New York Journal of Mathematics

New York J. Math. **19** (2013) 511–531.

C*-algebras generated by spherical hyperexpansions

Sameer Chavan

Abstract. Let T be a spherical completely hyperexpansive m-variable weighted shift on a complex, separable Hilbert space $\mathcal{H}$ and let $T^{\mathfrak{s}}$ denote its spherical Cauchy dual. We obtain the hyperexpansivity analog of the structure theorem of Olin–Thomson for the C*-algebra $C^*(T)$ generated by T, under the natural assumption that $T^{\mathfrak{s}}$ is commuting. If, in addition, the defect operator $I - T_1T_1^* - \cdots - T_mT_m^*$ is compact then we ensure exactness of the sequence of C*-algebras

$$0 \longmapsto \mathcal{C}(\mathcal{H}) \hookrightarrow C^*(T) \overset{\pi}{\longmapsto} C(\sigma_{ap}(T)) \longmapsto 0,$$

where $\mathcal{C}(\mathcal{H})$ stands for the ideal of compact operators on $\mathcal{H}$, and

$$\pi : C^*(T) \to C(\sigma_{ap}(T))$$

is the unital $*$-homomorphism defined by $\pi(T_i) = z_i$ $(i = 1, \ldots, m)$. This unifies and generalizes the results of Coburn, 1973/74 and Arveson, 1998. We further illustrate our results by exhibiting a one parameter family $\mathscr{F}$ of spherical completely hyperexpansive 2-tuples T_{ν_λ} acting on $P^2(\mu_\lambda)$ $(1 \le \lambda \le 2)$, where $d\mu_\lambda := d\nu_\lambda d\sigma$, ν_λ is a probability measure on $[0, 1]$, and σ is the normalized surface area measure on the unit sphere $\partial\mathbb{B}$. Interestingly, within the family $\mathscr{F}$, the Szegö 2-shift T_{ν_1} and the Drury–Arveson 2-shift T_{ν_2} occupy the extreme positions. We would like to emphasize that T_{ν_λ} is unitarily equivalent to the multiplication operator tuples in $P^2(\mu_\lambda)$ if and only if $\lambda = 1$.

Contents

Received August 2, 2012; revised August 3, 2013.

2010 *Mathematics Subject Classification.* Primary 47A13, 47B37, 46L05; Secondary 47B20, 46E20.

Key words and phrases. Drury–Arveson m-shift, subnormality, spherical hyperexpansivity, spherical Cauchy dual, Toeplitz algebra, boundary representation.

ISSN 1076-9803/2013

1. Introduction

Needless to say, the present paper has roots in [12]. This work may also be regarded as a sequel to [14], where the first author and R. Curto initiated the study of spherical Cauchy dual tuples. In the present paper, we continue our study of the spherical Cauchy dual tuples and use it to reveal the structure of the C*-algebras generated by spherical hyperexpansive multi-shifts. Our work relies heavily on the results of Bunce [10], Olin–Thomson [27], and Athavale [7], [8].

There is voluminous material on the C*-algebras of multiplication operator tuples (see, for instance, [15], [19], [4], [11], [24]). We would first like to mention here the work [19] on the C*-algebra generated by multi-variable weighted shifts. It turns out that the spherical completely hyperexpansive m-variable weighted shifts with *well-behaved weight functions* (not exactly in the sense of Curto and Muhly but quite close to it) are in abundance. We would also wish to mention the work [11, Section 5] on the C*-algebras generated by the multiplication operator tuples M_z in $P^2(d\mu)$ where $d\mu := d\nu d\sigma$, ν is a probability measure on $[0, 1]$ with the point 1 in the support of ν, and σ is the normalized surface area measure on the unit sphere $\partial\mathbb{B}$. The operator tuple T_ν, as discussed in Example 2.4 below, is precisely the spherical Cauchy dual of M_z, and provides many interesting examples of spherical complete hyperexpansions including the Szegö m-shift and the Drury–Arveson 2-shift. One of the main results, the hyperexpansivity analog of [27, Theorem 1], is new even for the Drury–Arveson 2-shift. In the proof of the main results, we implicitly employ the methods from the harmonic analysis of the completely alternating functions on the semigroup $\mathbb{N}^m$. This is why these results are not applicable to the Drury–Arveson m-shift if $m \geq 3$ [14, Example 6.3]. However, in the last section, we show that our methods can be modified to obtain the conclusions of Theorems 2.2 and 2.3 for the Drury–Arveson m-shift for any $m \geq 1$.

For a complex, infinite-dimensional, separable Hilbert space $\mathcal{H}$, let $B(\mathcal{H})$ denote the Banach algebra of bounded linear operators on $\mathcal{H}$. By a *commuting m-tuple* T on $\mathcal{H}$, we mean a tuple $(T_1, \ldots, T_m)$ of commuting bounded linear operators $T_1, \ldots, T_m$ on $\mathcal{H}$. For $T \in B(\mathcal{H})$, we interpret T^* to be $(T_1^*, \ldots, T_m^*)$, and T^p to be $T_1^{p_1} \cdots T_m^{p_m}$ for $p = (p_1, \ldots, p_m) \in \mathbb{N}^m$, where $\mathbb{N}$ stands for the set of nonnegative integers.

The reader is referred to [14] for the basics of the theory of generating m-tuples. In this paper, we are mainly interested in the spherical generating 1-tuples: Given a commuting m-tuple $T = (T_1, \ldots, T_m)$ on $\mathcal{H}$, the *spherical* *generating* 1*-tuple associated with* T is given by

$$Q_s(X) := \sum_{i=1}^{m} T_i^* X T_i \ (X \in B(\mathcal{H})).$$

Fix an integer $p \geq 1$. We say that T is a *spherical* p*-contraction (resp. spherical* p*-expansion)* if

$$\sum_{q\in\mathbb{N}, 0\leq q\leq p} (-1)^{|q|} \binom{p}{q} Q_s^q(I) \geq 0 \text{ (resp. } \leq 0). \tag{1.1}$$

T is a *spherical* p*-isometry* if equality occurs in (1.1). We say that T is a *spherical complete hypercontraction* (resp. *spherical complete hyperexpansion*) if T is a spherical p-contraction (resp. spherical p-expansion) for all positive integers p. In all the above definitions, if $p = 1$ then we drop the prefix 1- and if $m = 1$ then we drop the term spherical.

A spherical m-isometry is a spherical complete hyperexpansion if and only if $m = 2$. Also, a spherical 2-expansive T is subnormal if and only if it is a spherical isometry [14].

Recall that a commuting m-tuple $T = (T_1, \dots, T_m)$ on $\mathcal{H}$ is said to be *subnormal* if there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a commuting m-tuple $N = (N_1, \dots, N_m)$ of normal operators N_i in $\mathcal{B}(\mathcal{K})$ such that

$$N_i h = T_i h \text{ for every } h \in \mathcal{H} \text{ and } 1 \leq i \leq m.$$

A spherical complete hypercontraction is always subnormal [7].

An m-variable weighted shift $T = (T_1, \dots, T_m)$ with respect to an orthonormal basis $\{e_n\}_{n\in\mathbb{N}^m}$ of a Hilbert space $\mathcal{H}$ is defined by

$$T_i e_n := w_n^{(i)} e_{n+\epsilon_i} \ (1 \leq i \leq m),$$

where ϵ_i is the m-tuple with 1 in the ith place and zeros elsewhere. We indicate the m-variable weighted shift operator T with weight sequence

$$\left\{ w_n^{(i)} : 1 \leq i \leq m, n \in \mathbb{N}^m \right\}$$

by $T : \{w_n^{(i)}\}_{n\in\mathbb{N}^m}$. We always assume that the weight multi-sequence of T consists of positive numbers and that T is commuting.

Notice that T_i commutes with T_j if and only if $w_n^{(i)} w_{n+\epsilon_i}^{(j)} = w_n^{(j)} w_{n+\epsilon_j}^{(i)}$ for all $n \in \mathbb{N}^m$.

A rather special example of a spherical m-isometry is the Drury–Arveson m-shift ([22], [4], [23, Theorem 4.2]). The *Drury–Arveson* m*-shift* is the operator m-tuple $M_{z,m}$ of multiplication by the co-ordinate functions $z_1, \dots, z_m$ in the reproducing kernel Hilbert space associated with the positive definite kernel

$$\frac{1}{1 - z_1\overline{w}_1 - \cdots - z_m\overline{w}_m} \ (z, w \in \mathbb{B}),$$

where $\overline{u}$ denotes the complex conjugate of the complex number u, and $\mathbb{B}$ denotes the open unit ball in the m-dimensional hermitian space $\mathbb{C}^m$. $M_{z,m}$ can also be realized as the weighted shift with weight multi-sequence

$$\left\{\sqrt{\frac{n_i+1}{|n|+1}} : 1 \leq i \leq m,\ n \in \mathbb{N}^m\right\},$$

where $|n| := n_1 + \cdots + n_m$ $(n \in \mathbb{N}^m)$.

The Drury–Arveson m-shift is a spherical complete hyperexpansion if and only if $m = 2$.

Let $T = (T_1, \ldots, T_m)$ be a commuting m-tuple on $\mathcal{H}$ and let Q_s be the spherical generating 1-tuple associated with T. Suppose T is *jointly left-invertible,* that is, for some positive number α,

$$Q_s(I) = T_1^*T_1 + \cdots + T_m^*T_m \geq \alpha I.$$

Note that $Q_s(I)$ is invertible. We refer to the m-tuple $T^{\mathfrak{s}} := (T_1^{\mathfrak{s}}, \ldots, T_m^{\mathfrak{s}})$ as the *spherical Cauchy dual* to T, where

$$T_i^{\mathfrak{s}} := T_i(Q_s(I))^{-1}\ (i = 1, \ldots, m).$$

This notion, the spherical analog of the notion of the Cauchy dual operator [29], is introduced and studied in [14]. Here is one notational excuse. For future reference, it will be convenient to retain Shimorin's original notation T' for the Cauchy dual of T in case $m = 1$.

Notice that $T^{\mathfrak{s}}$ is jointly left-invertible if and only if so is T. In this case, $(T^{\mathfrak{s}})^{\mathfrak{s}} = T$. Since a spherical 2-expansion is a spherical expansion [14], its spherical Cauchy dual tuple is well-defined.

Let $T : \{w_n^{(i)}\}_{n \in \mathbb{N}^m}$ denote a commuting m-variable weighted shift. Set

$$\beta_n(T) := \left(\sum_{i=1}^{m} \left(w_n^{(i)}\right)^2\right)^{\frac{1}{2}}\ (n \in \mathbb{N}^m). \tag{1.2}$$

Suppose T is jointly left-invertible or equivalently $\inf_{n \in \mathbb{N}^m} \beta_n > 0$. It is easy to see that $T^{\mathfrak{s}}$ is an m-variable weighted shift with weight sequence

$$\left\{\frac{w_n^{(i)}}{\beta_n^2(T)} : 1 \leq i \leq m,\ n \in \mathbb{N}^m\right\}.$$

A routine calculation now shows that $T^{\mathfrak{s}}$ is commuting if and only if

$$\beta_{n+\epsilon(i)}(T) = \beta_{n+\epsilon(j)}(T)$$

for all $1 \leq i, j \leq m$ and for all $n \in \mathbb{N}^m$.

The spherical Cauchy dual $M_{z,m}^{\mathfrak{s}}$ of the Drury–Arveson m-shift $M_{z,m}$ to be referred as the *dual Drury–Arveson m-shift*, is the weighted shift operator tuple with weights

$$\left\{\frac{\sqrt{(n_i+1)(|n|+1)}}{|n|+m} : 1 \leq i \leq m,\ n \in \mathbb{N}^m\right\}.$$

Note that $M^{\mathfrak{s}}_{z,m}$ is commuting. One may employ Curto's six-point test [18] to reveal the interesting fact that the dual Drury–Arveson m-shift is jointly hyponormal. Interestingly, in the course of the proofs of the main results, we prove that the dual Drury–Arveson m-shift is indeed subnormal. The present paper is a part of the work, initiated in [12], and carried out in [13] and [14], to develop the theory of (spherical) complete hyperexpansions parallel to the classical theory of subnormals.

The paper is organized as follows. In Section 2, we state the main results of the paper (Theorems 2.2 and 2.3) and illustrate these results by exhibiting a family of multi-variable weighted shifts of which the Drury–Arveson 2-shift is a prototype. In Section 3, we present the proofs of the main results, which involve the following important steps:

- Characterization of spherical hyperexpansivity of multi-shift T in terms of hyperexpansivity of a *canonical* 1-variable shift T_β associated with T (Lemma 3.3).
- Subnormality of the spherical Cauchy dual (Proposition 3.4).
- Essential normality of spherical hyperexpansions (Proposition 3.7).

In Section 4, as an application to our main results, we obtain the following generalization of a fundamental lemma of Arveson [4, Lemma 7.13]: If the spherical Cauchy dual $T^{\mathfrak{s}}$ is commutative and $I - T_1T_1^* - \cdots - T_mT_m^*$ is compact for a spherical completely hyperexpansive m-variable weighted shift T, then the identity representation of the C*-algebra $C^*(T)$ generated by T is a boundary representation for the unital operator space generated by T if and only if T is not a spherical isometry (Proposition 4.1). One may attribute the absence of inner functions in the multiplier algebra of the Drury–Arveson m-shift $M_{z,m}$ ($m \geq 2$) to the harmonic analysis of the associated sequences $\{\beta_n(M_{z,m})\}_{n\in\mathbb{N}^m}$ (see (1.2) above). The last section is devoted to some possible generalizations and a couple of unsolved problems.

2. Statement and examples

Let T denote a jointly left-invertible commuting m-tuple. By *the C^*-algebra generated by* T (in symbol, $C^*(T)$), we mean the norm closure of all noncommutative polynomials in the $(2m)$-variables $T_1, \ldots, T_m, T_1^*, \ldots, T_m^*$ which fixes the origin. It is easy to see that our definition coincides with the standard definition of $C^*(T)$. Indeed, since T is jointly left-invertible, $C^*(T)$ contains $Q_s(I) = T_1^*T_1 + \cdots + T_m^*T_m$. A simple application of the Spectral Theorem now shows that $C^*(T)$ contains the inverse of $Q_s(I)$ (and hence the identity operator). This also proves the following elementary fact crucial for our investigations.

Lemma 2.1. *Let T denote a jointly left-invertible commuting m-tuple. If the spherical Cauchy dual $T^{\mathfrak{s}}$ is commuting then we have $C^*(T^{\mathfrak{s}}) = C^*(T)$.*

A *commutator ideal* $\mathscr{C}_T$ of $C^*(T)$ is the norm closed ideal of $C^*(T)$ generated by the set of all elements of the form $AB - BA$ $(A, B \in C^*(T))$. We use the symbol $\mathcal{C}(\mathcal{H})$ to denote the ideal of compact operators on $\mathcal{H}$.

For $T \in B(\mathcal{H})$, we reserve the symbols $\sigma(T), \sigma_{ap}(T), \sigma_p(T)$ for the joint (Taylor) spectrum, approximate point spectrum, point spectrum of T respectively. For the definitions and the basic theory of various spectra including the Taylor spectrum, the reader is referred to [17].

To state the main results of this paper, we need to introduce some more notations. Let $C(X)$ denote the C*-algebra of continuous functions on the compact Hausdorff space X, endowed with the sup norm $\|\cdot\|_\infty$. For $\mathcal{H} \subseteq \mathcal{K}$, let $P_{\mathcal{H}}$ stand for the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. Finally, let $z_1, \ldots, z_m$ be the co-ordinate functions in $\mathbb{C}^m$ given by

$$z_i(w_1, \ldots, w_m) = w_i, \ (w_1, \ldots, w_m) \in \mathbb{C}^m.$$

Theorem 2.2. *Let $T : \{w_n^{(i)}\}_{n \in \mathbb{N}^m}$ denote a spherical completely hyperexpansive m-variable weighted shift on $\mathcal{H}$. Assume that the spherical Cauchy dual $T^{\mathfrak{s}}$ is commuting. Then there exists a commuting m-tuple N consisting of normal operators on some Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ such that the diagram*

$$\begin{array}{ccccc} C^*(N) & & \xrightarrow{\quad\quad\psi\quad\quad} & & C(\sigma(N)) \\ \theta\big\downarrow & & & & \big\downarrow r \\ C^*(T) & \xrightarrow[q]{} & C^*(T)/\mathscr{C}_T & \xrightarrow[\phi]{} & C(\sigma_{ap}(T)) \end{array}$$

commutes, where:

(1) *$\psi : C^*(N) \to C(\sigma(N))$ is the isometric $*$-isomorphism such that*
$$\psi(N_i) = z_i \ (1 \le i \le m).$$
(2) *The restriction mapping $r : C(\sigma(N)) \to C(\sigma_{ap}(T))$ is a well-defined surjection.*
(3) *$\phi : C^*(T)/\mathscr{C}_T \to C(\sigma_{ap}(T))$ is the isometric $*$-isomorphism with*
$$\phi(T_i + \mathscr{C}_T) = z_i \ (1 \le i \le m).$$
(4) *$q : C^*(T) \to C^*(T)/\mathscr{C}_T$ is the quotient map.*
(5) *$\theta : C^*(N) \to C^*(T)$ given by $\theta(f(N)) = P_{\mathcal{H}} f(N)|_{\mathcal{H}}, \ f \in C(\sigma(N))$ is a (completely) positive linear mapping.*

Theorem 2.3. *Let $T : \{w_n^{(i)}\}_{n \in \mathbb{N}^m}$ denote a spherical completely hyperexpansive m-variable weighted shift on $\mathcal{H}$. Assume that the spherical Cauchy dual $T^{\mathfrak{s}}$ is commuting. Then the defect operator $I - T_1T_1^* - \cdots - T_mT_m^*$ is compact if and only if we have an exact sequence of C*-algebras*

$$0 \ \longmapsto \ \mathcal{C}(\mathcal{H}) \ \overset{i}{\hookrightarrow} \ C^*(T) \overset{\pi}{\longmapsto} \ C(\sigma_{ap}(T)) \ \longmapsto \ 0,$$

where $i : \mathcal{C}(\mathcal{H}) \hookrightarrow C^*(T)$ is the inclusion map and $\pi : C^*(T) \to C(\sigma_{ap}(T))$ is the unital-homomorphism defined by $\pi(T_i) = z_i$ $(i = 1, \ldots, m)$.*

Remark. Note that the compactness of $I - T_1T_1^* - \cdots - T_mT_m^*$ is necessary for the conclusion of Theorem 2.3. This follows from the fact that $\sigma_{ap}(T)$ is contained in the boundary of the unit ball (see Lemma 3.5 below).

We defer the proofs of Theorems 2.2 and 2.3 until the next section for two major reasons. Firstly, these are rather involved, at least conceptually. Secondly, as our proofs consist of several interesting observations of independent interest, a separate spot for their get-together is desirable!

In the remaining part of this section, we exhibit a family of spherical complete hyperexpansions of which the Drury–Arveson 2-shift is a prototype (cf. [11, Section 5]).

Example 2.4. For a probability measure ν on $[0,1]$, let

$$I_k := \int_{[0,1]} r^{2k} d\nu(r) \quad (k \in \mathbb{N}).$$

Consider the m-variable weighted shift $T_\nu = (T_{\nu 1}, \ldots, T_{\nu m})$ with weights

$$w_n^{(i)} := \sqrt{\frac{n_i+1}{|n|+m}}\sqrt{\frac{I_{|n|}}{I_{|n|+1}}} \quad (n \in \mathbb{N}^m, 1 \leq i \leq m).$$

Observe that $\beta_n^2(T_\nu) = \frac{I_{|n|}}{I_{|n|+1}}$ (see (1.2) of Section 1). It is now easy to see that the spherical Cauchy dual $T_\nu^{\mathfrak{s}}$ is commuting. In general, T_ν is not a spherical complete hyperexpansion. This is a consequence of the known fact that the Cauchy dual of a contractive subnormal weighted shift, if it exists, is not necessarily completely hyperexpansive [8] and Lemma 3.3 of the next section. In the next section, we provide a function-theoretic characterization of spherical completely hyperexpansive T_ν (refer to the remark following Lemma 3.3).

We claim that the compactness of defect operator $I - T_{\nu 1}T_{\nu 1}^* - \cdots - T_{\nu m}T_{\nu m}^*$ is equivalent to $\lim_{|n|\to\infty} \beta_n(T_\nu) = 1$. Notice that $T_{\nu 1}T_{\nu 1}^* + \cdots + T_{\nu m}T_{\nu m}^*$ is a diagonal operator with entries

$$\left(w_{n-\epsilon_1}^{(1)}\right)^2 + \cdots + \left(w_{n-\epsilon_m}^{(m)}\right)^2 = \frac{|n|}{|n|+m-1}\frac{I_{|n|-1}}{I_{|n|}} = \frac{|n|}{|n|+m-1}\beta_{|n|-1}^2(T_\nu),$$

with the convention that $w_{n-\epsilon_i}^{(i)} = 0$ if $n_i = 0$. By an application of the Cauchy–Schwarz inequality, $\beta_k^2(T_\nu) = \frac{I_k}{I_{k+1}}$ is easily seen to be a decreasing function of k. Since a diagonal operator $\{\lambda_n\}_{n\in\mathbb{N}^m}$ is compact if and only if $\lim_{|n|\to\infty} \lambda_n = 0$, the claim stands verified. We will see, as a consequence of Lemma 3.5(1) below, that $I - T_{\nu 1}T_{\nu 1}^* - \cdots - T_{\nu m}T_{\nu m}^*$ is compact if T_ν is a spherical 2-expansion.

For $\lambda \geq 1$, let ν_λ denote the probability measure on $[0,1]$ governed by

$$\frac{k!}{\lambda(\lambda+1)\cdots(k-1+\lambda)} = \int_{[0,1]} r^{2k} d\nu_\lambda(r) \ (k \in \mathbb{N}).$$

The existence of ν_λ is ensured by [20, Theorem 2.7]. Notice that T_{ν_λ} is the m-variable weighted shift with weight sequence

$$\sqrt{\frac{n_i+1}{|n|+m}}\sqrt{\frac{|n|+\lambda}{|n|+1}} \quad (n \in \mathbb{N}^m, 1 \leq i \leq m).$$

The following special cases are noteworthy:

(1) $\lambda = 1$: Notice that $d\nu_1(r)$ is the point mass at $\{1\}$ and that T_{ν_1} is nothing but the Szegö m-shift with weight sequence

$$\left\{\sqrt{\frac{n_i+1}{|n|+m}} : 1 \leq i \leq m, n \in \mathbb{N}^m\right\}.$$

(2) $\lambda = 2$: Notice that $d\nu_2(r)$ is the weighted Lebesgue measure $2rdr$ on $[0,1]$ and that T_{ν_2} is the m-variable weighted shift with weight sequence

$$\left\{\sqrt{\frac{n_i+1}{|n|+m}}\sqrt{\frac{|n|+2}{|n|+1}} : 1 \leq i \leq m, n \in \mathbb{N}^m\right\}.$$

If $m = 2$ then note that T_{ν_2} is nothing but the Drury–Arveson 2-shift.

Observe that $\lim_{|n|\to\infty} \beta_n(T_{\nu_\lambda}) = 1$. By the preceding discussion, the defect operator $I - T_{\nu_\lambda 1}T^*_{\nu_\lambda 1} - \cdots - T_{\nu_\lambda m}T^*_{\nu_\lambda m}$ is compact for any $\lambda \geq 1$.

3. Proofs of the structure theorems

Let $T : \{w_n^{(i)}\}_{n\in\mathbb{N}^m}$ be an m-variable weighted shift. Throughout this section, let $\beta_n(T)$ denote

$$\left(\sum_{i=1}^m \left(w_n^{(i)}\right)^2\right)^{\frac{1}{2}} \quad (n \in \mathbb{N}^m).$$

Whenever there is no ambiguity, we write simply β_n in place of $\beta_n(T)$. For ready reference, we record the following triviality.

Lemma 3.1. *Let $T : \{w_n^{(i)}\}_{n\in\mathbb{N}^m}$ denote a commuting m-variable weighted shift. Then the following statements are equivalent:*

(1) *The spherical Cauchy dual $T^{\mathfrak{s}}$ is commuting.*

(2) *The multi-sequence $\{\beta_n(T)\}_{n\in\mathbb{N}^m}$ satisfies*

$$\beta_{n+\epsilon_j}(T) = \beta_{n+\epsilon_i}(T) \ (1 \leq i,j \leq m, \ n \in \mathbb{N}^m),$$

where ϵ_i is the m-tuple with 1 in the ith place and zeros elsewhere.

(3) *The multi-sequence $\{\beta_n(T)\}_{n\in\mathbb{N}^m}$ satisfies*

$$\beta_n(T) = \beta_{|n|\epsilon_1}(T) \text{ for all } n \in \mathbb{N}^m,$$

where $|n| = n_1 + \cdots + n_m$.

Definition 3.2. Let $T : \{w_n^{(i)}\}_{n\in\mathbb{N}^m}$ be an m-variable weighted shift with respect to the orthonormal basis $\{e_n\}_{n\in\mathbb{N}^m}$ of $\mathcal{H}$. Consider the one-variable weighted shift $T_\beta : \{\beta_{k\epsilon_1}\}_{k\in\mathbb{N}}$ with respect to some another orthonormal basis $\{f_k\}_{k\in\mathbb{N}}$ of $\mathcal{H}$. We refer to T_β as the *shift associated with* T.

Clearly, T_β is a unitary invariant for T. Although, T_β is far from being complete, it can be used quite efficiently to characterize spherical contractions (resp. spherical expansions resp. spherical p-isometry).

The following simple lemma plays a key role in our investigations.

Lemma 3.3. *Let $T : \{w_n^{(i)}\}_{n\in\mathbb{N}^m}$ denote a commuting m-variable weighted shift with respect to the orthonormal basis $\{e_n\}_{n\in\mathbb{N}^m}$ of $\mathcal{H}$. Let T_β denote the shift associated with T with respect to the orthonormal basis $\{f_k\}_{n\in\mathbb{N}}$ of $\mathcal{H}$. If the spherical Cauchy dual $T^{\mathfrak{s}}$ is commuting, then for any positive integer p, the following statements are equivalent:*

1. *T is a spherical p-contraction (resp. spherical p-expansion, resp. spherical p-isometry).*
2. *T_β is a p-contraction (resp. p-expansion, resp. p-isometry).*

In particular, T is a spherical complete hypercontraction (resp. spherical complete hyperexpansion) if and only if T_β is a complete hypercontraction (resp. complete hyperexpansion).

Proof. Assume that $T^{\mathfrak{s}}$ is commuting. We contend that

$$\langle Q_s^k(I)e_n,\, e_n\rangle = \beta^2_{|n|\epsilon_1}\beta^2_{(|n|+1)\epsilon_1}\cdots\beta^2_{(|n|+k-1)\epsilon_1}\ (k\in\mathbb{N},\quad n\in\mathbb{N}^m), \tag{3.1}$$

where Q_s denote the spherical generating 1-tuple associated with T. We prove (3.1) by induction on $k\geq 1$. Clearly, $k=1$ is immediate from Lemma 3.1. Suppose that (3.1) holds true for $k\geq 1$. Since $Q_s^{k+1}(I) = \sum_{j=1}^m T_j^* Q_s^k(I)T_j$, one has

$$\begin{aligned}\langle Q_s^{k+1}(I)e_n,\, e_n\rangle &= \sum_{j=1}^m \langle Q_s^k(I)T_je_n,\, T_je_n\rangle\\ &= \sum_{j=1}^m \left(w_n^{(j)}\right)^2 \langle Q_s^k(I)e_{n+\epsilon_j},\, e_{n+\epsilon_j}\rangle\\ &= \sum_{j=1}^m \left(w_n^{(j)}\right)^2 \beta^2_{(|n|+1)\epsilon_1}\beta^2_{(|n|+2)\epsilon_1}\cdots\beta^2_{(|n|+k)\epsilon_1},\end{aligned}$$

where we used the induction hypothesis in the last step. The desired identity in (3.1) (with k replace by $k+1$) is immediate if we again apply Lemma 3.1.

Notice that a weighted shift $T : \{w_n^{(i)}\}$ is a spherical p-contraction (resp. spherical p-expansion resp. spherical p-isometry) if and only if

$$\sum_{q\in\mathbb{N},0\leq q\leq p} (-1)^{|q|}\binom{p}{q}\langle Q_s^q(I)e_n,\, e_n\rangle \geq 0 \quad (\text{resp. } \leq 0,\ \text{resp. } = 0)$$

for every $n \in \mathbb{N}^m$. Since $\langle Q^q_s(I)e_n, e_n\rangle = \|T^q_\beta f_{|n|}\|^2$ in view of the discussion in the previous paragraph, the desired equivalence is immediate. $\square$

Remark. Let T_ν be as in Example 2.4. It may be concluded from Example 2.4 that the shift $T_{\nu\beta}$ associated with T_ν admits the weight sequence $\{\beta_{k\epsilon_1}(T_\nu)\}_{k\in\mathbb{N}}$ given by

$$\frac{\int_{[0,1]} r^{2k} d\nu(r)}{\int_{[0,1]} r^{2(k+1)} d\nu(r)} \ (k \in \mathbb{N}).$$

It is easy to see from Lemma 3.3 that T_ν is a spherical complete hyperexpansion if and only if

$$\sum_{q\in\mathbb{N}, 0\le q\le p} (-1)^{|q|} \binom{p}{q} \frac{1}{\int_{[0,1]} r^{2q} d\nu(r)} \le 0$$

for every $p \geq 1$. The latter one is true if and only if $\left\{\frac{1}{\int_{[0,1]} r^{2k} d\nu(r)}\right\}_{k\in\mathbb{N}}$ is *completely alternating* (refer to [9] for the definition of completely alternating sequences).

Let T_{ν_λ} be as in the last paragraph of Example 2.4. It follows from the preceding discussion that T_{ν_λ} is a spherical complete hyperexpansion if and only if the sequence

$$\left\{\frac{\lambda(\lambda+1)\cdots(k-1+\lambda)}{k!}\right\}_{k\in\mathbb{N}}$$

is completely alternating. By [28, Example 2.3], this happens if and only if $1 \leq \lambda \leq 2$. Notice that the shift associated with the Szegö m-shift T_{ν_1} is the unilateral shift while the shift associated with the Drury–Arveson 2-shift T_{ν_2} (with $m = 2$) is the Dirichlet shift. The last observation has already been noted in the discussion prior to [14, Corollary 4.3].

The preceding lemma has numerous applications. For instance, it can be used to present an easy proof of the major half of [23, Theorem 4.2]. Further, the preceding lemma enables one to deduce various properties of spherical p-contractions (resp. spherical p-expansions resp. spherical p-isometries) from those which are known for p-contractions (resp. p-expansions resp. p-isometries) (see, for instance, the proof of Lemma 3.5 below).

3.1. Subnormality of spherical Cauchy dual. The proof of the next result relies heavily on the beautiful result [8, Proposition 6] of A. Athavale. It should be noted that this result relies heavily on a classical result of Schoenberg characterizing the negative definite kernels in terms of a one parameter family of positive definite kernels [9]. Ours may be considered as the spherical analog of Athavale's Theorem.

Proposition 3.4. *Let $T : \{w_n^{(i)}\}_{n\in\mathbb{N}^m}$ denote a spherical completely hyperexpansive m-variable weighted shift. If the spherical Cauchy dual $T^{\mathfrak{s}}$ is commuting then $T^{\mathfrak{s}}$ is subnormal.*

Proof. Suppose the spherical Cauchy dual $T^{\mathfrak{s}}$ is commuting. Since T is given to be a spherical complete hyperexpansion, by Lemma 3.3, the shift T_β associated with T is completely hyperexpansive. Hence, by [8, Proof of Proposition 6], the Cauchy dual $(T_\beta)'$ of T_β is completely hypercontractive. Now if Q_s (resp. P_s) denotes the spherical generating 1-tuple associated with T (resp. $T^{\mathfrak{s}}$) then

$$\frac{1}{(\beta_n(T))^2} = \langle Q_s(I)^{-1}e_n,\, e_n\rangle = \langle P_s(I)e_n,\, e_n\rangle = (\beta_n(T^{\mathfrak{s}}))^2.$$

It follows that $(T_\beta)'$ is indeed the shift $(T^{\mathfrak{s}})_\beta$ associated with $T^{\mathfrak{s}}$. Thus the shift $(T^{\mathfrak{s}})_\beta$ associated with $T^{\mathfrak{s}}$ is completely hypercontractive. By Lemma 3.3, $T^{\mathfrak{s}}$ must be a spherical complete hypercontraction. Thus [7, Theorem 5.2] is applicable, and hence $T^{\mathfrak{s}}$ is subnormal. □

3.2. Essential normality of spherical hyperexpansions.

Lemma 3.5. *Let $T : \{w_n^{(i)}\}_{n\in\mathbb{N}^m}$ denote a spherical 2-expansive m-variable weighted shift. Suppose the spherical Cauchy dual $T^{\mathfrak{s}}$ is commuting. Then the following statements are true:*

1. *$\beta_n \to 1$ as $|n| \to \infty$.*
2. *$T^{\mathfrak{s}}$ is a compact perturbation of T, that is, there exists an m-tuple K of compact operators such that $T = T^{\mathfrak{s}} + K$.*
3. *$\sigma_{ap}(T^{\mathfrak{s}}) = \sigma_{ap}(T) \subseteq \partial\mathbb{B}$, where $\partial\mathbb{B}$ is the unit sphere in $\mathbb{C}^m$.*

Proof. (1) Recall that the weight-sequence of a 2-expansive weighted shift converges to 1 [25]. Since $\beta_{k\epsilon_1}$ is the weight-sequence of the shift T_β associated with T and since T_β is 2-expansive (Lemma 3.3), one has

$$\beta_n = \beta_{|n|\epsilon_1} \to 1 \text{ as } |n| \to \infty$$

by an application of Lemma 3.1.

(2) Recall that $T^{\mathfrak{s}} = (T_1^{\mathfrak{s}}, \dots, T_m^{\mathfrak{s}})$ is the weighted shift with weight-sequence

$$v_n^{(i)} := \frac{w_n^{(i)}}{\beta_n^2}\ (1 \le i \le m,\ n \in \mathbb{N}^m).$$

To see (2), by the general theory, it suffices to check that

$$w_n^{(i)} - v_n^{(i)} \to 0 \text{ as } |n| \to \infty.$$

Since $\{w_n^{(i)}\}_{n\in\mathbb{N}^m}$ is bounded and since

$$w_n^{(i)} - v_n^{(i)} = w_n^{(i)}\left(1 - \frac{1}{\beta_n^2}\right),$$

this is immediate from (1).

(3) In view of (2), T and $T^{\mathfrak{s}}$ admit the same left Harte essential spectra. By [21, Theorem 2.10(a)], the approximate point spectrum is the union of the left Harte essential spectrum and the point spectrum. However, the point spectrum of a weighted shift is always empty. It follows that $\sigma_{ap}(T^{\mathfrak{s}}) = \sigma_{ap}(T)$.

To see the inclusion $\sigma_{ap}(T) \subseteq \partial\mathbb{B}$, let $\lambda = (\lambda_1, \ldots, \lambda_m) \in \sigma_{ap}(T)$. Note that if we replace T_i by λ_i and T_i^* by $\overline{\lambda}_i$ in the definition of spherical 2-expansivity, then one must have $\lambda \in \partial\mathbb{B}$. Validity of such a substitution is clear from the definition of the spherical 2-expansivity (see the discussion preceding [23, Lemma 3.2]). □

Recall that an operator S in $\mathcal{B}(\mathcal{H})$ is said to be *hyponormal* if the self-commutator $[S^*, S] := S^*S - SS^*$ of S is a positive operator.

Lemma 3.6. *Let $T : \{w_n^{(i)}\}_{n\in\mathbb{N}^m}$ be a spherical 2-hyperexpansive m-variable weighted shift such that $T^{\mathfrak{s}}$ is commuting. Suppose the defect operator*

$$I - T_1T_1^* - \cdots - T_mT_m^*$$

is compact and that $T_i^{\mathfrak{s}}$ is hyponormal for each $i = 1, \ldots, m$. Then T_i is essentially normal, that is, $T_i^*T_i - T_iT_i^*$ is compact for all $i = 1, \ldots, m$.

Proof. In view of the compactness of $I - T_1T_1^* - \cdots - T_mT_m^*$, one has

$$\lim_{|n|\to\infty} \left(w_{n-\epsilon_1}^{(1)}\right)^2 + \cdots + \left(w_{n-\epsilon_m}^{(m)}\right)^2 = 1. \tag{3.2}$$

Since T_i is a compact perturbation of $T_i^{\mathfrak{s}}$ (Lemma 3.5(2)), it suffices to check that $T_i^{\mathfrak{s}}$ is essentially normal or equivalently $v_n^{(i)} - v_{n-\epsilon_i}^{(i)} \to 0$ as $|n| \to \infty$, where $v_n^{(i)} := \frac{w_n^{(i)}}{\beta_n^2}$ $(1 \leq i \leq m,\ n \in \mathbb{N}^m)$. Since $T_i^{\mathfrak{s}}$ is hyponormal, one has

$$v_n^{(i)} - v_{n-\epsilon_i}^{(i)} \geq 0 \ (n \in \mathbb{N}^m,\ n_i \neq 0). \tag{3.3}$$

Observe that

$$\sum_{i=1}^m \left(v_n^{(i)}\right)^2 - \left(v_{n-\epsilon_i}^{(i)}\right)^2 = \frac{1}{\beta_n^2} - \frac{\sum_{i=1}^m \left(w_{n-\epsilon_i}^{(i)})^2\right)^2}{\beta_{n-\epsilon_i}^2} \longrightarrow 0$$

in view of (3.2) and Lemma 3.5(1). The essential normality of $T_i^{\mathfrak{s}}$ is now immediate from (3.3). □

Proposition 3.7. *Let $T : \{w_n^{(i)}\}_{n\in\mathbb{N}^m}$ be a spherical completely hyperexpansive m-variable weighted shift such that $T^{\mathfrak{s}}$ is commuting. Suppose the defect operator $I - T_1T_1^* - \cdots - T_mT_m^*$ is compact. Then T_i is essentially normal, that is, $T_i^*T_i - T_iT_i^*$ is compact for all $i = 1, \ldots, m$.*

Proof. As T is a spherical complete hyperexpansion, by Proposition 3.4, $T^{\mathfrak{s}}$ is subnormal. In particular, $T_i^{\mathfrak{s}}$ is hyponormal for all $i = 1, \ldots, m$ [16]. Now appeal to the preceding lemma. □

3.3. A commutative diagram of Olin and Thomson. Recall that the commutator ideal of the C*-algebra $C^*(T)$ generated by a commuting tuple T on $\mathcal{H}$ is denoted by $\mathscr{C}_T$.

Lemma 3.8. *Let S be a subnormal m-tuple on $\mathcal{H}$ and let N be a normal extension of S on some Hilbert space $\mathcal{K}$. Then the diagram*

$$\begin{array}{ccccc} C^*(N) & & \xrightarrow{\quad\psi\quad} & & C(\sigma(N)) \\ \Big\downarrow \theta & & & & \Big\downarrow r \\ C^*(S) & \xrightarrow[q]{} & C^*(S)/\mathscr{C}_S & \xrightarrow[\phi]{} & C(\sigma_{ap}(S)) \end{array}$$

commutes, where:

(1) *$\psi : C^*(N) \to C(\sigma(N))$ is the isometric $*$-isomorphism such that*
$$\psi(N_i) = z_i \ (1 \leq i \leq m).$$
(2) *The restriction mapping $r : C(\sigma(N)) \to C(\sigma_{ap}(S))$ is a well-defined surjection.*
(3) *$\phi : C^*(S)/\mathscr{C}_S \to C(\sigma_{ap}(S))$ is the isometric $*$-isomorphism with*
$$\phi(S_i + \mathscr{C}_S) = z_i \ (1 \leq i \leq m).$$
(4) *$q : C^*(S) \to C^*(S)/\mathscr{C}_S$ is the quotient map.*
(5) *$\theta : C^*(N) \to C^*(S)$ given by $\theta(f(N)) = P_{\mathcal{H}} f(N)|_{\mathcal{H}},\ f \in C(\sigma(N))$ is a (completely) positive linear mapping.*

Proof. The proof goes verbatim the one-variable situation [27, Proof of Theorem 1]. For the sake of completeness, we provide the essential details. For $p, q \in \mathbb{N}^m$, observe that
$$S^{*p} S^q = P_{\mathcal{H}} N^{*p} N^q.$$
It is now easy to see that $r \circ \psi$ and $\phi \circ q \circ \theta$ agree on the linear span of $\{N^{*p} N^q : p, q \in \mathbb{N}^m\}$. Let $f \in C(\sigma(N))$. By the Stone–Weierstrass Theorem, there exists a sequence $\{p_n\}_{n \in \mathbb{N}}$ of complex polynomials in z and $\overline{z}$ such that $\|p_n - f\|_\infty \to 0$, $\|p_n(N^*, N) - f(N)\| \to 0$ as $n \to \infty$. It follows that the diagram above is commutative provided all the maps involved are well-defined.

Verify that $\|p_n(S^*, S) - P_{\mathcal{H}} f(N)\| \to 0$ as $n \to \infty$, where $p_n(S^*, S)$ is given by Agler's hereditary functional calculus [1]. It is now clear that θ sends $C^*(N)$ into $C^*(S)$. Clearly, θ is positive. Since $\sigma_{ap}(S) \subseteq \sigma_{ap}(N) = \sigma(N)$, the restriction mapping r of (2) is well-defined. By the Tietze extension theorem, r is surjective as well. The proof is over if we invoke appropriate analog of [16, Corollary 12.4, Chapter II]. This is provided by a result of Bunce [10], which says that for any commuting m-tuple S of hyponormals, there exists an isometric $*$-isomorphism $\phi : C^*(S)/\mathscr{C}_S \to C(\sigma_{ap}(S))$ such that $\phi(S_i + \mathscr{C}_S) = z_i$ $(1 \leq i \leq m)$. This completes the proof of the lemma. □

We are now ready to prove Theorems 2.2 and 2.3.

Proof of Theorem 2.2. We just need to put all the pieces into the place. Suppose the spherical Cauchy dual $T^{\mathfrak{s}}$ is commuting. Then, by Proposition 3.4, $T^{\mathfrak{s}}$ is subnormal. Thus it admits a normal extension N on some Hilbert space $\mathcal{K}$. Now, the preceding lemma yields a commutative diagram for $T^{\mathfrak{s}}$. However, since $C^*(T^{\mathfrak{s}}) = C^*(T)$ (Lemma 2.1) and since $\sigma_{ap}(T^{\mathfrak{s}}) = \sigma_{ap}(T) \subseteq \partial\mathbb{B}$ (Lemma 3.5(3)), we obtain the desired commutative diagram. □

Proof of Theorem 2.3. We mention that the C*-algebra generated by a jointly left-invertible multi-variable weighted shift is irreducible [26, Corollary 13]. Suppose now that the defect operator $I - T_1T_1^* - \cdots - T_mT_m^*$ is compact. Then, by Proposition 3.7, T_i is essentially normal for all $i = 1, \ldots, m$. To show that the commutator ideal $\mathscr{C}_T$ of $C^*(T)$ coincides with the ideal $\mathcal{C}(\mathcal{H})$ of compact operators on $\mathcal{H}$, we imitate the proof of [16, Lemma 12.9, Chapter II]. Notice that $C^*(T)/\mathcal{C}(\mathcal{H})$ is generated by normal elements $T_1 + \mathcal{C}(\mathcal{H}), \ldots, T_m + \mathcal{C}(\mathcal{H})$, and hence it is abelian. In particular, $\mathcal{C}(\mathcal{H})$ contains $\mathscr{C}_T$. The inclusion $\mathcal{C}(\mathcal{H}) \subseteq \mathscr{C}_T$ follows from the fact that $\mathcal{C}(\mathcal{H})$ admits no nonzero characters (that is, multiplicative linear functionals sending I to 1) in view of

$$\mathscr{C}_T = \cap\{\rho^{-1}(0) : \rho \text{ is a character on } C^*(T)\}$$

(refer to the discussion following [10, Corollary 4]). The desired conclusion now follows from Theorem 2.2. □

4. Boundary representations for spherical hyperexpansions

It is not the only purpose of this section to extend some results of [4] to the setting of spherical hyperexpansions but also to pinpoint that several striking results (e.g. absence of inner functions in the multiplier algebra) in the theory of Drury–Arveson m-shift $M_{z,m}$ ($m \geq 2$) may be attributed to the harmonic analysis of the associated sequences $\{\beta_n(M_{z,m})\}_{n\in\mathbb{N}^m}$.

By a *unital operator space*, we mean a pair $\mathcal{S} \subseteq \mathcal{B}$ consisting of a linear subspace S of a unital C*-algebra $\mathcal{B}$, which contains the unit of $\mathcal{B}$ and generates $\mathcal{B}$ as a C*-algebra, $\mathcal{B} = C^*(\mathcal{S})$. An *irreducible representation* of $\mathcal{B}$ is a unital homomorphism $r : \mathcal{B} \to B(\mathcal{H})$ such that $r(\mathcal{B})$ is an irreducible subalgebra of $B(\mathcal{H})$. An irreducible representation $r : \mathcal{B} \to B(\mathcal{H})$ is said to be a *boundary representation* for $\mathcal{S}$ if $r|_{\mathcal{S}}$ has a unique completely positive linear extension to $\mathcal{B}$, namely r itself. Recall that ϕ from $\mathcal{B}$ into another C*-algebra $\mathcal{A}$ is *completely positive* if $\phi_n : M_n(\mathcal{B}) \to M_n(\mathcal{A})$ given by $\phi_n([a_{i,j}]) := [\phi(a_{i,j})]$, $[a_{i,j}] \in M_n(\mathcal{B})$, is positive for all $n \geq 1$.

Proposition 4.1. *Let $T : \{w_n^{(i)}\}_{n\in\mathbb{N}^m}$ denote a spherical completely hyperexpansive m-variable weighted shift on $\mathcal{H}$. Assume that the spherical Cauchy dual $T^{\mathfrak{s}}$ is commuting and that the defect operator $I - T_1T_1^* - \cdots - T_mT_m^*$ is compact. Then the following statements are equivalent:*

(1) *The identity representation of* $C^*(T)$ *is a boundary representation for the* $(d+1)$*-dimensional space* $\mathscr{L} :=$ *linear span* $\{I, T_1, \ldots, T_m\}$.
(2) T *is not a spherical isometry.*

Proof. (1) implies (2) Suppose that T is a spherical isometry. By [6, Proposition 2], T is necessarily a subnormal tuple. It is then not difficult to see that $\mathcal{H}$ is a subnormal module on $\sigma(T)$ in the sense of [24] (see, for example, [17, Theorem 7.7]). By [24, Theorem 3.2], the identity representation of $C^*(T)$ is not a boundary representation for $\mathscr{L}$.

(2) implies (1) We imitate the proof of [4, Lemma 7.13]. We already recorded in the proof of Theorem 2.3 that $C^*(T)$ is an irreducible C*-algebra containing the ideal $\mathcal{C}(\mathcal{H})$ of compact operators on $\mathcal{H}$. Hence, by the Arveson's Boundary Theorem [3], the identity representation of $C^*(T)$ is a boundary representation for $\mathscr{L}$ provided the quotient map $q : B(\mathcal{H}) \to B(\mathcal{H})/\mathcal{C}(\mathcal{H})$ is not isometric when promoted to the space of $m \times m$-matrices over $\mathscr{L}$. Consider the matrix A with rows

$$[T_1, 0, \ldots, 0], [0, T_2, 0, \ldots, 0], \ldots, [0, \ldots, 0, T_m].$$

By Theorem 2.3, the quotient map q sends T_i to the co-ordinate function z_i defined on $\sigma_{ap}(T)$ $(i = 1, \ldots, m)$. Thus $q(A)$ is the matrix with rows

$$[z_1, 0, \ldots, 0], [0, z_2, 0, \ldots, 0], \ldots, [0, \ldots, 0, z_m].$$

Since $\sigma_{ap}(T) \subseteq \partial\mathbb{B}$ (Lemma 3.5(3)), one has

$$\|q(A)\| = \sup\{\|q(A)(z)\| : z \in \sigma_{ap}(T)\} = 1.$$

Suppose now that T is not a spherical isometry. Since T is a spherical expansion, it follows that $\|A\| = \|T_1^*T_1 + \cdots + T_m^*T_m\| > 1$. In particular, $\|q(A)\| \neq \|A\|$. Thus the promoted quotient map is not isometric. □

There are several important consequences of the previous result (refer to [4], [24]). However, to avoid book-keeping, we list only one significant application.

Corollary 4.2. *Let* $T : \{w_n^{(i)}\}_{n \in \mathbb{N}^m}$ *denote a spherical completely hyperexpansive* m*-variable weighted shift on* $\mathcal{H}$*. Assume that the spherical Cauchy dual* $T^{\mathfrak{s}}$ *is commuting, that the defect operator* $I - T_1T_1^* - \cdots - T_mT_m^*$ *is compact, and that* T *is not a spherical isometry. Let* $R_1, R_2, \ldots,$ *be a finite or infinite sequence of operators on* $\mathcal{H}$*, which commutes with* T *and which satisfy*

$$R_1^*R_1 + R_2^*R_2 + \cdots = I.$$

Then each R_i *is a scalar multiple of the identity operator.*

Proof. We omit the deduction as it can be obtained along the lines of [4, Proposition 8.13] in view of the preceding result. □

5. Concluding remarks. Open problems

In this section, we first outline a possible approach to obtain a version of Theorem 2.2 for arbitrary spherical complete hyperexpansions. Observe that the conclusion of Theorem 2.2 holds true for any spherical complete hyperexpansion for which the spherical Cauchy dual consists of commuting hyponormals. Hence, it is natural to look for certain hyponormality property of spherical Cauchy dual tuples. The results from single-variable theory of hyperexpansions ([30], [12]) as well as Proposition 3.4 suggest the following conjecture:

Conjecture 5.1. *If the spherical Cauchy dual to a spherical* 2*-expansion is commuting then it is jointly hyponormal.*

For the definition and basic properties of jointly hyponormal tuples, the reader is referred to [5] and [18]. It is easy to see that a jointly hyponormal tuple is a spherical 2-contraction.

The next result supports Conjecture 5.1.

Proposition 5.2. *If* T *is a spherical* 2*-expansion then the* $m \times m$ *operator matrix*

$$\big([(T_j^{\mathfrak{s}})^*, T_i^{\mathfrak{s}}]\big)_{1 \le i,j \le m} \ge 0 \quad on \quad \{(T_1^{\mathfrak{s}}h, \ldots, T_m^{\mathfrak{s}}h) : h \in \mathcal{H}\}, \tag{5.1}$$

where $T^{\mathfrak{s}}$ *denotes the spherical Cauchy dual to* T*, and the symbol* $[A, B]$ *stands for the commutator* $AB - BA$ *of bounded linear operators* A *and* B*.*

In particular, the spherical Cauchy dual of spherical 2*-expansion is a spherical* 2*-contraction.*

Proof. Let Q_s be the spherical generating 1-tuple associated with T and let P_s denote the spherical generating 1-tuple associated with the spherical Cauchy dual $T^{\mathfrak{s}}$ of T. It is easy to see that the conclusion in (5.1) is equivalent to $P_s^2(I) \ge P_s(I)^2$. Because of the spherical 2-expansivity of T,

$$\|x\| \langle Q_s^2(I)x,\, x\rangle^{\frac{1}{2}} \;\le\; \frac{\|x\|^2 + \langle Q_s^2(I)x,\, x\rangle}{2} \;\le\; \langle Q_s(I)x,\, x\rangle \ (x \in \mathcal{H}).$$

Thus we obtain

$$\|x\|^2 \langle Q_s^2(I)x,\, x\rangle \;\le\; \langle Q_s(I)x,\, x\rangle^2 \tag{5.2}$$

for every $x \in \mathcal{H}$. Let $x := Q_s(I)^{-1}y$ for a nonzero $y \in \mathcal{H}$. Then observe that

$$\langle Q_s^2(I)x,\, x\rangle = \sum_{i,j=1}^{m} \|T_j T_i^{\mathfrak{s}} y\|^2, \tag{5.3}$$

which is positive since T and (hence) $T^{\mathfrak{s}}$ are jointly left-invertible. Further, by utilizing the identity $\sum_{j=1}^m T_j^* T_j^{\mathfrak{s}} = I$,

$$\begin{aligned}\langle Q_s(I)x,\, x\rangle &= \sum_{i=1}^m \|T_i^{\mathfrak{s}} y\|^2 = \sum_{i=1}^m \langle \sum_{j=1}^m T_j^* T_j^{\mathfrak{s}} T_i^{\mathfrak{s}} y,\, T_i^{\mathfrak{s}} y\rangle \\ &= \sum_{i=1}^m \sum_{j=1}^m \langle T_j^{\mathfrak{s}} T_i^{\mathfrak{s}} y,\, T_j T_i^{\mathfrak{s}} y\rangle \leq \sum_{i=1}^m \sum_{j=1}^m \|T_j^{\mathfrak{s}} T_i^{\mathfrak{s}} y\| \|T_j T_i^{\mathfrak{s}} y\| \\ &\leq \left(\sum_{i,j=1}^m \|T_j^{\mathfrak{s}} T_i^{\mathfrak{s}} y\|^2\right)^{\frac{1}{2}} \left(\sum_{i,j=1}^m \|T_j T_i^{\mathfrak{s}} y\|^2\right)^{\frac{1}{2}},\end{aligned}$$

by an application of the Cauchy–Schwarz inequality. Combining the last estimate with (5.2) and (5.3), we obtain

$$\|x\|^2 \leq \sum_{i,j=1}^m \|T_j^{\mathfrak{s}} T_i^{\mathfrak{s}} y\|^2 = \sum_{i,j=1}^m \|T_j^{\mathfrak{s}} T_i x\|^2 \text{ for every } x \in \mathcal{H}.$$

Equivalently, $Q_s \circ P_s(I) \geq I$. Since

$$P_s^2(I) = P_s(I) Q_s \circ P_s(I) P_s(I)$$

in view of $P_s(I) = Q_s(I)^{-1}$, the desired conclusion is immediate. □

We list here some problems which arise naturally out of our investigations. Notice that the defect operator $I - TT^*$ is always compact for any finitely multi-cyclic 2-expansion [12], and hence one would like to know whether $I - T_1 T_1^* - \cdots - T_m T_m^*$ is compact for any finitely multi-cyclic spherical 2-expansion for which $\sigma(T)$ is the closed unit ball. This is not true in case in case $\sigma(T)$ is a proper subset of the closed unit ball.

It may be concluded from Lemma 3.3 that the dual Drury–Arveson m-shift $M_{z,m}^{\mathfrak{s}}$ is a spherical complete hypercontraction if and only if the one-variable weighted shift with weight-sequence $\left\{\frac{k+1}{k+m}\right\}$ is a subnormal contraction. The latter one is true in view of [20, Theorem 2.7], and hence $M_{z,m}^{\mathfrak{s}}$ is subnormal. One may now imitate the proof of Theorem 2.3 to present an alternative proof of the first half of [4, Theorem 5.7]. Similarly, one may obtain a version of Theorem 2.2 for the Drury–Arveson m-shift. Our next result generalizes these facts substantially.

For compactly supported measure μ, let supp μ denote the support of μ.

Theorem 5.3. *Let σ denote the normalized surface area measure on the unit sphere $\partial\mathbb{B}$. For a probability measure ν on $[0,1]$ with the point 1 in supp ν, let $d\mu := d\nu d\sigma$. Let $N_z = (N_{z_1}, \ldots, N_{z_m})$ denote the operator tuple of multiplication by the co-ordinate functions $z_1, \ldots, z_m$ in $L^2(\mu)$. Let $P^2(\mu)$ denote the closure of the analytic polynomials in $L^2(\mu)$. Let $T_\nu = (T_{\nu 1}, \ldots, T_{\nu m})$ be* *the m-variable weighted shift with weight-sequence*

$$w_n^{(i)} := \sqrt{\frac{n_i+1}{|n|+m}}\sqrt{\frac{\int_{[0,1]} r^{2k} d\nu(r)}{\int_{[0,1]} r^{2(k+1)} d\nu(r)}} \quad (n \in \mathbb{N}^m, 1 \leq i \leq m)$$

with respect to the orthonormal basis $\left\{\frac{z^n}{\sqrt{\int_{\mathbb{B}} |z^n|^2 d\mu(z)}}\right\}_{n\in\mathbb{N}^m}$ *of* $P^2(\mu)$*. Then the diagram*

$$\begin{array}{ccccc} C^*(N_z) & & \xrightarrow{\quad\psi\quad} & & C(supp\ \mu) \\ \theta\downarrow & & & & \downarrow r \\ C^*(T_\nu) & \xrightarrow[q]{} & C^*(T_\nu)/\mathscr{C}_{T_\nu} & \xrightarrow[\phi]{} & C(\partial\mathbb{B}) \end{array}$$

commutes, where:

(1) $\psi : C^*(N_z) \to C(supp\ \mu)$ *is the isometric* $*$*-isomorphism such that*

$$\psi(N_{z_i}) = z_i \ (1 \leq i \leq m).$$

(2) *The restriction mapping* $r : C(supp\ \mu) \to C(\partial\mathbb{B})$ *is a well-defined surjection.*

(3) $\phi : C^*(T_\nu)/\mathscr{C}_{T_\nu} \to C(\partial\mathbb{B})$ *is the isometric* $*$*-isomorphism with*

$$\phi(T_{\nu i} + \mathscr{C}_{T_\nu}) = z_i \ (1 \leq i \leq m).$$

(4) $q : C^*(T_\nu) \to C^*(T_\nu)/\mathscr{C}_{T_\nu}$ *is the quotient map.*

(5) $\theta : C^*(N_z) \to C^*(T_\nu)$ *given by*

$$\theta(f(N_z)) = P_{\mathcal{H}} f(N_z)|_{P^2(\mu)}, \quad f \in C(\text{supp } \mu),$$

is a completely positive linear mapping.

In particular, $C^*(T_\nu)$ *is the Toeplitz* C^**-algebra, that is, the* C^**-algebra generated by all Toeplitz operators* $f \rightsquigarrow P_{P^2(\mu)}\phi f$ *on* $P^2(\mu)$ *with* ϕ *belonging to* $C(supp\ \mu)$*.*

Proof. To see the first half of the theorem, in view of the proofs of Theorems 2.2 and 2.3, it suffices to check the following assertions:

(a) The spherical Cauchy dual $T_\nu^{\mathfrak{s}}$ is subnormal.
(b) $\sigma(N_z) = \text{supp}(\mu)$ and $\sigma_{ap}(T_\nu) = \partial\mathbb{B}$.
(c) The commutator ideal $\mathscr{C}_{T_\nu}$ of $C^*(T_\nu)$ coincides with $\mathcal{C}(P^2(\mu))$.

(a) Let M_z denote the operator tuple of multiplication by the co-ordinate functions in $P^2(\mu)$. Observe that N_z is a normal extension of M_z. It has been noted in [11, p. 1458] that M_z admits the weight-sequence

$$v_n^{(i)} := \sqrt{\frac{n_i+1}{|n|+m}}\sqrt{\frac{\int_{[0,1]} r^{2(k+1)} d\nu(r)}{\int_{[0,1]} r^{2k} d\nu(r)}} \quad (n \in \mathbb{N}^m, 1 \leq i \leq m).$$

It is now easy to see that the spherical Cauchy dual $T_\nu^{\mathfrak{s}}$ is unitarily equivalent to M_z.

(b) Clearly, $\sigma(N_z) = \text{supp}(\mu)$. In view of $T_\nu^{\mathfrak{s}} = M_z$ as obtained in (a), it may be concluded from [11, Proposition 16] that $\sigma_{ap}(T_\nu^{\mathfrak{s}}) = \partial\mathbb{B}$. Also, by Lemma 3.5(3), $\sigma_{ap}(T_\nu) = \sigma_{ap}(T_\nu^{\mathfrak{s}})$. Thus we have $\sigma_{ap}(T) = \partial\mathbb{B}$.

(c) It has been noted in [11] that M_{z_i} is essentially normal for $i = 1, \ldots, m$. One may now argue as the proof of Theorem 2.3 to conclude that the the commutator ideal of $C^*(M_z)$ coincides with $\mathcal{C}(P^2(\mu))$. Since $C^*(T_\nu) = C^*(T_\nu^{\mathfrak{s}}) = C^*(M_z)$ in view of Lemma 2.1, the desired conclusion in (c) is immediate.

To see the remaining part, recall that $C^*(T_\nu) = C^*(M_z)$, and appeal to [11, Corollary 18]. □

The method of the proof of the last theorem may tempt one to address the following problem: Characterize m-variable weighted shifts (in particular, spherical m-isometric m-variable weighted shifts) which admit subnormal spherical Cauchy dual tuple.

We believe that the structure theory of spherical Cauchy dual tuples of spherical complete hyperexpansions (resp. spherical m-isometries) will have far reaching consequences for the spectral and function theory of spherical complete hyperexpansions (resp. spherical m-isometries).

Acknowledgements. I convey my sincere thanks to Dr. V. M. Sholapurkar, with whom I had a few fruitful conversations concerning the subject of the paper.

References

[1] Agler, Jim. An abstract approach to model theory. Surveys of some recent results in operator theory, Vol. II, 1–23, Pitman Res. Notes Math. Ser, 192. *Longman Sci. Tech., Harlow*, 1988. MR0976842 (90a:47016), Zbl 0788.47005.

[2] Arveson, William B. Subalgebras of C*-algebras. *Acta Math.* **123** (1969), 141–224. MR0253059 (40 #6274), Zbl 0194.15701.

[3] Arveson, William. Subalgebras of C*-algebras. II. *Acta Math.* **128** (1972) no. 3–4, 271–308. MR0394232 (52 #15035), Zbl 0245.46098.

[4] Arveson, William. Subalgebras of C^*-algebras. III. Multivariable operator theory. *Acta Math.* **181** (1998), no. 2, 159–228. MR1668582 (2000e:47013), Zbl 0952.46035, arXiv:funct-an/9705007, doi: 10.1007/BF02392585.

[5] Athavale, Ameer. On joint hyponormality of operators. *Proc. Amer. Math. Soc.* **103** (1988), no. 2, 417–423. MR0943059 (89f:47033), Zbl 0657.47028, doi: 10.1090/S0002-9939-1988-0943059-X.

[6] Athavale, Ameer. On the intertwining of joint isometries. *J. Operator Theory* **23** (1990), no. 2, 339–350. MR1066811 (91i:47029), Zbl 0738.47005.

[7] Athavale, Ameer. Model theory on the unit ball in $\mathbb{C}^n$. *J. Operator Theory* **27** (1992), no. 2, 347–358. MR1249651 (94i:47011), Zbl 0804.47005.

[8] Athavale, Ameer. On completely hyperexpansive operators. *Proc. Amer. Math. Soc.* **124** (1996), no. 12, 3745–3752. MR1353373 (97b:47017), Zbl 0863.47017, doi: 10.1090/S0002-9939-96-03609-X.

[9] Berg, Christian; Christensen, Jens Peter Reus; Ressel, Paul. Harmonic analysis on semigroups. Theory of positive definite and related functions. Graduate Texts in Mathematics, 100. *Springer-Verlag, New York*, 1984. x+289 pp. ISBN: 0-387-90925-7. MR0747302 (86b:43001), Zbl 0619.43001, doi: 10.1002/bimj.4710290410.

[10] Bunce, John. The joint spectrum of commuting nonnormal operators. *Proc. Amer. Math. Soc.* **29** (1971), 499–505. MR0283602 (44 #832), Zbl 0215.20903, doi: 10.1090/S0002-9939-1971-0283602-7.

[11] Chan K.; Čučković Željko. C*-algebras generated by a subnormal operator. *Tran. Amer. Math. Soc.* **351** (1999), no. 4, 1445–1460. MR1624093 (99f:46079), Zbl 0910.47016, doi: 10.1090/S0002-9947-99-02389-2.

[12] Chavan, Sameer. On operators Cauchy dual to 2-hyperexpansive operators. *Proc. Edinb. Math. Soc.* (2) **50** (2007), no. 3, 637–652. MR2360521 (2008k:47047), Zbl 1155.47024, doi: 10.1017/S0013091505001124.

[13] Chavan, Sameer. On operators Cauchy dual to 2-hyperexpansive operators: the unbounded case. *Studia Math.* **203** (2011), no. 2, 129–162. MR2784021 (2012e:47065), Zbl 1241.47019, doi: 10.4064/sm203-2-2.

[14] Chavan, Sameer; Curto, Raúl E. Operators Cauchy dual to 2-hyperexpansive operators: the multivariable case. *Integral Equations Operator Theory* **73** (2012), no. 4, 481–516. MR2956811, Zbl 06149102, doi: 10.1007/s00020-012-1986-4.

[15] Coburn, L. A. Singular integral operators and Toeplitz operators on odd spheres. *Indiana Univ. Math. J.* **23** (1973/74), 433–439. MR0322595 (48 #957), Zbl 0271.46052.

[16] Conway, John B. The theory of subnormal operators. Mathematical Surveys and Monographs, 36. *American Mathematical Society, Providence, RI*, 1991. xvi+436 pp. ISBN: 0-8218-1536-9. MR1112128 (92h:47026), Zbl 0743.47012.

[17] Curto, Raúl E. Applications of several complex variables to multiparameter spectral theory. Surveys of some recent results in operator theory, Vol. II, 25–90, Pitman Res. Notes Math. Ser., 192. *Longman Sci. Tech., Harlow*, 1988. MR0976843 (90d:47007), Zbl 0827.47005.

[18] Curto, Raúl E. Joint hyponormality: a bridge between hyponormality and subnormality. Operator theory: operator algebra and applications, Part 2, 69–91, Proc. Symposia Pure Math., 51. *Amer. Math. Soc., Providence, RI*, 1990. MR1077422 (91k:47049), Zbl 0713.47019.

[19] Curto, Raúl E.; Muhly, Paul S. C*-algebras of multiplication operators on Bergman spaces. *J. Funct. Anal.* **64** (1985), no. 3, 315–329. MR0813203 (86m:47044), Zbl 0583.46049, doi: 10.1016/0022-1236(85)90062-X.

[20] Curto, Raúl E.; Poon, Yiu; Yoon, Jasang. Subnormality of Bergman-like weighted shifts. *J. Math. Anal. Appl.* **308** (2005), no. 1, 334–342. MR2142422 (2005m:47059), Zbl 1072.47027, doi: 10.1016/j.jmaa.2005.01.028.

[21] Dash, A. T. Joint essential spectra. *Pacific J. Math.* **64** (1976), no. 1, 119–128. MR0425641 (54 #13595), Zbl 0329.47001.

[22] Drury, S. W. A generalization of von Neumann's inequality to the complex ball. *Proc. Amer. Math. Soc.* **68** (1978), no. 3, 300–304. MR0480362 (80c:47010), Zbl 0377.47016, doi: 10.1090/S0002-9939-1978-0480362-8.

[23] Gleason, Jim; Richter, Stefan. m-isometric commuting tuples of operators on a Hilbert space. *Integral Equations Operator Theory* **56** (2006), no. 2, 181–196. MR2264515 (2007g:47009), Zbl 1112.47003, doi: 10.1007/s00020-006-1424-6.

[24] Guo, Kunyu; Hu, Junyun; Xu, Xianmin. Toeplitz algebras, subnormal tuples and rigidity on reproducing $\mathbf{C}[z_1, \dots, z_d]$-modules. *J. Funct. Anal.* **210** (2004), no. 1, 214–247. MR2052120 (2005a:47007), Zbl 1066.47075, doi: 10.1016/j.jfa.2003.06.003.

[25] Jabłoński Zenon J.; Stochel, Jan. Unbounded 2-hyperexpansive operators. *Proc. Edinb. Math. Soc.* (2) **44** (2001), no. 3, 613–629. MR1875772 (2002k:47080), Zbl 0993.47003, doi: 10.1017/S001309159900139X.

[26] Jewell, Nicholas P.; Lubin, A. R. Commuting weighted shifts and analytic function theory in several variables. *J. Operator Theory* **1** (1979), no. 2, 207–223. MR0532875 (80c:47030), Zbl 0431.47016.

[27] Olin, Robert F.; Thomson, James E. Algebras generated by a subnormal operator. *Trans. Amer. Math. Soc.* **271** (1982), no. 1, 299–311. MR0648094 (84i:47063), Zbl 0485.47015, doi: 10.1090/S0002-9947-1982-0648094-6.

[28] Sholapurkar, V. M.; Athavale, Ameer. Completely and alternatingly hyperexpansive operators. *J. Operator Theory* **43** (2000), no. 1, 43–68. MR1740894 (2001a:47023), Zbl 0992.47012.

[29] Shimorin, Serguei. Wold-type decompositions and wandering subspaces for operators close to isometries. *J. Reine Angew. Math.* **531** (2001), 147–189. MR1810120 (2002c:47018), Zbl 0974.47014, doi: 10.1515/crll.2001.013.

[30] Shimorin, Serguei. Complete Nevanlinna–Pick property of Dirichlet-type spaces. *J. Funct. Anal.* **191** (2002), no. 2, 276–296. MR1911187 (2003c:47032), Zbl 1038.46022, doi: 10.1006/jfan.2001.3871.

Indian Institute of Technology Kanpur, Kanpur- 208016, India
chavan@iitk.ac.in

This paper is available via http://nyjm.albany.edu/j/2013/19-26.html.