100 Years in Seattle" conference.

In contrast with Grünbaum, we are certain that Wirsing rejected Helfenstein's argument for the right reasons: it contradicted Wirsing's own research. Wirsing refers to Helfenstein's work in two places. The main point is made in his *Abschnitt 4*, in which Wirsing proves a variant of our Theorem 1, and at the end he writes:

> Es bleibt die Frage, ob für irgendwelche c und X die analytische Fortsetzung von $\mathfrak{C}_1$ and $\mathfrak{C}_2$ zu geschlossen Kurven führen kann.
>
> Übrigens steht die Tatsache, daß die Doppelspeichenbedingung sich wenigstens lokal (in Umgebungen von T_1 and T_2) durch regulär analytische Kurven befriedigen läßt, im Gegensatz zu der Arbeit von Helfenstein [5], der durch 6-malige Differentiation der Funktionalgleichung bei T_1, T_2 zu einem Widerspruch gelangt und daraus auf die Nichtenxistenz einer 6-mal differenzierbaren D-Kurve schließt. Der Fehler dürfte in den unveröffentlichten Rechnungen liegen.

10. Asymptotic analysis beyond all orders

Shäfke and Volkmer [6] conducted a deep asymptotic analysis of the equichordal point problem, which implies that a 2e-curve may exist only for a finite set of values of c. In spirit, the paper continues the researches of Wirsing. Wirsing proved that if $c \to 1/2$ then the hypothetical 2e-curve must be extremely close to the circle of radius $1/2$ centered at O, closer than $C_\alpha\,|c-1/2|^\alpha$ where α is an arbitrarily large power, and C_α is a certain constant, depending on α. Thus the name "asymptotic analysis beyond all orders" which is sometimes used in reference to this kind of result.

Shäfke and Volkmer quantified this statement even further, expressing the leading asymptotics of the difference between the 2e-curve and the circle in terms of the exponential. In fact, the paper extends the representation of the arcs C_A and C_B in the discussion following Theorem 1 to the full angle θ in polar coordinates except one point (either zero or π) near which the arcs start rapidly oscillating, breaking continuity!

Appendix A. Maxima code showing Helfenstein's mistake

```
/* -*- Mode: Maxima -*-
 * Author: Marek Rychlik,  Date: January 20, 2012
 * This program finds the coefficients a[n] of the Helfenstein's
 * paper by expanding f(x) in a Taylor series about x=0
 * and comparing coefficients at like powers.
 * Additionally, it factors a[n], and finds b[n] obtained
 * from a[n] by substitution c = 1/2 + sqrt(z)/2.
 */
assume(c>0);
/* Define Taylor expansion of f(x) with generic coefficients a[k];
the meaning of a[k] is consistent with Helfenstein */
deftaylor(f(x), sum(a[k]*x^k,k,2,inf));
/* The denominator */
d(x,y):=sqrt(x^2+(c-y)^2);
ksi(x,y):=x/d(x,y)-x;
eta(x,y):=-(c-y)/d(x,y)+(1-y);
/* The functional equation */
eqn:f(ksi(x,f(x)))=eta(x,f(x));
/* The maximum derivative order we consider */
nlimit:10;
/* Expand functional equation in a Taylor series */
tay_eqn:taylor(eqn,x,0,nlimit);
/* Extract coefficients at relevant powers */
sys_eqn:makelist(coeff(tay_eqn,x,k),k,0,nlimit);
/* Solve for the coefficients a[k] */
soln:solve(sys_eqn,makelist(a[k],k,2,nlimit));
/* Assign coefficients */
for n:2 thru nlimit do (
  a[n]:ev(a[n],soln),
  a_factored[n]:factor(a[n])
);
/* Translate coefficients to TeX */
for n:2 thru nlimit step 2 do (
  tex('a[n] = a[n]),
  tex('a[n] = a_factored[n])
  );
/* Find the coefficients b[k] */
for n:2 thru nlimit step 2 do b[n]:ratsimp(ev(a[n],c=1/2+sqrt(z)/2));
/* Translate coefficients b[k] to TeX */
for n:2 thru nlimit step 2 do (
  tex('b[n] = b[n])
  );
```

Appendix B. Maxima code finding Fréchet derivative of G

```
/* -*- Mode: Maxima -*-
 * Author: Marek Rychlik,  Date: January 20, 2012
```

```
 * This program finds the derivative DG(x,y,c)
 * and simplified expression for DG(0,y,c).
 */
kill(all); assume(c>0); assume(y<c);
/* The denominator */
d(x,y,c):=sqrt(x^2+(c-y)^2);
ksi(x,y,c):=x/d(x,y,c)-x;
eta(x,y,c):=-(c-y)/d(x,y,c)+(1-y);
/* The mapping G */
G(x,y,c):=[ksi(x,y,c),eta(x,y,c),c];
/* The Frechet derivative of G */
A:matrix(diff(G(x,y,c),x),diff(G(x,y,c),y),diff(G(x,y,c),c));
/* Typeset the derivative as TeX */
tex(A);
/* Compute the Frechet derivative at a point of V */
B:ev(A,[x=0]);
/* Clean up by expanding into partial fractions */
B:partfrac(B,y);
/* TeX form of the derivative at a point  of V */
tex(B);
```

Appendix C. Maxima code verifying formula for inverse of G

```
/* -*- Mode: Maxima -*-
 * Author: Marek Rychlik,  Date: February 1, 2012
 * This program provides an arithmetical proof that
 * G_c and G_{1-c} are inverses of each other.
 */
/* The denominator */
kill(all);
assume(c>0,c<1,y<c,y-c+1<0,y+c>0,y+c-1>0);
/* The denominator */
d(x,y,c):=sqrt(x^2+(c-y)^2);
u(x,y,c):=x-x/d(x,y,c);
v(x,y,c):=y+(c-y)/d(x,y,c);
ksi(x,y,c):=-u(x,y,c);
eta(x,y,c):=1-v(x,y,c);
/* The map */
G(x,y,c):=[ksi(x,y,c),eta(x,y,c)];
/* The generic point near B */
Q:[x,y];
/* Point P is collinear with S and Q, such that dist(P,Q)=1 */
/* Note: P is expressed in xy-coordinates */
P:[u(x,y,c),v(x,y,c)];
/* Reflection of Q; also expressed in xy-coordinates */
P1:[-x, 1-y];
/* Q1 = G(Q) */
```

```
Q1:[ksi(x,y,c),eta(x,y,c)];
/* Checks that dist(Q1,P1)=1 */
/* Must print "true"! */
is(ratsimp(apply("+",(Q1-P1)^2))=1);
/* Checks that P1,Q1 and R are collinear */
/*(ksi-eta coordinate system) */
R:[0,1-c];
/* Must print "true"! */
is(ratsimp(determinant(matrix(P1-R,Q1-R)))=0);
```

References

[1] Helfenstein, Heinz G. Ovals with equichordal points. *J. London Math. Soc.* **31** (1956), 54–57. MR0075616 (17,778a), Zbl 0070.17504, doi: 10.1112/jlms/s1-31.1.54.

[2] Hirsch, M. W.; Pugh, C. C.; Shub, M. Invariant manifolds. Lecture Notes in Mathematics, 583. *Springer-Verlag, Berlin, New York*, 1977. ii+149 pp. MR0501173 (58 #18595), Zbl 0355.58009.

[3] Klee, Victor. Unsolved problems in intuitive geometry. Annotations by B. Grünbaum. http://www.math.washington.edu/~reu/papers/2010/mark/Unsolved_Problems_In_Intuitive_Geometry.pdf

[4] Maxima. A computer algebra system. http://maxima.sourceforge.net.

[5] Rychlik, Marek R. A complete solution to the equichordal point problem of Fujiwara, Blaschke, Rothe and Weizenböck. *Invent. Math.* **129** (1997), no. 1, 141–212. MR1464869 (98g:52005), Zbl 0882.51008, doi: 10.1007/s002220050161.

[6] Schäfke, R.; Volkmer, H. Asymptotic analysis of the equichordal problem. *J. Reine Angew. Math.* **425** (1992), 9–60. MR1151312 (93d:52003), Zbl 0781.53005, doi: 10.1515/crll.1992.425.9.

[7] Süss, W. Review of [1], Zentralblatt 70, 175–176: Zbl 0070.17504.

[8] Wirsing, Eduard. Zur Analytisität von Doppelspeichenkurven. *Arch. Math.* **9** (1958), 300–307. MR0103436 (21 #2205), Zbl 0083.38404, doi: 10.1007/BF01900594.

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