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Semistable symmetric spectra in $\mathbb{A}^1$-homotopy theory

Stephan Hähne and Jens Hornbostel

Abstract. We study semistable symmetric spectra based on quite general monoidal model categories, including motivic examples. In particular, we establish a generalization of Schwede's list of equivalent characterizations of semistability in the case of motivic symmetric spectra. We also show that the motivic Eilenberg–MacLane spectrum and the algebraic cobordism spectrum are semistable. Finally, we show that semistability is preserved under localization if some reasonable conditions — which often hold in practice — are satisfied.

Contents

1. Introduction 2
2. Semistability 4
2.1. The first generalization 5
2.2. The sign $(-1)_T$ and the action of the symmetric group 13
2.2.1. The sign of the simplicial circle 14
2.2.2. The sign $(-1)_{\mathbb{P}^1}$ of the projective line 15
2.3. Definition of the $\mathcal{M}$-action on stable homotopy groups 20
2.4. Some $\mathcal{M}$-isomorphisms between stable homotopy groups 23
2.5. Generalities concerning the $\mathcal{M}$-action 30
2.6. Criteria for semistability: the generalized theorem 31
3. Examples of semistable motivic symmetric spectra 35
3.1. The motivic Eilenberg–Mac Lane spectrum 35
3.2. The algebraic cobordism spectrum 37
4. The multiplicative structure on stable homotopy groups of symmetric ring spectra and its localizations 39
4.1. The multiplication on stable homotopy groups 41
4.2. Localization of ring spectra 47
References 54

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1. Introduction

A map between CW-spectra (or Bousfield–Friedlander-spectra) is a stable weak equivalence if and only if it induces an isomorphism on stable homotopy groups. This is not true if we replace spectra by symmetric spectra in general. However, there is a large class of symmetric spectra for which the stable homotopy groups (sometimes called the "naive stable homotopy groups" as they ignore the action of the symmetric groups) do coincide with the stable weak equivalences. This leads to the notion of *semistable* symmetric spectra, and these have been studied notably by Schwede [Sch07], [Sch08], [Sch12]. There are many equivalent ways to recognize them, and there are indeed many examples of symmetric spectra which are semistable (e.g., suspension spectra, Eilenberg–MacLane spectra, K-theory and various cobordism spectra). Any symmetric spectrum is weakly equivalent to a semistable one, and semistable spectra are very suitable both under theoretical and computational aspects.

The goal of this article is to study semistability for symmetric spectra based on other model categories than simplicial sets or topological spaces. Our main interest here are symmetric spectra based on motivic spaces as studied in [Hov2], [Ja2], which model the motivic stable homotopy category [Vo]. However, we state most results in greater generality so that they may be applied to other settings as well.

The results of this article may be divided in three families. First, we establish a long list of equivalent characterizations of semistability. Second, using one of these characterizations, we prove that important examples of motivic spectra, namely Eilenberg–Mac Lane spectra and algebraic cobordism, are indeed semistable motivic symmetric ring spectra. Third, we show that semistable ring spectra are particularly well-behaved under localization. Most of our results are generalizations of known results for symmetric spectra bases on simplicial sets, but at least some proofs considerably differ.

One of our motivations to study semistability for motivic symmetric ring spectra was our expectation that a motivic version of a theorem of Snaith [GS], [SØ] should lead to a motivic symmetric commutative ring spectrum representing algebraic K-theory, which then would fit in the framework of [Hor2]. Indeed, while the first author was writing [H], Röndigs, Spitzweck and Østvær were able to deduce this result carrying out a small part of the general theory established here, see Remark 2.48.

We now briefly recall the notion of semistability. For any symmetric spectrum X, the actions of Σ_n on X_n induce an action of the *injection monoid* $\mathcal{M}$ (that is the monoid of injective self-maps on $\mathbb{N}$) on $\pi_* X$. We say that X is *semistable* if this action is trivial. In general, the $\mathcal{M}$-action encodes additional information of the symmetric spectrum. See [Sch08, Example 3.4] for an example of symmetric spectra with isomorphic stable homotopy groups but having different $\mathcal{M}$-action.

The following theorem of Schwede provides a list of equivalent ways of describing semistable symmetric spectra based on simplicial sets. This is essentially [Sch07, Theorem I.4.44], see also [Sch08, Theorem 4.1] and [Sch12].

Theorem 1.1. *For any symmetric spectrum X in simplicial sets, the following conditions* (i)–(v) *are equivalent. If X is levelwise fibrant, then these are also equivalent to conditions* (vi)–(viii)*.*

(i) *There is a $\hat{\pi}_*$-isomorphism from X to an Ω-spectrum, that is an isomorphism of naive stable homotopy groups.*
(ii) *The tautological map $c : \hat{\pi}_k X \longrightarrow \pi_k X$ from naive to "true" homotopy groups is an isomorphism for all $k \in \mathbb{Z}$.*
(iii) *The action of $\mathcal{M}$ is trivial on all homotopy groups of X.*
(iv) *The cycle operator d acts trivially on all homotopy groups of X.*
(v) *The morphism $\lambda_X : S^1 \wedge X \longrightarrow \operatorname{sh} X$ is a $\hat{\pi}_*$-isomorphism.*
(vi) *The morphism $\tilde{\lambda}_X : X \longrightarrow \Omega(\operatorname{sh} X)$ is a $\hat{\pi}_*$-isomorphism.*
(vii) *The morphism $\lambda_X^\infty : X \longrightarrow R^\infty X$ is a $\hat{\pi}_*$-isomorphism.*
(viii) *The symmetric spectrum $R^\infty X$ is an Ω-spectrum.*

In order to generalize this theorem to other model categories $\mathcal{D}$, it seems natural to generalize the $\mathcal{M}$-action to appropriate stable homotopy groups in $\mathcal{D}$. However, in our first partial generalization Theorem 2.10 homotopy groups do not appear. They only do appear later in the full generalization, namely in Theorem 2.43. To state and prove the latter, we need to axiomatize the properties of the *sign* $(-1)_{S^1}$ on S^1 (see Definition 2.16). That is, we require that our circle object T has an automorphism $(-1)_T$ in $\mathrm{Ho}(\mathcal{D})$ satisfying the conditions of that definition. For our applications, it is thus crucial that the pointed motivic space $T = \mathbb{P}^1$ has a sign (see Proposition 2.24). We are then able to prove the full generalization of Schwede's theorem. The precise statement of this Main Theorem 2.43 looks rather technical at first glance and can be appreciated only after having read Section 2, so we don't reproduce it here.

In Section 3, we prove the following using the results of Section 2 (see Corollaries 3.5 and 3.9).

Theorem 1.2. *The symmetric motivic Eilenberg–Mac Lane spectrum $\mathbb{H}$ is semistable. The symmetric algebraic cobordism spectrum $\mathbb{MGL}$ of Voevodsky is semistable.*

The key to both proofs here, relating the rather abstract considerations of Section 2 to the algebraic geometry of these spectra, is that the Σ_n-actions that occur extend to GL_n-actions.

Section 4 generalizes [Sch07, Corollary I.4.69] about the localization of semistable symmetric ring spectra with respect to central elements. The following is a special case of our Theorem 4.11:

Theorem 1.3. *Let R be a level fibrant semistable motivic symmetric ring spectrum and $x : T^l \to R_m$ a central map. Then we can define a motivic*

symmetric ring spectrum $R[1/x]$ which is semistable, and the ring homomorphism $\pi^{\mathrm{mot}}_{*,*}(R) \xrightarrow{j_*} \pi^{\mathrm{mot}}_{*,*}(R[1/x])$ is a localization with respect to x.

This article is based on the diploma thesis of the first author [H] written under the direction of the second author. We thank Stefan Schwede for providing us with updates [Sch12] of his book project [Sch07] on symmetric spectra. As the structure and in particular the numbering are still subject to change, we only provide precise references to the version [Sch07]. We provide details rather than refering to [Sch12] when relying on arguments not contained in the version [Sch07] or in [Sch08].

We assume that the reader is familiar with model categories in general [Hi], [Hov1]. For symmetric spectra, we refer to [HSS], [Hov2] and [Sch07], [Sch12]. References for motivic spaces (that is simplicial presheaves on Sm/S for a noetherian base scheme S of finite Krull dimension) and motivic symmetric spectra include [MV], [Ja2] and [DLØRV]. It will be useful for the reader to have a copy of [Hov2] and [Sch07] at hand.

2. Semistability

In this section, we will generalize Theorem 1.1 in two ways. The first generalization (Theorem 2.10) applies to symmetric spectra based on a very general monoidal model category, but covers only part of the list of equivalent properties of Theorem 1.1. The second generalization (Theorem 2.43) applies to a slightly more restricted class of examples (in particular the motivic ones we are mainly interested in) and provides the "full" analog of Theorem 1.1. We will always assume that $\mathcal{D}$ is a monoidal model category, and that T is a cofibrant object of $\mathcal{D}$. If moreover $\mathcal{D}$ is cellular and left proper, then by [Hov2] (see also [Ja2]), we have both a level and a stable projective model structure on $\mathrm{Sp}(\mathcal{D},T)$, and similarly on $\mathrm{Sp}^\Sigma(\mathcal{D},T)$. We refer to [Hov2, Definition 4.1] for the definition of "almost finitely generated".

As usual, for a spectrum X we define sX by $(sX)_n = X_{n+1}$, $\Omega = \mathrm{Hom}(T,-)$, $\Theta := \Omega \circ s$ and $\Theta^\infty := \mathrm{colim}\,\Theta^k$. We write $\tilde{\sigma}^X_n$ for the adjoints of the structure maps σ^X_n of X, and J for a fibrant replacement functor in $\mathrm{Sp}(\mathcal{D},T)$. By definition, an Ω-spectrum is level-wise fibrant.

For some almost finitely generalized model categories stable weak equivalences may be characterized as follows [Hov2, Section 4]:

Theorem 2.1. *Assume that $\mathcal{D}$ is almost finitely generated, and that sequential colimits commute with finite products and with Ω. Then for any $A \in \mathrm{Sp}(\mathcal{D},T)$, the map $A \to \Theta^\infty JA$ is a stable equivalence into an Ω-spectrum. Moreover, for an f in $\mathrm{Sp}(\mathcal{D},T)$ the following are equivalent:*

- *f is a stable equivalence.*
- *For any levelwise fibrant replacement f' of f the map $\Theta^\infty f'$ is a level equivalence.*
- *There is a levelwise fibrant replacement f' of f such that the map $\Theta^\infty f'$ is a level equivalence.*

Proof. This is a special case of [Hov2, Theorem 4.12] with $U = \Omega$. □

2.1. The first generalization. We refer to [HSS] and [Sch07] for standard definitions and properties of symmetric spectra. We consider a closed symmetric monoidal model category $(\mathcal{D}, \wedge, S^0)$ with internal Hom-objects Hom. As above, let T be a cofibrant object in $\mathcal{D}$ and $\Omega = \mathrm{Hom}(T, -)$. We will consider the category of symmetric T-spectra $\mathrm{Sp}^\Sigma(\mathcal{D}, T)$ with the projective stable model structure of [Hov2]. As usual, we define an endofunctor sh on $\mathrm{Sp}^\Sigma(\mathcal{D}, T)$ by $\mathrm{sh}\, X_n = X_{1+n}$, where (following Schwede) the notation $1 + n$ emphasizes which action of Σ_n on X_{n+1} we consider. We further set $R := \Omega \circ \mathrm{sh}$ and $R^\infty := \mathrm{colim}\, R^k$. Recall also that there is a natural map $\lambda_X : X \wedge T \to \mathrm{sh}\, X$, which has an adjoint $\tilde{\lambda}_X : X \to RX = \Omega \circ \mathrm{sh}\, X$.

Lemma-Definition 2.2.

(i) *Let X be any object of $\mathcal{D}$. We inductively define*

$$\mathrm{ev}^n_X : \Omega^n X \wedge T^n \to X$$

by $\mathrm{ev}^1_X = \mathrm{ev}$ and $\mathrm{ev}^n_X = \mathrm{ev} \cdot (\mathrm{ev}^{n-1}_{\Omega X} \wedge T)$. Then the adjoint

$$\delta_{n,X} : \Omega^n X \to \mathrm{Hom}(T^n, X)$$

of ev^n_X is a natural isomorphism. Using this, we define for any $\tau \in \Sigma_n$ a natural transformation $\Omega^\tau : \Omega^n \to \Omega^n$:

$$\begin{array}{ccc} \Omega^n & \xrightarrow[\delta_{n,X}]{\cong} & \mathrm{Hom}(T^n, -) \\ \downarrow{\scriptstyle \Omega^\tau} & & \downarrow{\scriptstyle \mathrm{Hom}(\tau^{-1}, -)} \\ \Omega^n & \xrightarrow[\delta_{n,X}]{\cong} & \mathrm{Hom}(T^n, -) \end{array}$$

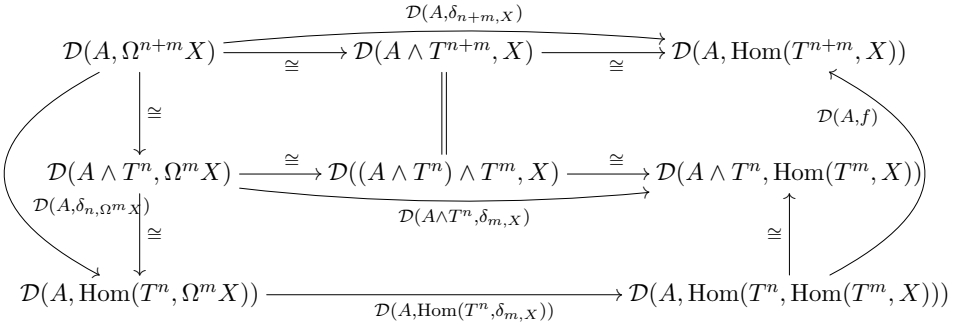
(ii) *If $(\tau_1, \tau_2) \in \Sigma_n \times \Sigma_m$, $(n, m \in \mathbb{N}_0)$, then $\Omega^{\tau_1+\tau_2}_X = \Omega^{\tau_1}_{\Omega^m X} \cdot \Omega^n \Omega^{\tau_2}_X$*

Proof. (i) Obvious.

(ii) Setting

$$f := \delta_{n+m,X} \cdot \delta^{-1}_{n,\Omega^m X} \cdot \mathrm{Hom}(T^n, \delta^{-1}_{m,X}) : \mathrm{Hom}(T^n, \mathrm{Hom}(T^m, X)) \to \mathrm{Hom}(T^{n+m}, X)$$

we may identify $\mathcal{D}(A, f)$ using the following commutative diagram:



Hence f is compatible with $\tau_1^{-1} : T^n \to T^n$ and $\tau_2^{-1} : T^m \to T^m$. By naturality $\delta_{n+m,X} \cdot \delta^{-1}_{n,\Omega^m X}$ is then compatible with τ_1^{-1}, and similarly (because $f = \delta_{n+m,X} \cdot \Omega^n \delta^{-1}_{m,X} \cdot \delta^{-1}_{n,\mathrm{Hom}(T^m,X)}$) the map $\delta_{n+m,X} \cdot \Omega^n \delta^{-1}_{m,X}$ is compatibe with τ_2^{-1}. The first compatibility imples $\Omega_X^{\tau_1+m} = \Omega^{\tau_1}_{\Omega^m X}$ and the second $\Omega_X^{\tau_1+m} = \Omega^n \Omega_X^{\tau_2}$, whence the claim. □

Lemma 2.3. *Let X be a symmetric T-spectrum and $\chi_{l,m} \in \Sigma_{l+m}$ permuting the blocks of the first l and the last m elements. Then for the structure maps of $\Omega^l X$, we have the equality $\tilde\sigma_n^{\Omega^l X} = \Omega^{\chi_{l,1}}_{X_{n+1}} \cdot \Omega^l \tilde\sigma_n^X$.*

For $R^\infty X$, we have $\tilde\sigma_n^{R^\infty X} = \mathrm{incl} \cdot \mathrm{colim}\, \tilde\sigma_n^{R^k X}$, with incl *being the map* $\mathrm{colim}(\Omega(R^k X)_{n+1}) \to \Omega(R^\infty X)_{n+1}$.

Proof. For $l = 1$, we have $\tilde\sigma_n^{\Omega X} = \Omega^{\chi_{1,1}}_{X_{n+1}} \cdot \Omega\tilde\sigma_n^X$, as by definition we have

$$\mathrm{ev}^1_{X_{n+1}} \cdot (\sigma_n^{\Omega X} \wedge T) = \sigma_n^X \cdot (\mathrm{ev}^1_{X_n} \wedge T) \cdot (\Omega X_n \wedge t_{T,T})$$

and thus

$$\begin{aligned}
&\mathrm{ev} \cdot [(\delta_{2,X_{n+1}} \cdot \tilde\sigma_n^{\Omega X}) \wedge T^2] \\
&= \mathrm{ev}^1_{X_{n+1}} \cdot (\mathrm{ev}^1_{\Omega X_{n+1}} \wedge T) \cdot (\tilde\sigma_n^{\Omega X} \wedge T^2) \\
&= \mathrm{ev}^1_{X_{n+1}} \cdot (\sigma_n^{\Omega X} \wedge T) \\
&= \sigma_n^X \cdot (\mathrm{ev}^1_{X_n} \wedge T) \cdot (\Omega X_n \wedge t_{T,T}) \\
&= \mathrm{ev}^1_{X_{n+1}} \cdot (\tilde\sigma_n^X \wedge T) \cdot (\mathrm{ev}^1_{X_n} \wedge T) \cdot (\Omega X_n \wedge t_{T,T}) \\
&= \mathrm{ev}^1_{X_{n+1}} \cdot (\mathrm{ev}^1_{\Omega X_{n+1}} \wedge T) \cdot (\Omega\tilde\sigma_n^X \wedge t_{T,T}) \\
&= \mathrm{ev} \cdot (\delta_{2,X_{n+1}} \wedge T^2) \cdot (\Omega\tilde\sigma_n^X \wedge t_{T,T}) \\
&= \mathrm{ev} \cdot (\mathrm{Hom}(T^2, X_{n+1}) \wedge t_{T,T}) \cdot [(\delta_{2,X_{n+1}} \Omega\tilde\sigma_n^X) \wedge T^2] \\
&= \mathrm{ev} \cdot (\mathrm{Hom}(t_{T,T}, X_{n+1}) \wedge T^2) \cdot [(\delta_{2,X_{n+1}} \Omega\tilde\sigma_n^X) \wedge T^2] \\
&= \mathrm{ev} \cdot [(\delta_{2,X_{n+1}} \Omega^{\chi_{1,1}}_{X_{n+1}} \Omega\tilde\sigma_n^X) \wedge T^2].
\end{aligned}$$

Induction over l then yields

$$\begin{aligned}
\tilde\sigma_n^{\Omega^{l-1}\Omega X} &= \Omega^{\chi_{l-1,1}}_{\Omega X_{n+1}} \cdot \Omega^{l-1} \tilde\sigma_n^{\Omega X} \\
&= \Omega^{\chi_{l-1,1}}_{\Omega X_{n+1}} \cdot \Omega^{l-1} (\Omega^{\chi_{1,1}}_{X_{n+1}} \cdot \Omega\tilde\sigma_n^X) \\
&= \Omega^{\chi_{l,1}}_{X_{n+1}} \cdot \Omega^l \tilde\sigma_n^X,
\end{aligned}$$

by Lemma 2.2 and $\chi_{l,1} = (\chi_{l-1,1} + 1) \cdot ((l-1) + \chi_{1,1})$.

The second claim follows as the adjoints of the maps already coincide on $(R^l X)_n \wedge T$, where they are $\sigma_n^{R^l X} = \mathrm{ev} \cdot (\tilde\sigma_n^{R^l X} \wedge T)$. □

In Sections 2.3 and 2.4 below (compare also [Sch07, Example I.4.17]), we will study in detail the action of the injection monoid $\mathcal{M}$ on $\underline{X}(\omega) \cong (\Theta^\infty X)_0$. In this section, we only need to know how the action of the cycle operator d relates to the map $\tilde\lambda$ (generalizing a result of [Sch12]).

Lemma 2.4. *For any symmetric T-spectrum X, the following triangle commutes:*

$$\begin{array}{ccc} (\Theta^\infty X)_0 & \xrightarrow{d} & (\Theta^\infty X)_0 \\ & \searrow^{(\Theta^\infty \tilde{\lambda}_X)_0} & \downarrow \cong \\ & & (\Theta^\infty \Omega \operatorname{sh} X)_0. \end{array}$$

Proof. The isomorphism on the right hand side is induced by

$$\Omega^{1+l} X_{1+l} \xrightarrow{\Omega^{\chi_{1,l}}_{X_{1+l}}} \Omega^{l+1} X_{1+l}.$$

In the diagram

$$\begin{array}{ccccc} \Omega^{1+l} X_{1+l} & \xrightarrow{\Omega^{1+l}\tilde{\sigma}} & \Omega^{1+l+1} X_{1+l+1} & & \\ \downarrow{\scriptstyle \Omega^{\chi_{1,l}}} & & \downarrow{\scriptstyle \Omega^{\chi_{1,l}}\Omega} & \searrow^{\Omega^{\chi_{1,l+1}}} & \\ \Omega^{l+1} X_{1+l} & \xrightarrow{\Omega^{l+1}\tilde{\sigma}} & \Omega^{l+2} X_{1+l+1} & \xrightarrow{\Omega^l \Omega^{\chi_{1,1}}} & \Omega^{l+1+1} X_{1+l+1} \end{array}$$

the lower composition equals $\Omega^l \tilde{\sigma}^{\Omega X}_{1+l}$ by Lemma 2.3. As $\Omega^{\chi_{1,l}}\Omega = \Omega^{\chi_{1,l}+1}$ and $\Omega^l \Omega^{\chi_{1,1}} = \Omega^{l+\chi_{1,1}}$ (Lemma 2.2) and $\chi_{1,l+1} = (l + \chi_{1,1}) \cdot (\chi_{1,l} + 1)$ everything commutes, hence the above maps are compatible with the structure maps. Finally, the diagram of the lemma is induced by the following commutative diagram:

$$\begin{array}{ccccc} \Omega^l X_l & \xrightarrow{\Omega^l \tilde{\sigma}} & \Omega^{l+1} X_{l+1} & \xrightarrow{\Omega^{\chi_{l,1}} \chi_{l,1}} & \Omega^{1+l} X_{1+l} \\ & \searrow^{\Omega^l \tilde{\sigma}} & & & \downarrow{\scriptstyle \Omega^{\chi_{1,l}}} \\ & & \Omega^{l+1} X_{l+1} & \xrightarrow{\Omega^{l+1} \chi_{l,1}} & \Omega^{l+1} X_{1+l}. \end{array}$$

□

Lemma 2.5. *For any symmetric T-spectrum X, there is an isomorphism* $\operatorname{sym}_{X,n} : (\Theta^\infty X)_n \cong (R^\infty X)_n$.

Proof. The isomorphism is induced by a sequence of compatible isomorphisms

$$\begin{array}{ccccccccc} X_n & \xrightarrow{\tilde{\sigma}} & \Omega X_{n+1} & \xrightarrow{\Omega\tilde{\sigma}} & \Omega^2 X_{n+2} & \xrightarrow{\Omega^2\tilde{\sigma}} & \dots & \longrightarrow & \Omega^l X_{n+l} \xrightarrow{\Omega^l\tilde{\sigma}} \\ \downarrow{\scriptstyle 1=\alpha_{0,n}} & & \downarrow{\scriptstyle \Omega\alpha_{1,n}} & & \downarrow{\scriptstyle \Omega^2\alpha_{2,n}} & & & & \downarrow{\scriptstyle \Omega^l\alpha_{l,n}} \\ X_n & \xrightarrow{\tilde{\lambda}} & \Omega X_{1+n} & \xrightarrow{\Omega\tilde{\lambda}} & \Omega^2 X_{2+n} & \xrightarrow{\Omega^2\tilde{\lambda}} & \dots & \longrightarrow & \Omega^l X_{l+n} \xrightarrow{\Omega^l\tilde{\lambda}} \end{array}$$

where $\alpha_{l,n}$ is a permutation which is inductively defined by the following commutative diagram:

$$\begin{array}{ccccc}
\Omega^l X_{n+l} & \xrightarrow{\Omega^l\tilde{\sigma}} & \Omega^{l+1} X_{n+l+1} & & \\
\downarrow{\scriptstyle \Omega^l\alpha_{l,n}} & & \downarrow{\scriptstyle \Omega^{l+1}(\alpha_{l,n}+1)} & \searrow^{\Omega^{l+1}\alpha_{l+1,n}} & \\
\Omega^l X_{l+n} & \xrightarrow[\Omega^l\tilde{\sigma}]{} & \Omega^{l+1} X_{l+n+1} & \xrightarrow[\Omega^{l+1}\chi_{l+n,1}]{} & \Omega^{l+1} X_{1+l+n}.
\end{array}$$

Here we use the Σ_{n+l}-equivariance of $\tilde{\sigma}$ and set $\alpha_{l+1,n} = \chi_{l+n,1} \cdot (\alpha_{l,n} + 1)$. Then by induction, it follows that $\alpha_{l,n} = \chi_{n,l} \cdot (n + \beta_l)$, where $\beta_l \in \Sigma_l$ is the reflection $\beta_l(i) = l + 1 - i$:

$$\begin{aligned}
\chi_{n,0} \cdot (n + \beta_0) &= \mathrm{id} \\
\chi_{l+n,1} \cdot (\alpha_{l,n} + 1) &= \chi_{l+n,1} \cdot ([\chi_{n,l} \cdot (n + \beta_l)] + 1) \\
&= \chi_{n,l+1} \cdot (n + \beta_{l+1}).
\end{aligned}$$

□

Corollary 2.6. *Assume that Ω commutes with sequential colimits. Then for any X in $\mathrm{Sp}^\Sigma(\mathcal{D}, T)$, the following diagram commutes:*

$$\begin{array}{ccccccc}
(\Theta^\infty \mathrm{sh}^n X)_0 & = & (\Theta^\infty X)_n & & \xrightarrow{\quad\mathrm{sym}_{X,n}\quad} & & (R^\infty X)_n \\
\downarrow{\scriptstyle d} & & & & & & \downarrow{\scriptstyle \tilde{\sigma}_n^{R^\infty X}} \\
(\Theta^\infty \mathrm{sh}^n X)_0 & = & (\Theta^\infty X)_n & \cong & \Omega(\Theta^\infty X)_{n+1} & \xrightarrow{\Omega\,\mathrm{sym}_{X,n+1}} & \Omega(R^\infty X)_{n+1}.
\end{array}$$

Proof. Using Lemma 2.3 it suffices to show that the following diagram commutes:

$$\begin{array}{ccccc}
\Omega^l X_{n+l} & \xrightarrow{\Omega^l\tilde{\sigma}} & \Omega^{l+1} X_{n+l+1} & \xrightarrow{\Omega^{\chi_{l,1}}(n+\chi_{l,1})} & \Omega^{1+l} X_{n+1+l} \\
\downarrow{\scriptstyle \Omega^l\alpha_{l,n}} & & \downarrow{\scriptstyle \Omega^l(\alpha_{l,n}+1)} & & \downarrow{\scriptstyle \Omega^{1+l}\alpha_{l,n+1}} \\
\Omega^l X_{l+n} & \xrightarrow{\Omega^l\tilde{\sigma}} & \Omega^{l+1} X_{l+n+1} & \xrightarrow{\Omega^{\chi_{l,1}}} & \Omega^{1+l} X_{l+n+1}.
\end{array}$$

This is the case as we have

$$\begin{aligned}
\alpha_{l,n+1} \cdot (n + \chi_{l,1}) &= \chi_{n+1,l} \cdot (n + 1 + \beta_l) \cdot (n + \chi_{l,1}) \\
&= \chi_{n+1,l} \cdot (n + \chi_{l,1}) \cdot (n + \beta_l + 1) \\
&= [\chi_{n,l} \cdot (n + \beta_l)] + 1 = \alpha_{l,n} + 1.
\end{aligned}$$

□

Lemma 2.7. *Let $X \in \mathrm{Sp}^\Sigma(\mathcal{D}, T)$. Then the maps $\tilde{\lambda}_{\mathrm{sh}\,X}$ and $\mathrm{sh}\,\tilde{\lambda}_X$ are equal in $\mathrm{Sp}^\Sigma(\mathcal{D}, T)$ up to a canonical isomorphism of the targets.*

Proof. We use the isomorphism $\Omega\,\mathrm{sh}(\mathrm{sh}\,X) \xrightarrow{\cong} \mathrm{sh}(\Omega\,\mathrm{sh}\,X)$ which is levelwise given by the Σ_n-equivariant map $\Omega X_{1+1+n} \xrightarrow{\Omega(\chi_{1,1}+n)} \Omega X_{1+1+n}$. This really

is a map in $\mathrm{Sp}(\mathcal{D}, T)$, as

$$\begin{array}{ccccc}
\Omega X_{1+1+n} & \xrightarrow{\Omega\tilde{\sigma}} & \Omega^2 X_{1+1+n+1} & \xrightarrow{\Omega^{\chi_{1,1}}} & \Omega^2 X_{1+1+n+1} \\
\downarrow{\scriptstyle \Omega(\chi_{1,1}+n)} & & \downarrow{\scriptstyle \Omega^2(\chi_{1,1}+n+1)} & & \downarrow{\scriptstyle \Omega^2(\chi_{1,1}+(n+1))} \\
\Omega X_{1+1+n} & \xrightarrow{\Omega\tilde{\sigma}} & \Omega^2 X_{1+1+n+1} & \xrightarrow{\Omega^{\chi_{1,1}}} & \Omega^2 X_{1+1+n+1}
\end{array}$$

commutes by Lemma 2.3. This yields a commutative diagram

$$\begin{array}{ccc}
\operatorname{sh} X & \xrightarrow{\tilde{\lambda}_{\operatorname{sh} X}} & \Omega \operatorname{sh}(\operatorname{sh} X) \\
& {\scriptstyle \operatorname{sh}\tilde{\lambda}_X} \searrow & \downarrow{\scriptstyle \cong} \\
& & \operatorname{sh}(\Omega \operatorname{sh} X)
\end{array}$$

as we have levelwise

$$\begin{array}{ccccc}
X_{1+n} & \xrightarrow{\tilde{\sigma}} & \Omega X_{1+n+1} & \xrightarrow{\Omega(1+\chi_{n,1})} & \Omega X_{1+1+n} \\
\| & & \| & & \downarrow{\scriptstyle \Omega(\chi_{1,1}+n)} \\
X_{1+n} & \xrightarrow{\tilde{\sigma}} & \Omega X_{1+n+1} & \xrightarrow{\Omega\chi_{1+n,1}} & \Omega X_{1+1+n}.
\end{array}$$

□

Lemma 2.8. *Let $X \in \mathrm{Sp}^{\Sigma}(\mathcal{D}, T)$. Then we have a natural isomorphism $(\Theta^\infty RX)_n \cong (\Theta^\infty X)_n$.*

Proof. The isomorphismus is induces by the following chain of compatible isomorphisms:

$$\begin{array}{ccccccccc}
\Omega X_{n+1} & \xrightarrow{\Omega\tilde{\sigma}^X_{n+1}} & \Omega^2 X_{n+2} & \xrightarrow{\Omega^2\tilde{\sigma}^X_{n+2}} & \cdots & \longrightarrow & \Omega^l X_{n+l} & \xrightarrow{\Omega^l\tilde{\sigma}^X_{n+l}} & \Omega^{l+1} X_{n+l+1} \xrightarrow{\Omega^{l+1}\tilde{\sigma}^X_{n+l+1}} \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \Omega^{\chi_{1,1}}} & & & & \downarrow{\scriptstyle \Omega^{\chi_{1,l-1}}} & & \downarrow{\scriptstyle \Omega^{\chi_{1,l}}} \\
\Omega X_{1+n} & \xrightarrow{\tilde{\sigma}^{RX}_n} & \Omega^2 X_{1+n+1} & \xrightarrow{\Omega\tilde{\sigma}^{RX}_{n+1}} & \cdots & \longrightarrow & \Omega^l X_{1+n+l-1} & \xrightarrow{\Omega^{l-1}\tilde{\sigma}^{RX}_{n+l-1}} & \Omega^{l+1} X_{1+n+l} \xrightarrow{\Omega^l\tilde{\sigma}^{RX}_{n+l}}
\end{array}$$

This diagram commutes as the following does and we have

$$\Omega^{l-1+\chi_{1,1}} = \Omega^{l-1}\Omega^{\chi_{1,1}}$$

(see Lemma-Definition 2.2):

$$\begin{array}{ccccc}
\Omega^l X_{n+l} & & \xrightarrow{\Omega^l\tilde{\sigma}^X_{n+l}} & & \Omega^{l+1} X_{n+l+1} \\
\downarrow{\scriptstyle \Omega^{\chi_{1,l-1}}} & & & \swarrow{\scriptstyle \Omega^{\chi_{1,l-1}+1}} & \downarrow{\scriptstyle \Omega^{\chi_{1,l}}} \\
\Omega^l X_{1+n+l-1} & \xrightarrow[\Omega^l\tilde{\sigma}^X_{1+n+l-1}]{} & \Omega^{l+1} X_{1+n+l} & \xrightarrow[\Omega^{l-1+\chi_{1,1}}]{} & \Omega^{l+1} X_{1+n+l}.
\end{array}$$

□

Proposition 2.9. *Let $(\mathcal{D}, \wedge, S^0)$ be as in Theorem 2.1. Then the endofunctor R preserves stable weak equivalences in $\mathrm{Sp}(\mathcal{D}, T)$ between level fibrant objects in $\mathrm{Sp}^\Sigma(\mathcal{D}, T)$.*

Proof. Let $f : X \to Y$ be a map in $\mathrm{Sp}^\Sigma(\mathcal{D}, T)$ between level fibrant objects which is a stable weak equivalence in $\mathrm{Sp}(\mathcal{D}, T)$. Then by assumption $\Theta^\infty f$ is a level equivalence. By Lemma 2.8, we have $(\Theta^\infty Rf)_l \cong (\Theta^\infty f)_l$ for all $l \in \mathbb{N}_0$. Hence $\Theta^\infty Rf$ is a level equivalence and RX, RY are level fibrant objects (Ω preserves fibrant objects), and consequently Rf is a stable weak equivalence again by assumption. □

We now establish a first incomplete generalization of Schwede's Theorem 1.1. Then we provide an example for $\mathcal{D}$ which satisfies the hypotheses.

Theorem 2.10. *Let $(\mathcal{D}, \wedge, S^0)$ be a symmetric monoidal model category and T a cofibrant object. Assume that for $\mathrm{Sp}(\mathcal{D}, T)$ the projective level model structure (see, e.g.,* [Hov2, Theorem 1.13]*) exists. Assume further that:*

(a) *For any map f in $\mathrm{Sp}(\mathcal{D}, T)$ the following are equivalent (compare also Theorem 2.1):*
   - *f is a stable equivalence.*
   - *For any level fibrant replacement f' of f, we have that $\Theta^\infty f'$ is a level equivalence.*
   - *There is a level fibrant replacement f' of f such that $\Theta^\infty f'$ is a level equivalence.*

(b) *Countable compositions of stable equivalences in $\mathrm{Sp}(\mathcal{D}, T)$ between level fibrant objects are stable equivalences in $\mathrm{Sp}(\mathcal{D}, T)$.*

(c) *Ω commutes with sequential colimits in $\mathcal{D}$ (see also Theorem 2.1).*

(d) *Sequential colimits of fibrant objects in $\mathcal{D}$ are fibrant.*

Let X be a symmetric spectrum in $\mathrm{Sp}^\Sigma(\mathcal{D}, T)$ which is levelwise fibrant. Then (i) *to* (iv) *below are equivalent, and* (v) *follows from these.*

(i) *There is a map in $\mathrm{Sp}^\Sigma(\mathcal{D}, T)$ from X to an Ω-spectrum which is a stable equivalence in $\mathrm{Sp}(\mathcal{D}, T)$.*

(ii) *The morphism $\tilde{\lambda}_X : X \longrightarrow RX$ is a stable equivalence in $\mathrm{Sp}(\mathcal{D}, T)$.*

(iii) *For all $n \in \mathbb{N}_0$, the cycle operator*

$$d_{\mathrm{sh}^n X} : (\Theta^\infty \mathrm{sh}^n X)_0 \to (\Theta^\infty \mathrm{sh}^n X)_0$$

is a weak equivalence.

(iv) *The symmetric spectrum $R^\infty X$ is an Ω-spectrum.*

(v) *The morphism $\lambda_X^\infty : X \longrightarrow R^\infty X$ in $\mathrm{Sp}^\Sigma(\mathcal{D}, K)$ is a stable equivalence in $\mathrm{Sp}(\mathcal{D}, T)$.*

Proof. (i)⇒(ii) Let $f : X \to Y$ be a map in $\mathrm{Sp}^\Sigma(\mathcal{D}, T)$ with Y being an Ω-spectrum, and such that f is a stable equivalence in $\mathrm{Sp}(\mathcal{D}, T)$. Then by Proposition 2.9, Rf is also a stable equivalence in $\mathrm{Sp}(\mathcal{D}, T)$. This implies that $\tilde{\lambda}_Y$ ($(\tilde{\lambda}_Y)_l = \chi^Y_{l,1} \cdot \tilde{\sigma}^Y_l$) is a level equivalence, and hence a stable

equivalence in $\mathrm{Sp}(\mathcal{D},T)$, follows by naturality of $\tilde{\lambda}$ that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \tilde{\lambda}_X} & & \downarrow{\scriptstyle \tilde{\lambda}_Y} \\ RX & \xrightarrow{Rf} & RY \end{array}$$

commutes, hence by the 2-out-of-3 axiom $\tilde{\lambda}_X$ is a stable equivalence in $\mathrm{Sp}(\mathcal{D},T)$.

(ii)$\Leftrightarrow$(iii) We have $d_{\mathrm{sh}^n X} \cong (\Theta^\infty \tilde{\lambda}_{\mathrm{sh}^n X})_0$ by Lemma 2.4 and $\tilde{\lambda}_{\mathrm{sh}^n X} \cong \mathrm{sh}^n \tilde{\lambda}_X$ by Lemma 2.7, and furthermore $(\Theta^\infty \mathrm{sh}^n)_0 \cong (\Theta^\infty)_n$, hence $d_{\mathrm{sh}^n X} \cong (\Theta^\infty \tilde{\lambda}_X)_n$. As Ω is a right Quillen functor on $\mathcal{D}$, both X (by assumption) and RX are level fibrant. Using (a) and the above isomorphism, we deduce that $\tilde{\lambda}_X$ is a stable equivalence in $\mathrm{Sp}(\mathcal{D},T)$ if for every $n \in \mathbb{N}_0$ the map $d_{\mathrm{sh}^n X}$ is a weak equivalence in $\mathcal{D}$.

(iii)$\Leftrightarrow$(iv) By Proposition 2.9, the maps $R^s\tilde{\lambda}_X$ are stable equivalences in $\mathrm{Sp}(\mathcal{D},T)$ for all $s \in \mathbb{N}_0$ between level fibrant objects (Ω is right Quillen). By (b), the inclusion λ_X^∞ is then a stable equivalence in $\mathrm{Sp}(\mathcal{D},T)$.

(ii)$\Rightarrow$(i) This follows from (ii)$\Rightarrow$(iv),(v). □

An important class of examples is given by almost finitely generated model categories:

Proposition 2.11. *Let $\mathcal{D}$ be a symmetric monoidal model category which is almost finitely generated, and let T be a cofibrant object of $\mathcal{D}$. Assume that sequential colimits commute with finite products, weak equivalences and Ω, and that the projective level model structure on $(\mathcal{D},T)$ exists. Then the couple $(\mathcal{D},T)$ satisfies the hypotheses of Theorem* 2.10*.*

Proof. (a) holds by 2.1.

(b) We show more generally that stable equivalences in $\mathrm{Sp}(\mathcal{D},T)$ are closed under sequential colimits. Using a standard reduction, it suffices to show that sequential colimits of stable equivalences between stably fibrant objects in $\mathrm{Sp}(\mathcal{D},T)$ are stable equivalences. But the stable model structure is a left Bousfield localization of the projective level model structure, hence stable equivalences between stably fibrant objects are level equivalences [Hi, Theorem 3.2.13, Prop. 3.4.1]. By assumption, those are preserved by sequential colimits (as these are defined level-wise), hence are stable equivalences again.

(c) holds by assumption.

(d) holds by [Hov2, Lemma 4.3]. □

We now consider the category $M.(S)$ of pointed simplicial presheaves on Sm/S for a given noetherian base scheme S (sometimes called the category of *motivic spaces*). Besides the injective [MV] and the projective motivic model structure, there is a third model structure introduced in [PPR1, Section A.3] and denoted by $M.^{cm}(S)$ which is convenient for our purposes.

(Recall also [MV], [Ja2] that there is a model structures on pointed simplicial sheaves $s\,\mathrm{Shv}(S)$. which is – via the sheafification a as a left Quillen functor – Quillen equivalent to the injective model structure on $M.(S)$.)

Corollary 2.12. *The assumptions of Theorem* 2.10 *are satisfied for the model category* $\mathcal{D} = M.^{cm}(S)$ *and for all cofibrant objects* T *for which* $\mathrm{Hom}(T, -)$ *commutes with sequential colimits (in particular for* $T = \mathbb{P}^1$*).*

Proof. The projective level model structure exists by [Hov2, Theorem 1.13]. The model category $M.^{cm}(S)$ is symmetric monoidal by [PPR1, Theorem A.17] and weak equivalences are stable under sequential colimits by [PPR1, Lemma A.18]. The claims about T and $\mathbb{P}^1$ follow from [PPR1, Lemma A.10] and [DRØ, Lemma 2.5]. To show that $M.^{cm}(S)$ is almost finitely generated, one shows that the model category $M.^{cs}(S)$ (see [PPR1, Section A.3]) is almost fintely generated, left proper and cellular. From this, one deduces that the left Bousfield-Hirschhorn localization $M.^{cm}(S)$ exists and is still almost finitely generated. See [NS] or [H, Propositions 2.20, 2.44 and 2.49] for further details. □

The model category sSet_* together with $T = S^1$ also satisfies the assumptions of Theorem 2.10. By Lemma 2.13 below, the map $d_{\mathrm{sh}^n X}$ is a weak equivalence for all $n \geq 0$ if and only if the cycle operator d induces bijections on all stable homotopy groups $\hat{\pi}_k(X), k \in \mathbb{Z}$. Moreover, the stable equivalences in $\mathrm{Sp}(\mathrm{sSet}_*, S^1)$ are precisely the $\hat{\pi}_*$-equivalences. Hence Theorem 2.10 really is a partial generalization of Theorem 1.1.

Lemma 2.13. *Let* $X \in \mathrm{Sp}^\Sigma(\mathrm{sSet}_*, S^1)$*. Then*

$$d_{\mathrm{sh}^n X} : (\Theta^\infty \mathrm{sh}^n X)_0 \to (\Theta^\infty \mathrm{sh}^n X)_0$$

is a weak equivalence if and only if the cycle operator d *induces bijections on all stable homotopy groups* $\hat{\pi}_{k-n}(X), k \in \mathbb{N}_0$*.*

Proof. Lemma 2.4 shows that $d_{\mathrm{sh}^n X}$ is a weak equivalence if and only if $\pi_k(d_{\mathrm{sh}^n X})$ is a bijection for all $k \geq 0$. Using [Sch07, Construction I.4.12] and Section 2.2.1, we see that $\pi_k(d_{\mathrm{sh}^n X})$ is isomorphic to the action of d on $\hat{\pi}_k(\mathrm{sh}^n X)$. We also have isomorphisms of $\mathcal{M}$-modules

$$\hat{\pi}_k(\mathrm{sh}^n X) \cong \hat{\pi}_{k-n}(X)(n)$$

(see Propostion 2.35, Remark 2.32 and Example 2.27). By tameness, d acts as an automorphism on $\hat{\pi}_{k-n}(X)(n)$ if and only if the $\mathcal{M}$-action on $\hat{\pi}_{k-n}(X)(n)$ is trivial. Again by tameness, this in turn holds if and only if the $\mathcal{M}$-action on $\hat{\pi}_{k-n}(X)$ is trivial, because then the filtration is bounded (see Lemma 2.39). This is also equivalent to d acting trivially on $\hat{\pi}_{k-n}(X)$. □

We now state a first version of our definition of semistability (see also Definition 2.31 and Remark 2.32):

Definition 2.14. Assume that the assumptions of Theorem 2.10 are satisfied and the projective level structure on $\mathrm{Sp}^\Sigma(\mathcal{D},T)$ exists, in particular the functorial fibrant approximation J^Σ. Then in this section, a symetric spectrum $X \in \mathrm{Sp}^\Sigma(\mathcal{D},T)$ is called *semistable* if $J^\Sigma X$ satisfies one (and hence all) of the above properties (i)–(iv).

Using this definition, we have (compare also [HSS, Proposition 5.6.5]):

Proposition 2.15. *Assume that the assumptions of Theorem* 2.10 *are satisfied and the projective level structure on* $\mathrm{Sp}^\Sigma(\mathcal{D},T)$ *exists. Let* $f : X \to Y$ *be a morphism in* $\mathrm{Sp}^\Sigma(\mathcal{D},T)$ *between semistable symmetric spectra, and assume that the forgetful functor* $U : \mathrm{Sp}^\Sigma(\mathcal{D},T) \to \mathrm{Sp}(\mathcal{D},T)$ *reflects stable equivalences. Then if* f *is a stable equivalence in* $\mathrm{Sp}^\Sigma(\mathcal{D},T)$*, then so is* $U(f)$ *in* $\mathrm{Sp}(\mathcal{D},T)$*.*

Proof. It is enough to show the claim for $J^\Sigma f$. Namely, $Z \to J^\Sigma Z$ is a natural level equivalence, hence we may replace f by $J^\Sigma f$ and assume that X and Y are level fibrant and the hypotheses of Theorem 2.10 hold for X and Y. In the commutative diagram in $\mathrm{Sp}^\Sigma(\mathcal{D},T)$

$$\begin{array}{ccc} X & \xrightarrow{\tilde\lambda^\infty_X} & R^\infty X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle R^\infty f} \\ Y & \xrightarrow{\tilde\lambda^\infty_Y} & R^\infty Y \end{array}$$

$R^\infty X$ and $R^\infty Y$ are Ω-spectra by assumption, and hence fibrant objects for the stable model structure on $\mathrm{Sp}^\Sigma(\mathcal{D},T)$. Also, $U(\tilde\lambda^\infty_X)$ and $U(\tilde\lambda^\infty_Y)$ are stable equivalences. Using the assumptions on U, we see that $\tilde\lambda^\infty_X$ and $\tilde\lambda^\infty_Y$ are stable equivalences in $\mathrm{Sp}^\Sigma(\mathcal{D},T)$. But f is a stable equivalence in $\mathrm{Sp}^\Sigma(\mathcal{D},T)$, hence by [Hi, Theorem 3.2.13] $R^\infty f$ is a level equivalence. Therefore $U(R^\infty f)$ (and thus $U(f)$) is a stable equivalence. □

The condition that U reflects stable equivalences is satisfied for $\mathcal{D} = M_\cdot^{cm}(S)$, because by [PPR1, Theorem A.5.6 and Theorem A.6.4] the stable equivalences for $\mathrm{Sp}(\mathcal{D},T)$ and $\mathrm{Sp}^\Sigma(\mathcal{D},T)$ in [Ja2] resp. [PPR1] coincide and for the stable equivalences in [Ja2] the condition is satisfied by [Ja2, Prop. 4.8].

Comparing Theorem 2.10 with Theorem 1.1, one notices that several things are missing. We will provide what is missing below (see Theorem 2.43).

2.2. The sign $(-1)_T$ and the action of the symmetric group. We now axiomatize some properties of the topological circle, in a way which is convenient for studying the $\mathcal{M}$-action on generalized stable homotopy groups. The following two subsections then discuss the two key examples, namely $T = S^1$ in pointed simplicial sets and $T = \mathbb{P}^1$ in pointed motivic spaces.

Let $(\mathcal{D}, \wedge, S^0)$ be a symmetric monoidal model category. Fix a cofibrant object T in $\mathcal{D}$ and set $T^n := T^{\wedge n}$.

Definition 2.16. A *sign* of T in $\mathcal{D}$ is an automorphism $(-1)_T$ of T in $\mathrm{Ho}(\mathcal{D})$ of order 2 with the following properties:

(i) For any $\tau \in \Sigma_n$, the permutation of factors $T^n \xrightarrow{\tau} T^n$ coincides with $|\tau|_T \wedge T^{n-1}$ in $\mathrm{Ho}(\mathcal{D})$ (the latter map is defined as T is cofibrant), where we set $|\tau|_T = (-1)_T$ if τ is an odd permutation and $|\tau|_T = 1$ otherwise. We call $|\tau|_T$ the *sign of the permutation* τ.

(ii) $T^2 \xrightarrow{(-1)_T \wedge 1_T} T^2$ coincides with $T^2 \xrightarrow{1_T \wedge (-1)_T} T^2$ in $\mathrm{Ho}(\mathcal{D})$.

2.2.1. The sign of the simplicial circle. Let $\mathcal{D} = \mathrm{sSet}_*$ with the usual smash product.

Definition 2.17. Fix a homeomorphism $h : |S^1| \cong S^1$. This yields a weak equivalence $\nu : S^1 \xrightarrow{\sim} \mathrm{Sing}(|S^1|) \xrightarrow{h} \mathrm{Sing}(\mathbb{R}^+)$ in sSet_*. The map $(-1)_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}, t \mapsto -t$ then induces the automorphism

$$(-1)_{S^1} = \nu^{-1} \cdot \mathrm{Sing}((-1)^+_{\mathbb{R}}) \cdot \nu$$

of S^1 in $\mathrm{Ho}(\mathrm{sSet}_*)$, which we call the *sign of* S^1, and which is obviously of order 2.

In particular $(-1)_{\mathbb{R}^+}$ has degree -1.

Lemma 2.18. *The above automorphism $(-1)_{S^1}$ is a sign of S^1.*

Proof. It is enough to check the properties of Definition 2.16 in $\mathrm{Ho}(\mathrm{Top}_*)$, that is after geometric realization. It also suffices to check the equalities after conjugation with the canonical isomorphism

$$|(S^1)^{\wedge n}| \to |S^1|^{\wedge n} \xrightarrow{h^{\wedge n}} (\mathbb{R}^+)^{\wedge n} \to \mathbb{R}^{n+}$$

(here we used that $-^+$ is strictly monoidal) Conjugation of

$$\tau : (S^1)^{\wedge n} \to (S^1)^{\wedge n}$$

then yields the map $\mathbb{R}^{n+} \xrightarrow{\tau^+} \mathbb{R}^{n+}$ because

$$\begin{array}{ccccccc} |(S^1)^{\wedge n}| & \cong & |S^1|^{\wedge n} & \cong & (\mathbb{R}^+)^{\wedge n} & \cong & \mathbb{R}^{n+} \\ \downarrow{\scriptstyle |\tau|} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau^+} \\ |(S^1)^{\wedge n}| & \cong & |S^1|^{\wedge n} & \cong & (\mathbb{R}^+)^{\wedge n} & \cong & \mathbb{R}^{n+} \end{array}$$

commutes ($-^+$ is symmetric monoidal).

After conjugation, the map $|(-1)_{S^1} \wedge (S^1)^{\wedge n-1}|$ yields

$$\mathbb{R}^{n+} \xrightarrow{\mathrm{diag}(-1,1,\ldots,1)^+} \mathbb{R}^{n+}$$

because the following diagram commutes (here we use relations between units and counit):

$$\begin{array}{ccccccc}
|S^1\wedge(S^1)^{\wedge n-1}| & \xrightarrow{\cong} & |S^1|\wedge|S^1|^{\wedge n-1} & \xrightarrow{\cong} & (\mathbb{R}^+)^{\wedge n} & \xrightarrow{\cong} & \mathbb{R}^{n+} \\
\downarrow{\scriptstyle |\nu\wedge 1|} & & \downarrow{\scriptstyle |\nu|\wedge 1} & & \| & & \\
|\operatorname{Sing}(\mathbb{R}^+)\wedge(S^1)^{n-1}| & \xrightarrow{\cong} & |\operatorname{Sing}(\mathbb{R}^+)|\wedge|S^1|^{\wedge n-1} & \xrightarrow{\sim} & (\mathbb{R}^+)^{\wedge n} & & \\
\downarrow{\scriptstyle |\operatorname{Sing}((-1)^+_{\mathbb{R}})\wedge 1|} & & \downarrow{\scriptstyle |\operatorname{Sing}((-1)^+_{\mathbb{R}})|\wedge 1} & & \downarrow{\scriptstyle (-1)^+_{\mathbb{R}}\wedge 1} & & \downarrow{\scriptstyle ((-1)_{\mathbb{R}}\times 1_{\mathbb{R}^{n-1}})^+} \\
|\operatorname{Sing}(\mathbb{R}^+)\wedge(S^1)^{\wedge n-1}| & \xrightarrow{\cong} & |\operatorname{Sing}(\mathbb{R}^+)|\wedge|S^1|^{\wedge n-1} & \xrightarrow{\sim} & (\mathbb{R}^+)^{\wedge n} & & \\
\uparrow{\scriptstyle |\nu\wedge 1|} & & \uparrow{\scriptstyle |\nu|\wedge 1} & & \| & & \\
|S^1\wedge(S^1)^{\wedge n-1}| & \xrightarrow{\cong} & |S^1|\wedge|S^1|^{\wedge n-1} & \xrightarrow{\cong} & (\mathbb{R}^+)^{\wedge n} & \xrightarrow{\cong} & \mathbb{R}^{n+}
\end{array}$$

and $\mathbb{R}^{2+}\xrightarrow{\operatorname{diag}(1,-1)^+}\mathbb{R}^{2+}$ is the conjugated map of $|S^1\wedge(-1)_{S^1}|$.

Now let $\tau\in\Sigma_n$ and $P_\tau\in\mathrm{GL}_n(\mathbb{R})$ the permutation matrix corresponding to τ. If τ is odd, then $\det P_\tau=-1=\det\operatorname{diag}(-1,1,...,1)$. Lemma 2.19 then implies that the maps $\tau^+:\mathbb{R}^{n+}\to\mathbb{R}^{n+}$ and $\operatorname{diag}(-1,1,...,1)^+:\mathbb{R}^{n+}\to\mathbb{R}^{n+}$ are equal in $\mathrm{Ho}(\mathrm{Top}_*)$, hence $\tau:(S^1)^{\wedge n}\to(S^1)^{\wedge n}$ and $(-1)_{S^1}\wedge(S^1)^{\wedge n-1}$ are also equal in $\mathrm{Ho}(\mathrm{sSet}_*)$. If τ is even , then $\det P_\tau=1=\det E_n$, and the maps $\tau:(S^1)^{\wedge n}\to(S^1)^{\wedge n}$ equals the identity on $(S^1)^{\wedge n}$ in $\mathrm{Ho}(\mathrm{sSet}_*)$. For the second condition, note that the diagonal matrices $\operatorname{diag}(-1,1)$ and $\operatorname{diag}(1,-1)$ have the same determinant, so by Lemma 2.19 the maps

$$\mathbb{R}^{2+}\xrightarrow{\operatorname{diag}(1,-1)^+}\mathbb{R}^{2+},$$

$$\mathbb{R}^{2+}\xrightarrow{\operatorname{diag}(-1,1)^+}\mathbb{R}^{2+},$$

are equal in $\mathrm{Ho}(\mathrm{Top}_*)$ and therefore also $(-1)_{S^1}\wedge S^1$ and $S^1\wedge(-1)_{S^1}$. □

We have just used the following:

Lemma 2.19. *The topological group* $\mathrm{GL}_n(\mathbb{R})$ *has two path components (corresponding to the sign of the determinant). If* $A,B\in\mathrm{GL}_n(\mathbb{R})$ *have determinants with the same sign, then the two pointed maps*

$$\mathbb{R}^{n+}\xrightarrow{A^+,B^+}\mathbb{R}^{n+}$$

are equal in $\mathrm{Ho}(\mathrm{Top}_*)$.

Proof. Well-known. □

2.2.2. The sign $(-1)_{\mathbb{P}^1}$ of the projective line. We have a pushout diagram (both in Sm/S and in $s\operatorname{Shv}(S)$.)

$$\begin{array}{ccc}
\mathbb{G}_{m\,S}=D_+(T_0T_1)\times S & \xrightarrow{i'_1} & \mathbb{A}^1_S=D_+(T_1)\times S \\
\downarrow{\scriptstyle i'_0} & & \downarrow{\scriptstyle i_1} \\
\mathbb{A}^1_S=D_+(T_0)\times S & \xrightarrow{i_0} & \mathbb{P}^1_S.
\end{array}$$

The base point of $\mathbb{P}^1_{\mathbb{Z}}$ is the closed immersion $\mathrm{Spec}(\mathbb{Z}) \xrightarrow{0} \mathbb{A}^1_{\mathbb{Z}} \xrightarrow{i_1} \mathbb{P}^1_{\mathbb{Z}}$ and its base change $S \to \mathbb{P}^1_S$ is the base point of $\mathbb{P}^1_S$. The latter map induces a base point map $\mathbb{P}^1_S\colon \infty : * = S \to \mathbb{P}^1_S$ which is closed for the *cm*-model structure above (see Corollary 2.12). For that model structure, $(\mathbb{P}^1_S, \infty)$ is a cofibrant pointed motivic space which we denote by $\mathbb{P}^1$ from now on. Similarly, we write $\mathbb{G}_m$ for the *cm*-cofibrant pointed motivic space $(\mathbb{G}_{m\,S}, 1)$.

Now we define the sign of $\mathbb{P}^1$. (See also [Mo, 6.1 The element ϵ] for the sign of $\mathbb{P}^1$ and its behaviour with respect to $\mathbb{P}^1 \cong S^1 \wedge \mathbb{G}_m$.)

Definition 2.20. The automorphism $\mathbb{P}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}}$ given by the graded isomorphism $\mathbb{Z}[T_0, T_1] \to \mathbb{Z}[T_0, T_1], T_0 \mapsto -T_0, T_1 \mapsto T_1$ is denoted by $(-1)_{\mathbb{P}^1_{\mathbb{Z}}}$, and similarly $(-1)_{\mathbb{P}^1_S} = (-1)_{\mathbb{P}^1_{\mathbb{Z}}} \times S$ for the base change of that automorphism to Sm/S. The following lemma shows that $(-1)_{\mathbb{P}^1_S}$ induces an automorphism $(-1)_{\mathbb{P}^1_S}$ on $\mathbb{P}^1$, which we denote by $(-1)_{\mathbb{P}^1}$, and call it the *sign of* $\mathbb{P}^1$.

Lemma 2.21. *The automorphism $(-1)_{\mathbb{P}^1_{\mathbb{Z}}}$ is the morphism induced by (the push-outs of) the following diagram:*

$$\begin{array}{ccccc}
D_+(T_0) & \longleftarrow & D_+(T_0T_1) & \longrightarrow & D_+(T_1) \\
\downarrow{\scriptstyle \frac{T_1}{T_0}\mapsto -\frac{T_1}{T_0}} & & \downarrow{\scriptstyle \frac{T_1}{T_0}\mapsto -\frac{T_1}{T_0}} & & \downarrow{\scriptstyle \frac{T_0}{T_1}\mapsto -\frac{T_0}{T_1}} \\
D_+(T_0) & \longleftarrow & D_+(T_0T_1) & \longrightarrow & D_+(T_1).
\end{array}$$

Consequently, the diagram

$$\begin{array}{ccc}
\mathbb{A}^1_S \cup_{\mathbb{G}_{m\,S}} \mathbb{A}^1_S & \longrightarrow & \mathbb{P}^1_S \\
\downarrow{\scriptstyle (-1)\cup_{(-1)}(-1)} & & \downarrow{\scriptstyle (-1)_{\mathbb{P}^1_S}} \\
\mathbb{A}^1_S \cup_{\mathbb{G}_{m\,S}} \mathbb{A}^1_S & \longrightarrow & \mathbb{P}^1_S
\end{array}$$

commutes where (-1) on coordinates is given by $T \mapsto -T$. Hence $(-1)_{\mathbb{P}^1_S}$ respects the base point ∞.

Proof. Straightforward. For the last claim, use the first one and that $\mathrm{Spec}(\mathbb{Z}[T]) \to \mathrm{Spec}(\mathbb{Z}[T]), T \mapsto -T$ maps the point $T = 0$ to itself. □

From now on, we replace the motivic space $(\mathbb{P}^1)^{\wedge n}$ by the weakly equivalent $\mathbb{A}^n_S/((\mathbb{A}^n_{\mathbb{Z}} - 0) \times S)$. On the latter, we consider the usual $\mathrm{GL}_{n\,S}$-action and relate it to the sign of $\mathbb{P}^1$.

Lemma 2.22.

(i) *There is a zig-zag of weak equivalences in $M.(S)$ between the pointed spaces $\mathbb{P}^1$ and $\mathbb{A}^1_S/_{i'_0}\mathbb{G}_{m\,S}$.*

(ii) *Via this zig-zag, the pointed map $(-1)_{\mathbb{P}^1_S}$ corresponds to the map $(-1)_{\mathbb{A}^1_S}/(-1)_{\mathbb{G}_{m\,S}}$.*

Proof. (i) We have a commutative diagram

$$\begin{array}{ccccc} \mathbb{A}^1_S & \xleftarrow[i'_0]{} & \mathbb{G}_{m\,S} & \xrightarrow{i'_1} & \mathbb{A}^1_S \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \sim} \\ \mathbb{A}^1_S & \xleftarrow[i'_0]{} & \mathbb{G}_{m\,S} & \longrightarrow & S \end{array}$$

The map $\mathbb{G}_{m\,S} \xrightarrow{i'_0} \mathbb{A}^1_S$ is a monomorphism, and the vertical maps are weak equivalences. As the injective model structure is left proper, the induces map $f : \mathbb{A}^1_S \amalg_{\mathbb{G}_{m\,S}} \mathbb{A}^1_S \to \mathbb{A}^1_S \amalg_{\mathbb{G}_{m\,S}} * = \mathbb{A}^1_S/_{i'_0}\mathbb{G}_{m\,S}$ is a weak equivalence, too. The motivic space $\mathbb{A}^1_S \amalg_{\mathbb{G}_{m\,S}} \mathbb{A}^1_S$ is pointed by $S \xrightarrow{0} \mathbb{A}^1_S \xrightarrow{\mathrm{incl}_1} \mathbb{A}^1_S \amalg_{\mathbb{G}_{m\,S}} \mathbb{A}^1_S$, and with this choice f is a pointed map. The induced map $\mathbb{A}^1_S \amalg_{\mathbb{G}_{m\,S}} \mathbb{A}^1_S \xrightarrow{(i_0,i_1)} \mathbb{P}^1_S$ is a motivic weak equivalence, as it is an isomorphism after sheafification [Mo, Lemma 2.1.13]. It is pointed as $(i_0, i_1) \cdot \mathrm{incl}_1 \cdot 0 = i_1 \cdot 0$.

(ii) The squares

$$\begin{array}{ccc} \mathbb{A}^1_S \amalg_{\mathbb{G}_{m\,S}} \mathbb{A}^1_S & \longrightarrow & \mathbb{A}^1_S \amalg_{\mathbb{G}_{m\,S}} * \\ {\scriptstyle (-1)_{\mathbb{A}^1_S} \amalg_{(-1)_{\mathbb{G}_{m\,S}}} (-1)_{\mathbb{A}^1_S}}\downarrow & & \downarrow{\scriptstyle (-1)_{\mathbb{A}^1_S} \amalg_{(-1)_{\mathbb{G}_{m\,S}}} 1_*} \\ \mathbb{A}^1_S \amalg_{\mathbb{G}_{m\,S}} \mathbb{A}^1_S & \longrightarrow & \mathbb{A}^1_S \amalg_{\mathbb{G}_{m\,S}} * \end{array}$$

and

$$\begin{array}{ccc} \mathbb{A}^1_S \amalg_{\mathbb{G}_{m\,S}} \mathbb{A}^1_S & \longrightarrow & \mathbb{P}^1_S \\ {\scriptstyle (-1)_{\mathbb{A}^1_S} \amalg_{(-1)_{\mathbb{G}_{m\,S}}} (-1)_{\mathbb{A}^1_S}}\downarrow & & \downarrow{\scriptstyle (-1)_{\mathbb{P}^1_S}} \\ h(\mathbb{A}^1_S) \amalg_{\mathbb{G}_{m\,S}} \mathbb{A}^1_S & \longrightarrow & \mathbb{P}^1_S \end{array}$$

commute by Lemma 2.21. □

For any $S \to \mathrm{Spec}(\mathbb{Z})$, we consider the usual actions

$$\mu : \mathrm{GL}_{n\,S} \times_S \mathbb{A}^n_S \to \mathbb{A}^n_S$$

(on the open subscheme $(\mathbb{A}^n - 0) \times S$ as well) and homomorphisms

$$\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{Aut}_{\mathrm{Sch}_S}(\mathbb{A}^n_S),$$
$$\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{Aut}_{M.(S)}(\mathbb{A}^n_S/(\mathbb{A}^n - 0) \times S).$$

Above, as usual, we have identified smooth varieties and the associated simplicially constant (pre-)sheaf given by the Yoneda embedding. However, to avoid confusion when it comes to base points, we will write

$$h. : Sm/S \to M.(S)$$

for the composition of the Yoneda embedding with adding a disjoint base point.

The above induces a map

$$h.(\mathrm{GL}_{n\,S}) \wedge [\mathbb{A}^n_S/((\mathbb{A}^n - 0) \times S)] \to [\mathbb{A}^n_S/((\mathbb{A}^n - 0) \times S)],$$

and for any $A \in \mathrm{GL}_n(\mathbb{Z})$ the diagram

$$\begin{array}{ccc}
[\mathbb{A}^n_S/((\mathbb{A}^n - 0) \times S)] & & \\
\downarrow \cong & \searrow^{A} & \\
h.(S) \wedge [\mathbb{A}^n_S/((\mathbb{A}^n - 0) \times S)] & & \\
\downarrow {\scriptstyle h.(A)\wedge 1} & & \\
h.(\mathrm{GL}_{n\,S}) \wedge [\mathbb{A}^n_S/((\mathbb{A}^n - 0) \times S)] & \longrightarrow & [\mathbb{A}^n_S/((\mathbb{A}^n - 0) \times S)]
\end{array}$$

commutes. Precomposition with the monomorphism $\Sigma_n \to \mathrm{GL}_n(\mathbb{Z})$ yields the above Σ_n-actions on $\mathbb{A}^n_S$ and on $\mathbb{A}^n_S/((\mathbb{A}^n - 0) \times S)$.

Lemma 2.23.

(i) *There is a Σ_n-equivariant map*

$$f : [\mathbb{A}^1_S/\mathbb{G}_{m\,S}]^{\wedge n} \to \mathbb{A}^n_S/((\mathbb{A}^n_{\mathbb{Z}} - 0) \times S)$$

in $M.(S)$ which is a motivic equivalence.

(ii) *The diagram*

$$\begin{array}{ccc}
\mathbb{A}^1_S/\mathbb{G}_{m\,S} \wedge [\mathbb{A}^1_S/\mathbb{G}_{m\,S}]^{\wedge n-1} & \longrightarrow & \mathbb{A}^n_S/((\mathbb{A}^n_{\mathbb{Z}} - 0) \times S) \\
\downarrow {\scriptstyle (-1)_{\mathbb{A}^1_S/\mathbb{G}_{m\,S}}\wedge 1} & & \downarrow {\scriptstyle \mathrm{diag}(-1,1,\ldots,1)} \\
\mathbb{A}^1_S/\mathbb{G}_{m\,S} \wedge [\mathbb{A}^1_S/\mathbb{G}_{m\,S}]^{\wedge n-1} & \longrightarrow & \mathbb{A}^n_S/((\mathbb{A}^n_{\mathbb{Z}} - 0) \times S)
\end{array}$$

commutes, and similarly for $\mathrm{diag}(1, \ldots, 1, -1)$.

Proof. (i) We have a commutative diagram

$$\begin{array}{ccc}
\coprod_{i=0}^{n-1} (\mathbb{A}^1_S)^{\times i} \times \mathbb{G}_{m\,S} \times (\mathbb{A}^1_S)^{\times n-(i+1)} & \longrightarrow & (\mathbb{A}^1_S)^{\times n} \\
\downarrow \cong & & \downarrow \cong \\
\coprod_{i=0}^{n-1} \mathbb{A}^{1\times_S i}_S \times_S \mathbb{G}_{m\,S} \times_S \mathbb{A}^{1\times_S n-(i+1)}_S & \longrightarrow & \mathbb{A}^{1\times_S n}_S \\
\downarrow & & \downarrow \cong \\
(\mathbb{A}^n - 0) \times S & \longrightarrow & \mathbb{A}^n_S
\end{array}$$

in which the vertical maps are Σ_n-equivariant. The horizontal maps induce the desired map f on the quotients. To see that f is a weak equivalence, it suffices to show [Ja1, Lemma 2.6] that its sheafification is an isomorphism. Using the adjunction $a : M.(S) \rightleftarrows s\,\mathrm{Shv}.(S) : i$, this reduces to show

that for all $\mathcal{F} \in s\,\mathrm{Shv}.(S)$ the induced map $M.(S)(f, i(\mathcal{F}))$ is a bijection. The familiy of open immersions

$$\{\mathbb{A}_S^{1\times_S i} \times_S \mathbb{G}_{m\,S} \times_S \mathbb{A}_S^{1\times_S n-(i+1)} \hookrightarrow ((\mathbb{A}^n - 0) \times S); 0 \leq i \leq n-1\}$$

is a Zariski covering, hence a Nisnevich covering. Therefore, in the diagram

$$\begin{array}{ccccc}
\mathcal{F}(\mathbb{A}_S^n) & \longrightarrow & \mathcal{F}((\mathbb{A}^n - 0) \times S) & \longleftarrow & \mathcal{F}(S) \\
\downarrow \cong & & \downarrow & & \downarrow 1 \\
\mathcal{F}(\mathbb{A}_S^{1\times_S n}) & \longrightarrow & \prod_{i=0}^{n-1} \mathcal{F}(\mathbb{A}_S^{1\times_S i} \times_S \mathbb{G}_{m\,S} \times_S \mathbb{A}_S^{1\times_S n-(i+1)}) & \longleftarrow & \mathcal{F}(S)
\end{array}$$

the middle vertical map is injective. It follows that the induced map on pull-backs is bijective, and that one coincides with $M.(S)(f, i(\mathcal{F}))$.

(ii) The first diagram above is compatible with the corresponding maps for $\mathrm{diag}(-1, 1, \ldots, 1)$. (Apply the monomorphism $((\mathbb{A}^n - 0) \times S) \hookrightarrow \mathbb{A}_S^n$ to see this for the lower left map.) □

The above together with Lemma 2.19 below leads to the main result of this subsection:

Proposition 2.24. *The automorphism $(-1)_{\mathbb{P}^1}$ is a sign of $\mathbb{P}^1$ in $M_\cdot^{cm}(S)$.*

Proof. By Definition 2.20, the automorphism $(-1)_{\mathbb{P}^1}$ has order 2. Using that the smash product of weak equivalences in $M.(S)$ is again a weak equivalence, as well as Lemmas 2.22 and 2.23, the required properties of Definition 2.16 follow from the following:

(i) Let $\tau \in \Sigma_n$ be a permutation. Then, in $\mathrm{Ho}(M_\cdot^{cm}(S))$ the automorphism induced by τ on $\mathbb{A}_S^n/((\mathbb{A}_{\mathbb{Z}}^n - 0) \times S)$ equals $\mathrm{diag}(-1, 1, \ldots, 1)$ if τ is an odd permutation, and the identity if τ is even.

(ii) The automorphisms $\mathrm{diag}(-1, 1)$ and $\mathrm{diag}(1, -1)$ of $\mathbb{A}_S^2/(\mathbb{A}_{\mathbb{Z}}^2 - 0) \times S$ are equal in $\mathrm{Ho}(M_\cdot^{cm}(S))$.

Using Lemma 2.25(ii) these in turn follow from

(i) $\det P_\tau = \begin{cases} \det \mathrm{diag}(-1, 1, \ldots, 1) & \tau \text{ is odd} \\ \det \mathrm{diag}(1, \ldots, 1) & \tau \text{ is even.} \end{cases}$

(ii) $\det \mathrm{diag}(-1, 1) = \det \mathrm{diag}(1, -1)$. □

Lemma 2.25. *Let $A_0, A_1 \in \mathrm{GL}_n(\mathbb{Z})$ two matrices with $A_1 A_0^{-1} \in \mathrm{SL}_n(\mathbb{Z})$. Via the inclusion $\mathrm{GL}_n(\mathbb{Z}) \hookrightarrow \mathrm{GL}_n(\mathcal{O}_S(S)) \cong \mathrm{Sch}_S(S, \mathrm{GL}_{n,S})$, these matrices induce morphisms $\underline{A_0}, \underline{A_1} : S \to \mathrm{GL}_{n,S}$ in Sm/S.*

(i) *There is a map $f : \mathbb{A}_S^1 \to \mathrm{GL}_{n,S}$ in Sm/S with $f \cdot i_l = \underline{A_l}$ for $l = 0, 1$, where $i_l : S \to \mathbb{A}_S^1$ are the morphisms represented by $\overline{0}$ and*

1 *in* $\mathcal{O}_S(S)$*:*

$$\begin{array}{ccc} S \amalg S & \xrightarrow{(A_0,A_1)} & \mathrm{GL}_{n,S} \\ {\scriptstyle (i_0,i_1)}\downarrow & \overset{f}{\nearrow} & \\ \mathbb{A}^1_S & & \end{array}$$

(ii) *For any pointed motivic space* E *and* $\mu : h.(\mathrm{GL}_{n\,S}) \wedge E \to E$ *a map in* $M.(S)$*, the endomorphisms on* E *induced by* $\underline{A_0}$ *and* $\underline{A_1}$ *are equal in* $\mathrm{Ho}(M_.^{cm}(S))$*.*

Proof. (i) By adjunction, a map $f : \mathbb{A}^1_S \to \mathrm{GL}_{n,S}$ in Sm/S corresponds uniquely to a matrix $\tilde{A} = \tilde{A}(f) \in \mathrm{GL}_n(\mathcal{O}_S(S)[T])$. On global sections, i_l is given by $\mathcal{O}_S(S)[T] \to \mathcal{O}_S(S), T \mapsto l$. Therefore, the condition that a lift f exists corresponds to the equalities $\tilde{A}(l) = A_l$ for $l = 0, 1$, where $\tilde{A}(l)$ is the image of $\tilde{A}$ under $\mathrm{GL}_n(\mathcal{O}_S(S)[T]) \to \mathrm{GL}_n(\mathcal{O}_S(S)), T \mapsto l$. We may assume that $A_0 = E$ is the unit matrix and $A_1 \in \mathrm{SL}_n(\mathbb{Z})$. (If the couple $(E,\, A_1 \cdot A_0^{-1})$ allows for a lift $\tilde{A} \in \mathrm{GL}_n(\mathcal{O}_S(S)[T])$, then $\tilde{A} \cdot A_0 \in \mathrm{GL}_n(\mathcal{O}_S(S)[T])$ is a lift for $(A_0,\, A_1)$ with A_0 constant with respect to T.) We may further assume that A_1 is an elementary matrix, as $T \mapsto l$ is multiplicative. Namely, if $\tilde{A}$ is a lift of (A_0, A_1) and $\tilde{B}$ is a lift of (B_0, B_1), then $\tilde{A}\tilde{B}$ is a lift of (A_0B_0, A_1B_1). Finally, for $A_0 = E$ and $A_1 = E_{k,l}(a)$ an elementary matrix with $a \in \mathbb{Z}$, we may choose $\tilde{A} := E_{k,l}(aT) \in \mathrm{GL}_n(\mathcal{O}_S(S)[T])$ as a lift.

(ii) If $pr : \mathbb{A}^1_S \to S$ is the projection, we have

$$h.(pr) \cdot h.(i_l) = h.(pr \cdot i_l) = h.(1_S) = 1_{h.(S)}, \quad l = 0, 1.$$

As $h.(pr)$ is a motivic weak equivalence, $h.(i_0)$ and $h.(i_1)$ are isomorphic in the motivic homotopy category and hence [MV, Lemma 3.2.13] so are $h.(i_l) \wedge E, l = 0, 1$. Now the claim follows by $\underline{A_l} = f \cdot i_l$. □

2.3. Definition of the $\mathcal{M}$-action on stable homotopy groups. *From now on, we will make the following standard assumptions: Let* $(\mathcal{D}, \wedge, S^0)$ *be a pointed symmetric monoidal model category. There is a monoidal left Quillen functor* [Hov1, Def. 4.2.16] $i : \mathrm{sSet}_* \hookrightarrow \mathcal{D}$ *with right adjoint* $j : \mathcal{D} \to \mathrm{sSet}_*$*. We choose a cofibrant object* T *in* $\mathcal{D}$ *such that* $- \wedge T$ *preserves weak equivalences. Moreover, we assume that* T *is a cogroup object in* $\mathrm{Ho}(\mathcal{D})$*. (This is the case if, e.g.,* $T \simeq S^1 \wedge B$ *for some object* B *of* $\mathcal{D}$*.) Finally, we fix a class* $\mathcal{B}$ *of cofibrant objects in* $\mathcal{D}$*.*

For the category $M.(S)$, we will take i to be the functor mapping a simplicial set to a constant simplicial presheaf, and j the evaluation on the terminal object $S \in Sm/S$. The condition that is T cofibrant is equivalent to require that the functor $- \wedge T$ preserves cofibrations, as then $i(S^0) \wedge T \cong S^0 \wedge T \cong T$ is also cofibrant. The functor $- \wedge T$ induces a functor on $\mathrm{Ho}(\mathcal{D})$.

Definition 2.26. Let E be a T-spectrum in $\mathcal{D}$. Then for all $q \in \mathbb{Z}, V \in \mathcal{B}$, the abelian groups (see also Lemma 2.28)

$$\operatorname*{colim}_{m\geq 0, q+m\geq 1}(\cdots \to [V \wedge T^{q+m}, E_m] \xrightarrow{\sigma_*(-\wedge T)} [V \wedge T^{q+m+1}, E_{m+1}] \to \cdots)$$

are called the *stable homotopy groups* of E, and will be denoted by $\pi_q^V(E)$. They are functors $\mathrm{Sp}(\mathcal{D}, T) \to Ab$.

Example 2.27.

(i) For $\mathcal{D} = \mathrm{sSet}_*$, $T = S^1$ and $\mathcal{B} = \{S^0\}$, one recovers the definition of the usual (naive, that is forgetting the Σ_n-action) stable homotopy groups (denoted by $\hat{\pi}_*$ in [Sch12]): $\pi_q^{S^0}(E) \cong \hat{\pi}_q(E)$.

(ii) For $\mathcal{D} = M_{\cdot}^{cm}(S), T = \mathbb{P}^1$ and

$$\mathcal{B} = \{S^r \wedge h_\cdot(U) \wedge \mathbb{G}_m^{\wedge s} \mid r, s \in \mathbb{N}_0, U \in Sm/S\},$$

the groups $\pi_q^V(E)$ are the motivic stable homotopy groups of E. In particular,

$$\pi_q^{S^r \wedge \mathbb{G}_m^{\wedge s}}(E) \cong \pi^{\mathrm{mot}}_{q+r+s,q+s}(E)(U)$$

(note that $\mathcal{B}$ consists of cm-cofibrant objects).

Lemma 2.28. *Consider two objects A and X in $\mathcal{D}$ with A cofibrant, and $V \in \mathcal{B}$. Then $V \wedge T^2 \wedge A$ has an abelian cogroup structure, and the corresponding group structure on $[V \wedge T^2 \wedge A, X]$ is compatible with $- \wedge T$.*

Proof. As T is a cogroup object by assumption, T^2 and more generally $A' := V \wedge T^2 \wedge A$ is an abelian cogroup object with comultiplication

$$\begin{aligned} V \wedge T^2 \wedge A \xrightarrow{V\wedge\mu\wedge T\wedge A} & V \wedge (T \vee T) \wedge T \wedge A \\ & \cong [V \wedge T \wedge T \wedge A] \wedge [V \wedge T \wedge T \wedge A]. \end{aligned}$$

As the comultiplication on $A' \wedge T$ is given by

$$A' \wedge T \xrightarrow{\mu_{A'}\wedge T} (A' \vee A') \wedge T \cong (A' \wedge T) \vee (A' \wedge T),$$

the compatibility with $- \wedge T$ follows. □

Definition 2.29. Let $f : E \to F$ be a map of T-spectra in $\mathcal{D}$. Then f is called a $\pi^{\mathcal{B}}$-stable equivalence if the induced maps

$$\pi_q^V(f) : \pi_q^V(E) \to \pi_q^V(F)$$

are isomorphisms for all $q \in \mathbb{Z}, V \in \mathcal{B}$.

We now turn to the $\mathcal{M}$-action. Let $\mathcal{I}$ be the category of finite sets and injective maps, and $\mathcal{M}$ the "injection monoid" (see [Sch07], [Sch08] and Definition 2.34 below). Recall (see [Sch07, Section 4.2], [Sch08, Section 1.2]) that there are functors from symmetric spectra to Ab-valued $\mathcal{I}$-functors and from $\mathcal{I}$-functors to (tame) $\mathcal{M}$-modules, mapping X to $\underline{X}$ and further to $\underline{X}(\omega)$.

We still make the above assumptions, and also assume that T has a sign. The following definition generalizes [Sch08, 1.2 Construction, Step 1].

Proposition-Definition 2.30. *Let $q \in \mathbb{Z}$ and $V \in \mathcal{B}$. For any symmetric spectrum X in $\mathcal{D}$, we define a functor $\underline{X} : \mathcal{I} \to Ab$ for any symmetric T-spectrum X in $\mathcal{D}$ and then obtain (see above) an $\mathcal{M}$-action on its evaluation at ω, $\pi_q^V(X)$, which is precisely the group $\pi_q^V(X)$ of Definition 2.26. In more detail, any m in $\mathcal{I}$ is mapped to $[V \wedge T^{q+m}, X_m]$ (see Lemma 2.28) if $q+m \geq 2$, and to 0 otherwise. For $f : m \to n$ a morphism in $\mathcal{I}$ (hence $n \geq m$) we choose a permutation $\gamma \in \Sigma_n$ with $f = \gamma_{|m}$. Then $\underline{X}(f)$ is the composition*

$$\begin{array}{ccc} [V \wedge T^{q+m}, X_m] & \xrightarrow{\sigma_*^{n-m}(-\wedge T^{n-m})} & [V \wedge T^{q+n}, X_n] \\ & & \Big\downarrow{\scriptstyle (V \wedge |\gamma|_T \wedge T^{q+n-1})^* \gamma_*} \\ & & [V \wedge T^{q+n}, X_n] \end{array}$$

if $q+m \geq 2$, and 0 otherwise.

Proof. The map $V \wedge |\gamma|_T \wedge T^{q+n-1}$ is defined as V and T are cofibrant. The above composition is a group homomorphism as the group structure is compatible with $- \wedge T$ (Lemma 2.28), and we have

$$V \wedge |\gamma|_T \wedge T^{q+n-1} = V \wedge T \wedge |\gamma|_T \wedge T^{q+n-2}$$

by Definition 2.16.

The functor $\underline{X}$ is well-defined on morphisms: Consider $\gamma, \gamma' \in \Sigma_n$ with $\gamma_{|m} = \gamma'_{|m}$. Then there is a $\tau \in \Sigma_{n-m}$ with $\gamma'^{-1}\gamma = m + \tau$ and the claim $\underline{X}(\gamma) = \underline{X}(\gamma')$ is equivalent to showing that the two compositions

$$\begin{array}{ccc} [V \wedge T^{q+m}, X_m] & \xrightarrow{\sigma_*^{n-m}(-\wedge T^{n-m})} & [V \wedge T^{q+n}, X_n] \\ & & {\scriptstyle (V \wedge |m+\tau|_T \wedge T^{q+n-1})^*} \Big\downarrow {\scriptstyle (1_m \times \tau)_*} \\ & & [V \wedge T^{q+n}, X_n] \end{array}$$

are equal. Let $n \geq m$ (otherwise there is nothing to prove). By Definition 2.16, we have

$$|m+\tau|_T \wedge T^{q+n-1} = T^{q+m} \wedge |\tau|_T \wedge T^{n-m-1} = T^{q+m} \wedge \tau_T$$

in $\mathrm{Ho}(\mathcal{D})$. Applying $V \wedge -$ and using the equivariance of

$$(m+\tau) \cdot \sigma^{n-m} = \sigma^{n-m} \cdot (X_m \wedge \tau),$$

the equality follows from the equality of the following two compositions:

$$\begin{array}{ccc} [V \wedge T^{q+m}, X_m] & \xrightarrow{(-\wedge T^{n-m})} & [V \wedge T^{q+m} \wedge T^{n-m}, X_m \wedge T^{n-m}] \\ & & \downarrow{\scriptstyle (X_m \wedge \tau)_* \quad [(V \wedge T^{q+m} \wedge \tau)^*]} \\ & & [V \wedge T^{q+m} \wedge T^{n-m}, X_m \wedge T^{n-m}]. \end{array}$$

A straighforward computation involving that

$$\operatorname{sgn}(\delta \cdot (\gamma + (n' - n))) = \operatorname{sgn}(\delta) \cdot \operatorname{sgn}(\gamma)$$

shows that $\underline{X}$ is indeed a functor. Finally, as the inclusion $m \to m+1$ corresponds to $\sigma_*(- \wedge T)$, $\underline{X}(\omega)$ is indeed $\pi_q^V(X)$ as claimed. □

For $\mathcal{D} = \mathrm{sSet}_*, T = S^1$, this coincides with the definition of [Sch08], because $|(-1)_{S^1}|$ is isomorphic to a self-map on S^1 of degree -1. For $\mathcal{D} = M_\cdot^{cm}(S)$ and $T = \mathbb{P}^1$, note that being semistable does not depend on the $\mathbb{A}^1$-local model structure (projective, injective, cm...), but only on the motivic homotopy category $\mathrm{Ho}(\mathcal{D})$.

We are now able to state our key definition.

Definition 2.31 (Compare [Sch08, Theorem 4.1])**.** Let $\mathcal{D}$ be as above and fix a class $\mathcal{B}$ of cofibrant objects. A symmetric T-spectrum X is called *semistable*, if the $\mathcal{M}$-action (see Definition 2.30) is trivial on all homotopy groups of X appearing in Definition 2.29.

Remark 2.32. Note that this definition heavily depends on the choice of $\mathcal{B}$. If the $\pi^{\mathcal{B}}$-stable equivalences coincide with the stable equivalences in $\mathrm{Sp}(\mathcal{D}, T)$, then under the assumptions of Theorem 2.43 the two definitions of semistability coincide. This holds in particular for $\mathcal{D} = M_\cdot^{cm}(S)$ (see above and Proposition 2.45 below), and $\mathcal{B}$ as in the example above.

Lemma 2.33. *Let $f : X \to Y$ be a $\pi^{\mathcal{B}}$-stable equivalence in $\mathrm{Sp}^\Sigma(\mathcal{D}, T)$. Then $\pi_q^V(f)$ is an isomorphism of $\mathcal{M}$-objects. In particular: X is semistable if and only if Y is semistable.*

Proof. By Definition, the map $\pi_q^V(f)$ commutes with the $\mathcal{M}$-action and by assumption the map is an isomorphism. □

2.4. Some $\mathcal{M}$-isomorphisms between stable homotopy groups. We keep the assumptions of the previous section, and assume that T has a sign. Recall [Sch07], [Sch08] the definition of the cycle operator and of tameness:

Definition 2.34.
- Let $\mathcal{M}$ be the set of all self injections of $\mathbb{N}$. This is a monoid under composition, the so-called *injection monoid*.
- The injective map $d : \mathbb{N} \to \mathbb{N}$ given by $x \mapsto x + 1$ is called the *cycle operator*.

- As usual, we sometimes consider $\mathcal{M}$ as a category with a single object. A $\mathcal{M}$-object W in $\mathcal{D}$ is a functor $W : \mathcal{M} \to \mathcal{D}$, and we have the category $Func(\mathcal{M}, \mathcal{D})$ of $\mathcal{M}$-objetcs in $\mathcal{D}$. If $\mathcal{D}$ is the category of sets resp- abelian groups, we call these objects $\mathcal{M}$-modules resp. $\mathcal{M}$-sets.
- Let $n \in \mathbb{N}_0$. The injective map $\mathcal{M} \to \mathcal{M}$, given by mapping f to the map
$$x \mapsto \begin{cases} x & x \leq n \\ f(x-n) & x > n, \end{cases}$$
is denoted by $n + -$ or $-(n)$. For W any $\mathcal{M}$-object, note that $W(n)$ is the $\mathcal{M}$-object with underlying object W and the $\mathcal{M}$-action restricted along $n + -$.
- Now assume further that $\mathcal{D}$ has a forgetful functor to the category of sets. Let ϕ be an $\mathcal{M}$-action on an object W in $\mathcal{D}$. Then we sometimes write fx for $[\phi(f)](x)$ if the $\mathcal{M}$-action is understood. For any $f \in \mathcal{M}$ let $|f| := \min\{i \geq 0; f(i+1) \neq i+1\}$. An element $x \in W$ *has filtration* n if for all $f \in \mathcal{M}$ with $|f| \geq n$ we have $fx = x$. We write $W^{(n)}$ for the subset of all elements of *filtration* n. The $\mathcal{M}$-action on W is *tame* if $W = \bigcup_{n \geq 0} W^{(n)}$. If $\mathcal{D}$ fas a forgetful functor to abelian groups, then $W^{(n)}, n \geq 0$ are abelian groups as well.

The stable homotopy groups of $\operatorname{sh} X$, $T \wedge X$ and ΩX may be expressed through the stable homotopy groups of X. The following generalizes [Sch08, Examples 3.10 and 3.11].

Proposition 2.35. *Let X be a T-spectrum in $\mathcal{D}$ and $q \in \mathbb{Z}, V \in \mathcal{B}$. Then we have the following isomorphisms of groups. They are compatible with the sign of T, and if X is a symmetric spectrum they also respect the $\mathcal{M}$-action:*

(i) $\pi^V_{q+1}(\operatorname{sh} X) \cong \pi^V_q(X)(1)$.

(ii) $\pi^V_q(\Omega X) \cong \pi^V_{q+1}(X)$, *if X is level-fibrant and T is cofibrant.*

(iii) $\pi^V_q(X) \xrightarrow{T\wedge -} \pi^V_{1+q}(T \wedge X)$.

Proof. We first establish the isomorphisms.

(i) Easy.

(ii) As X_m is fibrant and $V \wedge T^{q+m}$ is cofibrant, we have isomorphisms:

$$[V \wedge T^{q+m}, \Omega X_m] \xrightarrow{\alpha_{V\wedge T^{q+m}, X_m}} [V \wedge T^{q+m} \wedge T, X_m] \xrightarrow{(V\wedge \chi_{1,q+m})^*} [V \wedge T \wedge T^{q+m}, X_m],$$

compatible with the structure maps, that is the diagram

$$\begin{array}{ccccc} [V \wedge T^{q+m}, \Omega X_m] & \xrightarrow{\alpha_{V\wedge T^{q+m}, X_m}} & [V \wedge T^{q+m} \wedge T, X_m] & \xrightarrow{(V\wedge \chi_{1,q+m})^*} & [V \wedge T \wedge T^{q+m}, X_m] \\ \downarrow{\scriptstyle \sigma^{\Omega X}_*(-\wedge T)} & & & & \downarrow{\scriptstyle \sigma^X_*(-\wedge T)} \\ [V \wedge T^{q+m+1}, \Omega X_{m+1}] & \xrightarrow{\alpha_{V\wedge T^{q+m+1}, X_{m+1}}} & [V \wedge T^{q+m+1} \wedge T, X_{m+1}] & \xrightarrow{(V\wedge \chi_{1,q+m+1})^*} & [V \wedge T \wedge T^{q+m+1}, X_{m+1}] \end{array}$$

commutes. Now for any $f : V \wedge T^{q+m} \to \Omega X_m$ in $\mathrm{Ho}(\mathcal{D})$, we have

$$\begin{aligned}\alpha_{V\wedge T^{q+m+1},X_{m+1}}(\sigma^{\Omega X}\cdot(f\wedge T)) &= \mathrm{ev}\cdot([\sigma^{\Omega X}\cdot(f\wedge T)]\wedge T)\\ &= \sigma^X\cdot(\mathrm{ev}_X\wedge T)\cdot(1\wedge\chi_{1,1})\cdot(f\wedge T^2)\\ &= \sigma^X\cdot(\mathrm{ev}_X\wedge T)\cdot(f\wedge\chi_{1,1}).\end{aligned}$$

Thus under the lower left composition, f maps to

$$\begin{aligned}&\sigma^X\cdot(\mathrm{ev}_X\wedge T)\cdot(f\wedge\chi_{1,1})\cdot(V\wedge\chi_{1,q+m+1})\\ &=\sigma^X\cdot(\mathrm{ev}_X\wedge T)\cdot(f\wedge T^2)\cdot(V\wedge\chi_{1,q+m}\wedge T),\end{aligned}$$

and to

$$\begin{aligned}&\sigma^X\cdot([\alpha_{V\wedge T^{q+m},X_m}(f)\cdot(V\wedge\chi_{1,q+m})]\wedge T)\\ &=\sigma^X\cdot([\mathrm{ev}\cdot(f\wedge T)\cdot(V\wedge\chi_{1,q+m})]\wedge T)\end{aligned}$$

under the upper right composition. This yields the claimed bijection. Using Lemma 2.28 (respectively Definition 2.16), we see that $\alpha_{V\wedge T^{q+m},X_m}$ (respectively $(V\wedge\chi_{1,q+m})^*$) is a group homomorphism.

(iii) As $T\wedge -$ preserves weak equivalences in $\mathcal{D}$, it induces maps

$$\begin{array}{ccc}[V\wedge T^{q+m},X_m] & \xrightarrow{T\wedge -} & [T\wedge V\wedge T^{q+m},T\wedge X_m]\\ & & \Big\downarrow{\scriptstyle (t_{V,T}\wedge T^{q+m})^*}\\ & & [V\wedge T\wedge T^{q+m},T\wedge X_m],\end{array}$$

which are obviously compatible with the structure maps. For any

$$f : V\wedge T^{q+m}\to X_m$$

in $\mathrm{Ho}(\mathcal{D})$, the diagram

$$\begin{array}{ccccc}V\wedge T\wedge T^{q+m} & \xrightarrow{t_{V,T}\wedge T^{q+m}} & T\wedge V\wedge T^{q+m} & \xrightarrow{T\wedge f} & T\wedge X_m\\ & \searrow^{t_{T,T^{q+m}}} & \uparrow{\scriptstyle t_{V\wedge T^{q+m},T}} & & \uparrow{\scriptstyle t_{T,X_m}}\\ & & V\wedge T^{q+m}\wedge T & \xrightarrow{f\wedge T} & X_m\wedge T\end{array}$$

commutes, therefore the map above equals the composition

$$\begin{array}{ccc}[V\wedge T^{q+m},X_m] & \xrightarrow{-\wedge T} & [V\wedge T^{q+m}\wedge T,X_m\wedge T]\\ & & \Big\downarrow{\scriptstyle (V\wedge\chi_{1,q+m})^*t_{T,X_m*}}\\ & & [V\wedge T\wedge T^{q+m},T\wedge X_m].\end{array}$$

Arguing as in (ii), we see this is a group homomorphism. Passing to the colimit yields the desired map $T\wedge(-)=(T\wedge(-))_X$. By naturality, any level equivalence $X^c\to X$ in $\mathrm{Sp}(\mathcal{D},T)$ induces an isomorphism between the maps $(T\wedge -)_X$ and $(T\wedge -)_{X^c}$. Choosing X^c to be level cofibrant, we may

assume that X is level cofibrant itself when showing that $(T \wedge -)_X$ is an isomorphism.

To see injectivity, assume that there is some f in the kernel, and that f is represented by some element in $[V \wedge T^{q+m}, X_m]$. Then contemplating the commutative diagram

$$\begin{array}{ccccc}
[V \wedge T^{q+m}, X_m] & \xrightarrow{-\wedge T} & [V \wedge T^{q+m} \wedge T, X_m \wedge T] & \xrightarrow{\sigma_*} & [V \wedge T^{q+m} \wedge T, X_{m+1}] \\
\downarrow{\scriptstyle T\wedge -} & \nearrow{\scriptstyle t^*_{V\wedge T^{q+m},T}\cdot t_{T,X_m*}} & & & \uparrow{\scriptstyle \sigma_*} \\
[T \wedge V \wedge T^{q+m}, T \wedge X_m] & \xrightarrow[(t_{V,T}\wedge T^{q+m})^*]{} & [V \wedge T \wedge T^{q+m}, T \wedge X_m] & \xrightarrow[(V\wedge \chi_{q+m,1})^*\cdot t_{T,X_m*}]{} & [V \wedge T^{q+m} \wedge T, X_m \wedge T]
\end{array}$$

we see that it has to be zero in the upper right corner, showing injectivity as claimed.

To obtain inverse images, consider the composition

$$\begin{array}{ccc}
[V \wedge T^{1+q+m}, T \wedge X_m] & \xrightarrow{(V\wedge \chi_{q+m,1})^*\cdot t_{T,X_m*}} & [V \wedge T^{(q+m)+1}, X_m \wedge T] \\
& & \downarrow{\scriptstyle \sigma_*} \\
& & [V \wedge T^{q+m+1}, X_{m+1}].
\end{array}$$

It remains to show $\sigma_*^{T\wedge X}(- \wedge T)$ is the result of composing this with the map above. This will rely on the existence of the sign on T. Let

$$f : V \wedge T^{1+q+m} \to T \wedge X_m$$

be a map in $\mathrm{Ho}(\mathcal{D})$. Then we have

$$\begin{aligned}
&[(t_{V,T} \wedge 1)^* \cdot (T \wedge -)] \cdot [\sigma_* \cdot t_{T,X_m*} \cdot (V \wedge \chi_{q+m,1})^*](f) \\
&= [(t_{V,T} \wedge 1)^* \cdot (T \wedge -)](\sigma \cdot t_{T,X_m} \cdot f \cdot (V \wedge \chi_{q+m,1})) \\
&= T \wedge (\sigma \cdot t_{T,X_m} \cdot f \cdot (V \wedge \chi_{q+m,1})) \cdot (t_{V,T} \wedge T^{q+m+1}) \\
&= \sigma^{T\wedge X} \cdot (T \wedge t_{T,X_m}) \cdot (T \wedge f) \cdot (t_{V,T} \wedge \chi_{q+m,1}).
\end{aligned}$$

Let us first consider

$$\begin{aligned}
&(T \wedge t_{T,X_m}) \cdot (T \wedge f) \\
&= (T \wedge t_{T,X_m}) \cdot (t_{T,T} \wedge X_m)^2 \cdot (T \wedge f) \\
&= [(T \wedge t_{T,X_m}) \cdot (t_{T,T} \wedge X_m)] \cdot [((-1)_T \wedge T \wedge X_m) \cdot (T \wedge f)] \\
&= t_{T,T\wedge X_m} \cdot (T \wedge f) \cdot ((-1)_T \wedge V \wedge T^{1+q+m}) \\
&= (f \wedge T) \cdot t_{T,V\wedge T^{1+q+m}} \cdot ((-1)_T \wedge V \wedge T^{1+q+m}).
\end{aligned}$$

Because

$$\begin{aligned}
&t_{T,V\wedge T^{1+q+m}} \cdot ((-1)_T \wedge V \wedge T^{1+q+m}) \cdot (t_{V,T} \wedge \chi_{q+m,1}) \\
&= t_{T,V\wedge T^{1+q+m}} \cdot (t_{V,T} \wedge \chi_{q+m,1}) \cdot (V \wedge (-1)_T \wedge T^{1+q+m}) \\
&= (V \wedge \tau_{1,1+q+m+1}) \cdot (V \wedge (-1)_T \wedge T^{1+q+m})
\end{aligned}$$

$$= (V \wedge (-1)_T \wedge T^{1+q+m}) \cdot (V \wedge (-1)_T \wedge T^{1+q+m}) = 1$$

we finally obtain

$$[(t_{V,T} \wedge 1)^* \cdot (T \wedge -)] \cdot [\sigma_* \cdot t_{T,X_m *} \cdot (V \wedge \chi_{q+m,1})^*](f) = \sigma^{T \wedge X} \cdot (f \wedge T).$$

Here $\tau_{1,1+q+m+1} \in \Sigma_{1+q+m+1}$ is the permutation interchanging $1+q+m+1$ and 1.

We now turn to the $\mathcal{M}$-action. Let $f : \mathbb{N} \to \mathbb{N}$ be injective, $\max(f(m)) = n$ and $\gamma \in \Sigma_n$ with $\gamma_{|m} = f|^n_m$. Concerning (i), for $1 + \gamma \in \Sigma_{1+m}$ we have $(1+\gamma)_{|1+n} = (1+f)|^{1+n}_{1+m}$ and the diagram

$$\begin{array}{ccc}
[V \wedge T^{(q+1)+m}, (\operatorname{sh} X)_m] & = & [V \wedge T^{q+(1+m)}, X_{1+m}] \\
\downarrow{\scriptstyle \sigma_*^{n-m} \cdot (- \wedge T^{n-m})} & & \downarrow{\scriptstyle \sigma_*^{(n+1)-(m+1)} \cdot (- \wedge T^{(n+1)-(m+1)})} \\
[V \wedge T^{(q+1)+n}, (\operatorname{sh} X)_n] & = & [V \wedge T^{q+(1+n)}, X_{1+n}] \\
\downarrow{\scriptstyle (V \wedge |\gamma|_T \wedge 1)^* \cdot \gamma_*} & & \downarrow{\scriptstyle (V \wedge |1+\gamma|_T \wedge 1)^* \cdot (1+\gamma)_*} \\
[V \wedge T^{(q+1)+n}, (\operatorname{sh} X)_n] & = & [V \wedge T^{q+(1+n)}, X_{1+n}]
\end{array}$$

commutes as $\operatorname{sgn}(\gamma) = \operatorname{sgn}(1+\gamma)$. But the right hand side is precisely the $\mathcal{M}$-action on $\pi^V_q(X)(1)$.

As the maps in (ii) and (iii) commute levelwise with $\sigma_*^{n-m} \cdot (- \wedge T^{n-m})$, it remains to show that they also commute with maps of the form

$$(V \wedge |\gamma|_T \wedge 1)^* \cdot \gamma_*.$$

For (ii), consider the diagram

$$\begin{array}{ccc}
[V \wedge T^{q+m}, \Omega X_n] & \xrightarrow{(1 \wedge |\gamma|_T \wedge 1)^* \cdot (\Omega \gamma)_*} & [V \wedge T^{q+m}, \Omega X_n] \\
\downarrow{\scriptstyle - \wedge T} & & \downarrow{\scriptstyle - \wedge T} \\
[V \wedge T^{q+m} \wedge T, \Omega X_n \wedge T] & \xrightarrow{(1 \wedge |\gamma|_T \wedge 1 \wedge T)^* \cdot (\Omega \gamma \wedge T)_*} & [V \wedge T^{q+m} \wedge T, \Omega X_n \wedge T] \\
\downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{ev}} \\
[V \wedge T^{q+m} \wedge T, X_n] & \xrightarrow{(1 \wedge |\gamma|_T \wedge 1 \wedge T)^* \cdot \gamma_*} & [V \wedge T^{q+m} \wedge T, X_n] \\
\downarrow{\scriptstyle (V \wedge \chi_{1,q+m})^*} & & \downarrow{\scriptstyle (V \wedge \chi_{1,q+m})^*} \\
[V \wedge T \wedge T^{q+m}, X_n] & \xrightarrow{(1 \wedge T \wedge |\gamma|_T \wedge 1)^* \cdot \gamma_*} & [V \wedge T \wedge T^{q+m}, X_n]
\end{array}$$

which commutes by the naturality of ev and $t_{-,-}$). In the last row, we have

$$1 \wedge T \wedge |\gamma|_T \wedge 1 = 1 \wedge |\gamma|_T \wedge 1$$

by Definition 2.16. As $\alpha = \mathrm{ev}\cdot(-\wedge T)$ the compatibility with the $\mathcal{M}$-action follows. For (iii), we consider the commutative diagram

$$\begin{array}{ccc}
[V\wedge T^{q+m}, X_n] & \xrightarrow{(1\wedge|\gamma|_T\wedge 1)^*\cdot\gamma_*} & [V\wedge T^{q+m}, X_n] \\
\downarrow{\scriptstyle (T\wedge -)} & & \downarrow{\scriptstyle (T\wedge -)} \\
[T\wedge V\wedge T^{q+m}, T\wedge X_n] & \xrightarrow{(T\wedge V\wedge|\gamma|_T\wedge 1)^*\cdot(T\wedge\gamma)_*} & [T\wedge V\wedge T^{q+m}, T\wedge X_n] \\
\downarrow{\scriptstyle (t_{V,T}\wedge 1)^*} & & \downarrow{\scriptstyle (t_{V,T}\wedge 1)^*} \\
[V\wedge T\wedge T^{q+m}, T\wedge X_n] & \xrightarrow{(V\wedge T\wedge|\gamma|_T\wedge 1)^*\cdot(T\wedge\gamma)_*} & [V\wedge T\wedge T^{q+m}, T\wedge X_n].
\end{array}$$

Here for the last row we have $V\wedge T\wedge|\gamma|_T\wedge T^{q+m-1} = V\wedge|\gamma|_T\wedge T^{q+m}$ by Definition 2.16, hence the third isomorphism also respects the $\mathcal{M}$-action.

The compatibility with the sign is shown by a similar argument. □

The proposition implies that the class of semistable spectra is stable under various operations (compare [Sch08, Section 4], [Sch12]):

Corollary 2.36. *Assume that $\mathrm{Sp}^\Sigma(\mathcal{D}, T)$ has a levelwise fibrant replacement functor. Then for any symmetric T-spectrum in $\mathcal{D}$, the following are equivalent:*

- *X is semistable.*
- *$T\wedge X$ is semistable.*
- *$\Omega J^\Sigma X$ is semistable.*
- *$\mathrm{sh}\, X$ is semistable.*

Proof. Most of this follows directly from Proposition 2.35. Concerning $\mathrm{sh}\, X$ it remains to show that for a tame $\mathcal{M}$-modul W the $\mathcal{M}$-action is trivial if and only if it is trivial on $W(1)$. But if the $\mathcal{M}$-action is trivial on $W(1)$, then W has filtration ≤ 1 and thus by Lemma 2.39 below the $\mathcal{M}$-action is trivial. □

Definition 2.37. Let X be a levelwise fibrant symmetric T-spectrum. We denote the composition of the $\mathcal{M}$-isomorphisms (i) and (ii) of Proposition 2.35 by $\alpha : \pi_q^V(RX) \cong \pi_q^V(X)(1)$.

The following will be used when proving Theorem 2.43:

Proposition 2.38. *Let X be a symmetric T-spectrum. The action of $d\in\mathcal{M}$ is isomorphic to the action of λ_X on stable homotopy groups, i.e., the square*

$$\begin{array}{ccc}
\pi_q^V(X) & \xrightarrow{d_*} & \pi_q^V(X)(1) \\
{\scriptstyle (-1)^q_T\cdot(T\wedge -)}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
\pi_{1+q}^V(T\wedge X) & \xrightarrow{\pi_{1+q}^V(\lambda_X)} & \pi_{1+q}^V(\mathrm{sh}\, X)
\end{array}$$

commutes. If X is levelwise fibrant, the for all $n \in \mathbb{N}_0$ the squares

$$\begin{array}{ccc} \pi_q^V(X) & \xrightarrow{d_*} & \pi_q^V(X)(1) \\ {\scriptstyle \tilde{\lambda}_{X\,*}\cdot(-1)_T^q}\downarrow{\scriptstyle \cong} & & \| \\ \pi_q^V(RX) & \xrightarrow{\alpha} & \pi_q^V(X)(1) \end{array} \quad \text{and} \quad \begin{array}{ccc} \pi_q^V(R^nX) & \xrightarrow{\alpha^n} & \pi_q^V(X)(n) \\ \downarrow{\scriptstyle \pi_q^V(\Omega^n\tilde{\lambda}_{\mathrm{sh}^n X})\cdot(-1)_T^q} & & \downarrow{\scriptstyle d_*} \\ \pi_q^V(R^{n+1}X) & \xrightarrow{\alpha^{n+1}} & [\pi_q^V(X)(n)](1) \end{array}$$

commute as well, the right d_* is the action of $d(n)$ on the underlying sets $\pi_q^V(X)$ (see Definition 2.34). In particular, the action of $d(n)$ on $\pi_q^V(X)$ is isomorphic to the map $\pi_q^V(R^n\tilde{\lambda}_X)$.

Proof. Let $f : V \wedge T^{q+m} \to X_m$ be a morphism in $\mathrm{Ho}(\mathcal{D})$. The first square commutes because

$$\begin{aligned} &[\lambda_{X,m*} \cdot (V \wedge (-1)_T^q \wedge 1)^* \cdot [(t_{V,T} \wedge T^{q+m})^* \cdot (T \wedge -)]](f) \\ &= (\chi_{m,1} \cdot \sigma_m^X \cdot t_{T,X_m}) \cdot (T \wedge f) \cdot (t_{V,T} \wedge T^{q+m}) \cdot (V \wedge (-1)_T^q \wedge 1) \\ &= \chi_{m,1} \cdot \sigma_m^X \cdot (f \wedge T) \cdot t_{T,V\wedge T^{q+m}} \cdot (t_{V,T} \wedge T^{q+m}) \cdot (V \wedge (-1)_T^q \wedge 1) \\ &= \chi_{m,1} \cdot \sigma_m^X \cdot (f \wedge T) \cdot (V \wedge \chi_{1,q+m}) \cdot (V \wedge (-1)_T^q \wedge 1) \\ &= \chi_{m,1} \cdot \sigma_m^X \cdot (f \wedge T) \cdot (V \wedge (-1)_T^{q+m} \wedge 1) \cdot (V \wedge (-1)_T^q \wedge 1) \\ &= \chi_{m,1} \cdot \sigma_m^X \cdot (f \wedge T) \cdot (V \wedge (-1)_T^m \wedge 1) \\ &= [\chi_{m,1*} \cdot (V \wedge (-1)_T^m \wedge 1)^* \cdot \sigma_*^X \cdot (- \wedge T)](f) = d_*(f). \end{aligned}$$

And similarly for the second square,

$$\begin{aligned} &[[(V \wedge \chi_{1,q+m})^* \cdot \alpha] \cdot \tilde{\lambda}_{X,m\,*} \cdot (V \wedge (-1)_T^q \wedge 1)^*](f) \\ &= \mathrm{ev} \cdot ([\tilde{\lambda}_{X,m} \cdot f \cdot (V \wedge (-1)_T^q \wedge 1)] \wedge T) \cdot (V \wedge \chi_{1,q+m}) \\ &= \chi_{m,1} \cdot \sigma_m^X \cdot (f \wedge T) \cdot (V \wedge \chi_{1,q+m}) \cdot (V \wedge (-1)_T^q \wedge 1) = d_*(f). \end{aligned}$$

Finally, following Schwede we observe that the commutativity of the third square follows from the second. To see this, consider the large commutative (note that the isomorphisms are compatible with the sign by Proposition 2.35) diagram

$$\begin{array}{ccccccc} & & & \overset{\alpha^n}{\frown} & & & \\ \pi_q^V(\Omega^n \mathrm{sh}^n X) & & \xrightarrow{\cong} & & \pi_{q+n}^V(\mathrm{sh}^n X) & \xrightarrow{\cong} & \pi_q^V(X) \\ \downarrow{\scriptstyle (\Omega^n\tilde{\lambda}_{\mathrm{sh}^n X})_*\cdot(-1)_T^q} & & & \swarrow{\scriptstyle (\tilde{\lambda}_{\mathrm{sh}^n X})_*\cdot(-1)_T^q} & \searrow{\scriptstyle d_*} & & \downarrow{\scriptstyle d(n)} \\ \pi_q^V(\Omega^n \mathrm{sh}^n RX) & \xrightarrow{\cong} & \pi_{q+n}^V(R\,\mathrm{sh}^n X) & & \xrightarrow[\cong]{\alpha} & \pi_{q+n}^V(\mathrm{sh}^n X) \xrightarrow{\cong} & \pi_q^V(X). \\ & & & \underset{\alpha^{n+1}}{\smile} & & & \end{array}$$

The last claim follows from Lemma 2.7 by which the morphisms $\tilde{\lambda}_{\mathrm{sh}^n X}$ and $\mathrm{sh}^n \tilde{\lambda}_X$ are isomorphic in $\mathrm{Sp}^\Sigma(\mathcal{D}, T)$. □

2.5. Generalities concerning the $\mathcal{M}$-action.

Lemma 2.39 (Schwede)**.** *Let W be a tame $\mathcal{M}$-module.*

(i) *Any element of $\mathcal{M}$ acts injectively on W.*
(ii) *If the filtration on W is bounded, then W is a trivial $\mathcal{M}$-module.*
(iii) *If $d \in \mathcal{M}$ acts surjectively on W, then W is a trivial $\mathcal{M}$-module.*
(iv) *If W is a finitely generated abelian group, then W is a trivial $\mathcal{M}$-module.*

Proof. See [Sch08, Lemma 2.3]. □

Lemma 2.40. *Let $F : \mathcal{I} \to \mathcal{D}$ be a functor and assume that $\mathcal{D}$ has sequential colimits and a forgetful functor to the category of sets. Then, if any element of $F(\omega)$ is in the image of some inclusion map $\mathrm{incl}_m^{F(\omega)}$, $F(\omega)$ is tame.*

Proof. It suffices to show that any $x \in F(\omega)$ arising via

$$y \in F(\mathbf{l}), \quad x = \mathrm{incl}_l^{F(\omega)}(y)$$

has filtration $\leq l$. Consider $f \in \mathcal{M}$ with $|f| = l$. By definition of $F(f)$, we have $F(f) \cdot \mathrm{incl}_l^{F(\omega)} = F(f_{|\mathbf{l}}) = \mathrm{incl}_l^{F(\omega)} \cdot F(1_{\mathbf{l}}) = \mathrm{incl}_l^{F(\omega)}$ as f restrics to $1_{\mathbf{l}}$. This yields $F(f)(x) = x$, so x has filtration $\leq l$. □

The next result describes several general properties of the construction which [Sch12] applies to the functors $\hat{\pi}_k$.

Proposition-Definition 2.41. *Let $\mathcal{D}$ be a category and $\mathcal{F}$ a class of functors from $\mathcal{D}$ to the category of $\mathcal{M}$-sets. Let $\mathcal{C} = \mathcal{D}_{\mathcal{F}}$ be the full subcategory of $\mathcal{D}$ of those X for which the $\mathcal{M}$-action on $F(X)$ is trivial for all $f \in \mathcal{F}$.*

(i) *For any $X \in \mathcal{D}, F \in \mathcal{F}$, consider the set $\tilde{F}(X)$ of natural transformations of functors $\mathcal{C} \to Set$ from $\mathcal{D}(X, -)$ to F. Then $\tilde{F}$ is a functor from $\mathcal{D}$ to $\mathcal{M}$-sets.*
(ii) *$\mathcal{M}$ acts trivially on $\tilde{F}(X)$.*
(iii) *There is a natural map $c_X : F(X) \to \tilde{F}(X)$ of $\mathcal{M}$-sets.*
(iv) *An object X of $\mathcal{D}$ is in $\mathcal{C}$ if and only if $c_X : F(X) \to \tilde{F}(X)$ is bijective (or equivalently injective).*

Proof. (i) Let $f : X_1 \to X_2$ be a map in $\mathcal{D}$, $g \in \tilde{F}(X_1)$ and $k : X_2 \to Y$ a map in $\mathcal{D}$ with Y in $\mathcal{C}$. The natural transformation g maps kf to an element $g'_Y(k) := g_Y(kf) \in F(Y)$. By naturality of g the assignment $k \mapsto g'_Y(k)$ is natural in Y. Hence we obtain a map $\tilde{F}(f) : \tilde{F}(X_1) \to \tilde{F}(X_2), g \mapsto g'$, and one easily verifies that $\tilde{F}$ is a functor. Now let $w \in \mathcal{M}$ and $g \in \tilde{F}(X_1)$. Then the composition $F_{|\mathcal{C}}(w) \cdot g \in \tilde{F}(X_1)$ is a natural transformation, thus defining an $\mathcal{M}$-action on $\tilde{F}(X_1)$. For any $f : X_1 \to X_2$ in $\mathcal{D}$ we then have $[w_* \tilde{F}(f)](g) = F_{|\mathcal{C}}(w) \cdot g(- \cdot f) = [F_{|\mathcal{C}}(w) \cdot g](- \cdot f) = [\tilde{F}(f) w_*](g)$. Therefore $\tilde{F}(f)$ respects the $\mathcal{M}$-action.

(ii) Let $w \in \mathcal{M}, g \in \tilde{F}(X)$ and $k : X \to Y$ with Y in $\mathcal{M}$. Then $\mathcal{M}$ acts trivially on $F(Y)$ and we have $[(F_{|\mathcal{C}}(w) \cdot g)(k)] = w_*[g(k)] = g(k)$, so $\mathcal{M}$ acts trivially on $\tilde{F}(X)$ as well.

(iii) The map c_X sents $x \in F(X)$ to the natural transformation

$$k \mapsto [F(k)](x) \quad (k : X \to Y,\ Y \text{ in } \mathcal{C}),$$

which is natural in X. For $w \in \mathcal{M}$ we have

$$\begin{aligned}[w_*(c_X(x))](k) &= [F_{|\mathcal{C}}(w) \cdot c_X(x)](k) = F_{|\mathcal{C}}(w)([F(k)](x)) \\ &= [F(k)](w_*(x)) = [c_X(w_*(x))](k)\end{aligned}$$

as F is compatible with $\mathcal{M}$. Hence c_X is a map of $\mathcal{M}$-sets.

(iv) Now let X in $\mathcal{C}$. By Yoneda $\mathrm{ev}_{1_X} : \tilde{F}(X) \to F(X)$ an 1_X is bijective with inverse c_X ($\mathrm{ev}_{1_X} \cdot c_X = 1_{F(X)}$). Conversely, if c_X is injective, then by (ii) and (iii) the action of $\mathcal{M}$ on $F(X)$ trivial, hence X is in $\mathcal{B}$. □

One can show that for $\mathcal{D}$ the category of symmetric spectra based on simplicial sets and $\mathcal{F}$ the set of stable homotopy groups $\hat{\pi}_k, k \in \mathbb{Z}$ the above definition of $\widetilde{\hat{\pi}_k}$ is isomorphic to the definition of the "true" stable homotopy groups. Later we will also need the follwing standard result.

Corollary 2.42. *Assume that fibrant objects $\mathcal{D}$ are closed under sequential colimits, and the functors j, $\mathrm{Hom}(T, -)$ and $\mathrm{Hom}(A, -)$ for all $A \in \mathcal{B}$ preserve sequential colimits. Then for any sequential diagram $X^\bullet$ in $\mathrm{Sp}(\mathcal{D}, T)$ the map*

$$\operatorname*{colim}_{n \geq 0} \pi_q^V(X^n) \xrightarrow{\mathrm{incl}_*} \pi_q^V(\operatorname{colim} X^\bullet)$$

is an isomorphism for all $q \in \mathbb{Z}, V \in \mathcal{B}$.

2.6. Criteria for semistability: the generalized theorem. We keep the hypotheses of Section 2.3. We now extend Theorem 2.10 (under additional assumptions), which simultaneously generalizes Schwede's Theorem 1.1.

Theorem 2.43. *Let $(\mathcal{D}, \wedge, S^0)$ be a pointed symmetric monoidal model category with a cofibrant object T, such that $- \wedge T$ preserves weak equivalences and T has a sign. Let $i : \mathrm{sSet}_* \to \mathcal{D}$ be a monoidal left Quillen functor with adjoint j. Let $\mathcal{B}$ be a class of cofibrant objects in $\mathcal{D}$. Moreover, assume that fibrant objects in $\mathcal{D}$ are closed under sequential colimits and that j, $\mathrm{Hom}(T, -)$ and $\mathrm{Hom}(A, -)$ for all $A \in \mathcal{B}$ preserve sequential colimits. Then for any T-spectrum X in $\mathcal{D}$ the following are equivalent:*

(i) *X is semistable (see Definition 2.31).*
(ii) *The cycle operator d (see Definition 2.34) acts surjectively on all stable homotopy groups.*
(iii) *The map $\lambda_X : T \wedge X \to \mathrm{sh}\, X$ is a $\pi^{\mathcal{B}}$- stable equivalence.*

If the class $\{\pi_q^V; q \in \mathbb{Z}, V \in \mathcal{B}\}$ *of functors from* $\mathrm{Sp}^\Sigma(\mathcal{D}, T)$ *to* $\mathcal{M}$*-sets satisfies the assumptions of Proposition-Definition* 2.41*, then* (i) *is also equivalent to*

(i′) *The map* $c_X : \pi_q^V(X) \to \widetilde{\pi_q^V}(X)$ *(Definition* 2.41*) is a bijection for all* $q \in \mathbb{Z}, V \in \mathcal{B}$*.*

If X *is level fibrant, then* (i)–(iii) *are also equivalent to:*

(iv) *The map* $\tilde{\lambda}_X : X \to RX$ *is a* $\pi^\mathcal{B}$*-stable equivalence.*

(v) *The map* $\tilde{\lambda}_X^\infty : X \to R^\infty X$ *is a* $\pi^\mathcal{B}$*-stable equivalence.*

(vi) *The symmetric spectrum* $R^\infty X$ *is semistable.*

Now consider the following conditions:

(1) *The projective level model structure on* $\mathrm{Sp}(\mathcal{D}, T)$ *exists and the conditions* (a) *and* (b) *of Theorem* 2.10 *are satisfied.*

(2) *The projective level model structure on* $\mathrm{Sp}^\Sigma(\mathcal{D}, T)$ *exists (in particular there is a level fibrant replacement functor* $1 \to J^\Sigma$*.*

(3) $\pi^\mathcal{B}$*-stable equivalences coincide with stable equivalences in* $\mathrm{Sp}(\mathcal{D}, T)$*.*

If (1)–(3) *hold, then* (i)–(iii) *are equivalent to* (viii)*, below, and if* X *is also level fibrant all above conditions are equivalent to* (vii):

(vii) *The symmetric spectrum* $R^\infty X$ *is an* Ω*-spectrum.*

(viii) *There is a* $\pi^\mathcal{B}$*-stable equivalence* X *to an* Ω*-spectrum.*

In any case, we always have the implications (viii)⇒(i) *and* (vii)⇒(vi)*.*

Proof. (i)⇔(ii) By definition (ii) follows from (i). Because of tameness (see Lemma 2.40), Lemma 2.39(iii) shows the converse.

(ii)⇔(iii) This follows from the first commutative diagram in Proposition 2.38.

(i)⇔(i′) follows from Proposition 2.41 and Definition 2.31.

(viii)⇒(ii) For any Ω-spectrum Z, $\tilde{\lambda}_Z$ is a level equivalence and hence a $\pi^\mathcal{B}$-stable equivalence. By (iv)⇒(ii)⇒(i) it follows that Ω-spectra are semistable. Lemma 2.33 then shows that X is semistable.

(vii)⇒(vi) We saw in (viii)⇒(ii) that Ω-spectra are semistable.

Now assume that X is level fibrant.

(ii)⇔(iv) By the second commutative square in Proposition 2.38, (iv) is equivalent to d acting bijectively on all $\pi^\mathcal{B}$-stable homotopy groups of X. Now use (i)⇔(ii).

(iv)⇒(v) As $\tilde{\lambda}_X$ is a $\pi^\mathcal{B}$-stable equivalence, so are $R^n\tilde{\lambda}_X, n \in \mathbb{N}_0$ as Ω and sh preserve $\pi^\mathcal{B}$-stable equivalences according to Proposition 2.35(i),(ii), By Corollary 2.42, the map $\pi_q^V(\tilde{\lambda}_X^\infty)$ is isomorphic to the inclusion

$$\pi_q^V(X) \xrightarrow{\mathrm{incl}_0} \operatorname*{colim}_{n\geq 0} \pi_q^V(R^n X).$$

But all the maps $\pi_q^V(R^n\tilde{\lambda}_X), n \in \mathbb{N}_0$ are isomorphisms, hence so is the inclusion and thus $\tilde{\lambda}_X^\infty$ is a $\pi^\mathcal{B}$-stable equivalence.

(v)⇒(ii) The maps $\pi_q^V(R^n\tilde{\lambda}_X), n \in \mathbb{N}_0$ are injective, because by Proposition 2.38 they are isomorphic to the action of $d(n)$ on $\pi_q^V(X)$, which again by Lemma 2.39 and 2.40 is injective. The inclusion

$$\pi_q^V(X) \xrightarrow{\text{incl}} \operatorname*{colim}_{n\geq 0} \pi_q^V(R^n X)$$

is an isomorphism, as it is isomorphic to $\pi_q^V(\tilde{\lambda}_X^\infty)$ (Corollary 2.42). As all maps in the sequential diagram $\pi_q^V(R^\bullet X)$ are injective, they must be surjective. Hence d acts surjectively on $\pi_q^V(X)$.

(iv)⇒(vi) As (iv) implies (v) and (ii), hence also (i), Lemma 2.33 shows that $R^\infty X$ is semistable.

(vi)⇒(i) We saw above ((v)⇒(ii)) that $\pi_q^V(\tilde{\lambda}_X^\infty) : \pi_q^V(X) \to \pi_q^V(R^\infty X)$ is injective and compatible with the $\mathcal{M}$-action. As the $\mathcal{M}$-action on $\pi_q^V(R^\infty X)$ is trivial, so is its restriction to $\pi_q^V(X)$.

Finally, we assume that hypotheses (1)–(3) are satisfied.

(iv)⇒(vii) By hypothesis $\tilde{\lambda}_X$ is a stable equivalence in $\mathrm{Sp}(\mathcal{D}, T)$. The implication (ii)⇒(iv) in Theorem 2.10 then yields the claim.

(i)⇒(viii) We have a $\pi^{\mathcal{B}}$-stable equivalence $X \to J^\Sigma X =: Y$ in $\mathrm{Sp}^\Sigma(\mathcal{D}, T)$. Using Lemma 2.33 we see that is $J^\Sigma X$ semistable, so the implications (i)⇒(v),(vii) show that $\tilde{\lambda}_Y^\infty : Y \to R^\infty Y$ is a $\pi^{\mathcal{B}}$-stable equivalence and $R^\infty Y$ an Ω-spectrum. □

Example 2.44. For suspension spectra $\Sigma^\infty L$ the map $\lambda_{\Sigma^\infty L}$ is already levelwise an isomorphism, as the structure maps σ are identities. Hence suspension spectra are semistable.

The above Theorem 2.43 is designed to apply notably to the motivic model category $M_\cdot^{cm}(S)$:

Proposition 2.45. *All assumptions (except for those preceding* (i′)*) of Theorem* 2.43 *are satisfied for* $\mathcal{D} = M_\cdot^{cm}(S), T = \mathbb{P}^1$,

$$\mathcal{B} = \{S^r \wedge \mathbb{G}_m^s \wedge U_+ \mid r, s \geq 0, U \in Sm/S\}.$$

Proof. Most of this has been proved in Corollary 2.12 already. Subsection 2.2.2 shows that $\mathbb{P}^1$ has a sign, and the projective level model structure on $\mathrm{Sp}^\Sigma(\mathcal{D}, T)$ is established in [Hov2, Theorem 8.2]. The $\pi^{\mathcal{B}}$-equivalences coincide with the stable equivalences in $\mathrm{Sp}(\mathcal{D}, T)$ by [Ja2, Section 3.2]. □

Sometimes sequential colimits preserve semistability:

Proposition 2.46. *Let* $X^\bullet$ *be a sequential diagram in* $\mathrm{Sp}^\Sigma(\mathcal{D}, T)$ *and assume that the hypotheses of Corollary* 2.42 *hold. If all* $X^n, n \in \mathbb{N}_0$ *are semistable, then so is* $\operatorname{colim} X^\bullet$.

Proof. Following Corollary 2.42, we have an isomorphism

$$\operatorname{colim} \pi_q^V(X^\bullet) \to \pi_q^V(\operatorname{colim} X^\bullet).$$

Now the maps $\pi_q^V(X^n) \xrightarrow{\mathrm{incl}_*} \pi_q^V(\operatorname{colim} X^\bullet)$ respect the $\mathcal{M}$-action and the sets $\pi_q^V(X^n)$, $n \in \mathbb{N}_0$ have trivial $\mathcal{M}$-action. As colimits preserve identities, $\mathcal{M}$ acts trivially on $\pi_q^V(\operatorname{colim} X^\bullet)$ as well. □

For $\mathcal{D} = \mathrm{Top}_*, T = S^1$ a special class of semistable spectra is given by orthogonal spectra (see [Sch08, Example 3.2].) These include not only suspension spectra, but also various Thom spectra. This is related to the following criteria:

Proposition 2.47. *A symmetric spectrum X is semistable if one of the following conditions hold:*

(i) *For any $q \in \mathbb{Z}$ and $V \in \mathcal{B}$ there is an $l \geq 0$ such that the inclusion map $[V \wedge T^{q+l}, X_l] \to \pi_q^V(X)$ is surjective. This holds, in particular, if the stable homotopy groups stabilize, i.e.,*

$$[V \wedge T^{q+n}, X_n] \to [V \wedge T^{q+n+1}, X_{n+1}]$$

is an isomorphism for $n \gg 0$.

(ii) *Even permutations on X_l induce identities in $\mathrm{Ho}(\mathcal{D})$.*

(iii) *The stable homotopy groups $\pi_q^V(X)$ are finitely generated abelian groups for all $q \in \mathbb{Z}$ and $V \in \mathcal{B}$.*

Proof. (i) According to Lemma 2.40 the filtration on $\pi_q^V(X)$ is bounded, hence by Lemma 2.39(ii) the $\mathcal{M}$-action on $\pi_q^V(X)$ is trivial.

(ii) We show that d acts trivially on $\pi_q^V(X)$. The following observation is crucial: For any even $n \in \mathbb{N}_0$ the map

$$[V \wedge T^{q+n+1}, T_{n+1}] \xrightarrow{\chi_{n,1*}(V \wedge |\chi_{n,1}|_T \wedge 1)^*} [V \wedge T^{q+n+1}, T_{n+1}]$$

is the identity. This is because $\chi_{n,1}$ is even, hence $|\chi_{n,1}|_T = 1$ (Definition 2.16), and $\chi_{n,1*}$ is the identity by assumption. Any element in $\pi_q^V(X)$ is (stably) represented by some $f \in [V \wedge T^{q+n}, T_n]$ with $n \in \mathbb{N}_0$ even. Therefore $d[f] = [\chi_{n,1*}(V \wedge |\chi_{n,1}|_T \wedge 1)^* \cdot \iota_*(f)] = [\iota_*(f)] = [f]$. Thus d acts trivially. Following Lemma 2.39, the $\mathcal{M}$-action on $\pi_q^V(X)$ is trivial.

(iii) By the tameness of $\pi_q^V(X)$ (use Lemma 2.40), this follows from Lemma 2.39(iv). □

Remark 2.48. The result in [RSØ, Proposition 3.2] provides exactly the same criterion as Propostion 2.47(ii).

The motivic stable homotopy category contains various spectra X which come with a natural action of the general linear group. If this action is compatible with the action of the symmetric group, then X is semistable:

Corollary 2.49. *Let E be a symmetric T-spectrum. Assume that for any $n \in \mathbb{N}_0$ there is an E'_n in $M.(S)$ with Σ_n-action, a zig-zag of Σ_n-equivariant maps between E_n and E'_n which are motivic weak equivalences and a map*

$$h.(\mathrm{GL}_{n\,S}) \wedge E'_n \to E'_n$$

in $M_\cdot(S)$ such that this linear action restrics to the given Σ_n-action on E'_n. Then E is semistable.

Proof. Let $\mathcal{D} = M_\cdot^{cm}(S)$ and $\tau \in \Sigma_n$ even with permutation matrix P_τ. By Lemma 2.25 we know that $\underline{P_\tau}$ and $\underline{\mathrm{id}}$ induce the same endomorphism on E'_n in $\mathrm{Ho}(\mathcal{D})$, and the latter is the identity by assumption. Hence any even permutation acts trivially on E'_n (in $\mathrm{Ho}(\mathcal{D})$) as it is conjugated to the action on E_n. Now apply Proposition 2.47(ii). □

Remark 2.50. In fact, one may define the notion of a motivic linear spectrum, using the canonical action of GL_n on $\mathbb{A}^n$ and the canonical isomorphisms $(\mathbb{A}^n/(\mathbb{A}^n - 0)) \wedge (\mathbb{A}^m/(\mathbb{A}^m - 0)) \cong (\mathbb{A}^{n+m}/(\mathbb{A}^{n+m} - 0))$ (see [MV, Proposition 3.2.17]). Then the forgetful functor from motivic linear spectra to motivic spectra with the projective, flat,... model structure should create a projective, flat... monoidal model structure on motivic linear spectra. Moreover, this forgetful functor has a right adjoint for formal reasons (see, e.g., [MMSS, Proposition 3.2]), and this Quillen adjunction is expected to be a Quillen equivalence. Motivic linear spectra will be a convenient framework for equivariant stable motivic homotopy theory.

3. Examples of semistable motivic symmetric spectra

In [RSØ] it is shown that algebraic K-theory may be represented by an explicit semistable motivic spectrum. In this section, we discuss two further examples. *In the following section, we only consider the motivic case, that is $\mathcal{D} = M_\cdot^{cm}(S), T = \mathbb{P}^1$, $\mathcal{B} = \{S^r \wedge \mathbb{G}_m^s \wedge U_+ | r, s \geq 0, U \in Sm/S\}$ as in Proposition* 2.45.

3.1. The motivic Eilenberg–Mac Lane spectrum. In [DRØ, Example 3.4], the motivic Eilenberg–Mac Lane spectrum is defined as the evaluation of a certain motivic functor on smash powers of T (see [DRØ, Abschnitt 3]). According to [DRØ, Lemma 4.6] this represents integral motivic cohomology, and this is the description we will use.

In general, consider a functor $H : M_\cdot(S) \to M_\cdot(S)$ with the following properties: First, there are natural functors

$$H_{A,B} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(H(A), H(B))$$

compatible with the composition and such that restriction to S and zero-simplices is just H on morphisms. Second, H maps motivic weak equivalences between projective cofibrant objects (see [DRØ, Section 2.1]) to motivic weak equivalences. We will see below that these two properties are sufficient to define a semistable motivic symmetric spectrum. To obtain the motivic Eilenberg–Mac Lane spectrum as in [DRØ, Example 3.4], we must take $H = u \circ \mathbb{Z}_{tr}$ where u denotes forgetting the transfers, and the second property holds by [DRØ, S. 524].

Let $\tilde{T}$ be a projective cofibrant replacement of $\mathbb{G}_m \wedge S^1$.

Definition 3.1. The *motivic Eilenberg–Mac Lane spectrum* $\mathbb{H}$ is the symmetric $\tilde{T}$-spectrum with $\mathbb{H}_n := H(\tilde{T}^n)$, Σ_n acting by permutation of the smash functors and structure maps $\mathbb{H}_n \wedge \tilde{T} \to \mathbb{H}_{n+1}$ adjoint to

$$\tilde{T} \xrightarrow{\text{unit}} \operatorname{Hom}(\tilde{T}^n, \tilde{T}^n \wedge \tilde{T}) \xrightarrow{H_{\tilde{T}^n, \tilde{T}^n \wedge \tilde{T}}} \operatorname{Hom}(H(\tilde{T}^n), H(\tilde{T}^n \wedge \tilde{T})).$$

Note that the compositions $\sigma_n^{\mathbb{H}\, l} : \mathbb{H}_n \wedge \tilde{T}^l \to \mathbb{H}_{n+l}$ of the structure maps are adjoint to $\tilde{T}^l \xrightarrow{\text{unit}} \operatorname{Hom}(\tilde{T}^n, \tilde{T}^n \wedge \tilde{T}^l) \xrightarrow{H} \operatorname{Hom}(H(\tilde{T}^n), H(\tilde{T}^{n+l}))$ because H is compatible with compositions on Hom, hence $\Sigma_n \times \Sigma_l$-equivariant.

The following lemmas show that $\mathbb{H}$ satisfies the assumptions of Corollary 2.49.

Lemma 3.2. *There is a zigzag of Σ_n-equivariant maps between $\tilde{T}^{\wedge n}$ and $T^n := h.(\mathbb{A}^n_S)//h.((\mathbb{A}^n - 0)_S)$, and this is a zigzag of motivic weak equivalences between projectively cofibrant pointed objects.*

Proof. Using Lemma 2.22, Lemma 2.23 and [MV, Lemma 3.2.13] we obtain the desired zigzag

$$\begin{array}{ccccc} h.(\mathbb{A}^n_S)//h.((\mathbb{A}^n - 0)_S) & \xrightarrow{\sim} & h.(\mathbb{A}^n_S)/h.((\mathbb{A}^n - 0)_S) & & \\ & & \uparrow \sim & & \\ & & (\mathbb{A}^1/\mathbb{G}_m)^{\wedge n} & \simeq & (\mathbb{G}_m \wedge S^1)^{\wedge n} \xleftarrow{\sim} \tilde{T}^n. \end{array}$$

if we replace everything projectively cofibrant. Choosing a functorial replacement, it is Σ_n-equivariant as well. □

Lemma 3.3. *There is a zigzag of motivic weak equivalences which are Σ_n-invariant between $\mathbb{H}_n$ and $H(h.(\mathbb{A}^n_S)//h.((\mathbb{A}^n - 0)_S))$.*

Proof. The zigzag of weak equivalences follows from Lemma 3.2 and the second above property of H, and equivariance follows from the first property. □

Lemma 3.4.

(i) *There is a map $h.(\mathrm{GL}_{n\,S}) \wedge T^n \to T^n$ extending the Σ_n-action on T^n.*
(ii) *There is a map $h.(\mathrm{GL}_{n\,S}) \wedge H(T^n) \to H(T^n)$ extending the Σ_n-action on $H(T^n)$.*

Proof. (i) We have a commutative diagram

$$\begin{array}{ccc} h.(\mathrm{GL}_{n\,S}) \wedge h.((\mathbb{A}^n - 0)_S) & \xrightarrow{\mu} & h.((\mathbb{A}^n - 0)_S) \\ \downarrow{\scriptstyle 1 \wedge h.(\mathrm{incl})} & & \downarrow{\scriptstyle h.(\mathrm{incl})} \\ h.(\mathrm{GL}_{n\,S}) \wedge h.(\mathbb{A}^n_S) & \xrightarrow{\mu} & h.(\mathbb{A}^n_S) \end{array}$$

where the maps μ extend the Σ_n-action. As the smash product commutes with colimits, the diagram induces a map

$$h.(\mathrm{GL}_{n\,S}) \wedge T^n \to T^n$$

extending the Σ_n-action.

(ii) The map in the first part is adjoint to a map

$$h.(\mathrm{GL}_{n\,S}) \to \mathrm{Hom}(T^n, T^n)$$

whose composition with H_{T^n,T^n} is adjoint to a map

$$h.(\mathrm{GL}_{n\,S}) \wedge H(T_2^n) \to H(T_2^n).$$

The latter extends the Σ_n-action because $H_{T^n,T^n}(S)$ is the map

$$M.(S)(T^n, T^n) \to M.(S)(H(T^n), H(T^n))$$

and the Σ_n-action on $H(T^n)$ is induced by the one on T^n. □

Corollary 3.5. *The motivic Eilenberg–MacLane spectrum $\mathbb{H}$ is semistable.*

Proof. This follows from Lemma 3.3, Lemma 3.4 and Corollary 2.49. □

3.2. The algebraic cobordism spectrum. [Vo, Abschnitt 6.3] gave the first definition of the algebraic cobordism spectrum. In [PY, Section 6.5] (see also [PPR2, Section 2.1]) it is shown how to construct it as a motivic symmetric commutative ring spectrum. We only care about the underlying motivic symmetric spectrum $\mathbb{MGL}$ (see Definition 3.8 below) and will show that it is semistable.

Recall the following definition of [MV]. Let X be an S-scheme and

$$\xi : E \to X$$

a vector bundle. Then the zero section $z(\xi) : X \to E$ of ξ is a closed immersion, and the *Thom space* $\mathrm{Th}(\xi)$ of ξ is the pointed motivic space $a[h.(E)/(h.(U(\xi))]$.

Lemma 3.6.

(i) *Let A be an S-scheme. Then $U(1_A) = \emptyset$, and there is a natural motivic pointed weak equivalence $h.(A) \to \mathrm{Th}(1_A)$.*

(ii) *Let X, X' be two S-schemes with vector bundles*

$$\xi : V \to X,$$
$$\xi' : V' \to X'.$$

Then $U(\xi \times_S \xi') = pr_1^{-1}(\xi) \cup pr_2^{-1}(\xi')$. Furthermore, we have a motivic pointed weak equivalence

$$\mathrm{Th}(\xi) \wedge \mathrm{Th}(\xi') \xrightarrow{\simeq} \mathrm{Th}(\xi \times_S \xi')$$

which is associative and commutes with the permutation of ξ *and* ξ'*. The composition* $h.(A) \wedge \mathrm{Th}(\xi) \to \mathrm{Th}(1_A) \wedge \mathrm{Th}(\xi) \to \mathrm{Th}(1_A \times_S \xi)$ *is denoted by* $\mathrm{Th}_{A,\xi}$*. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Th}(\xi) & & \\ \downarrow \cong & \searrow^{\mathrm{Th}(\cong)} & \\ h.(S) \wedge \mathrm{Th}(\xi) & \xrightarrow{\mathrm{Th}_{S,\xi}} & \mathrm{Th}(1_S \times_S \xi). \end{array}$$

Proof. Straightforward. □

Considering schemes as functors on commutative rings [DG, 4.4 Comparison Theorem in I, §1], we define Grassmannian schemes $\mathrm{Gr}(d,n)$ in the usual way (see [DG, I, §1, 3.4 and I, §2, 4.4]). The tautological bundle is denoted by $\xi_{n,d} : \tau(d,n) \to \mathrm{Gr}(d,n)$.

Lemma-Definition 3.7. *For* $m, n \geq 0$ *there is a commutative diagramm of* GL_n*-equivariant maps*

$$\begin{array}{ccc} \tau(n,nm) & \longrightarrow & \tau(n,n(m+1)) \\ \downarrow \xi_{n,nm} & & \downarrow \xi_{n,n(m+1)} \\ \mathrm{Gr}(n,nm) & \longrightarrow & \mathrm{Gr}(n,n(m+1)). \end{array}$$

The induced morphism $\xi_{n,nm} \to \xi_{n,n(m+1)}$ *will be denoted by* $\upsilon_{n,m}$*. Then* $U(\xi_{n,nm})$ *is mapped to* $U(\xi_{n,n(m+1)})$*.*

Proof. Straightforward. □

As before, we may restrict the GL_n-action to a Σ_n-action. Then we are ready for the definition of $\mathbb{MGL}$. Recall that $\overline{T}$ is the Thom space of the trivial line bundle on S.

Definition 3.8. The symmetric *algebraic cobordism spectrum* $\mathbb{MGL}$ is the underlying $\overline{T}$-Spectrum of the following motivic commutative ring spectrum:

- The sequence of motivic spaces

$$\mathbb{MGL}_n := \operatorname*{colim}_{m \geq 1}(\cdots \to \mathrm{Th}(\xi^S_{n,nm}) \xrightarrow{\mathrm{Th}(\upsilon_{n,m})} \mathrm{Th}(\xi^S_{n,n(m+1)}) \to \cdots),\ n \geq 0$$

  with the induced Σ_n-action.
- $\Sigma_n \times \Sigma_p$-equivariant multiplication maps

$$\mu_{n,p} : \mathbb{MGL}_n \wedge \mathbb{MGL}_p \to \mathbb{MGL}_{n+p}, n, p \geq 0$$

  induced by

$$\begin{aligned} \mathrm{Th}(\xi^S_{n,nm}) \wedge \mathrm{Th}(\xi^S_{p,pm}) &\to \mathrm{Th}(\xi^S_{n,nm} \times_S \xi^S_{p,pm}) \\ &\xrightarrow{\mathrm{Th}(\mu_{n,p,m})} \mathrm{Th}(\xi^S_{n+p,(n+p)m}). \end{aligned}$$

- Σ_n-equivariant unit maps $\iota_n : \overline{T}^n \to \mathbb{MGL}_n, n \geq 0$ which for $n \geq 1$ are given by the compositions

$$\overline{T}^n \cong \mathrm{Th}(\xi^S_{1,1})^{\wedge n} \to \mathrm{Th}(\xi^{S\times_S n}_{1,1}) \to \mathrm{Th}(\xi^S_{n,n}) \to \mathbb{MGL}_n$$

(and for $n = 0$ by $S^0 = h.(S) \to \mathrm{Th}(1_S) \cong \mathrm{Th}(\xi^S_{0,0}) \xrightarrow{\cong} \mathbb{MGL}_0$).

Now the semistability of $\mathbb{MGL}$ follows from the above discussion and (again) Corollary 2.49.

Corollary 3.9. *The motivic symmetric spectrum $\mathbb{MGL}$ is semistable.*

Proof. We have a morphism $a_{\mathbb{MGL}_n} : h.(\mathrm{GL}^S_n) \wedge \mathbb{MGL}_n \to \mathbb{MGL}_n$ in $M.(S)$ induced by the following commutative diagram:

$$\begin{array}{ccc}
h.(\mathrm{GL}^S_n) \wedge \mathrm{Th}(\xi^S_{n,nm}) & \xrightarrow{1\wedge \mathrm{Th}(\upsilon_{n,m})} & h.(\mathrm{GL}^S_n) \wedge \mathrm{Th}(\xi^S_{n,n(m+1)}) \\
\downarrow{\scriptstyle \mathrm{Th}_{\mathrm{GL}^S_n,\xi^S_{n,nm}}} & & \downarrow{\scriptstyle \mathrm{GL}^S_n,\xi^S_{n,n(m+1)}} \\
\mathrm{Th}(\mathrm{GL}^S_n \times_S \xi^S_{n,nm}) & \xrightarrow{\mathrm{Th}(\mathrm{GL}^S_n \times_S \upsilon_{n,m})} & \mathrm{Th}(\mathrm{GL}^S_n \times_S \xi^S_{n,n(m+1)}) \\
\downarrow{\scriptstyle \mathrm{Th}(a^S_{n,m})} & & \downarrow{\scriptstyle \mathrm{Th}(a^S_{n,m+1})} \\
\mathrm{Th}(\xi^S_{n,nm}) & \xrightarrow{\mathrm{Th}(\upsilon_{n,m})} & \mathrm{Th}(\xi^S_{n,n(m+1)}).
\end{array}$$

Here the top square commutes by naturality (see Lemma 3.6) and the bottom square by functoriality of Thom spaces and the GL_n-equivariance in Lemma 3.7. Now for $\tau \in \Sigma_n$ and $S \xrightarrow{f_\tau} \mathrm{GL}^S_n$ the associated matrix, the following square commutes (see Lemma 3.6):

$$\begin{array}{ccc}
\mathrm{Th}(\xi^S_{n,nm}) & & \\
\downarrow{\scriptstyle \cong} \quad \searrow^{\mathrm{Th}(\cong)} & & \searrow^{\mathrm{Th}(\tau_*)} \\
h.(S) \wedge \mathrm{Th}(\xi^S_{n,nm}) & \xrightarrow{\mathrm{Th}_{S,\xi^S_{n,nm}}} & \mathrm{Th}(S \times_S \xi^S_{n,nm}) \\
\downarrow{\scriptstyle h.(f_\tau)\wedge 1} & & \downarrow{\scriptstyle \mathrm{Th}(f_\tau \times_S 1)} \\
h.(\mathrm{GL}^S_n) \wedge \mathrm{Th}(\xi^S_{n,nm}) & \xrightarrow{\mathrm{Th}_{\mathrm{GL}^S_n,\xi^S_{n,nm}}} & \mathrm{Th}(\mathrm{GL}^S_n \times_S \xi^S_{n,nm}) \xrightarrow{\mathrm{Th}(a^S_{n,m})} \mathrm{Th}(\xi^S_{n,nm}).
\end{array}$$

Thus $h.(\mathrm{GL}^S_n) \wedge \mathbb{MGL}_n \to \mathbb{MGL}_n$ extends the Σ_n-action on $\mathbb{MGL}_n$, and the semistability follows from Corollary 2.49. □

4. The multiplicative structure on stable homotopy groups of symmetric ring spectra and its localizations

In this section, we will prove a generalization of [Sch07, Corollary I.4.69]. More precisely, we will show that the localization $R[1/x]$ (see below) of a semistable symmetric ring spectrum R with respect to a suitable x is again

semistable and the map $j : R \to R[1/x]$ behaves as expected on stable homotopy groups (see Section 4.2).

Throughout this section, we assume the following: The assumptions of Section 2.3 hold and T has a sign. The smash product in $\mathcal{D}$ preserves weak equivalences, which is the case for simplicial sets and motivic spaces by [MV, Lemma 3.2.13]. *We also assume that there is a commutative monoid N with zero, for any $r \in N$ a cofibrant object $\mathbb{S}^r$ and isomorphisms*

$$s_{r_1,r_2} : \mathbb{S}^{r_1+r_2} \to \mathbb{S}^{r_1} \wedge \mathbb{S}^{r_2}$$

in $\mathcal{D}$ for all $r_1, r_2 \in N$ such that the following hold:

- *There is an isomorphism $\cong^{\mathbb{S}^0}\colon \mathbb{S}^0 \cong S^0$.*
- *$s_{-,-}$ is associative.*
- *There are isomorphisms $s_{0,r} \cong l_{\mathbb{S}^r}^{-1}$ and $s_{r,0} \cong \rho_{\mathbb{S}^r}^{-1}$ (via $\mathbb{S}^0 \cong S^0$) (here l and ρ are the obvious structure morphisms, see* [Hov1, Chapter 4]*).*

Finally, we assume that there is a class of cofibrant objects $\mathcal{B}'$ in $\mathcal{D}$ with $\mathcal{B} = \{\mathbb{S}^r \wedge U | r \in N, U \in \mathcal{B}'\}$.

Example 4.1. The standard example is, of course, $N = \mathbb{N}_0$ and $\mathbb{S}^r = S^r = (S^1)^{\wedge r}$ together with the identities $S^{r_1+r_2} = S^{r_1} \wedge S^{r_2}$ (recall that the simplicial spheres are in $\mathcal{D}$ via i by assumption). If $\mathcal{D} = M_\cdot(S)$ and $\mathcal{B} = \{S^r \wedge \mathbb{G}_m^s \wedge U_+ | r, s \geq 0, U \in Sm/S\}$ as above, we may also consider $N = \mathbb{N}_0^2$ and $\mathbb{S}^r = S^{r'} \wedge \mathbb{G}_m^{\wedge r''}$ with $r = (r', r'')$ and the isomorphisms given by the obvious permutations. Note that in general $\mathbb{S}$ and T may be completely unrelated, but in the motivic case that we care about they are the same.

Definition 4.2. For any symmetric T-spectrum X we set

$$\pi_{r,q}^U(X) := \pi_q^{\mathbb{S}^r \wedge U}(X),$$

for alle $r \in N, U \in \mathcal{B}', q \in \mathbb{Z}$. We further set $\mathbb{S}^{t_{r,r'}} = s_{r,r'}^{-1} t_{\mathbb{S}^r,\mathbb{S}^{r'}} s_{r,r'}$ and obtain maps $t_{r',r} : \pi_{r'+r,q}^U(X) \to \pi_{r+r',q}^U(X)$ induced by the maps

$$[\mathbb{S}^{r'} \wedge \mathbb{S}^r \wedge U \wedge T^{q+m}, X_m] \xrightarrow{(\mathbb{S}^{t_{r,r'}} \wedge T \wedge T^{q+m})^*} [\mathbb{S}^r \wedge \mathbb{S}^{r'} \wedge U \wedge T^{q+m}, X_m].$$

In particular, we have $t_{0,r} = t_{r,0} = \mathrm{id}$ as $l_{\mathbb{S}^r} \circ t_{\mathbb{S}^r,S^0} = \rho_{\mathbb{S}^r}$.

In the motivic case, one of the indices is of course redundant. Namely, if $\mathbb{S}^r = S^{r'} \wedge \mathbb{G}_m^{\wedge r''}$ (hence $r = (r', r'')$ and $U = S^0$, we have

$$\pi_{r,q}^U(X) \cong \pi_{q+r'+r'',q+r'}^{\mathrm{mot}}(X),$$

where we used Voevodsky's indexing on the right hand side.

4.1. The multiplication on stable homotopy groups. The following generalizes the multiplication of stable homotopy groups for usual symmetric ring spectra (see, e.g., [Sch07, Section I.4.6]). The sign $(-1)_T^{q'n}$ below will be used to show that the product is compatible with stabilization. See [Sch07, Definition I.1.3] (resp. its obvious generalization) for the definition of a (commutative) symmetric ring spectra. In particular, for any symmetric ring spectrum R we have maps $\mu_{n,m} : R_n \times R_m \to R_{n+m}$. Recall also the definition of central elements $x : T^{l+m} \to R_m$ of [Sch07, Proposition I.4.61(i)]. Those are stable under smash multiplication: if $y : T^{k+n} \to R_n$ is another central element, then $\mu \circ (x \wedge y)$ is also central. If R is commutative, then of course all maps $T^{l+m} \to R_m$ are central.

Lemma 4.3. *Let R be a semistable symmetric T-ring spectrum. Then for any cofibrant objects U, V in $\mathcal{D}$ and $r, r' \in N, q, q' \in \mathbb{Z}$, there is a natural (in R, U, V) biadditive map*

$$m^{U,V,R}_{r,q,r',q'} : \pi^U_{r,q}(R) \times \pi^V_{r',q'}(R) \to \pi^{U\wedge V}_{r+r',q+q'}(R)$$

induced by

$$\begin{aligned}\cdot : [\mathbb{S}^r \wedge U \wedge T^{q+n}, R_n] \times [\mathbb{S}^{r'} \wedge V \wedge T^{q'+n'}, R_{n'}] \\ \to [\mathbb{S}^{r+r'} \wedge U \wedge V \wedge T^{q+q'+n+n'}, R_{n+n'}].\end{aligned}$$

This pairing maps (f, g) to the composition

$$\begin{aligned}&\mathbb{S}^{r+r'} \wedge U \wedge V \wedge T^{q+q'+n+n'} \\ &\xrightarrow{s_{r,r'} \wedge U \wedge V \wedge (-1)_T^{q'n} \wedge 1} \mathbb{S}^r \wedge \mathbb{S}^{r'} \wedge U \wedge V \wedge T^{q+n} \wedge T^{q'+n'} \\ &\xrightarrow{\eta^{r,r',U,V}_{q+n,q'+n'}} (\mathbb{S}^r \wedge U \wedge T^{q+n}) \wedge (\mathbb{S}^{r'} \wedge V \wedge T^{q'+n'}) \\ &\xrightarrow{f \wedge g} R_n \wedge R_{n'} \xrightarrow{\mu_{n,n'}} R_{n+n'}\end{aligned}$$

with $\eta^{r,r',U,V}_{q+n,q'+n'}$ being the obvious permutation of smash functors.

The product is associative, that is the square

$$\begin{array}{ccc} \pi^U_{r,q}(R) \times \pi^V_{r',q'}(R) \times \pi^W_{r'',q''}(R) & \xrightarrow{m\times 1} & \pi^{U\wedge V}_{r+r',q+q'}(R) \times \pi^W_{r'',q''}(R) \\ \downarrow{\scriptstyle 1\times m} & & \downarrow{\scriptstyle m} \\ \pi^U_{r,q}(R) \times \pi^{V\wedge W}_{r'+r'',q'+q''}(R) & \xrightarrow{1\times m} & \pi^{U\wedge V\wedge W}_{r+r'+r'',q+q'+q''}(R) \end{array}$$

commutes. It is compatible with the sign $(-1)_T$ in both variables, namely we have

$$(-1)_T(f \cdot g) = ((-1)_T f) \cdot g = f \cdot ((-1)_T g)$$

If the ring spectrum R is commutative, then the multiplication on stable homotopy groups is commutative, that is the square

$$\begin{array}{ccc} \pi^U_{r,q}(R) \times \pi^V_{r',q'}(R) & \xrightarrow{m^{U,V}} & \pi^{U\wedge V}_{r+r',q+q'}(R) \\ \downarrow t & & \searrow^{(-1)^{q'q}_T t_{r',r}} \\ \pi^V_{r',q'}(R) \times \pi^U_{r,q}(R) & \xrightarrow{m^{V,U}} \pi^{V\wedge U}_{r'+r,q'+q}(R) & \xrightarrow{t^*_{U,V}} \pi^{U\wedge V}_{r'+r,q'+q}(R) \end{array}$$

also commutes. Finally, if $f : \mathbb{S}^r \wedge U \wedge T^{q+n} \to R_n$ is central in $\mathcal{D}$, then $t^*_{U,V} \circ (g \cdot [f]) = (-1)^{q'q}_T t_{r',r}([f] \cdot g)$.

Proof.

Biadditivity. The product is biadditive already before stabilization. This is a long, but straightforward verification.

Associativity. We show that the product is associative already before stabilization. (The symbol η below denotes various obvious isomorphisms.) Let

$$\begin{aligned} f &\in [\mathbb{S}^r \wedge U \wedge T^{q+n}, R_n], \\ g &\in [\mathbb{S}^{r'} \wedge V \wedge T^{q'+n'}, R_{n'}], \\ h &\in [\mathbb{S}^{r''} \wedge W \wedge T^{q''+n''}, R_{n''}]. \end{aligned}$$

Then we have

$$\begin{aligned} f \cdot (g \cdot h) &= \mu_{n,n'+n''} \circ (f \wedge (g \cdot h)) \circ \eta^{r,r'+r'',U,V\wedge W}_{q+n,q'+q''+n'+n''} \\ &\quad \circ (s_{r,r'+r''} \wedge 1 \wedge (-1)^{(q'+q'')n}_T \wedge 1) \\ &= \mu_{n,n'+n''} \circ (f \wedge [\mu_{n',n''} \circ (g \wedge h) \circ \eta^{r',r'',V,W}_{q'+n',q''+n''} \\ &\quad \circ (s_{r',r''} \wedge 1 \wedge (-1)^{q''n'}_T \wedge 1)]) \\ &\quad \circ \eta^{r,r'+r'',U,V\wedge W}_{q+n,q'+q''+n'+n''} \circ (s_{r,r'+r''} \wedge 1 \wedge (-1)^{(q'+q'')n}_T \wedge 1) \\ &= [\mu_{n,n'+n''} \circ (1 \wedge \mu_{n',n''})] \circ (f \wedge g \wedge h) \\ &\quad \circ (1 \wedge [\eta^{r',r'',V,W}_{q'+n',q''+n''} \circ (s_{r',r''} \wedge 1 \wedge (-1)^{q''n'}_T \wedge 1)]) \\ &\quad \circ \eta^{r,r'+r'',U,V\wedge W}_{,q+n,q'+q''+n'+n''} \circ (s_{r,r'+r''} \wedge 1 \wedge (-1)^{(q'+q'')n}_T \wedge 1) \\ &= [\mu_{n,n'+n''} \circ (1 \wedge \mu_{n',n''})] \circ (f \wedge g \wedge h) \\ &\quad \circ [(1 \wedge \eta^{r',r'',V,W}_{q'+n',q''+n''}) \circ \eta^{r,r'+r'',U,V\wedge W}_{q+n,q'+q''+n'+n''}] \\ &\quad \circ [1 \wedge (s_{r',r''} \wedge 1 \wedge T^{q+n} \wedge (-1)^{q''n'}_T \wedge 1) \\ &\quad \circ (s_{r,r'+r''} \wedge 1 \wedge (-1)^{(q'+q'')n}_T \wedge 1)] \\ &= [\mu_{n,n'+n''} \circ (1 \wedge \mu_{n',n''})] \circ (f \wedge g \wedge h) \\ &\quad \circ [(1 \wedge \eta^{r',r'',V,W}_{q'+n',q''+n''}) \circ \eta^{r,r'+r'',U,V\wedge W}_{q+n,q'+q''+n'+n''}] \end{aligned}$$

$$
\begin{aligned}
&\circ [1 \wedge (s_{r',r''} \wedge 1 \wedge (-1)_T^{q''n'} \wedge 1) \\
&\circ (s_{r,r'+r''} \wedge 1 \wedge (-1)_T^{(q'+q'')n} \wedge 1)] \\
= &[\mu_{n,n'+n''} \circ (1 \wedge \mu_{n',n''})] \circ (f \wedge g \wedge h) \\
&\circ [(1 \wedge \eta^{r',r'',V,W}_{q'+n',q''+n''}) \circ \eta^{r,r'+r'',U,V\wedge W}_{q+n,q'+q''+n'+n''}] \\
&\circ (((1 \wedge s_{r',r''})s_{r,r'+r''}) \wedge 1 \wedge (-1)_T^{q''n'+(q'+q'')n} \wedge 1).
\end{aligned}
$$

Here the second last equality uses Definition 2.16, which yields

$$
\begin{aligned}
&\mathbb{S}^{r+r'+r''} \wedge U \wedge V \wedge W \wedge T^{q+n} \wedge (-1)_T^{q''n'} \wedge T^{q'+q''+n'+n''-1} \\
&\qquad = \mathbb{S}^{r+r'+r''} \wedge U \wedge V \wedge W \wedge (-1)_T^{q''n'} \wedge T^{q+q'+q''+n+n'+n''-1}.
\end{aligned}
$$

A similar computation (slightly easier, Definition 2.16 is not used here) shows that

$$
\begin{aligned}
&(f \cdot g) \cdot h \\
&= [\mu_{n+n',n''} \circ (\mu_{n,n'} \wedge 1)] \circ (f \wedge g \wedge h) \circ [(\eta^{r,r',U,V}_{q+n,q'+n'} \wedge 1) \\
&\quad \circ \eta^{r+r',r'',U\wedge V,W}_{q+q'+n+n',q''+n''}] \circ (((s_{r,r'} \wedge 1)s_{r+r',r''}) \wedge 1 \wedge (-1)_T^{q'n+q''(n+n')} \wedge 1).
\end{aligned}
$$

As R is associative, we have $\mu_{n,n'+n''} \circ (1 \wedge \mu_{n',n''}) = \mu_{n+n',n''} \circ (\mu_{n,n'} \wedge 1)$. Moreover

$$
(1 \wedge \eta^{r',r'',V,W}_{q'+n',q''+n''}) \circ \eta^{r,r'+r'',U,V\wedge W}_{q+n,q'+q''+n'+n''} = (\eta^{r,r',U,V}_{q+n,q'+n'} \wedge 1) \circ \eta^{r+r',r'',U\wedge V,W}_{q+q'+n+n',q''+n''}
$$

as both sides are induced by permutations, and finally

$$
\begin{aligned}
q'n + q''(n+n') &= q''n' + (q'+q'')n, \\
(s_{r,r'} \wedge 1)s_{r+r',r''} &= (1 \wedge s_{r',r''})s_{r,r'+r''}.
\end{aligned}
$$

Compatibility with stabilization. We show that the unstable product above is compatible with the stabilization $\iota_* := \sigma_* \cdot (- \wedge T)$ in both variables. For the second variable, we must show that

$$
\begin{array}{ccc}
[\mathbb{S}^r \wedge U \wedge T^{q+n}, R_n] \times [\mathbb{S}^{r'} \wedge V \wedge T^{q'+n'}, R_{n'}] & & \\
\downarrow{\scriptstyle 1\times\iota_*} & \searrow{\scriptstyle \cdot} & \\
[\mathbb{S}^r \wedge U \wedge T^{q+n}, R_n] \times [\mathbb{S}^{r'} \wedge V \wedge T^{q'+n'+1}, R_{n'+1}] & & [\mathbb{S}^{r+r'} \wedge U \wedge V \wedge T^{q+q'+n+n'}, R_{n+n'}] \downarrow{\scriptstyle \iota_*} \\
& \searrow{\scriptstyle \cdot} & \\
& & [\mathbb{S}^{r+r'} \wedge U \wedge V \wedge T^{q+q'+n+n'+1}, R_{n+n'+1}]
\end{array}
$$

commutes. For $f \in [\mathbb{S}^r \wedge U \wedge T^{q+n}, R_n]$, and

$$
c = l_T \circ l_{S^0 \wedge T} \circ (\cong \wedge 1) : \mathbb{S}^0 \wedge S^0 \wedge T \to T,
$$

we have

$$\begin{aligned}
&[f \cdot (\iota_1 \circ c)] \circ (\mathbb{S}^r \wedge \rho_U^{-1} \wedge T^{q+n+1}) \\
&= \mu_{n,1} \circ (f \wedge (\iota_1 \circ c)) \circ \eta^{r,0,U,S^0}_{q+n,0+1} \circ (s_{r,0} \wedge 1) \circ (\mathbb{S}^r \wedge \rho_U^{-1} \wedge T^{q+n+1}) \\
&= \mu_{n,1} \circ (f \wedge \iota_1) = \sigma_n \circ (f \wedge T) = \iota_*(f)
\end{aligned}$$

because of

$$\begin{aligned}
&(1 \wedge ([l_T l_{S^0 \wedge T} \circ ((\mathbb{S}^0 \to S^0) \wedge 1)]) \\
&\quad \circ (\mathbb{S}^r \wedge [(t_{\mathbb{S}^0, U \wedge T^{q+n}} \wedge S^0)(\mathbb{S}^0 \wedge U \wedge t_{S^0, T^{q+n}})] \wedge T) \circ (s_{r,0} \wedge \rho_U^{-1} \wedge T^{q+n+1}) \\
&= (1 \wedge l_T l_{S^0 \wedge T}) \circ (\mathbb{S}^r \wedge [(t_{S^0, U \wedge T^{q+n}} \wedge S^0)(S^0 \wedge U \wedge t_{S^0, T^{q+n}})] \wedge T) \\
&\quad \circ (\rho_{\mathbb{S}^r}^{-1} \wedge \rho_U^{-1} \wedge T^{q+n+1}) = 1.
\end{aligned}$$

Applying this to g and $f \cdot g$, together with associativity and naturality we obtain

$$\begin{aligned}
\iota_*(f \cdot g) &= [(f \cdot g) \cdot (\iota_1 c)] \circ (1 \wedge \rho_{U \wedge V}^{-1} \wedge 1) \\
&= [f \cdot (g \cdot (\iota_1 c))] \circ (1 \wedge U \wedge \rho_V^{-1} \wedge 1) \\
&= f \cdot [(g \cdot (\iota_1 c)) \circ (1 \wedge \rho_V^{-1} \wedge 1)] \\
&= f \cdot \iota_*(g).
\end{aligned}$$

This yields a map $[\mathbb{S}^r \wedge U \wedge T^{q+n}, R_n] \times \pi^U_{r',q'}(R) \to \pi^U_{r+r',q+q'}(R)$.

The first variable is more subtle. By

$$\begin{aligned}
\iota_*(f) \cdot g &= [(f \cdot \iota_1 c) \circ (\mathbb{S}^r \wedge \rho_U^{-1} \wedge T^{q+n+1})] \cdot g \\
&= [(f \cdot \iota_1 c) \cdot g] \circ (1 \wedge \rho_U^{-1} \wedge 1) \\
&= [f \cdot (\iota_1 c \cdot g)] \circ (1 \wedge l_V^{-1} \wedge 1) \\
&= f \cdot [(\iota_1 c \cdot g) \circ (\mathbb{S}^{r'} \wedge l_V^{-1} \wedge T^{1+q'+n'})]
\end{aligned}$$

and the above it suffices to show that $[(\iota_1 c \cdot g) \circ (\mathbb{S}^{r'} \wedge l_V^{-1} \wedge T^{1+q'+n'})] = [g]$ in $\pi^V_{r',q'}(R)$. For this, we first note that

$$(\iota_1 c \cdot g) \circ (\mathbb{S}^{r'} \wedge l_V^{-1} \wedge T^{1+q'+n'}) = \chi_{n',1} \circ \iota_*(g) \circ (1 \wedge (-1)^{n'}_T \wedge 1)$$

by the following computation:

$$\begin{aligned}
&(\iota_1 c \cdot g) \circ (\mathbb{S}^{r'} \wedge l_V^{-1} \wedge T^{1+q'+n'}) \\
&= \mu_{1,n'} \circ (\iota_1 \wedge 1) \circ (c \wedge g) \circ \eta^{0,r',S^0,V}_{0+1,q'+n'} \circ (s_{0,r'} \wedge 1 \wedge (-1)^{q'}_T \wedge 1) \\
&\quad \circ (\mathbb{S}^{r'} \wedge l_V^{-1} \wedge T^{1+q'+n'}) \\
&\underset{\text{central}}{=} [\chi_{n',1} \circ \mu_{n',1} \circ (1 \wedge \iota_1) \circ t_{T,R_{n'}}] \circ (c \wedge g) \circ \eta^{0,r',S^0,V}_{0+1,q'+n'} \\
&\quad \circ (s_{0,r'} \wedge l_V^{-1} \wedge T^{1+q'+n'}) \circ (1 \wedge (-1)^{q'}_T \wedge 1) \\
&= \chi_{n',1} \circ \mu_{n',1} \circ (g \wedge \iota_1) \circ [t_{T,\mathbb{S}^{r'} \wedge V \wedge T^{q'+n'}} \circ (c \wedge 1) \circ \eta^{0,r',S^0,V}_{0+1,q'+n'}
\end{aligned}$$

$$\begin{aligned}
&\circ (s_{0,r'} \wedge l_V^{-1} \wedge T^{1+q'+n'})] \circ (1 \wedge (-1)_T^{q'} \wedge 1) \\
&= \chi_{n',1} \circ \mu_{n',1} \circ (g \wedge \iota_1) \circ (\mathbb{S}^{r'} \wedge V \wedge t_{T,T^{q'+n'}}) \circ (1 \wedge (-1)_T^{q'} \wedge 1) \\
&= \chi_{n',1} \circ \iota_*(g) \circ (\mathbb{S}^{r'} \wedge V \wedge (-1)_T^{q'+n'} \wedge T^{q'+n'}) \circ (1 \wedge (-1)_T^{q'} \wedge 1) \\
&= \chi_{n',1} \circ \iota_*(g) \circ (1 \wedge (-1)_T^{n'} \wedge 1).
\end{aligned}$$

Stabilizing this, we obtain $[\chi_{n',1} \circ \iota_*(g) \circ (1 \wedge (-1)_T^{n'} \wedge 1)] = d[\iota_*(g)] = [g]$, because d acts trivially by semistability. Hence we have

$$\begin{aligned}
[\iota_*(f) \cdot g] &= [f \cdot \chi_{n',1} \circ \iota_*(g) \circ (1 \wedge (-1)_T^{n'} \wedge 1)] \\
&= f \cdot [\chi_{n',1} \circ \iota_*(g) \circ (1 \wedge (-1)_T^{n'} \wedge 1)] = f \cdot [g] = [f \cdot g].
\end{aligned}$$

Compatibility with signs. This follows by the naturality of the permutation map $\eta^{r,r',U,V}_{q+n,q'+n'}$ together with the second property of Definition 2.16:

$$\begin{aligned}
&[(\mathbb{S}^r \wedge U \wedge (-1)_T \wedge T^{q+n-1}) \wedge 1] \circ \eta^{r,r',U,V}_{q+n,q'+n'} \\
&\qquad = \eta^{r,r',U,V}_{q+n,q'+n'} \circ (1 \wedge (-1)_T \wedge T^{q+n-1} \wedge T^{q'+n'}) \\
&[1 \wedge (\mathbb{S}^{r'} \wedge U \wedge (-1)_T \wedge T^{q'+n'-1})] \circ \eta^{r,r',U,V}_{q+n,q'+n'} \\
&\qquad = \eta^{r,r',U,V}_{q+n,q'+n'} \circ (1 \wedge T^{q+n} \wedge (-1)_T \wedge T^{q'+n'-1}) \\
&1 \wedge T^{q+n} \wedge (-1)_T \wedge T^{q'+n'-1} = 1 \wedge (-1)_T \wedge T^{q+n-1} \wedge T^{q'+n'}.
\end{aligned}$$

Commutativity. We have a commutative diagram

$$\begin{array}{ccc}
\mathbb{S}^{r+r'} \wedge U \wedge V \wedge T^{q+q'+n+n'} & \xrightarrow{\alpha} & \mathbb{S}^{r'+r} \wedge U \wedge V \wedge T^{q'+q+n'+n} \\
\downarrow{\scriptstyle s_{r,r'} \wedge 1 \wedge (-1)_T^{q'n} \wedge 1} & & \downarrow{\scriptstyle s_{r',r} \wedge 1 \wedge t_{U,V} \wedge (-1)_T^{qn'} \wedge 1} \\
\mathbb{S}^r \wedge \mathbb{S}^{r'} \wedge U \wedge V \wedge T^{q+n} \wedge T^{q'+n'} & \xrightarrow{\beta} & \mathbb{S}^{r'} \wedge \mathbb{S}^r \wedge V \wedge U \wedge T^{q'+n'} \wedge T^{q+n} \\
\downarrow{\scriptstyle \eta^{r,r',U,V}_{q+n,q'+n'}} & & \downarrow{\scriptstyle \eta^{r',r,V,U}_{q'+n',q+n}} \\
(\mathbb{S}^r \wedge U \wedge T^{q+n}) \wedge (\mathbb{S}^{r'} \wedge V \wedge T^{q'+n'}) & \xrightarrow{\gamma} & (\mathbb{S}^{r'} \wedge V \wedge T^{q'+n'}) \wedge (\mathbb{S}^r \wedge U \wedge T^{q+n})
\end{array},$$

where $\beta = t_{\mathbb{S}^r,\mathbb{S}^{r'}} \wedge t_{U,V} \wedge t_{T^{q+n},T^{q'+n'}}$, $\gamma = t_{\mathbb{S}^r \wedge U \wedge T^{q+n},\mathbb{S}^{r'} \wedge V \wedge T^{q'+n'}}$, and

$$\alpha = \mathbb{S}^{t_{r,r'}} \wedge U \wedge V \wedge [((-1)_T^{qn'} \wedge 1) t_{T^{q+n},T^{q'+n'}} ((-1)_T^{q'n} \wedge 1)],$$

for which we have

$$\begin{aligned}
&((-1)_T^{qn'} \wedge 1) t_{T^{q+n},T^{q'+n'}} ((-1)_T^{q'n} \wedge 1) \\
&= (-1)_T^{qn'+(q+n)(q'+n')q'n} \wedge T^{q+q'+n+n'-1} \\
&= (-1)_T^{qq'+nn'} \wedge T^{q+q'+n+n'-1}.
\end{aligned}$$

If $f : \mathbb{S}^r \wedge U \wedge T^{q+n} \to R_n$ is central (e.g., if R is commutative), we have $\chi_{n,n'} \circ \mu_{n,n'} \circ (f \wedge 1) = \mu_{n',n} \circ (1 \wedge f) \circ t_{\mathbb{S}^r \wedge U \wedge T^{q+n},R_{n'}}$. Together with the

above commutative diagram, for $g \in [\mathbb{S}^{r'} \wedge V \wedge T^{q'+n'}, R_{n'}]$ we then deduce

$$\begin{aligned}
&\chi_{n,n'}(f \cdot g)(1 \wedge (-1)_T^{nn'} \wedge 1) \\
&= \chi_{n,n'} \circ \mu_{n,n'} \circ (f \wedge g) \circ \eta^{r,r',U,V}_{q+n,q'+n'} \circ (s_{r,r'} \wedge 1 \wedge (-1)_T^{q'n} \wedge 1) \\
&\quad \circ (1 \wedge (-1)_T^{nn'} \wedge 1) \\
&= \mu_{n',n} \circ (g \wedge f) \circ [t_{\mathbb{S}^r \wedge U \wedge T^{q+n}, \mathbb{S}^{r'} \wedge V \wedge T^{q'+n'}} \circ \eta^{r,r',U,V}_{q+n,q'+n'} \\
&\quad \circ (s_{r,r'} \wedge 1 \wedge (-1)_T^{q'n} \wedge 1)] \circ (1 \wedge (-1)_T^{nn'} \wedge 1) \\
&= \mu_{n',n} \circ (g \wedge f) \circ [\eta^{r',r,V,U}_{q'+n',q+n} \circ (s_{r',r} \wedge 1 \wedge t_{U,V} \wedge (-1)_T^{qn'} \wedge 1) \\
&\quad \circ (\mathbb{S}^{t_{r,r'}} \wedge U \wedge V \wedge [(-1)_T^{qq'+nn'} \wedge 1])] \circ (1 \wedge (-1)_T^{nn'} \wedge 1) \\
&= (g \cdot f)(\mathbb{S}^{t_{r,r'}} \wedge 1 \wedge (-1)_T^{qq'} \wedge 1).
\end{aligned}$$

As R is semistable, this implies $[f \cdot g] = [\chi_{n,n'}(f \cdot g)(1 \wedge (-1)_T^{nn'} \wedge 1)]$, which yields commutativity. □

To obtain an internal product on stable homotopy groups, we assume from now on that there are natural transformations $\mathrm{diag}^U : U \to U \wedge U$ and $\omega_U : U \to S^0$ for any $U \in \mathcal{B}'$ making U a commutative comonoid in $\mathcal{D}$.

Example 4.4. In sSet_* or $M_\cdot(S)$ we have

$$\mathrm{diag} : K_+ \xrightarrow{\mathrm{diag}} (K \times K)_+ \cong K_+ \wedge K_+$$

for any K in sSet or $M(S)$.

We set $l^2_{T^l} := l_{T^l} \circ (S^0 \wedge l_{T^l})$ and define c_l to be the map

$$\mathbb{S}^0 \wedge U \wedge T^l \xrightarrow{\cong \wedge \omega_U \wedge T^l} S^0 \wedge S^0 \wedge T^l \xrightarrow{l^2_{T^l}} T^l.$$

In particular, $c_l \wedge T^n = c_{l+n}$, and $c_l = \mathrm{id}$ if $U = S^0$.

Proposition-Definition 4.5 (multiplicative structure on stable homotopy groups)**.** *Let R be a semistabile symmetric T-ring spectrum. Then we have a natural (in R) structure of a $N \times \mathbb{Z}$-graded ring on*

$$\pi^U_{*,*}(R) := \oplus_{(r,q) \in N \times \mathbb{Z}} \pi^U_{r,q}(R),$$

induced by taking colimits of the following biadditive maps ($q+n, q'+n' \geq 1$ as usual):

$$\cdot : [\mathbb{S}^r \wedge U \wedge T^{q+n}, R_n] \times [\mathbb{S}^{r'} \wedge U \wedge T^{q'+n'}, R_{n'}] \to [\mathbb{S}^{r+r'} \wedge U \wedge T^{q+q'+n+n'}, R_{n+n'}].$$

Here a pair (f, g) is mapped to the composition

$$\begin{aligned}
&\mathbb{S}^{r+r'} \wedge U \wedge T^{q+q'+n+n'} \\
&\xrightarrow{s_{r,r'} \wedge \mathrm{diag}^U \wedge (-1)_T^{q'n} \wedge 1} \mathbb{S}^r \wedge \mathbb{S}^{r'} \wedge U \wedge U \wedge T^{q+n} \wedge T^{q'+n'} \\
&\xrightarrow{\mathbb{S}^r \wedge t_{\mathbb{S}^{r'} \wedge U, U \wedge T^{q+n}} \wedge T^{q'+n'}} \mathbb{S}^r \wedge U \wedge T^{q+n} \wedge \mathbb{S}^{r'} \wedge U \wedge T^{q'+n'}
\end{aligned}$$

$$\xrightarrow{f\wedge g} R_n \wedge R_{n'} \xrightarrow{\mu_{n,n'}} R_{n+n'}$$

$(q+n, q'+n' \geq 1)$.

The product is compatible with the signs, and graded commutative if R is commutative:

$$f \cdot g = (-1)_T^{q\cdot q'} t_{r',r}(g \cdot f),$$

for any $f \in \pi^U_{r,q}(R)$ and $g \in \pi^U_{r',q'}(R)$. For the latter equality it suffices that f is represented by a central map.

Proof. The multiplication decomposes as the external product of Lemma 4.3 and the diagonal:

$$\begin{array}{ccc} \pi^U_{r,q}(R) \times \pi^U_{r',q'}(R) & \xrightarrow{m^{U,U,R}_{r,q,r',q'}} & \pi^{U\wedge U}_{r+r',q+q'}(R) \\ & \searrow^{m^{U,R}_{r,q,r',q'}} & \downarrow{\scriptstyle \pi^{\mathrm{diag}^U}_{r+r',q+q'}(R)} \\ & & \pi^U_{r+r',q+q'}(R) \end{array}$$

because the map

$$\begin{aligned} &\mathbb{S}^r \wedge \mathbb{S}^{r'} \wedge V \wedge U \wedge T^{q+n} \wedge T^{q'+n'} \\ &\qquad \xrightarrow{1\wedge t_{V,U}\wedge 1} \mathbb{S}^r \wedge \mathbb{S}^{r'} \wedge U \wedge V \wedge T^{q+n} \wedge T^{q'+n'} \\ &\qquad \xrightarrow{\eta^{r,r',U,V}_{q+n,q'+n'}} \mathbb{S}^r \wedge U \wedge T^{q+n} \wedge \mathbb{S}^{r'} \wedge V \wedge T^{q'+n'} \end{aligned}$$

coincides with $\mathbb{S}^r \wedge t_{\mathbb{S}^{r'}\wedge V,U\wedge T^{q+n}} \wedge T^{q'+n'}$, because diag^- is cocommutative. As it is also coassociative, the product is also associative. Compatibility with the signs is clear, and commutativity follows from

$$(\mathrm{diag}^U)^* \circ t^*_{U,U} = (t_{U,U} \circ \mathrm{diag}^U)^* = (\mathrm{diag}^U)^*.$$

Another computation using the previous lemma shows that

$$[f] = [\iota_*(f)] = [f] \cdot [\iota_1 c_1]$$

and (note that $\iota_1 c_1$ is central)

$$[f] \cdot [\iota_1 c_1] = (-1)_T^{0\cdot q} t_{0,r}([\iota_1 c_1] \cdot [f]) = [\iota_1 c_1] \cdot [f]. \qquad \square$$

4.2. Localization of ring spectra. We are now ready to define the localization of a symmetric ring spectrum with respect to a central map, generalizing [Sch07, Example I.4.65]:

Proposition-Definition 4.6. *Let R be a symmetric ring spectrum and $x : T^l \to R_m$ a central map. Then we define a symmetric ring spectrum $R[1/x]$ together with a map of symmetric ring spectra $j : R \to R[1/x]$ as*

follows. Levelwise, we set $R[1/x]_p = \mathrm{Hom}(T^{lp}, R_{(1+m)p})$. *There are maps* $\Delta_{s,p} : \Sigma_p \to \Sigma_{sp}$,

$$\Delta_{s,p}(\gamma)(i + s \cdot (j-1)) = i + s \cdot (\gamma(j) - 1),$$

$1 \leq i \leq s$, $1 \leq j \leq p$ *permuting the* p *summands of* $sp = s + s + \cdots + s$. *Now* Σ_p *via* $\Delta_{l,p}$ *acts on* T^{lp}, *then via* $\Delta_{1+m,p}$ *on* $R_{(1+m)p}$ *and finally by conjugation on* $\mathrm{Hom}(T^{lp}, R_{(1+m)p})$. *Hence the square*

$$\begin{array}{ccc} R[1/x]_p \wedge T^{lp} & \xrightarrow{\gamma \wedge 1} & R[1/x]_p \wedge T^{lp} \\ \downarrow{\scriptstyle 1 \wedge \Delta_{l,p}(\gamma^{-1})} & & \downarrow{\scriptstyle \mathrm{ev}} \\ R[1/x]_p \wedge T^{lp} & \xrightarrow{\mathrm{ev}} & R_{(1+m)p} \xrightarrow{\Delta_{1+m,p}(\gamma)} R_{(1+m)p} \end{array}$$

is commutative. The multiplication

$$\mu_{p,q} : R[1/x]_p \wedge R[1/x]_q \to R[1/x]_{p+q}$$

is by definition the adjoint of

$$\begin{array}{ccc} R[1/x]_p \wedge R[1/x]_q \wedge T^{l(p+q)} & \xrightarrow{1 \wedge t_{R[1/x]_q, T^{lp}} \wedge 1} & R[1/x]_p \wedge T^{lp} \wedge R[1/x]_q \wedge T^{lq} \\ & & \downarrow{\scriptstyle \mathrm{ev} \wedge \mathrm{ev}} \\ & & R_{(1+m)p} \wedge R_{(1+m)q} \\ & & \downarrow{\scriptstyle \mu_{(1+m)p,(1+m)q}} \\ & & R_{(1+m)(p+q)}. \end{array}$$

The unit of $R[1/x]$ *is the composition of the unit of* R *with* j. *The map* j *is defined by* $j_p : R_p \to R[1/x]_p$ *being the adjoint to*

$$R_p \wedge T^{lp} \xrightarrow{1 \wedge x^p} R_p \wedge R_{mp} \xrightarrow{\mu_{p,mp}} R_{p+mp} \xrightarrow{\xi_{m,p}} R_{(1+m)p}.$$

Here x^p *means of course* $T^{lp} \xrightarrow{x^{\wedge p}} R_m^{\wedge p} \xrightarrow{\mu_{m,m,\ldots,m}} R_{mp}$, *and* $x^0 = \iota_0^R$. *The permutation* $\xi_{m,p} \in \Sigma_{(1+m)p}$ *is defined as follows:*

$$\xi_{m,p}(k) = \begin{cases} 1 + (1+m) \cdot (k-1) & \text{if } 1 \leq k \leq p \\ 1 + j + (1+m)(i-1) & \text{if } k = p + mi + j \\ & \quad \text{with } 1 \leq i \leq p, 1 \leq j \leq m. \end{cases}$$

Proof. Again, this is very long but essentially straightforward. To show the required properties (the multiplication maps are equivariant, the multiplication is associative, the claims about the unit and about j) one shows them for the adjoints. For example: for the equivariance of the μ, let

$(\gamma,\delta)\in\Sigma_p\times\Sigma_q\subseteq\Sigma_{p+q}$. We must show that $(\gamma+\delta)\cdot\mu_{p,q}^{R[1/x]}=\mu_{p,q}^{R[1/x]}\cdot(\gamma\wedge\delta)$. The left hand side is adjoint to

$$\Delta_{1+m,p+q}(\gamma+\delta)\cdot\widehat{\mu_{p,q}^{R[1/x]}}\cdot(R[1/x]_p\wedge R[1/x]_q\wedge\Delta_{1+m,p+q}(\gamma+\delta)^{-1}).$$

The right hand side is adjoint to $\widehat{\mu_{p,q}^{R[1/x]}}\cdot(\gamma\wedge\delta\wedge T^{l(p+q)})$, and one shows that these adjoints coincide. The claims about j also use the fact that central elements are stable under multiplication. □

The next results will be used to prove the Main Theorem 4.11.

Lemma 4.7. *Let $\gamma\in\Sigma_p$ and $s\in\mathbb{N}$. Then $\operatorname{sgn}(\Delta_{s,p}(\gamma))=\operatorname{sgn}(\gamma)^s$.*

Proof. The map $\Delta_{s,p}:\Sigma_p\to\Sigma_{sp}$ is a group homomorphism by definition, so we only need to show the claim for the generators $(\sigma_i=\tau_{i,i+1},1\le i\le p-1)$. For those, we have $\Delta_{s,p}(\sigma_i)=(s(i-1)+\chi_{s,s}+s(p-(i+1)))$ and thus $\operatorname{sgn}(\Delta_{s,p}(\sigma_i))=\operatorname{sgn}(\chi_{s,s})=(-1)^{s^2}=(-1)^s=\operatorname{sgn}(\sigma_i)^s$. □

Corollary 4.8. *For any $f\in\pi_{r,q}^U(R[1/x])$ we have*

$$f\cdot j_*([xc_l])=j_*[(xc_l)]\cdot((-1)_T^{(l-m)q}f).$$

Proof. One checks that j_mx and hence j_mxc_l is central and that

$$[j_mxc_l]=j_*([xc_l]).$$

Now the claim follows from the commutativity claim in Proposition 4.5 and $t_{0,r}=\mathrm{id}$. □

Lemma 4.9. *Let R be a symmetric T-ring spectrum and $x:T^l\to R_m$ a central map in $\mathcal{D}$. Let $f:\mathbb{S}^r\wedge U\wedge T^{q+n}\to R[1/x]_n$ be a map in $\mathrm{Ho}(\mathcal{D})$ and $\hat f:=\mathrm{ev}\circ(f\wedge T^{ln}):\mathbb{S}^r\wedge U\wedge T^{q+n}\wedge T^{ln}\to R_{(1+m)n}$. Then for*

$$\iota_*^\alpha(f)=\sigma_n^{\alpha,R[1/x]}\circ(f\wedge T^\alpha):\mathbb{S}^r\wedge U\wedge T^{q+n}\wedge T^\alpha\to R[1/x]_{n+\alpha},$$

$\alpha\in\mathbb{N}$, we have for the associated map

$$\begin{aligned}\widehat{\iota_*^\alpha(f)}&:=\mathrm{ev}\circ(\iota_*^\alpha(f)\wedge T^{l(n+\alpha)})\\&=(1+\xi_{m,\alpha})\circ\mu^R_{(1+m)n+\alpha,m\alpha}\circ(\iota_*^\alpha(\hat f)\wedge x^\alpha)\\&\quad\circ(\mathbb{S}^r\wedge U\wedge T^{q+n}\wedge t_{T^\alpha,T^{ln}}\wedge T^{l\alpha}).\end{aligned}$$

Proof. Because of $\sigma^{\alpha,R[1/x]}=\mu_{n,\alpha}^{R[1/x]}\circ(R[1/x]_n\wedge\iota_\alpha^{R[1/x]})$, $\iota_\alpha^{R[1/x]}=j_\alpha\circ\iota_\alpha^R$ and $\sigma^{\alpha,R}=\mu_{n,\alpha}^R\circ(R_n\wedge\iota_\alpha^R)$ we have for the associated map

$$\begin{aligned}\widehat{\iota_*^\alpha(f)}&=\mathrm{ev}\circ(\iota_*^\alpha(f)\wedge T^{l(n+\alpha)})\\&=\mathrm{ev}\circ([\mu_{n,\alpha}^{R[1/x]}\circ(f\wedge(j_\alpha\circ\iota_\alpha^R))]\wedge T^{l(n+\alpha)})\\&=\mu_{(1+m)n,(1+m)\alpha}\circ((\mathrm{ev}\circ(f\wedge T^{ln}))\wedge(\mathrm{ev}\circ((j_\alpha\circ\iota_\alpha^R)\wedge T^{l\alpha})))\\&\quad\circ(1\wedge t_{T^\alpha,T^{ln}}\wedge 1)\end{aligned}$$

$$
\begin{aligned}
&= \mu_{(1+m)n,(1+m)\alpha} \circ (\hat{f} \wedge (\xi_{m,\alpha} \circ \mu_{\alpha,m\alpha} \circ (\iota_\alpha^R \wedge x^\alpha)) \circ (1 \wedge t_{T^\alpha,T^{ln}} \wedge 1) \\
&= (1 + \xi_{m,\alpha}) \circ \mu_{(1+m)n,(1+m)\alpha} \circ (1 \wedge \mu_{\alpha,m\alpha}) \circ (\hat{f} \wedge \iota_\alpha^R \wedge x^\alpha) \\
&\quad \circ (1 \wedge t_{T^\alpha,T^{ln}} \wedge 1) \\
&= (1 + \xi_{m,\alpha}) \circ \mu_{(1+m)n+\alpha,m\alpha} \circ (\mu_{(1+m)n,\alpha} \wedge 1) \circ (\hat{f} \wedge \iota_\alpha^R \wedge x^\alpha) \\
&\quad \circ (1 \wedge t_{T^\alpha,T^{ln}} \wedge 1) \\
&= (1 + \xi_{m,\alpha}) \circ \mu_{(1+m)n+\alpha,m\alpha} \circ (\iota_*^\alpha(\hat{f}) \wedge x^\alpha) \circ (1 \wedge t_{T^\alpha,T^{ln}} \wedge 1). \qquad \square
\end{aligned}
$$

Lemma 4.10. *Let R be a levelwise fibrant semistabile symmetric T-ring spectrum and $x : T^l \to R_m$ a central map. Then for any*

$$f, g \in [\mathbb{S}^r \wedge U \wedge T^{q+n}, R[1/x]_n]$$

with $\hat{f} = (-1)_T^\nu(\xi \circ \hat{g})$ for some fixed $\nu \in \mathbb{Z}, \xi \in \Sigma_{(1+m)n}$, we have

$$[f] = ((-1)_T^\nu |\xi|_T)[g]$$

in $\pi_{r,q}^U(R[1/x])$.

Proof. As R is semistable, there is an $\alpha \in \mathbb{N}$ for which $\iota_*^\alpha(|\xi|_T(\xi \circ \hat{g})) = \iota_*^\alpha(\hat{g})$, hence

$$\iota_*^\alpha(\hat{f}) = \iota_*^\alpha((-1)_T^\nu(\xi \circ \hat{g})) = (-1)_T^\nu |\xi|_T(\iota_*^\alpha(|\xi|_T(\xi \circ \hat{g}))) = (-1)_T^\nu |\xi|_T(\iota_*^\alpha(\hat{g})).$$

Applying Lemma 4.9 we deduce

$$\widehat{\iota_*^\alpha(f)} = (-1)_T^\nu |\xi|_T \widehat{\iota_*^\alpha(g)} = \widehat{v}$$

with $v = [(-1)_T^\nu |\xi|_T] \iota_*^\alpha(g)$. As R is levelwise fibrant, the map

$$
\begin{aligned}
&[\mathbb{S}^r \wedge U \wedge T^{q+n+\alpha}, \mathrm{Hom}(T^{l(n+\alpha)}, R_{(1+m)(n+\alpha)})] \\
&\qquad \xrightarrow{\mathrm{ev} \circ (- \wedge T^{l(n+\alpha)})} [\mathbb{S}^r \wedge U \wedge T^{q+n+\alpha+l(n+\alpha)}, R_{(1+m)(n+\alpha)}]
\end{aligned}
$$

is bijective. Therefore we have $\iota_*^\alpha(f) = [(-1)_T^\nu |\xi|_T] \iota_*^\alpha(g)$. $\square$

We are now able to state the Main Theorem of this section, which is a generalization of [Sch07, Corollary I.4.69]. (The definition of c_l is before Proposition-Definition 4.5.)

Theorem 4.11. *Assume that the standard assumptions of the beginning of Section 4 hold (these are satisfied, e.g., in the motivic case by Proposition 2.45). Let R be a levelwise fibrant semistable symmetric T-ring spectrum and $x : T^l \to R_m$ a central map. Then $R[1/x]$ is semistable, and for all $U \in \mathcal{B}'$ the ring homomorphism $\pi_{*,*}^U(R) \xrightarrow{j_*} \pi_{*,*}^U(R[1/x])$ is a $[xc_l]$-localization.*

Proof. Semistability: Using Theorem 2.43 it suffices to show that the cycle operator d acts trivially on $\pi_{r,q}^U(R[1/x])$. Let $f \in [\mathbb{S}^r \wedge U \wedge T^{q+n}, R[1/x]_n]$ represent an element in $\pi_{r,q}^U(R[1/x])$. After stabilization, we may assume that n is even. Then df is represented by $\chi_{n,1} \circ \iota_*(f)$ as $|\chi_{n,1}|_T = 1$. It

remains to show that after stabilization $[\chi_{n,1} \circ \iota_*(f)] = [f] = [\iota_*(f)]$. This reduces to the following: For $f \in [\mathbb{S}^r \wedge U \wedge T^{q+n}, R[1/x]_n]$ and $\gamma \in \Sigma_n$ with $|\gamma|_T = 1$ we have $[\gamma \circ f] = [f]$ in $\pi^U_{r,q}(R)$. To show this, consider the adjoint $\widehat{\gamma \circ f} = \mathrm{ev} \circ ((\gamma \circ f) \wedge T^{ln}) = \Delta_{1+m,n}(\gamma) \circ \hat{f} \circ (1 \wedge \Delta_{l,n}(\gamma)^{-1})$ (compare Proposition-Definition 4.6). As $|\gamma|^s_T = 1$ is the sign of $\Delta_{s,n}(\gamma)$ (Lemma 4.7), we obtain $\widehat{\gamma \circ f} = |\Delta_{1+m,n}(\gamma)|_T(\Delta_{1+m,n}(\gamma) \circ \hat{f})$ by Definition 2.16. Applying Lemma 4.10 yields $[\gamma \circ f] = [f]$ as claimed.

Localization: By Proposition-Definition 4.5, we know that $\pi^U_{*,*}(R[1/x])$ is a ring and $j_* : \pi^U_{*,*}(R) \to \pi^U_{*,*}(R[1/x])$ a ring homomorphism. It remains to show that j_* is a $[xc_l]$-localization. For this, we will check that the three conditions of Proposition 4.12 are satisfied (note that the Ore condition holds by Corollary 4.8). First, we show that $j_*([xc_l])$ is a unit in $\pi^U_{*,*}(R[1/x])$. The map $j_m \circ x \circ c_l$ represents $j_*([xc_l])$ and this element has $(\pm 1)_T[yc_{1+m}]$ as a left inverse (up to sign). Here $y : T^{1+m} \to R[1/x]_{1+l}$ denotes the adjoint to $\mu_{1+m+l,ml} \circ (\iota_{1+m+l} \wedge x^l)$. We now show that $[(yc_{1+m}) \cdot (j_m x c_l)]$ equals (up to sign) the unit in $\pi^U_{*,*}(R)$. By definition

$$\begin{aligned}
f &:= (yc_{1+m}) \cdot (j_m x c_l) \\
&= \mu^{R[1/x]}_{1+l,m} \circ (y \wedge (j_m x)) \circ (c_{1+m} \wedge c_l) \circ (\mathbb{S}^0 \wedge t_{\mathbb{S}^0 \wedge U, U \wedge T^{1+m}} \wedge T^l) \\
&\quad \circ (s_{0,0} \wedge \mathrm{diag}^U \wedge (-1)^{(l-m)(1+l)}_T \wedge 1) \\
&= \mu^{R[1/x]}_{1+l,m} \circ (y \wedge (j_m x)) \circ (l^2_{T^{1+m}} \wedge l^2_{T^l}) \circ (S^0 \wedge t_{S^0 \wedge S^0, S^0 \wedge T^{1+m}} \wedge T^l) \\
&\quad \circ ([l^{-1}_{S^0} \cong^{\mathbb{S}^0}] \wedge ([\omega_U \wedge \omega_U] \circ \mathrm{diag}^U) \wedge (-1)^{(l-m)(1+l)}_T \wedge 1).
\end{aligned}$$

Using $(\omega_U \wedge \omega_U) \circ \mathrm{diag}^U = (\omega_U \wedge S^0) \circ \rho_U^{-1} = \rho^{-1}_{S^0} \circ \omega_U$ and

$$\begin{aligned}
l^2_{T^{1+m+l}} &= S^0 \wedge S^0 \wedge T^{1+m+l} \cong S^0 \wedge S^0 \wedge S^0 \wedge S^0 \wedge T^{1+m+l} \\
&\cong S^0 \wedge S^0 \wedge T^{1+m} \wedge S^0 \wedge S^0 \wedge T^l \cong T^{1+m} \wedge T^l
\end{aligned}$$

we get $f = \mu^{R[1/x]}_{1+l,m} \circ (y \wedge (j_m x)) \circ ((-1)^{(l-m)(1+l)}_T \wedge 1) \circ c_{1+m+l}$.

The adjoint of f is ($c := c_{1+m+l} \wedge T^{l(1+l+m)}, a := (1+m)(1+l)$):

$$\begin{aligned}
\hat{f} &= \mathrm{ev} \circ (f \wedge T^{l(1+l+m)}) \\
&= \mu_{(1+m)(1+l),(1+m)m} \circ ((\mathrm{ev} \circ (y \wedge T^{l(1+l)})) \wedge (\mathrm{ev} \circ (j_m x \wedge T^{lm}))) \\
&\quad \circ (1 \wedge t_{T^l, T^{l(1+l)}} \wedge 1) \circ ((-1)^{(l-m)(1+l)}_T \wedge 1) \circ c \\
&= \mu_{a,(1+m)m} \circ ((\mu_{1+m+l,ml} \circ (\iota_{1+m+l} \wedge x^l)) \wedge (\xi_{m,mm} \circ x^{1+m})) \\
&\quad \circ (1 \wedge t_{T^l, T^{l(1+l)}} \wedge 1) \circ ((-1)^{(l-m)(1+l)}_T \wedge 1) \circ c \\
&= (a + \xi_{m,mm}) \circ [\mu_{a,m+mm} \circ (\mu_{1+m+l,ml} \wedge 1)] \circ ((\iota_{1+m+l} \wedge x^l) \wedge x^{1+m}) \\
&\quad \circ (1 \wedge t_{T^l, T^{l(1+l)}} \wedge 1) \circ ((-1)^{(l-m)(1+l)}_T \wedge 1) \circ c \\
&= (a + \xi_{m,mm}) \circ [\mu_{1+m+l,ml+m+mm} \circ (1 \wedge \mu_{ml,m+mm})]
\end{aligned}$$

$$\begin{aligned}
&\circ (\iota_{1+m+l} \wedge (x^l \wedge x^{1+m})) \circ (1 \wedge t_{T^l,T^{l(1+l)}} \wedge 1) \circ ((-1)_T^{(l-m)(1+l)} \wedge 1) \circ c \\
&= (a + \xi_{m,mm}) \circ \mu_{1+m+l,m(1+l+m)} \circ (\iota_{1+m+l} \wedge x^{1+l+m}) \circ c \\
&\quad \circ (1 \wedge t_{T^l,T^{l(1+l)}} \wedge 1) \circ (1 \wedge (-1)_T^{(l-m)(1+l)} \wedge 1)
\end{aligned}$$

where we used $\mu_{sm,tm} \circ (x^s \wedge x^t) = x^{s+t}$, i.e., the associativity of R.

The unit $[\iota_1^{R[1/x]} c_1]$ in $\pi^U_{*,*}(R)$ is also represented by

$$g := \iota_*^{l+m}(\iota_1^{R[1/x]} c_1) = \iota_{1+l+m}^{R[1/x]} \circ c_{1+l+m}$$

which is adjoint to $\hat{g} = \xi_{m,1+l+m} \circ \mu_{1+l+m,m(1+l+m)} \circ (\iota_{1+l+m} \wedge x^{1+l+m}) \circ c$. Therefore

$$\hat{f} = \xi' \circ \hat{g} \circ (1 \wedge t_{T^l,T^{l(1+l)}} \wedge 1) \circ (1 \wedge (-1)_T^{(l-m)(1+l)} \wedge 1) = (-1)_T^{\nu} \xi' \circ \hat{g}$$

with $\xi' = (a + \xi_{m,mm}) \circ \xi^{-1}_{m,1+l+m}$ and $\nu = l^2(1+l) + (l-m)(1+l)$. Applying 4.10 yields $[f] = ((-1)_T^{\nu} |\xi'|_T)[g]$ and finally

$$(((-1)_T^{\nu} |\xi'|_T)[y c_{1+m}]) \cdot j_*([x c_l]) = ((-1)_T^{\nu} |\xi'|_T)[f] = [g] = 1$$

in $\pi^U_{r,q}(R[1/x])$. By Corollary 4.8 $j_*([x c_l])$ has then also a right inverse.

The second condition amounts to showing that for any $z \in \pi^U_{*,*}(R[1/x])$ — repesented by some $f \in [\mathbb{S}^r \wedge U \wedge T^{q+n}, R[1/x]_n]$ — there is some $u \in \pi^U_{*,*}(R)$ and some $p \in \mathbb{N}$ satisfying $z \cdot j_*([x c_l])^{\cdot p} = j_*((\pm 1)_T u)$. For u we choose $\hat{f}$ as representative and set $p = n$. Then $j_*(u)$ is represented by $g := j_{(1+m)n} \circ \hat{f}$ which is adjoint to

$$\hat{g} := \mathrm{ev} \circ (g \wedge T^{l(1+m)n}) = \xi_{m,(1+m)n} \circ \mu_{(1+m)n,m(1+m)n} \circ (\hat{f} \wedge x^{(1+m)n}).$$

The element $z \cdot j_*([x c_l])^{\cdot n} = z \cdot j_*([x c_l]^{\cdot n})$ is represented by

$$h := f \cdot (j_{mn} \circ (x c_l)^{\cdot n}),$$

where $(x c_l)^{\cdot n}$ is given by $x^n \circ ((-1)_T^{(l-m)m(n-1)n/2} \wedge 1) \circ c_{nl}$, as we show by induction:

$$\begin{aligned}
&(x^n \circ ((-1)_T^{(l-m)m(n-1)n/2} \wedge 1) \circ c_{nl}) \cdot (x c_l) \\
&= \mu^R_{mn,m} \circ ((x^n \circ ((-1)_T^{(l-m)m(n-1)n/2} \wedge 1)) \wedge x) \\
&\quad \circ ((-1)_T^{(l-m)mn} \wedge 1) \circ c_{(n+1)l} \\
&= x^{n+1} \circ ((-1)_T^{(l-m)m[(n-1)n/2+n]} \wedge 1) \circ c_{(n+1)l}
\end{aligned}$$

(cf. also the computation of f above). Furthermore,

$$\begin{aligned}
h = \mu_{n,mn}^{R[1/x]} &\circ (f \wedge (j_{mn} \circ (x c_l)^{\cdot n})) \circ (\mathbb{S}^r \wedge t_{\mathbb{S}^0 \wedge U, U \wedge T^{q+n}} \wedge T^{ln}) \\
&\circ (s_{r,0} \wedge \mathrm{diag}^U \wedge (-1)_T^{(l-m)n^2} \wedge 1)
\end{aligned}$$

implies

$$\hat{h} = \mathrm{ev} \circ (h \wedge T^{l(n+mn)})$$

$$
\begin{aligned}
&= \mu^R_{(1+m)n,(1+m)mn} \circ [(\mathrm{ev} \circ (f \wedge T^{ln})) \wedge (\mathrm{ev} \circ ((j_{mn} \circ (xc_l)^{\cdot n}) \wedge T^{lmn}))] \\
&\quad \circ (1 \wedge t_{\mathbb{S}^0 \wedge U \wedge T^{ln}, T^{ln}} \wedge 1) \circ (\mathbb{S}^r \wedge t_{\mathbb{S}^0 \wedge U, U \wedge T^{q+n}} \wedge 1) \\
&\quad \circ (s_{r,0} \wedge \mathrm{diag}^U \wedge (-1)_T^{(l-m)n^2} \wedge 1) \\
&= \mu^R_{(1+m)n,(1+m)mn} \\
&\quad \circ [\hat{f} \wedge (\xi_{m,mn} \circ \mu^R_{mn,mmn} \\
&\qquad \circ ((x^n \circ ((-1)_T^{(l-m)m(n-1)n/2} \wedge 1) \circ c_{nl}) \wedge x^{mn}))] \\
&\quad \circ (1 \wedge t_{\mathbb{S}^0 \wedge U \wedge T^{ln}, T^{ln}} \wedge 1) \circ (\mathbb{S}^r \wedge t_{\mathbb{S}^0 \wedge U, U \wedge T^{q+n}} \wedge 1) \\
&\quad \circ (s_{r,0} \wedge \mathrm{diag}^U \wedge (-1)_T^{(l-m)n^2} \wedge 1) \\
&= (a' + \xi_{m,mn}) \circ \mu^R_{(1+m)n,(1+m)mn} \circ (\hat{f} \wedge (\mu^R_{mn,mmn} \circ (x^n \wedge x^{mn})) \\
&\quad \circ (1 \wedge (-1)_T^{(l-m)n^2 + (l-m)m(n-1)n/2 + (ln)^2} \wedge 1) \\
&= (a' + \xi_{m,mn}) \circ \mu^R_{(1+m)n,(1+m)mn} \circ (\hat{f} \wedge x^{(1+m)n}) \\
&\quad \circ (1 \wedge (-1)_T^{(l-m)[n^2 + m(n-1)n/2] + (ln)^2} \wedge 1),
\end{aligned}
$$

where $a' := (1+m)n$. Here the second last step uses

$$
\begin{aligned}
&\mathbb{S}^r \wedge [(U \wedge T^{q+n} \wedge T^{ln} \wedge [l^2_{T^{ln}} (\cong^{\mathbb{S}^0} \wedge \omega_U \wedge T^{ln})]) \\
&\quad (U \wedge T^{q+n} \wedge t_{\mathbb{S}^0 \wedge U \wedge T^{ln}, T^{ln}}) (t_{\mathbb{S}^0 \wedge U, U \wedge T^{q+n}} \wedge T^{ln} \wedge T^{ln})] \wedge T^{lmn} \\
&\quad \circ (s_{r,0} \wedge \mathrm{diag}^U \wedge T^{\tilde{a}}) \\
&= \mathbb{S}^r \wedge [(U \wedge T^{q+n} \wedge t_{T^{ln}, T^{ln}})(U \wedge T^{q+n} \wedge l^2_{T^{ln}} \wedge T^{ln})(t_{S^0 \wedge S^0, U \wedge T^{q+n}} \wedge T^{ln})] \\
&\quad \wedge T^{lmn} \circ (\rho^{-1}_{\mathbb{S}^r} \wedge ((\omega_U \wedge U)\ \mathrm{diag}^U) \wedge T^{\tilde{a}}) \\
&= \mathbb{S}^r \wedge [(U \wedge T^{q+n} \wedge (-1)_T^{(ln)^2} \wedge 1) \circ (U \wedge T^{q+n} \wedge l^2_{T^{ln}} \wedge T^{ln}) \\
&\qquad \circ (t_{S^0 \wedge S^0, U \wedge T^{q+n}} \wedge T^{ln})] \wedge T^{lmn} \circ (\rho^{-1}_{\mathbb{S}^r} \wedge l_U^{-1} \wedge T^{\tilde{a}}) \\
&= \mathbb{S}^r \wedge U \wedge (-1)_T^{(ln)^2} \wedge T^{\tilde{a}-1}
\end{aligned}
$$

with $\tilde{a} := (q+n) + ln + l(n+mn)$. Hence $\hat{h}$ and $\hat{g}$ only differ by a permutation and a sign, and Lemma 4.10 then implies

$$z \cdot j_*([xc])^{\cdot p} = [h] = (\pm 1)_T[g] = (\pm 1)_T j_*(u) = j_*((\pm 1)_T u).$$

It remains to verify the third condition: for any $[f], [g] \in \pi^U_{*,*}(R)$ with $j_*([f]) = j_*([g])$, we have $[f] \cdot [xc_l]^{\cdot n} = [g] \cdot [xc_l]^{\cdot n}$ for some $n \in \mathbb{N}$. We may assume that $f, g \in [\mathbb{S}^r \wedge U \wedge T^{q+n}, R_n]$ and that $j_n \circ f = j_n \circ g$. Using $(xc_l)^{\cdot n} = x^n \circ ((-1)_T^{(l-m)m(n-1)n/2} \wedge 1) \circ c_{nl}$ we obtain

$$
\begin{aligned}
f \cdot (xc_l)^{\cdot n} &= \mu^R_{n,mn} \circ (f \wedge x^n) \circ (-1)_T^{(l-m)[m(n-1)n/2 + n^2]} \\
&= \xi^{-1}_{m,n} \circ \widehat{j_n \circ f} \circ (-1)_T^{(l-m)[m(n-1)n/2 + n^2]},
\end{aligned}
$$

as

$$\begin{aligned}
&\mathbb{S}^r \wedge [(U \wedge T^{q+n} \wedge l^2_{T^{ln}}) \circ (U \wedge T^{q+n} \wedge \cong^{\mathbb{S}^0} \wedge \omega_U \wedge T^{ln}) \\
&\quad \circ t_{\mathbb{S}^0 \wedge U, U \wedge T^{q+n}} \wedge T^{ln}] \circ (s_{r,0} \wedge \operatorname{diag}^U \wedge (-1)_T^{(l-m)n^2} \wedge 1) \\
&= \mathbb{S}^r \wedge [(U \wedge T^{q+n} \wedge l^2_{T^{ln}}) \circ t_{S^0 \wedge S^0, U \wedge T^{q+n}} \wedge T^{ln}] \\
&\quad \circ (\rho^{-1}_{\mathbb{S}^r} \wedge ([\omega_U \wedge U] \operatorname{diag}^U) \wedge (-1)_T^{(l-m)n^2} \wedge 1) \\
&= \mathbb{S}^r \wedge [(U \wedge T^{q+n} \wedge l^2_{T^{ln}}) \circ t_{S^0 \wedge S^0, U \wedge T^{q+n}} \wedge T^{ln}] \\
&\quad \circ (\rho^{-1}_{\mathbb{S}^r} \wedge l_U^{-1} \wedge (-1)_T^{(l-m)n^2} \wedge 1) \\
&= 1 \wedge (-1)_T^{(l-m)n^2} \wedge 1.
\end{aligned}$$

This also holds for g, thus $f \cdot (xc_l)^{\cdot n} = g \cdot (xc_l)^{\cdot n}$ and hence

$$[f] \cdot [xc_l]^{\cdot n} = [f \cdot (xc_l)^{\cdot n}] = [g \cdot (xc_l)^{\cdot n}] = [g] \cdot [xc_l]^{\cdot n}$$

as desired. □

We have used the following standard criterion for localizations above:

Proposition 4.12. *Let M, N be two rings and $x \in M$. Assume that for any $x_1 \in M$ there is an $x_2 \in M$ with $x_1 x = x x_2$ (Ore condition). Assume further that there is a ring homomorphism $j : M \to N$ satisfying the following:*

(i) *There are $y, y' \in N$ with $yj(x) = 1$ and $j(x)y' = 1$.*
(ii) *For all $z \in N$ there is some $p \in \mathbb{N}$ and some $u \in M$ with*

$$zj(x)^p = j(u).$$

(iii) *For all $a, b \in M$ with $j(a) = j(b)$ there is an $n \in \mathbb{N}$ with $ax^n = bx^n$.*

Then j is an $[x]$-localization. If M and N are graded, then j is a graded ring homomorphism. If moreover x is homogenous, then it suffices to check the above conditions for homogenous elements x_1, x_2, a and b.

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(Stephan Hähne) Groenenweg 79, 22549 Hamburg
`StephanHP@gmx.de`

(Jens Hornbostel) Bergische Universität Wuppertal, FB C, Mathematik und Informatik, Gaussstrasse 20, 42119 Wuppertal
`hornbostel@math.uni-wuppertal.de`

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