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Computation of the λ_u-function in JB^*-algebras

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Abstract. Motivated by the work of Gert K. Pedersen on a geometric function, which is defined on the unit ball of a C^*-algebra and called the λ_u-function, the present author recently initiated a study of the λ_u-function in the more general setting of JB^*-algebras. He used his earlier results on the geometry of the unit ball to investigate certain convex combinations of elements in a JB^*-algebra and to obtain analogues of some related C^*-algebra results, including a formula to compute λ_u-function on invertible elements in a JB^*-algebra. The main purpose in this article is to investigate the computation of the λ_u-function on noninvertible elements in the unit ball of a JB^*-algebra. Additional results that relate the λ_u-function to convex combinations, unitary rank, and distance to the invertibles in the C^*-algebra setting are generalized to the JB^*-algebra context. Results of G. K. Pedersen and M. Rørdam are generalized. An open problem is presented.

Contents

1. Introduction and preliminaries

Inspired by the work of R. M. Aron and R. H. Lohman [2], G. K. Pedersen [11] studied a geometric function, called the λ_u-function, which is defined on the unit ball of a C^*-algebra. In a recent paper [22], we initiated a study of the λ_u-function in the more general setting of JB^*-algebras (originally, called Jordan C^*-algebras [24]).

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In [17, 22], we discussed two related set-valued functions $\mathcal{V}(x)$ and $\mathcal{S}(x)$ defined on the closed unit ball of a unital JB^*-algebra, which play a significant role in the study of the λ_u-function. Using our earlier results on the geometry of the unit ball (cf. [16, 18, 20, 21]), we obtained JB^*-algebra analogues of certain C^*-algebra results due to G. K. Pedersen, C. L. Olsen and M. Rørdam. Besides other related results, we have shown that $\sup \mathcal{S}(x) = (\inf \mathcal{V}(x))^{-1}$ if $\mathcal{V}(x) \neq \emptyset$ (see [22, Theorem 2.5]); $\mathcal{V}(x) \cap [1,2) \neq \emptyset$ if and only if x is invertible (see [22, Corollary 2.8]); and for any invertible element x in the closed unit ball, $\lambda_u(x) = (\inf \mathcal{V}(x))^{-1} = \frac{1}{2}(1 + \|x^{-1}\|^{-1})$ satisfying $x = \lambda_u(x)u_1 + (1 - \lambda_u(x))u_2$ for some unitary elements u_1, u_2 (see [22, Theorem 2.5 and Corollary 2.10]).

In this article, we continue the study of the λ_u-function in the general setting of JB^*-algebras (of course, the λ_u-function is not defined in the context of more general JB^*-triple systems (cf. [7, 23]), which have no unitary elements). Our main goal here is to obtain some formulae to compute the λ_u-function for noninvertible elements of the closed unit ball in a JB^*-algebra. We compute the functions $\mathcal{V}(x)$, $\mathcal{S}(x)$ for noninvertible elements and make some estimates on $\inf \mathcal{V}(x)$ in terms of the distance, $\alpha(x)$, from x to the set of invertible elements in a unital JB^*-algebra.

Further, we introduce a condition, called the Λ_u-condition, which is satisfied by all C^*-algebras and all finite-dimensional JB^*-algebras. For JB^*-algebras satisfying the Λ_u-condition, we obtain sharper bounds for estimates of $\inf \mathcal{V}(x)$, together with estimates of distances to the invertibles and to the unitaries. We also obtain the formula $\lambda_u(x) = \frac{1}{2}(1 - \alpha(x))$ for all noninvertible elements x in the closed unit ball. In the course of our analysis, we prove several results on convex combinations, unitary rank, and distance to the invertibles related to the λ_u-function. These include the extension of some other results on C^*-algebras, due to G. K. Pedersen and M. Rørdam, to general JB^*-algebras. We shall conclude the article with a discussion on JB^*-algebras satisfying the Λ_u-condition and by formulating an open problem.

Our notation and terminology are standard and are the same as those found in [22] and [5, 8]. We recall that a commutative (but not necessarily associative) algebra $\mathcal{J}$ with product "$\circ$" is called a Jordan algebra if for all $x, y \in \mathcal{J}$, $x^2 \circ (x \circ y) = (x^2 \circ y) \circ x$. For any fixed element x in a Jordan algebra $\mathcal{J}$, the x-homotope $\mathcal{J}_{[x]}$ of $\mathcal{J}$ is the Jordan algebra consisting of the same elements and linear space structure as $\mathcal{J}$ but with a different product, "$\cdot_x$", defined by $a \cdot_x b = \{axb\}$ for all a, b in $\mathcal{J}_{[x]}$. Here, $\{pqr\}$ denotes the usual Jordan triple product defined in the Jordan algebra $\mathcal{J}$ by $\{pqr\} = (p \circ q) \circ r - (p \circ r) \circ q + (q \circ r) \circ p$.

An element x in a Jordan algebra $\mathcal{J}$ with unit e is said to be invertible if there exists (necessarily unique) element $x^{-1} \in \mathcal{J}$, called the inverse of x, such that $x \circ x^{-1} = e$ and $x^2 \circ x^{-1} = x$. The set of all invertible elements in the unital Jordan algebra $\mathcal{J}$ is denoted by $\mathcal{J}_{\text{inv}}$. In this case, we have $x \cdot_{x^{-1}} y = y$, and so x acts as the unit in the homotope $\mathcal{J}_{[x^{-1}]}$ of $\mathcal{J}$. Henceforth, the homotope $\mathcal{J}_{[x^{-1}]}$ will be called the *x-isotope* of $\mathcal{J}$ and denoted by $\mathcal{J}^{[x]}$ (cf. [8]). It is well known that the x-isotope $\mathcal{J}^{[x]}$ of a Jordan algebra $\mathcal{J}$ need not be isomorphic to $\mathcal{J}$ (cf. [9, 7]). However, some important features of Jordan algebras are unaffected by the process of forming isotopes (see [18, Lemma 4.2 and Theorem 4.6]).

A real or complex Jordan algebra $(\mathcal{J}, \circ)$ is called a *Banach Jordan algebra* if there is a complete norm $\|\cdot\|$ on $\mathcal{J}$ satisfying $\|a \circ b\| \leq \|a\|\|b\|$; if, in addition, $\mathcal{J}$ has unit e with $\|e\| = 1$, then $\mathcal{J}$ is called a *unital* Banach Jordan algebra. A complex Banach Jordan algebra $\mathcal{J}$ with involution "$*$" is called a JB^*-algebra if $\|\{xx^*x\}\| = \|x\|^3$ for all $x \in \mathcal{J}$. It follows that $\|x^*\| = \|x\|$ for all elements x of a JB^*-algebra (cf. [26]). The class of JB^*-algebras was introduced by Kaplansky in 1976 and it includes all C^*-algebras as a proper subclass (cf. [24]). For basic theories of Banach Jordan algebras and JB^*-algebras, we refer to [1, 4, 14, 23, 24, 25, 26]. Throughout this note, $\mathcal{J}$ will denote a unital JB^*-algebra unless stated otherwise. A unital JB^*-algebra $\mathcal{J}$, is said to be of topological stable rank 1 (in short, $tsr1$) if $\mathcal{J}_{\text{inv}}$ is norm dense in $\mathcal{J}$. Such JB^*-algebras have been recently studied by the present author in [18]. All complex spin factors and all finite-dimensional JB^*-algebras are of $tsr1$. Additional properties of JB^*-algebras of $tsr1$ are developed in [18].

An invertible element u in a unital JB^*-algebra $\mathcal{J}$ is called *unitary* if $u^{-1} = u^*$. We denote the set of all unitary elements of the JB^*-algebra $\mathcal{J}$ by $\mathcal{U}(\mathcal{J})$ and its convex hull by $\text{co}\,\mathcal{U}(\mathcal{J})$. If $u \in \mathcal{U}(\mathcal{J})$ then the u-isotope $\mathcal{J}^{[u]}$ is called a unitary isotope of $\mathcal{J}$. It is well known (see [7, 3, 18]) that for any unitary element u in a unital JB^*-algebra $\mathcal{J}$, the unitary isotope $\mathcal{J}^{[u]}$ is a JB^*-algebra with u as its unit with respect to the original norm and the involution "$*_u$" defined by $x^{*_u} = \{ux^*u\}$. Like invertible elements, the set of unitary elements in a unital JB^*-algebra $\mathcal{J}$ is invariant on passage to isotopes of $\mathcal{J}$ (cf. [18, Theorem 4.2 (ii) and Theorem 4.6]).

A self-adjoint element x (which means $x^* = x$) is called *positive* in $\mathcal{J}$ if its spectrum $\sigma_{\mathcal{J}}(x) := \{\lambda \in \mathbb{C} : x - \lambda e \notin \mathcal{J}_{\text{inv}}\}$ is contained in the set of nonnegative real numbers, where $\mathbb{C}$ denotes the field of complex numbers. Every element in a finite-dimensional JB^*-algebra $\mathcal{J}$ is positive in some unitary isotope of $\mathcal{J}$ (cf. [18, Theorem 5.9]). One of the main results (namely, Theorem 4.12) in [18] states that every invertible element x of a unital JB^*-algebra $\mathcal{J}$ is positive in the unitary isotope $\mathcal{J}^{[u]}$ of $\mathcal{J}$, where the unitary u is given by the usual polar decomposition $x = u|x|$ of x considered as an operator in the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on certain Hilbert space $\mathcal{H}$; indeed, the same unitary u is the unitary approximant of x, meaning that $\text{dist}(x,\mathcal{U}(\mathcal{J})) = \|x - u\|$. More generally, $\text{dist}(y,\mathcal{U}(\mathcal{J})) = \|y - e\|$ for any positive element in a JB^*-algebra $\mathcal{J}$ with unit e. In [17, 18], we obtained some formulae to compute $\text{dist}(x,\mathcal{U}(\mathcal{J}))$ including the cases when $x \in (\mathcal{J})_1$, the closed unit ball of $\mathcal{J}$, when $\mathcal{J}$ is finite-dimensional, and when $\mathcal{J}$ is of $tsr1$. In general, one may not have unitary approximants for elements even in the case of von Neumann algebras (for such an example, see [10]).

In [17, 18], the author observed some interesting properties of the distance function $\alpha(x)$, in the context of JB^*-algebras. Here, we continue studying the function $\alpha(x)$ and we investigate its connections with the convex hull $\mathrm{co}\,\mathcal{U}(\mathcal{J})$ of the unitaries. We connect it with the unitary rank $u(x)$ of an element x — which is the least integer n such that x can be expressed as a convex combination of n unitary elements in $\mathcal{J}$; $u(x) = \infty$ otherwise — and with the λ_u-function.

2. The $\boldsymbol{\lambda_u}$-function

We begin this section by recalling (from [22]) the following construction of the functions $\mathcal{V}(x)$, $\mathcal{S}(x)$, and $\lambda_u(x)$ at elements x of the closed unit ball $(\mathcal{J})_1$ in a unital JB^*-algebra $\mathcal{J}$: for each number $\delta \geq 1$,

$$\mathrm{co}_\delta\,\mathcal{U}(\mathcal{J}) := \left\{ \delta^{-1} \sum_{i=1}^{n-1} u_i + \delta^{-1}(1+\delta-n)u_n : u_j \in \mathcal{U}(\mathcal{J}), j = 1, \ldots, n \right\}$$

where n is the integer given by $n - 1 < \delta \leq n$;

$$\mathcal{V}(x) := \{\delta \geq 1 : x \in \mathrm{co}_\delta\,\mathcal{U}(\mathcal{J})\};$$

$$\mathcal{S}(x) := \{0 \leq \lambda \leq 1 : x = \lambda v + (1-\lambda)y \ \ \textit{with} \ \ v \in \mathcal{U}(\mathcal{J}), y \in (\mathcal{J})_1\};$$

and

$$\lambda_u(x) := \sup \mathcal{S}(x).$$

Before presenting further results involving these constructions, it may be helpful to recall some of our results from [22]. Part (i) of the following theorem extends a C^*-algebra result due to Rørdam (see [13, Proposition 3.1]). The proof given in [22, Theorem 2.2] follows his argument with suitable changes necessitated by the nonassociativity of Jordan algebras.

Theorem 2.1. *Let $\mathcal{J}$ be a unital JB^*-algebra and let $x \in (\mathcal{J})_1$.*

(i) *Let $\|\gamma x - u_o\| \leq \gamma - 1$ for some $\gamma \geq 1$ and some $u_o \in \mathcal{U}(\mathcal{J})$. Let $(\alpha_2, \ldots, \alpha_m) \in \mathbb{R}^{m-1}$ with $0 \leq \alpha_j < \gamma^{-1}$ and $\gamma^{-1} + \sum_{j=2}^m \alpha_j = 1$. Then there exist unitaries $u_1, \ldots, u_m$ in $\mathcal{J}$ such that*

$$x = \gamma^{-1} u_1 + \sum_{j=2}^{m} \alpha_j u_j.$$

Moreover, $(\gamma, \infty) \subseteq \mathcal{V}(x)$.

(ii) *If $(\gamma, \infty) \subseteq \mathcal{V}(x)$ then for all $r > \gamma$ there is $u_1 \in \mathcal{U}(\mathcal{J})$ such that $\|rx - u_1\| \leq r - 1$.*

This immediately gives the following result (cf. [22, Corollary 2.3]).

Corollary 2.2. *For any unital JB^*-algebra $\mathcal{J}$, $\operatorname{co}_\gamma \mathcal{U}(\mathcal{J}) \subseteq \operatorname{co}_\delta \mathcal{U}(\mathcal{J})$ whenever $1 \le \gamma \le \delta$. Thus, for each $x \in (\mathcal{J})_1$, $\mathcal{V}(x)$ is either empty or equal to $[\gamma, \infty)$ or (γ, ∞) for some $\gamma \ge 1$.*

The following result gives some interesting relationship between the sets $\mathcal{S}(x)$ and $\mathcal{V}(x)$; in particular, $(\inf \mathcal{V}(x))^{-1} = \sup \mathcal{S}(x)$ if $\mathcal{V}(x) \neq \emptyset$:

Theorem 2.3 ([22, Theorem 2.5])**.** *Let $\mathcal{J}$ be a unital JB^*-algebra and let $x \in (\mathcal{J})_1$. Then:*

(i) *If $\lambda \in \mathcal{S}(x)$ and $\lambda > 0$ then $(\lambda^{-1}, \infty) \subseteq \mathcal{V}(x)$.*
(ii) *If $\delta \in \mathcal{V}(x)$ then $\delta^{-1} \in \mathcal{S}(x)$.*
(iii) *$\lambda_u(x) = 0$ if and only if $\mathcal{V}(x) = \emptyset$.*
(iv) *If $\lambda_u(x) > 0$ then $\mathcal{S}(x) = [0, \lambda_u(x))$ or $[0, \lambda_u(x)]$.*
(v) *If $\lambda_u(x) > 0$ and if $0 < \lambda < \lambda_u(x)$ then $\lambda^{-1} \in \mathcal{V}(x)$.*
(vi) *If $\lambda_u(x) > 0$ then $(\inf \mathcal{V}(x))^{-1} = \lambda_u(x)$.*
(vii) *If $\inf(\mathcal{V}(x)) \in \mathcal{V}(x)$ then $\lambda_u(x) \in \mathcal{S}(x)$.*

As the next example shows, $\lambda_u(x) \in \mathcal{S}(x)$ may not imply $\inf \mathcal{V}(x) \in \mathcal{V}(x)$.

Example 2.4. Let $\mathcal{J} = \mathcal{C}_{\mathbb{C}}(\Delta)$ be the algebra of all complex-valued continuous functions on the closed unit disk Δ in the complex plane $\mathbb{C}$. For any integer $n \ge 2$, let the functions $f_n \in \mathcal{C}_{\mathbb{C}}(\Delta)$ be given by $f_n(z) = (1 - \frac{1}{n})z + \frac{1}{n}$. Then $\lambda_u(f_n) \in \mathcal{S}(f_n)$ but $\inf \mathcal{V}(f_n) \notin \mathcal{V}(f_n)$.

Indeed, since $f_n = \frac{1}{n}e + (1 - \frac{1}{n})g$ where $e \in \mathcal{U}(\mathcal{J})$, $g \in (\mathcal{J})_1$ are given by $e(z) = 1$ and $g(z) = z$ for all $z \in \Delta$, we have $\lambda_u(f_n) \ge \frac{1}{n}$. Suppose $\lambda_u(f_n) > \frac{1}{n}$. Then by Part (v) of Theorem 2.3, $(\frac{1}{n})^{-1} \in \mathcal{V}(f_n)$ so that $n \in \mathcal{V}(f_n)$. This contradicts the fact that the unitary rank $u(f_n) \neq n$ (cf. [17, Example 2.5]). Therefore, $\lambda_u(f_n) = \frac{1}{n}$. Hence, $\lambda_u(f_n) \in \mathcal{S}(f_n)$. But, by Part (vi) of Theorem 2.3, $\inf \mathcal{V}(f_n) = (\lambda_u(f_n))^{-1} = n \notin \mathcal{V}(f_n)$.

The following example shows the existence of an element x in a C^*-algebra of $tsr1$ with $\lambda_u(x) > 0$ but $\inf \mathcal{V}(x) \notin \mathcal{V}(x)$:

Example 2.5. Let $\mathcal{J} = \mathcal{C}_{\mathbb{C}}((\mathbb{N} \cup \{\infty\}))$ be the C^*-algebra of all convergent complex sequences, where $\mathbb{N}$ denotes the set of natural numbers (cf. [18, Remark 5.11]). If $f \in (\mathcal{J})_1^\circ$ (the open unit ball of $\mathcal{J}$) is defined by

$$f(n) = \begin{cases} (2n)^{-1} e^{\frac{1}{2} i \pi n} & \text{if } n \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

then $\inf \mathcal{V}(f) \notin \mathcal{V}(f)$ even though $\mathcal{J}$ is of $tsr1$.

This is because $\mathcal{J}$ is of $tsr1$ by [12, Proposition 1.7]. Since $f \in (\mathcal{J})_1^\circ$, we get $f \in \operatorname{co}_{2^+} \mathcal{U}(\mathcal{J})$ by [15, Theorem 11], where

$$\operatorname{co}_{2^+} \mathcal{U}(\mathcal{J}) = \Big\{ x \in \mathcal{J} : \text{for each } \epsilon > 0,\ x \text{ has convex decomposition} \\ \sum_{i=1}^{3} \alpha_i u_i \text{ with } u_i \in \mathcal{U}(\mathcal{J}), \alpha_3 < \epsilon \Big\}.$$

Hence, $\inf \mathcal{V}(x) = 2$ by [17, Theorem 30]. However, $u(x) > 2$ by [6, Remark 19]. Thus, $\inf \mathcal{V}(x) \notin \mathcal{V}(x)$ by [17, Lemma 24].

From Part (iii) of Theorem 2.3, we get the following connections among $\alpha(x)$, $\mathcal{V}(x)$ and $\lambda_u(x)$ (cf. [22, Corollary 2.6]):

Corollary 2.6. *For any $x \in (\mathcal{J})_1 \setminus \mathcal{J}_{\text{inv}}$, the following statements are equivalent:*

(i) $\alpha(x) < 1 \Rightarrow \mathcal{V}(x) \neq \emptyset$.
(ii) $\lambda_u(x) = 0 \Rightarrow \alpha(x) = 1$.
(iii) $\alpha(x) < 1 \Rightarrow \lambda_u(x) > 0$.

For the elements x with $\mathcal{V}(x) \cap [1, 2) \neq \emptyset$, we know the following relations among $\text{dist}(x, \mathcal{U}(\mathcal{J}))$, $\mathcal{V}(x)$ and $\mathcal{S}(x)$ (see [22, Theorem 2.7]):

Theorem 2.7. *Let $0 \leq \gamma < \frac{1}{2}$. Let $\mathcal{J}$ be a unital JB^*-algebra and let $x \in (\mathcal{J})_1$. Then the following statements are equivalent:*

(i) $\text{dist}(x, \mathcal{U}(\mathcal{J})) \leq 2\gamma$.
(ii) $x \in \gamma\mathcal{U}(\mathcal{J}) + (1 - \gamma)\mathcal{U}(\mathcal{J})$.
(iii) $(1 - \gamma)^{-1} \in \mathcal{V}(x)$.
(iv) $1 - \gamma \in \mathcal{S}(x)$.

This leads us to the following characterizations of the invertible elements in the unit ball; for such elements x, we obtain $\inf \mathcal{V}(x) = 2(1 + \|x^{-1}\|^{-1})^{-1}$ (see [22, Corollary 2.8]):

Corollary 2.8. *Let $\mathcal{J}$ be a unital JB^*-algebra and let $x \in (\mathcal{J})_1$. Then:*

(a) *The following statements are equivalent:*
  (i) *x is invertible.*
  (ii) *$x \in \gamma\mathcal{U}(\mathcal{J}) + (1 - \gamma)\mathcal{U}(\mathcal{J})$ for some $0 \leq \gamma < \frac{1}{2}$.*
  (iii) *$\text{dist}(x, \mathcal{U}(\mathcal{J})) \leq 2\gamma$ for some $0 \leq \gamma < \frac{1}{2}$.*
  (iv) *$1 - \gamma \in \mathcal{S}(x)$ for some $0 \leq \gamma < \frac{1}{2}$.*
  (v) *$(1 - \gamma)^{-1} \in \mathcal{V}(x)$ for some $0 \leq \gamma < \frac{1}{2}$.*
  (vi) *$\lambda \in \mathcal{V}(x)$ for some $1 \leq \lambda < 2$.*

(b) *Moreover, if x is invertible then $\inf \mathcal{V}(x) = 2(1 + \|x^{-1}\|^{-1})^{-1}$ and $\mathcal{V}(x) = [2(1 + \|x^{-1}\|^{-1})^{-1}, \infty)$.*

Using Part (b) of Corollary 2.8 together with Theorem 2.3 and Theorem 2.7, one can easily deduce that for any invertible element x in the closed unit ball of a unital JB^*-algebra $\mathcal{J}$, $\lambda_u(x) = \frac{1}{2}(1 + \|x^{-1}\|^{-1})$ and $x = \lambda_u(x)u_1 + (1 - \lambda_u(x))u_2$ for some $u_1, u_2 \in \mathcal{U}(\mathcal{J})$ (cf. [22, Corollary 2.10]).

We proceed to obtain formulae for $\lambda_u(x)$ when x is a noninvertible element of the closed unit ball in a unital JB^*-algebra. We shall also compute $\mathcal{V}(x)$ and $\mathcal{S}(x)$ for such elements, and we shall derive some estimates of $\inf \mathcal{V}(x)$ in terms of $\alpha(x)$.

The following result gives an upper bound for $\lambda_u(x)$ on noninvertible elements x of the closed unit ball:

Theorem 2.9. *Let $\mathcal{J}$ be the JB^*-algebra and let $x \in (\mathcal{J})_1$ be noninvertible. Then $\lambda_u(x) \leq \frac{1}{2}(1-\alpha(x))$. Further, if $\alpha(x) = 1$ then $\lambda_u(x) = 0$.*

Proof. Since $\|x\| \leq 1$, $\alpha(x) \leq 1$ and so the inequality is true if $\lambda_u(x) = 0$. Next, suppose $\lambda_u(x) > 0$ and $x = \lambda u + (1-\lambda)v$ with $u \in \mathcal{U}(\mathcal{J})$, $v \in (\mathcal{J})_1$ and $0 < \lambda \leq 1$. If $1 \geq \lambda > \frac{1}{2}$ then $1 \leq \lambda^{-1} < 2$, hence $x = \lambda u + (1-\lambda)v$ gives $\|\lambda^{-1}x - u\| < 1$ and so $\lambda^{-1}x$ is invertible in the isotope $\mathcal{J}^{[u]}$ by [18, Lemma 2.1]. So that $x \in \mathcal{J}_{\mathrm{inv}}$ by [18, Lemma 4.2]; this contradicts the hypothesis. Therefore, $0 < \lambda \leq \frac{1}{2}$. Thus we have

$$\|x - \lambda(u+v)\| = \|(1-2\lambda)v\| \leq 1 - 2\lambda. \tag{1}$$

For any positive integer n, $\|(1-\frac{1}{n})v\| < 1$, so that $u + (1-\frac{1}{n})v$ is invertible in $\mathcal{J}^{[u]}$ again by [18, Lemma 2.1], and hence it is in $\mathcal{J}_{\mathrm{inv}}$ as above. Hence, $\alpha(x) \leq \|x - \lambda(u + (1-\frac{1}{n})v)\|$ for all positive integers $n \in \mathbb{N}$, and so

$$\alpha(x) \leq \|x - \lambda(u+v)\|. \tag{2}$$

From (1) and (2), we conclude that $\lambda_u(x) \leq \frac{1}{2}(1-\alpha(x))$.

Further, if $\alpha(x) = 1$, then $\lambda_u(x) \leq \frac{1}{2}(1-\alpha(x))$ gives $\lambda_u(x) \leq 0$, and hence $\lambda_u(x) = 0$ because $\lambda_u(x) \geq 0$. □

We now give the following extension of [17, Theorem 34] for noninvertible elements of the open unit ball in a general JB^*-algebra; the norm 1 case will be discussed in the next section. For the invertible elements of the unit ball, see Corollary 2.8.

Theorem 2.10. *Let $\mathcal{J}$ be a unital JB^*-algebra and $x \in (\mathcal{J})_1^\circ \setminus \mathcal{J}_{\mathrm{inv}}$ (so that $\alpha(x) < 1$). Then $\mathcal{V}(x) \neq \emptyset$. Further:*

(i) *$\mathcal{V}(x) = [(\lambda_u(x))^{-1}, \infty)$ or $\mathcal{V}(x) = ((\lambda_u(x))^{-1}, \infty)$.*
(ii) *$u(x) = n$ if $n \neq (\lambda_u(x))^{-1}$ given by $n - 1 < (\lambda_u(x))^{-1} \leq n$.*
(iii) *$u(x) = n$ or $u(x) = n+1$ if $n = (\lambda_u(x))^{-1}$.*

In any case, for each $0 < \epsilon \leq 1$, there exist $u_1, \ldots, u_{n+1} \in \mathcal{U}(\mathcal{J})$ such that $x = (\epsilon + n)^{-1}(u_1 + \cdots + u_n + \epsilon u_{n+1})$. Moreover, $0 < (u(x))^{-1} \leq \lambda_u(x) \leq \frac{1}{2}(1-\alpha(x))$. Hence $x \in \mathrm{co}_{n+}\mathcal{U}(\mathcal{J})$.

Proof. By [20, Theorem 2.3], $u(x) < \infty$. Since $u(x) = \min(\mathcal{V}(x) \cap \mathbb{N})$ by [17, Lemma 24], we have $\mathcal{V}(x) \neq \emptyset$. Hence, $\lambda_u(x) = (\inf \mathcal{V}(x))^{-1}$ by Theorem 2.3. Thus, Part (i) follows from Corollary 2.2. However, the other parts follow easily from Part (i) and Theorem 2.9. □

We close the section with the following realization:

Corollary 2.11. *Let $\mathcal{J}$ be a JB^*-algebra of tsr1 and let $x \in (\mathcal{J})_1^\circ \setminus \mathcal{J}_{\mathrm{inv}}$. Then*

$$\mathcal{V}(x) = [2, \infty) \quad \textit{or} \quad \mathcal{V}(x) = (2, \infty).$$

Proof. By Theorem 2.10,

$$\mathcal{V}(x) = [(\lambda_u(x))^{-1}, \infty) \quad \text{or} \quad \mathcal{V}(x) = ((\lambda_u(x))^{-1}, \infty).$$

Since $\mathcal{J}$ is of $tsr1$ and $x \in (\mathcal{J})_1^\circ$, we get $x \in \mathrm{co}_{2+}\mathcal{U}(\mathcal{J})$ by [15, Theorem 11]. Hence, $2+\epsilon \in \mathcal{V}(x)$ for all $0<\epsilon \leq 1$. Now, since $\alpha(x)=0$, we get

$$2 \leq (\lambda_u(x))^{-1} = \inf \mathcal{V}(x) \leq 2$$

by Theorem 2.3 and Theorem 2.9. □

3. The Λ_u-condition

In the previous section, we observed several facts about convex combinations of unitaries in relation to the λ_u-function. We now introduce a condition on a general JB^*-algebra, called the Λ_u-condition. Under this condition, more precise assertions about the λ_u-function can be made. In particular, for any element x in a JB^*-algebra satisfying the Λ_u-condition, we have $\mathcal{V}(x) \neq \emptyset$ and $\lambda_u(x) > 0$ whenever $\alpha(x) < 1$. We shall observe some interesting characterizations of the Λ_u-condition, which in turn would give the bound of $\inf \mathcal{V}(x)$ for elements x with $\alpha(x) < 1$.

Definition 3.1. We say that a unital JB^*-algebra satisfies the Λ_u-condition if and only if every noninvertible unit vector $y \in \mathcal{J}$ with $\lambda_u(y) = 0$ satisfies $\alpha(y) = 1$.

It may be noted that for any $x \in (\mathcal{J})_1^\circ$, we have $\mathcal{V}(x) \neq \emptyset$ by [20, Theorem 2.3] and [17, Lemma 24]. Hence, $\lambda_u(x) \neq 0$ by Theorem 2.3. Here, it is worth recalling from Theorem 2.9 that $\lambda_u(x) = 0$ if $\alpha(x) = 1$ with $x \in (\mathcal{J})_1$. Thus, in any unital JB^*-algebra satisfying the Λ_u-condition, we have $\lambda_u(x) = 0$ if and only if $\alpha(x) = 1$.

Example 3.2. Any finite-dimensional JB^*-algebra and all unital C^*-algebras satisfy the Λ_u-condition by [17, Theorem 34]and [11, Theorem 5.1]), respectively.

The Λ_u-condition is good enough to guarantee an appropriate JB^*-algebra analogue of [13, Theorem 3.3], and hence that of [11, Theorem 5.1].

Theorem 3.3. *Suppose the unital JB^*-algebra $\mathcal{J}$, satisfies the Λ_u-condition and let $x \in (\mathcal{J})_1$ be noninvertible with $\alpha(x) < 1$. Then:*

(i) $\lambda_u(x) > 0$.
(ii) $\mathcal{V}(x) = [(\lambda_u(x))^{-1}, \infty)$ *or* $\mathcal{V}(x) = ((\lambda_u(x))^{-1}, \infty)$.
(iii) $u(x) = n$ *if* $n \neq (\lambda_u(x))^{-1}$ *given by* $n - 1 < (\lambda_u(x))^{-1} \leq n$.
(iv) $u(x) = n$ *or* $u(x) = n+1$ *if* $n = (\lambda_u(x))^{-1}$.

In either case, for each $0 < \epsilon \leq 1$, there exist $u_1, \ldots, u_{n+1} \in \mathcal{U}(\mathcal{J})$ such that $x = (\epsilon + n)^{-1}(u_1 + \cdots + u_n + \epsilon u_{n+1})$, hence $x \in \mathrm{co}_{n+}\mathcal{U}(\mathcal{J})$.

Proof. If $\|x\| = 1$ then from Corollary 2.6 we get $\mathcal{V}(x) \neq \emptyset$ since $\alpha(x) < 1$. Hence, assertion (ii) follows for $\|x\| = 1$ from Theorem 2.3 and Corollary 2.2. In the case $\|x\| < 1$, assertion (ii) follows from Theorem 2.10. The remaining assertions can easily be deduced from the assertion (i). □

Corollary 3.4. *Let $\mathcal{J}$ be a unital JB^*-algebra satisfying the Λ_u-condition. Then:*

(i) $(\mathcal{J})_1 \setminus \mathrm{co}\,\mathcal{U}(\mathcal{J}) \subseteq \{y \in \mathcal{J} : \|y\| = \alpha(y) = 1\}$.
(ii) *If $\alpha(x) < 1$ for all $x \in (\mathcal{J})_1$ then $(\mathcal{J})_1 = \mathrm{co}\,\mathcal{U}(\mathcal{J})$.*
(iii) *If $\mathcal{J}$ is of $tsr1$ then $(\mathcal{J})_1 = \mathrm{co}\,\mathcal{U}(\mathcal{J})$.*

Proof. (i) If $x \in (\mathcal{J})_1 \setminus \mathrm{co}\,\mathcal{U}(\mathcal{J})$, then $\|x\| = 1$ (because $\|x\| < 1$ gives $x \in \mathrm{co}\,\mathcal{U}(\mathcal{J})$ by [20, Thorem 2.3]) and $\alpha(x) = 1$ (for otherwise, $\lambda_u(x) > 0$ so that $\mathcal{V}(x) \neq \emptyset$ by Theorem 2.3, and hence $x \in \mathrm{co}\,\mathcal{U}(\mathcal{J})$). Thus,

$$(\mathcal{J})_1 \setminus \mathrm{co}\,\mathcal{U}(\mathcal{J}) \subseteq \{y \in \mathcal{J} : \|y\| = \alpha(y) = 1\}.$$

(ii) Since $\alpha(x) < 1$ for all $x \in \mathcal{J}$, $\{y \in \mathcal{J} : \|y\| = \alpha(y) = 1\}$ is the empty set and hence $(\mathcal{J})_1 = \mathrm{co}\,\mathcal{U}(\mathcal{J})$ by assertion (i).

(iii) As $\mathcal{J}$ is of $tsr1$, $\alpha(x) = 0$ for all $x \in \mathcal{J}$. So the result follows from assertion (ii). □

The next result provides motivation for the subsequent results.

Corollary 3.5. *Suppose the unital JB^*-algebra $\mathcal{J}$ satisfies the Λ_u-condition, and let $x \in (\mathcal{J})_1$ be noninvertible with $\alpha(x) < 1$. Then $\lambda_u(x) > 0$ and so $\mathcal{V}(x) \neq \emptyset$. Moreover:*

(i) $((\lambda_u(x))^{-1}, \infty) \subseteq \mathcal{V}(x)$.
(ii) $(\lambda_u(x))^{-1} = \inf(\mathcal{V}(x))$.
(iii) *If $\lambda > (\lambda_u(x))^{-1}$, then there is $u \in \mathcal{U}(\mathcal{J})$ with $\|\lambda x - u\| \leq \lambda - 1$.*

Proof. Since $\mathcal{J}$ satisfies the Λ_u-condition and since $\alpha(x) < 1$, $\lambda_u(x) > 0$. Now, the result follows from Theorem 2.1 and Theorem 2.3. □

Next, we see if we can identify $\inf \mathcal{V}(x)$ in terms of $\alpha(x)$. For any noninvertible element x of the closed unit ball in a unital JB^*-algebra $\mathcal{J}$ with $\alpha(x) < 1$, the number β_x is defined by $\beta_x = 2(1 - \alpha(x))^{-1}$:

Theorem 3.6. *Let $\mathcal{J}$ be a unital JB^*-algebra and suppose $x \in (\mathcal{J})_1$ with $\alpha(x) < 1$. Then the following conditions are equivalent:*

(Λ_1) $(\beta_x, \infty) \subseteq \mathcal{V}(x)$.
(Λ_2) $(\lambda_u(x))^{-1} = \inf \mathcal{V}(x) = \beta_x$.
(Λ_3) *For all $\gamma > \beta_x$, there exists $u \in \mathcal{U}(\mathcal{J})$ such that $\|\gamma x - u\| \leq \gamma - 1$.*
(Λ_4) $\lambda_u(x) \geq \beta_x^{-1}$.

Proof. (Λ_1) ⇒ (Λ_2): By [17, Theorem 30], $\mathcal{V}(x) \subseteq [\beta_x, \infty)$. Then, by the condition (Λ_1), $\inf \mathcal{V}(x) = \beta_x$. Hence, the required equality follows from Theorem 2.3.

(Λ_2) ⇒ (Λ_3): See [17, Theorem 30].

(Λ_3) ⇒ (Λ_4): Let $\gamma > \beta_x$. Then, by the condition (Λ_3), there exists $u \in \mathcal{U}(\mathcal{J})$ such that $\|\gamma x - u\| \leq \gamma - 1$. Then, by Theorem 2.1, $(\gamma, \infty) \subseteq \mathcal{V}(x)$ so that $\inf \mathcal{V}(x) \leq \gamma$. Hence, by Theorem 2.3, $\lambda_u(x) \geq \gamma^{-1}$. It follows that $\lambda_u(x) \geq \beta_x^{-1}$.

(Λ_4) $\Rightarrow$ (Λ_1): Let $\gamma > \beta_x$. Then, by the condition (Λ_4), $0 < \gamma^{-1} < \beta_x^{-1} \leq \lambda_u(x)$. Thus, $\gamma^{-1} \in \mathcal{S}(x)$, and so $(\gamma, \infty) \subseteq \mathcal{V}(x)$ by the assertion (i) of Theorem 2.3. It follows that $(\beta_x, \infty) \subseteq \mathcal{V}(x)$. □

Corollary 3.7. *Let $\mathcal{J}$ be a unital JB^*-algebra and $x \in (\mathcal{J})_1$ with $\alpha(x) < 1$ satisfy any of the conditions (Λ_1)–(Λ_4). If $\alpha(x) < 1 - \frac{2}{m}$ then $u(x) \leq m$.*

Proof. As $\alpha(x) < 1 - \frac{2}{m}$, $m > 2(1 - \alpha(x))^{-1}$. Hence, for the case when $x \notin \mathcal{J}_{\text{inv}}$, we have by [17, Theorem 30] that $m \in \mathcal{V}(x)$, or equivalently, $u(x) \leq m$. If $x \in \mathcal{J}_{\text{inv}}$ then we get from Corollary 2.8 that $m \in \mathcal{V}(x)$ since $m \geq 2$, and hence $u(x) \leq m$. □

Corollary 3.8. *Let $\mathcal{J}$ be a unital JB^*-algebra of $tsr1$ and let x be a non-invertible element of $(\mathcal{J})_1$. Let $0 < \epsilon \leq 1$. If x satisfies any one of the conditions (Λ_1)–(Λ_4), then there exist unitaries u_1, u_2 and u_3 in $\mathcal{J}$ such that $x = (2+\epsilon)^{-1}(u_1 + u_2 + \epsilon u_3)$.*

Proof. Since $\mathcal{J}$ is of $tsr1$, $\alpha(x) = 0$ for all $x \in \mathcal{J}$. If the JB^*-algebra $\mathcal{J}$ satisfies any one of the conditions (Λ_1)–(Λ_4), then for each noninvertible $x \in (\mathcal{J})_1$ we have $(2, \infty) \subseteq \mathcal{V}(x)$ since $\alpha(x) = 0$ gives $2 + \epsilon > 2(1 + \alpha(x))^{-1}$ for any $\epsilon \in (0, 1]$. This proves the result. □

Remark 3.9. [15, Theorem 11] states the same fact for elements of $(\mathcal{J})_1^\circ$.

If in Theorem 3.6 we restrict x to be of norm 1, then we obtain more equivalent conditions in the following result:

Theorem 3.10. *Let $\mathcal{J}$ be a unital JB^*-algebra and let $x \in \mathcal{J} \setminus \mathcal{J}_{\text{inv}}$ with $\|x\| = 1$ and $\alpha(x) < 1$. Then the following are equivalent:*

(i) (Λ_1) *holds for x.*
(ii) (Λ_2) *holds for x.*
(iii) (Λ_3) *hold for x.*
(iv) (Λ_4) *holds for x.*
(v) (Λ_1) *holds for each rx with $0 < r \leq 1$.*
(vi) (Λ_2) *holds for each rx with $0 < r \leq 1$.*
(vii) (Λ_3) *holds for each rx with $0 < r \leq 1$.*
(viii) (Λ_4) *holds for each rx with $0 < r \leq 1$.*
(ix) *If $y \in \text{Sp}(x)$ (the linear span of x) and $\|y\| > \alpha(y) + 2$, then*

$$\|y - u\| \leq \|y\| - 1$$

for some $u \in \mathcal{U}(\mathcal{J})$.

Moreover, if any one of the above conditions (i) to (ix) holds for all $y \in \mathcal{J} \setminus \mathcal{J}_{\text{inv}}$ with $\|y\| = 1$ and $\alpha(y) < 1$, then $\mathcal{J}$ satisfies the Λ_u-condition.

Proof. We first establish the equivalence of the listed conditions. By Theorem 3.6, (i)–(iv) are equivalent. It is clear that $rx \in (\mathcal{J})_1$; and by [18, Lemma 6.2], $\alpha(rx) = r\alpha(x) < 1$ (as $\alpha(x) < 1$) for each $0 < r \leq 1$. Hence, again by Theorem 3.6, (v)–(viii) are equivalent. Next, we show (ii) $\Leftrightarrow$ (vi), (iv) $\Rightarrow$ (ix) and (ix) $\Rightarrow$ (i).

(ii) $\Leftrightarrow$ (vi): Of course, (vi) $\Rightarrow$ (ii). Conversely, suppose

$$(\lambda_u(x))^{-1} = \inf \mathcal{V}(x) = \beta_x.$$

Let r be any fixed number such that $0 < r < 1$. Then $rx \in (\mathcal{J})_1^\circ \setminus \mathcal{J}_{\text{inv}}$; so that $\lambda_u(rx) \leq \beta_{rx}^{-1}$ by Theorem 2.9. Let $\lambda > \beta_x$. Then, by the condition (ii) and Corollary 2.2, $\lambda \in \mathcal{V}(x)$ so that $x \in \text{co}_\lambda \mathcal{U}(\mathcal{J})$. Hence, there exist $u_1, \dots, u_n \in \mathcal{U}(\mathcal{J})$ with $n - 1 < \lambda \leq n \in \mathbb{N}$ such that

$$x = \lambda^{-1}(u_1 + \cdots + u_{n-1} + (1 + \lambda - n)u_n),$$

so that

$$rx = r\lambda^{-1}(u_1 + \cdots + u_{n-1} + (1 + \lambda - n)u_n) + \frac{1-r}{2}u_1 + \frac{1-r}{2}(-u_1).$$

This implies

$$\begin{aligned}\lambda_u(rx) \geq r\lambda^{-1} + \frac{1-r}{2} &= r\beta_x^{-1} + \frac{1-r}{2} + r\lambda^{-1} - r\beta_x^{-1} \\ &= \frac{1}{2}(1 - r\alpha(x)) + r(\lambda^{-1} - \beta_x^{-1}) = \beta_{rx}^{-1} + r(\lambda^{-1} - \beta_x^{-1}).\end{aligned}$$

Hence, $\lambda_u(rx) \geq \beta_{rx}^{-1} + r(\lambda^{-1} - \beta_x^{-1})$ for all $\lambda > \beta_x$. Thus, $\lambda_u(rx) = \beta_{rx}^{-1}$.

(iv) $\Rightarrow$ (ix): Let $y \in \text{Sp}(x)$ with $\|y\| > \alpha(y) + 2$. Clearly, $\|y\|^{-1}$ exists and satisfies

$$\|y\|^{-1} < \frac{\|y\|^{-1}}{2}(\|y\| - \alpha(y)) = \frac{1}{2}(1 - \alpha(\|y\|^{-1}y)).$$

Since $x = \|y\|^{-1}y$, we get by (iv) that

$$\|y\|^{-1} < \frac{1}{2}(1 - \alpha(x)) \leq \lambda_u(x).$$

Then, by Theorem 2.3, for $\lambda = \|y\|^{-1}$ there exist $u \in \mathcal{U}(\mathcal{J})$ and $v \in (\mathcal{J})_1$ such that $x = \lambda u + (1 - \lambda)v$. Hence, $\|x - \lambda u\| \leq 1 - \lambda$ as $\lambda \leq 1$ (in fact, $\lambda \leq \frac{1}{2}$ as $\lambda = \|y\|^{-1} < \frac{1}{\alpha(x)+2} \leq \frac{1}{2}$). Thus, $\|y - u\| \leq \|y\| - 1$.

(ix) $\Rightarrow$ (i): For any $\gamma > 2(1-\alpha(x))^{-1}$, we have $\|\gamma x\| - \alpha(\gamma x) = \gamma - \gamma\alpha(x) > 2$ so that $\|\gamma x\| > \alpha(\gamma x) + 2$. Hence, by (ix), $\|\gamma x - u\| \leq \|\gamma x\| - 1$ for some $u \in \mathcal{U}(\mathcal{J})$. So, by Theorem 2.1, $(\gamma, \infty) \subseteq \mathcal{V}(x)$. Thus, $(\beta_x, \infty) \subseteq \mathcal{V}(x)$.

Finally, suppose $x \in \mathcal{J} \setminus \mathcal{J}_{\text{inv}}$ with $\|x\| = 1$ and $\lambda_u(x) = 0$. Then $\alpha(x) = 1$: for otherwise, $\alpha(x) < 1$ would give $\lambda_u(x) \neq 0$ by (iv); a contradiction. However, all of the conditions (i) to (ix) are equivalent as seen above. □

We close this section by observing the following fact about the norm 1 noninvertible elements in a JB^*-algebra of $tsr1$.

Corollary 3.11. *Let $\mathcal{J}$ be a unital JB^*-algebra of $tsr1$ and $x \in \mathcal{J} \setminus \mathcal{J}_{\text{inv}}$ with $\|x\| = 1$. If x satisfies any of the conditions* (i)–(ix) *given in Theorem* 3.10, *then:*

(i) $\mathcal{V}(x) = [2, \infty)$ *or* $\mathcal{V}(x) = (2, \infty)$.
(ii) $u(x) = 2$ *or* $u(x) = 3$.

Further, for each $\epsilon \in (0,1]$*, there are unitaries* $u_1, \ldots, u_3 \in \mathcal{J}$ *such that* $x = (2+\epsilon)^{-1}(u_1 + u_2 + \epsilon u_3)$*. Hence,* $x \in \mathrm{co}_{2+}\mathcal{U}(\mathcal{J})$*.*

Proof. For this, we only have to show that $(2,\infty) \subseteq \mathcal{V}(x)$. Suppose $\gamma > 2$. Then $\|\gamma x\| = \gamma > 2$ and hence, by the condition (ix) in Theorem 3.10, there exists some unitary $u \in \mathcal{U}(\mathcal{J})$ such that $\|\gamma x - u\| = \|\gamma x\| - 1 = \gamma - 1$. Then, by Theorem 2.1, $(\gamma, \infty) \subseteq \mathcal{V}(x)$. We conclude that $(2,\infty) \subseteq \mathcal{V}(x)$. □

4. An open problem

The following question remains unanswered:

Does every JB^*-algebra satisfy the Λ_u-condition?

As noted in the previous section, every unital C^*-algebra satisfies the Λ_u-condition. This fact follows immediately from a result due to G. K. Pedersen: $\lambda_u(x) = \frac{1}{2}(1 - \alpha(x))$ *for* $\|x\| \leq 1$ *with* $\alpha(x) < 1$ (see [11, Theorem 5.1]). We do not know if an appropriate analogue of [11, Theorem 5.1] holds for general JB^*-algebras. The proof of this result for C^*-algebras given in [11] by Pedersen depends fundamentally on another result [13, Theorem 2.1], due to M. Rørdam, which may be expressed as follows: *for any element* T *of a* C^**-algebra* $\mathcal{U}$*, if* $a > \alpha(T)$ *then there is an invertible element* S *in* $\mathcal{U}$ *such that* $V(I - E_a) = S(I - E_a)$*, where* V *is a partial isometry in the polar decomposition of* T *and* E_a *denotes the spectral projection corresponding to the interval* $[0,a]$ *for* $|T|$. We do not know if this holds for a general JB^*-algebra but we will show that the proof given in [13] for the C^*-algebra case does not work in the setting of the finite-dimensional JB^*-algebra, $\mathcal{M}_2^s(\mathbb{C})$, consisting of all 2×2 complexified symmetric matrices.

Recall the following steps in the proof of [13, Theorem 2.1]: For $0 < b < a$, let f and g be continuous functions defined on the interval $[0,\infty]$ by

$$f(t) = \begin{cases} b^{-1} & \text{if } t \leq b, \\ t^{-1} & \text{otherwise,} \end{cases} \quad \text{and} \quad g(t) = \begin{cases} 0 & \text{if } t \leq b, \\ \frac{t-b}{a-b} & \text{if } b < t \leq a, \\ 1 & \text{otherwise.} \end{cases}$$

Choose b such that $\alpha(T) < b < a$ and $A \in \mathcal{U}_{\mathrm{inv}}$ such that $\|T^* - A\| < b$. Let $B = Af(|T^*|)$, $C = (1 - BV)g(|T|)$ and $D = I - C$. Then the required element S is given by $S = B^{-1}D$.

Example 4.1. Let $\mathcal{J}$ be the JB^*-algebra $\mathcal{M}_2^s(\mathbb{C})$ and $T = \begin{bmatrix} i & i+1 \\ i+1 & 2 \end{bmatrix}$. Then $|T| = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$, $|T^*| = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$ so that $f(|T^*|) = \frac{1}{18}\begin{bmatrix} 8 & -i-1 \\ i-1 & 7 \end{bmatrix}$ and $g(|T|) = \frac{1}{3}\begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$. It is easy to see that $\alpha(T) = 0$ (cf. [18, Theorem 5.2]). Let $a = 3$. Choosing $b = 2$ we have $\alpha(T) < b < a$. We take $A = \begin{bmatrix} 1-i & 1-i \\ 1-i & 3 \end{bmatrix} \in \mathcal{J}$. Then A is invertible and satisfies

$\|T^* - A\| = \|I\| < b$, so that $B = Af(|T^*|) = \frac{1}{18}\begin{bmatrix} 8-6i & 5-7i \\ 5-5i & 19 \end{bmatrix}$ with the inverse $B^{-1} = \frac{3+i}{30}\begin{bmatrix} 19 & 7i-5 \\ 5i-5 & 8-6i \end{bmatrix}$. Next, the polar decomposition $T = V|T|$ gives $V = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}$, so we calculate $C = (I - BV)g(|T|) = \frac{1}{9}\begin{bmatrix} -i & -i-1 \\ -i-1 & -2 \end{bmatrix}$. Hence, $D = I - C = \frac{1}{9}\begin{bmatrix} 9+i & 1+i \\ 1+i & 11 \end{bmatrix}$. Thus,

$$S = B^{-1}D = \frac{1}{90}\begin{bmatrix} 152+74i & -68i+84i \\ -50+30i & 100-40i \end{bmatrix}$$

is not in the algebra $\mathcal{J}$, unfortunately.

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