

# Infinitesimal local boundary dilatation attained by asymptotical extremal

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ABSTRACT. In this paper, we prove the existence of an asymptotical extremal in an infinitesimal equivalence class as a locally extremal representative at a boundary point.

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## 1. Introduction

Let  $S$  be a plane domain with at least two boundary points. The Teichmüller space  $T(S)$  is the space of equivalence classes of quasiconformal maps  $f$  from  $S$  to a variable domain  $f(S)$ . Two quasiconformal maps  $f$  from  $S$  to  $f(S)$  and  $g$  from  $S$  to  $g(S)$  are equivalent if there is a conformal map  $c$  from  $f(S)$  onto  $g(S)$  and a homotopy through quasiconformal maps  $h_t$  mapping  $S$  onto  $g(S)$  such that  $h_0 = c \circ f$ ,  $h_1 = g$  and  $h_t(p) = c \circ f(p) = g(p)$  for every  $t \in [0, 1]$  and every  $p$  in the boundary of  $S$ . Denote by  $[f]$  the Teichmüller equivalence class of  $f$ ; also sometimes denote the equivalence class by  $[\mu]$  where  $\mu$  is the Beltrami differential of  $f$ .

Denote by  $\text{Bel}(S)$  the Banach space of Beltrami differentials

$$\mu = \mu(z)d\bar{z}/dz$$

on  $S$  with finite  $L^\infty$ -norm and by  $M(S)$  the open unit ball in  $\text{Bel}(S)$ .

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The cotangent space to  $T(S)$  at the basepoint is the Banach space  $Q(S)$  of integrable holomorphic quadratic differentials on  $S$  with  $L^1$ -norm

$$\|\varphi\| = \iint_S |\varphi(z)| \, dx dy < \infty.$$

In what follows, let  $Q^1(S)$  denote the unit sphere of  $Q(S)$ .

Two Beltrami differentials  $\mu$  and  $\nu$  in  $\text{Bel}(S)$  are said to be infinitesimally equivalent if

$$\iint_S (\mu - \nu)\varphi \, dx dy = 0, \text{ for any } \varphi \in Q(S).$$

The tangent space  $Z(S)$  of  $T(S)$  at the basepoint is defined as the quotient space of  $\text{Bel}(S)$  under the equivalence relations. Denote by  $[\mu]_Z$  the equivalence class of  $\mu$  in  $Z(S)$ .

$Z(S)$  is a Banach space and its standard sup-norm is defined by

$$\|[\mu]_Z\| := \sup_{\varphi \in Q^1(S)} \text{Re} \iint_S \mu \varphi \, dx dy = \inf\{\|\nu\|_\infty : \nu \in [\mu]_Z\}.$$

Define the (infinitesimal) boundary dilatation  $b([\mu]_Z)$  of  $[\mu]_Z$  to be the infimum over all elements in the equivalence class  $[\mu]_Z$  of the quantity  $b^*(\nu)$ . Here  $b^*(\nu)$  is the infimum over all compact subsets  $E$  contained in  $S$  of the essential supremum of the the Beltrami differential  $\nu$  as  $z$  varies over  $S - E$ .

Define  $h^*(\mu)$  to be the infimum over all compact subsets  $E$  contained in  $S$  of the essential supremum norm of the Beltrami differential  $\mu(z)$  as  $z$  varies over  $S \setminus E$  and  $h([\mu])$  to be the infimum of  $h^*(\nu)$  taken over all representatives  $\nu$  of the class  $[\mu]$ .

Let  $p$  be a point on  $\partial S$  and let  $\mu \in M(S)$ . Define

$$h_p([\mu]) = \inf \{h_p^*(\nu) : \nu \in [\mu]\},$$

to be the boundary dilatation of  $[\mu]$  at  $p$ , where

$$h_p^*(\mu) = \inf \left\{ \text{esssup}_{z \in U \cap S} |\mu(z)| : \right. \\ \left. U \text{ is an open neighborhood in } \mathbb{C} \text{ containing } p \right\}.$$

If  $\mu \in M(S)$ , define

$$h_p([\mu]) = \inf \{h_p^*(\nu) : \nu \in [\mu]\}.$$

It was proved by Fehlmann [3] for the unit disk and by Lakic [5] for the plane domains that

$$h([\mu]) = \max_{p \in \partial S} h_p([\mu]).$$

For  $\mu \in \text{Bel}(S)$ , we use  $b_p^*(\mu)$  to denote  $h_p^*(\mu)$  and define

$$b_p([\mu]_Z) = \inf \{b_p^*(\nu) : \nu \in [\mu]_Z\}$$

to be the boundary dilatation of  $[\mu]_Z$  at  $p$ . The parallel result

$$b([\mu]_Z) = \max_{p \in \partial S} b_p([\mu]_Z)$$

for the plane domains was proved by Lakic in [5].

The following problem was proposed by F. Gardiner and N. Lakic in ([4], page 335) as an open problem.

**Problem 1.** *Let  $p$  be a boundary point of a plane domain  $S$ , and let  $\tau \in T(S)$ . Is there a locally extremal Beltrami differential  $\mu$  representing the class  $\tau$  at the point  $p$ ? That is, can we find a Beltrami differential  $\mu \in M(S)$  such that  $\tau = [\mu]$  and  $h_p^*(\mu) = h_p(\tau)$ ?*

The problem was partly solved by Cui and Qi in [2] and the answer is affirmative when  $S$  is the unit disk  $\Delta$ . Recently, the author strengthened their result in [7] by showing that  $h_p^*(\mu) = h_p(\tau)$  can be attained by an asymptotically extremal representative  $\mu \in \tau$ . Naturally, the problem has its counterpart in the infinitesimal case. That is:

**Problem 2.** *Let  $p$  be a boundary point of a plane domain  $S$ , and let  $\tau \in Z(S)$ . Is there a locally extremal Beltrami differential  $\mu$  representing the class  $\tau$  at the point  $p$ ? That is, can we find a Beltrami differential  $\mu \in \text{Bel}(S)$  such that  $\tau = [\mu]_Z$  and  $b_p^*(\mu) = b_p(\tau)$ ?*

Generally,  $\mu \in \text{Bel}(S)$  is called an asymptotical extremal in  $[\mu]_Z$  if

$$b^*(\mu) = b([\mu]_Z).$$

In this paper, we prove that the local boundary dilatation can be attained by an asymptotical extremal which gives an affirmative answer to Problem 2 in a stronger sense.

**Theorem 1.** *Let  $p$  be a boundary point of the unit disk  $\Delta$  and let  $\tau \in Z(\Delta)$ . Then for any given  $\epsilon > 0$ , there is an asymptotically extremal Beltrami differential  $\mu \in \tau$  such that  $\|\mu\|_\infty < \|\tau\| + \epsilon$  and  $b_p^*(\mu) = b_p(\tau)$ .*

The method used here can also be used to deal with some more general cases.

## 2. Deformation of Beltrami differentials

In this section, we deform a Beltrami differential to obtain a new equivalent Beltrami differential whose essential supremum can be controlled properly. The following infinitesimal main inequality (see [1]) is needed.

**Theorem A.** *Let  $\mu, \nu \in M(S)$ . Suppose  $\mu$  and  $\nu$  are infinitesimally equivalent. Then*

$$(2.1) \quad \iint_S |\varphi|(1 - |\mu|^2) dx dy \leq \iint_S |\varphi| \left| 1 - \mu \frac{\varphi}{|\varphi|} \right|^2 \frac{\left| 1 + \nu \frac{\varphi}{|\varphi|} \frac{1 - \bar{\mu} \frac{\bar{\varphi}}{|\varphi|}}{1 - \mu \frac{\varphi}{|\varphi|}} \right|^2}{1 - |\nu|^2} dx dy$$

for all  $\varphi \in Q(S)$ .

The following lemma is Proposition 1 of Chapter 15 in [4].

**Lemma 2.1.** *For every  $\tau \in Z(S)$  and every  $\epsilon > 0$  there exists a representative  $\eta$  in  $\tau$  such that  $\|\eta\|_\infty < \|\tau\| + \epsilon$  and  $b^*(\eta) = b(\tau)$ .*

Suppose  $\{J_n : n \in \mathbb{N}\}$  is a sequence of Jordan domains in  $\Delta$  with the properties:

- (1)  $\Delta \setminus \bar{J}$  is simply-connected where  $J = J_0$ .
- (2)  $J_{n+1} \subsetneq J_n$  and  $J_n \setminus \bar{J}_{n+1}$  is simply-connected for all  $n \geq 0$ .
- (3)  $\lim_{n \rightarrow \infty} \bar{J}_n$  is a boundary point  $\zeta \in \partial\Delta$ .

Set  $U_n = \Delta \setminus \bar{J}_n$  for  $n \in \mathbb{N}$ . It is easy to see that  $U_n$  is simply-connected.

**Theorem 2.** *Let  $\nu \in \text{Bel}(\Delta)$  and let  $J, J_n$  given as the above. Then for every given  $\epsilon > 0$ , there exists some  $n \in \mathbb{N}$  and  $\mu \in \text{Bel}(\Delta)$  such that:*

- (1)  $\mu \in [\nu]_Z$ .
- (2)  $\mu(z) = \nu(z)$  restricted on  $J_n$ .
- (3)  $\|\mu|_{U_n}\|_\infty \leq \max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon$ .

**Proof.** Since  $Z(\Delta)$  is a Banach space, without loss of generality, we can assume that  $\|\nu\|_\infty < 1 - \epsilon$  for small  $\epsilon > 0$ . Regard  $[\nu|_{U_n}]_Z$  as a point in the space  $Z(U_n)$ . Then there is an infinitesimal extremal  $\mu_n$  in  $[\nu|_{U_n}]_Z$  such that  $\|\mu_n\|_\infty = \|[\nu|_{U_n}]_Z\|$ . It is obvious that  $\|\mu_n\|_\infty \leq \|\nu\|_\infty < 1$ . If for some  $n$ ,  $\|\mu_n\|_\infty \leq \max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon$ , then

$$(2.2) \quad \tilde{\mu}_n(z) := \begin{cases} \mu_n(z), & z \in U_n, \\ \nu(z), & z \in \bar{J}_n \end{cases}$$

is the required Beltrami differential.

Now, we assume that  $\|\mu_n\|_\infty > \max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon$  holds for all  $n \in \mathbb{N}$ . Then  $\|[\nu|_{U_n}]_Z\| > b([\nu|_{U_n}]_Z)$  and consequently by the infinitesimal frame mapping theorem (see Theorem 2.4 in [6]) of Reich,  $\mu_n$  is a Teichmüller differential, i.e.,  $\mu_n = k_n \bar{\varphi}_n / |\varphi_n|$  ( $0 < k_n < 1$ ), where  $\varphi_n \in Q^1(U_n)$ .

*Claim.*  $\varphi_n$  converges to 0 uniformly on any compact subset of  $\Delta$  as  $n \rightarrow \infty$ .

Note the condition  $\lim_{n \rightarrow \infty} \bar{J}_n = \zeta \in \partial\Delta$ . We may assume, by contradiction, that there is  $\varphi_0 \in Q(\Delta)$ ,  $\varphi_0 \not\equiv 0$  and a subsequence  $\{n_j\}$  of  $\mathbb{N}$  with  $n_j < n_{j+1}$  such that  $\varphi_{n_j} \rightarrow \varphi_0$  as  $j \rightarrow \infty$ . We may choose a subsequence of  $\mu_{n_j}$ , also denoted by itself, such that  $k_{n_j} \rightarrow k_0$  as  $j \rightarrow \infty$ . Thus, the Teichmüller differential  $\mu_{n_j}$  converges to  $\mu_0 = k_0 \bar{\varphi}_0 / |\varphi_0|$  in  $\Delta$ .

Observe that  $\|\tilde{\mu}_{n_j}\|_\infty \leq \|\nu\|_\infty$  for all  $j > 0$ . We have  $\mu_0 \in [\nu]_Z$  and hence  $\mu_0$  is a Teichmüller extremal in  $[\nu]_Z$ . On the other hand, the assumption that  $\|\mu_n\|_\infty > \max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon$  holds for all  $n \in \mathbb{N}$  implies  $k_0 \geq \|[\nu]_Z\| + \epsilon$ . This gives rise to a contradiction. The proof of Claim is completed.

Fix a positive integer  $N$ . By the definition of boundary dilatation, we have

$$b([\nu|_{U_N}]_Z) \leq \max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\}.$$

By Lemma 2.1, there exists a Beltrami differential  $\nu_N \in [\nu|_{U_N}]_Z$  such that  $b^*(\nu_N) = b([\nu|_{U_N}]_Z)$ . So, there is a compact subset  $E \subset U_N$  such that

$$|\nu_N(z)| \leq \max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \frac{\epsilon}{2}$$

for almost all  $z \in U_N \setminus E$ .

For any  $n > N$ , let

$$\tilde{\nu}_n(z) := \begin{cases} \nu_N(z), & z \in U_N, \\ \nu(z), & z \in U_n \setminus U_N. \end{cases}$$

Then  $\tilde{\nu}_n \in [\nu|_{U_n}]_Z (= [\mu_n|_{U_n}]_Z)$ . We apply the infinitesimal main inequality (2.1) on  $U_n$  and get

$$\begin{aligned} & \iint_{U_n} |\varphi_n|(1 - |\mu_n|^2) dx dy \\ & \leq \iint_{U_n} |\varphi_n| \left| 1 - \mu_n \frac{\varphi_n}{|\varphi_n|} \right|^2 \frac{\left| 1 + \tilde{\nu}_n \frac{\varphi_n}{|\varphi_n|} \frac{1 - \mu_n \frac{\overline{\varphi_n}}{|\varphi_n|}}{1 - \mu_n \frac{\varphi_n}{|\varphi_n|}} \right|^2}{1 - |\tilde{\nu}_n|^2} dx dy \\ & \leq \iint_{U_n} |\varphi_n| \left| 1 - \mu_n \frac{\varphi_n}{|\varphi_n|} \right|^2 \frac{1 + |\tilde{\nu}_n|}{1 - |\tilde{\nu}_n|} dx dy. \end{aligned}$$

Notice that  $\mu_n = k_n \overline{\varphi_n} / |\varphi_n|$ . We have

$$\iint_{U_n} |\varphi_n|(1 - k_n^2) dx dy \leq \iint_{U_n} |\varphi_n|(1 - k_n)^2 \frac{1 + |\tilde{\nu}_n|}{1 - |\tilde{\nu}_n|} dx dy.$$

Thus,

$$\begin{aligned} \frac{1 + k_n}{1 - k_n} & \leq \iint_{U_n} |\varphi_n| \frac{1 + |\tilde{\nu}_n|}{1 - |\tilde{\nu}_n|} dx dy \\ & \leq \iint_E |\varphi_n| \frac{1 + |\tilde{\nu}_n|}{1 - |\tilde{\nu}_n|} dx dy + \iint_{U_n \setminus E} |\varphi_n| \frac{1 + |\tilde{\nu}_n|}{1 - |\tilde{\nu}_n|} dx dy. \end{aligned}$$

Choose  $\tilde{\epsilon} > 0$  such that

$$\frac{1 + (\max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon/2)}{1 - (\max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon/2)} + \tilde{\epsilon} \leq \frac{1 + (\max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon)}{1 - (\max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon)}.$$

Since  $\varphi_n$  converges to 0 on  $E$  as  $n \rightarrow \infty$ ,

$$\iint_E |\varphi_n| \frac{1 + |\tilde{\nu}_n|}{1 - |\tilde{\nu}_n|} dx dy \leq \tilde{\epsilon}$$

holds for all sufficiently large  $n$ . On the other hand, by the definition of  $\tilde{\nu}_n$ , we have

$$\iint_{U_n \setminus E} |\varphi_n| \frac{1 + |\tilde{\nu}_n|}{1 - |\tilde{\nu}_n|} dx dy \leq \frac{1 + (\max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon/2)}{1 - (\max\{\|[\nu]_Z\|, \|\nu|_J\|_\infty\} + \epsilon/2)}.$$

Hence we get

$$\frac{1+k_n}{1-k_n} \leq \frac{1+(\max\{\|\nu\|_Z, \|\nu|_J\|_\infty\} + \epsilon/2)}{1-(\max\{\|\nu\|_Z, \|\nu|_J\|_\infty\} + \epsilon/2)} + \tilde{\epsilon}$$

and consequently,

$$k_n \leq \max\{\|\nu\|_Z, \|\nu|_J\|_\infty\} + \epsilon,$$

which completes the proof of Theorem 2.  $\square$

Unlike the Teichmüller equivalence class, the notion of the boundary map is lost for the infinitesimal equivalence classes. The gluing method used in [2] does not apply to prove our Theorem 2.

### 3. Proof of Theorem 1

We prove Theorem 1 by gluing Beltrami differentials in a suitable way. By Lemma 2.1, we only need to prove Theorem 1 in the case  $b_p(\tau) < b(\tau) := b$ . Put  $\delta = b(\tau) - b_p(\tau)$ . Define  $J_r = \{z \in \Delta : |z - p| < r\}$  for small  $r \in (0, 2)$  and  $U_r = \Delta \setminus \bar{J}_r$ .

*Step 1.* By the definition of boundary dilatation, there is a Beltrami differential  $\nu_1 \in \tau$  such that

$$b_p^*(\nu_1) \leq b_p(\tau) + \frac{\delta}{23}.$$

By the definition of  $b_p^*(\nu_1)$ , there is some  $r_1 > 0$  such that

$$|\nu_1(z)| \leq b_p(\tau) + \frac{\delta}{2} < b, \text{ a.e. } z \in J_{r_1}.$$

Applying Theorem 2, we can find some  $r'_1 < r_1$  and a Beltrami differential  $\mu_1 \in \tau$  such that,  $\mu_1(z) = \nu_1(z)$  restricted on  $J_{r'_1}$ ,  $\|\mu_1|_{J_{r'_1}}\|_\infty \leq b_p(\tau) + \frac{\delta}{2^2}$  and

$$\|\mu_1|_{U_{r'_1}}\|_\infty < \max\{\|\tau\|, \|\nu_1|_{J_{r_1}}\|_\infty\} + \frac{\epsilon}{2} = \|\tau\| + \frac{\epsilon}{2}.$$

It is not hard to see that  $b^*([\mu_1|_{U_{r'_1}}]_Z) = b$ . By Lemma 2.1, we can choose  $\eta_1 \in [\mu_1|_{U_{r'_1}}]_Z$  such that  $b^*(\eta_1) = b$  and  $\|\eta_1\|_\infty < \|\tau\| + \epsilon$ .

*Step 2.* Consider  $\nu_1(z)$  on  $J_{r'_1}$  and choose a Beltrami differential  $\nu_2 \in [\nu_1|_{J_{r'_1}}]_Z$  such that

$$b_p^*(\nu_2) \leq b_p(\tau) + \frac{\delta}{2^4}.$$

There is some  $r_2 < r'_1$  such that

$$|\nu_2(z)| \leq b_p(\tau) + \frac{\delta}{2^2}, \text{ a.e. } z \in J_{r_2}.$$

Again applying Theorem 2 on  $J_{r'_1}$ , we can find some  $r'_2 < r_2$  and a Beltrami differential  $\mu_2 \in [\nu_2|_{J_{r'_1}}]_Z$  such that,  $\mu_2(z) = \nu_2(z)$  restricted on  $J_{r'_2}$  and

$$\begin{aligned} \|\nu_2|_{J_{r'_2}}\|_\infty &\leq b_p(\tau) + \frac{\delta}{2^3}, \\ \|\mu_2|_{J_{r'_1} \setminus J_{r'_2}}\|_\infty &\leq \max\{\|[\nu_2|_{J_{r'_1}}]_Z\|, \|\nu_2|_{J_{r_2}}\|_\infty\} + \frac{\delta}{2^2} \\ &= \max\{\|[\nu_1|_{J_{r'_1}}]_Z\|, \|\nu_2|_{J_{r_2}}\|_\infty\} + \frac{\delta}{2^2} \\ &\leq b_p(\tau) + \frac{\delta}{2^2} + \frac{\delta}{2^2} = b_p(\tau) + \frac{\delta}{2}. \end{aligned}$$

*Step 3.* Following the construction in Step 2, we get two sequences  $\{r_n\}$  and  $\{r'_n\}$  and two sequences of Beltrami differentials  $\{\mu_n\}$  and  $\{\nu_n\}$  ( $n \geq 2$ ) with the following conditions:

- (i)  $r_n < r'_{n-1} < r_{n-1}$  and  $\lim_{n \rightarrow \infty} r_n = \lim_{n \rightarrow \infty} r'_n = 0$ .
- (ii)  $\nu_n \in [\nu_{n-1}|_{J_{r'_{n-1}}}]_Z$  and

$$(3.1) \quad b_p^*(\nu_n) \leq b_p(\tau) + \frac{\delta}{2^{n+2}},$$

$$(3.2) \quad |\nu_n(z)| \leq b_p(\tau) + \frac{\delta}{2^n}, \text{ a.e. } z \in J_{r_n}.$$

- (iii)  $\mu_n \in [\nu_{n-1}|_{J_{r'_{n-1}}}]_Z$ ,  $\mu_n(z) = \nu_n(z)$  restricted on  $J_{r'_n}$  and

$$(3.3) \quad \|\nu_n|_{J_{r'_n}}\|_\infty \leq b_p(\tau) + \frac{\delta}{2^{n+1}},$$

$$\begin{aligned} (3.4) \quad \|\mu_n|_{J_{r'_{n-1}} \setminus J_{r'_n}}\|_\infty &\leq \max\{\|[\nu_n|_{J_{r'_{n-1}}}]_Z\|, \|\nu_n|_{J_{r_n}}\|_\infty\} + \frac{\delta}{2^n} \\ &= \max\{\|[\nu_{n-1}|_{J_{r'_{n-1}}}]_Z\|, \|\nu_n|_{J_{r_n}}\|_\infty\} + \frac{\delta}{2^n} \\ &\leq b_p(\tau) + \frac{\delta}{2^n} + \frac{\delta}{2^n} = b_p(\tau) + \frac{\delta}{2^{n-1}}. \end{aligned}$$

Finally, we define

$$\mu(z) := \begin{cases} \eta_1(z), & z \in \Delta \setminus \bar{J}_{r'_1}, \\ \mu_2(z), & z \in \bar{J}_{r'_1} \setminus J_{r'_2}, \\ \vdots \\ \mu_n(z), & z \in \bar{J}_{r'_{n-1}} \setminus J_{r'_n}, \\ \vdots \end{cases}$$

Then  $\mu \in \tau$ . Inequality (3.4) indicates that  $b_p^*(\mu) = b_p(\tau)$ . The choice of  $\eta_1$  together with (3.4) gives  $\|\mu\|_\infty < \|\tau\| + \epsilon$ . It is clear that  $b^*(\mu) = b(\tau)$  and hence  $\mu$  is an asymptotical extremal.

The proof of Theorem 1 is completed.  $\square$

At last, we note that the following corollary follows from the above proof readily.

**Corollary 3.1.** *Let  $p_1, p_2, \dots, p_n$  be boundary points of the unit disk  $\Delta$  and let  $\tau \in Z(\Delta)$ . Then for any given  $\epsilon > 0$ , there is an asymptotically extremal Beltrami differential  $\mu \in \tau$  such that  $\|\mu\|_\infty < \|\tau\| + \epsilon$  and  $b_{p_j}^*(\mu) = b_{p_j}(\tau)$  for all  $1 \leq j \leq n$ .*

There even exists an asymptotical extremal in  $[\mu]_Z$  assuming local extremal boundary dilatations at infinitely many boundary points whose essential supremum is properly controlled as well.

**Theorem 3.** *Let  $\{p_m\}$  be a sequence of mutually different boundary points of the unit disk  $\Delta$  and let  $\tau \in Z(\Delta)$ . Then for any given  $\epsilon > 0$ , there is an asymptotically extremal Beltrami differential  $\mu \in \tau$  such that  $\|\mu\|_\infty < \|\tau\| + \epsilon$  and  $b_{p_m}^*(\mu) = b_{p_m}(\tau)$  for all  $m$ .*

**Proof.** We use an inductive procedure. Let  $m \geq 1$ . For any given  $\epsilon > 0$ , by Corollary 3.1 (actually by Theorem 1), there is an asymptotically extremal Beltrami differential  $\mu_m \in \tau$  such that

$$(3.5) \quad \|\mu_m\|_\infty < \|\tau\| + \sum_{j=1}^m \frac{\epsilon}{2^j}$$

and  $b_{p_j}^*(\mu_m) = b_{p_j}(\tau)$  for all  $1 \leq j \leq m$ .

Choose a small neighborhood of  $p_{m+1}$  in  $\Delta$ , say

$$B_{m+1} := \{z \in \Delta : |z - p_{m+1}| < \rho_{m+1}\},$$

where  $\rho_{m+1}$  is sufficiently small such that  $p_{m+1}$  is the only point of  $\{p_n\}$  that is contained in  $\bar{B}_{m+1}$ .

Restrict  $\mu_m$  on  $B_{m+1}$ . By Theorem 1, there is an asymptotically extremal Beltrami differential  $\tilde{\mu}_m \in [\mu_m|_{B_{m+1}}]_Z$  such that

$$(3.6) \quad \|\tilde{\mu}_m\|_\infty < \|[\mu_m|_{B_{m+1}}]_Z\| + \frac{\epsilon}{2^{m+1}}$$

and  $b_{p_{m+1}}^*(\tilde{\mu}_m) = b_{p_{m+1}}([\mu_m|_{B_{m+1}}]_Z)$ .

Combining (3.5) and (3.6), we have

$$(3.7) \quad \|\tilde{\mu}_m\|_\infty < \|\tau\| + \sum_{j=1}^{m+1} \frac{\epsilon}{2^j}.$$

Put

$$\mu_{m+1}(z) := \begin{cases} \mu_m(z), & z \in \Delta \setminus B_{m+1}, \\ \tilde{\mu}_m(z), & z \in B_{m+1}. \end{cases}$$

It is easy to check that

$$\|\mu_{m+1}\|_\infty < \|\tau\| + \sum_{j=1}^{m+1} \frac{\epsilon}{2^j}$$



and  $b_{p_j}^*(\mu_{m+1}) = b_{p_j}(\tau)$  for all  $1 \leq j \leq m + 1$ .

Thus, we can obtain two sequences  $\{\mu_m\}$  and  $\{B_m\}$  with the above conditions. Let

$$\mu(z) := \begin{cases} \mu_1(z), & z \in \Delta \setminus \bigcup_{j=2}^{\infty} B_j, \\ \mu_2(z), & z \in B_2, \\ \vdots & \\ \mu_m(z), & z \in B_m, \\ \vdots & \end{cases}$$

Then  $\mu \in \tau$  is the desired asymptotically extremal Beltrami differential.  $\square$

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