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On Φ-Mori modules

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Abstract. In this paper we introduce the concept of Mori module. An R-module M is said to be a Mori module if it satisfies the ascending chain conditon on divisorial submodules. Then we introduce a new class of modules which is closely related to the class of Mori modules. Let R be a commutative ring with identity and set

$$\mathbb{H} = \{M \mid M \text{ is an } R\text{-module and } \mathrm{Nil}(M) \text{ is a divided prime submodule of } M\}.$$

For an R-module $M \in \mathbb{H}$, set

$$T = (R \setminus Z(M)) \cap (R \setminus Z(R)),$$
$$\mathfrak{T}(M) = T^{-1}(M),$$
$$P := [\mathrm{Nil}(M) :_R M].$$

In this case the mapping $\Phi : \mathfrak{T}(M) \longrightarrow M_P$ given by $\Phi(x/s) = x/s$ is an R-module homomorphism. The restriction of Φ to M is also an R-module homomorphism from M in to M_P given by $\Phi(m/1) = m/1$ for every $m \in M$. A nonnil submodule N of M is Φ-divisorial if $\Phi(N)$ is divisorial submodule of $\Phi(M)$. An R-module $M \in \mathbb{H}$ is called Φ-Mori module if it satisfies the ascending chain condition on Φ-divisorial submodules. This paper is devoted to study the properties of Φ-Mori modules.

Contents

1. Introduction

We assume throughout this paper all rings are commmutative with $1 \neq 0$ and all modules are unitary. Let R be a ring with identity and $\mathrm{Nil}(R)$ be the set of nilpotent elements of R. Recall from [Dobb76] and [Bada99-b], that a prime ideal P of R is called a divided prime ideal if $P \subset (x)$ for

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every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of R. Badawi in [Bada99-a], [Bada00], [Bada99-b], [Bada01], [Bada02] and [Bada03] investigated the class of rings

$$\mathcal{H} = \{R \mid R \text{ is a commutative ring with } 1 \neq 0 \text{ and } \text{Nil}(R) \text{ is a divided prime ideal of } R\}.$$

Anderson and Badawi in [AB04] and [AB05] generalized the concept of Prüfer, Dedekind, Krull and Bezout domain to context of rings that are in the class $\mathcal{H}$. Also, Lucas and Badawi in [BadaL06] generalized the concept of Mori domains to the context of rings that are in the class $\mathcal{H}$. Let R be a ring, $Z(R)$ the set of zero divisors of R and $S = R \setminus Z(R)$. Then $T(R) := S^{-1}R$ denoted the total quotient ring of R. We start by recalling some background material. A nonzero divisor of a ring R is called a regular element and an ideal of R is said to be regular if it contains a regular element. An ideal I of a ring R is said to be a nonnil ideal if $I \nsubseteq \text{Nil}(R)$. If I is a nonnil ideal of $R \in \mathcal{H}$, then $\text{Nil}(R) \subset I$. In particular, it holds if I is a regular ideal of a ring $R \in \mathcal{H}$. Recall from [AB04] that for a ring $R \in \mathcal{H}$, the map $\phi : T(R) \longrightarrow R_{\text{Nil}(R)}$ given by $\phi(a/b) = a/b$, for $a \in R$ and $b \in R \setminus Z(R)$, is a ring homomorphism from $T(R)$ into $R_{\text{Nil}(R)}$ and ϕ restricted to R is also a ring homomorphism from R into $R_{\text{Nil}(R)}$ given by $\phi(x) = x/1$ for every $x \in R$.

For a nonzero ideal I of R let $I^{-1} = \{x \in T(R) : xI \subseteq R\}$ and let $I_\nu = (I^{-1})^{-1}$. It is obvious that $II^{-1} \subseteq R$. An ideal I of R is called invertible, if $II^{-1} = R$ and also I is called divisorial ideal if $I_\nu = I$. I is said to be a divisorial ideal of finite type if $I = J_\nu$ for some finitely generated ideal J of R. A Mori domain is an integral domain that satisfies the ascending chain condition on divisorial ideals. Lucas in [Luc02], generalized the concept of Mori domains to the context of commutative rings with zero divisors. According to [Luc02] a ring is called a Mori ring if it satisfies a.c.c on divisorial regular ideals. Let $R \in \mathcal{H}$. Then a nonnil ideal I of R is called ϕ-invertible if $\phi(I)$ is an invertible ideal of $\phi(R)$. A nonnil ideal I is ϕ-divisorial if $\phi(I)$ is a divisorial ideal of $\phi(R)$ [BadaL06]. Recall from [BadaL06] that R is called ϕ-Mori ring if it satisfies a.c.c on ϕ-divisorial ideals.

Let R be a ring and M be an R-module. Then M is a multiplication R-module if every submodule N of M has the form IM for some ideal I of R. If M be a multiplication R-module and N a submodule of M, then $N = IM$ for some ideal I of R. Hence $I \subseteq (N :_R M)$ and so $N = IM \subseteq (N :_R M)M \subseteq N$. Therefore $N = (N :_R M)M$ [Bar81]. Let M be a multiplication R-module, $N = IM$ and $L = JM$ be submodules of M for ideals I and J of R. Then, the product of N and L is denoted by $N.L$ or NL and is defined by IJM [Ame03]. An R-module M is called a cancellation module if $IM = JM$ for two ideals I and J of R implies $I = J$ [Ali08-a]. By [Smi88, Corollary 1 to Theorem 9], finitely generated faithful multiplication modules are cancellation modules. It follows that if M is a finitely generated faithful multiplication R-module, then $(IN :_R M) = I(N :_R M)$ for all ideals I of R and all submodules N of M. If R is an integral domain and M a faithful multiplication R-module, then M is a finitely generated R-module [ES98]. Let M be an R-module and set

$$\begin{aligned} T &= \{t \in S : \text{ for all } m \in M, tm = 0 \text{ implies } m = 0\} \\ &= (R \setminus Z(M)) \cap (R \setminus Z(R)). \end{aligned}$$

Then T is a multiplicatively closed subset of R with $T \subseteq S$, and if M is torsion-free then $T = S$. In particular, $T = S$ if M is a faithful multiplication R-module [ES98, Lemma 4.1]. Let N be a nonzero submodule of M. Then we write $N^{-1} = (M :_{R_T} N) = \{x \in R_T : xN \subseteq M\}$ and $N_\nu = (N^{-1})^{-1}$. Then N^{-1} is an R-submodule of R_T, $R \subseteq N^{-1}$ and $NN^{-1} \subseteq M$. We say that N is invertible in M if $NN^{-1} = M$. Clearly $0 \neq M$ is invertible in M. Following [Ali08-a], a submodule N of M is called a divisorial submodule of M in case $N = N_\nu M$. We say that N is a divisorial submodule of finite type if $N = L_\nu M$ for some finitely generated submodule L of M. Let R be a ring and M a finitely genetated faithful multiplication R-module. Let N be a submodule of M, then it is obviously that, N is a divisorial submodule of finite type if and only if $[N :_R M]$ is a divisorial ideal of finite type. If M is a finitely generated faithful multiplication R-module, then $N_\nu = (N :_R M)$. Consequently, $M_\nu = R$. Let M be a finitely generated faithful multiplication R-module, N a submodule of M and I an ideal of R. Then N is a divisorial submodule of M if and only if $(N :_R M)$ is a divisorial ideal of R. Also I is divisorial ideal of R if and only if IM is a divisorial submodule of M [Ali09-a]. If N is an invertible submodule of a faithful multiplication module M over an integral domain R, then $(N :_R M)$ is invertible and hence is a divisorial ideal of R. So N is a divisorial submodule of M [Ali09-a]. If R is an integral domain, M a faithful multiplication R-module and N a nonzero submodule of M, then $N_\nu = (N :_R M)_\nu$ [Ali09-a, Lemma 1]. We say that a submodule N of M is a radical submodule of M if $N = \sqrt{N}$, where $\sqrt{N} = \sqrt{(N :_R M)}M$.

Let M be an R-module. An element $r \in R$ is said to be zero divisor on M if $rm = 0$ for some $0 \neq m \in M$. The set of zero divisors of M is denoted by $Z_R(M)$ (briefly, $Z(M)$). It is easy to see that $Z(M)$ is not necessarily an ideal of R, but it has the property that if $a, b \in R$ with $ab \in Z(M)$, then either $a \in Z(M)$ or $b \in Z(M)$. A submodule N of M is called a nilpotent submodule if $[N :_R M]^n N = 0$ for some positive integer n. An element $m \in M$ is said to be nilpotent if Rm is a nilpotent submodule of M [Ali08-b]. We let $\mathrm{Nil}(M)$ to denote the set of all nilpotent elements of M; then $\mathrm{Nil}(M)$ is a submodule of M provided that M is a faithful module, and if in addition M is multiplication, then $\mathrm{Nil}(M) = \mathrm{Nil}(R)M = \bigcap P$, where the intersection runs over all prime submodules of M, [Ali08-b, Theorem 6]. If M contains no nonzero nilpotent elements, then M is called a reduced R-module. A submodule N of M is said to be a nonnil submodule if $N \nsubseteq \mathrm{Nil}(M)$. Recall that a submodule N of M is prime if whenever $rm \in N$ for some $r \in R$ and $m \in M$, then either $m \in N$ or $rM \subseteq N$. If N is a prime submodule of M, then $p := [N :_R M]$ is a prime ideal of R. In this case we say that N is a p-prime submodule of M. Let N be a submodule of multiplication R-module M, then N is a prime submodule of M if and only if $[N :_R M]$ is a prime ideal of R if and only if $N = pM$ for some prime ideal p of R with $[0 :_R M] \subseteq p$, [ES98, Corollary 2.11]. Recall from [Ali09-b] that a prime submodule P of M is called a divided prime submodule if $P \subset Rm$ for every $m \in M \setminus P$; thus a divided prime submodule is comparable to every submodule of M.

Now assume that $T^{-1}(M) = \mathfrak{T}(M)$. Set

$$\mathbb{H} = \{M \mid M \text{ is an } R\text{-module and } \mathrm{Nil}(M) \text{ is a divided prime submodule of } M\}.$$

For an R-module $M \in \mathbb{H}$, $\mathrm{Nil}(M)$ is a prime submodule of M. So

$$P := [\mathrm{Nil}(M) :_R M]$$

is a prime ideal of R. If M is an R-module and $\mathrm{Nil}(M)$ is a proper submodule of M, then $[\mathrm{Nil}(M) :_R M] \subseteq Z(R)$. Consequently,

$$R \setminus Z(R) \subseteq R \setminus [\mathrm{Nil}(M) :_R M].$$

In particular, $T \subseteq R \setminus [\mathrm{Nil}(M) :_R M]$ [Yous]. Recall from [Yous] that we can define a mapping $\Phi : \mathfrak{T}(M) \longrightarrow M_P$ given by $\Phi(x/s) = x/s$ which is clearly an R-module homomorphism. The restriction of Φ to M is also an R-module homomorphism from M in to M_P given by $\Phi(m/1) = m/1$ for every $m \in M$. A nonnil submodule N of M is said to be Φ-invertible if $\Phi(N)$ is an invertible submodule of $\Phi(M)$ [MY]. An R-module M is called a Nonnil-Noetherian module if every nonnil submodule of M is finitely genetated [Yous]. In this paper, we define concept of a Mori module and obtain some properties of this module. Then we introduce a generalization of ϕ-Mori rings.

2. Mori modules

Definition 2.1. Let R be a ring and M be an R-module. Then M is said to be a Mori module if it satisfies on divisorial submodules.

It is clear that, if M is a Noetherian R-module, then M is a Mori R-module.

Theorem 2.2. *Let R be an integral domain and M a faithful multiplication R-module. Then M is a Mori module if and only if R is a Mori domain.*

Proof. Let M be a Mori module and $\{I_m\}$ be an ascending chain of divisorial ideals of R. Then $\{(I_m)M\}$ is an ascending chain of divisorial submodules of M. Thus there exists an integer $n \geq 1$ such that $(I_n)M = (I_m)M$ for each $m \geq n$. Hence $[(I_n)M :_R M] = [(I_m)M :_R M]$ and so $I_n = I_m$ for each $m \geq n$. Therefore R is a Mori domain.

Conversely, let R be a Mori ring and $\{N_m\}$ be an ascending chain of divisorial submodules of M. Thus $\{[N_m :_R M]\}$ is an ascending chain of divisorial ideals of R. Then there exists an integer $n \geq 1$ such that $[N_n :_R M] = [N_m :_R M]$ for each $m \geq n$. Hence $[N_n :_R M]M = [N_m :_R M]M$ and so $N_n = N_m$. Therefore M is a Mori module. □

Theorem 2.3. *Let R be an integral domain and M a faithful multiplication R-module. Then M is a Mori module if and only if for every strictly descending chain of divisorial submodule $\{N_m\}$ of M, $\bigcap N_m = (0)$.*

Proof. Let M is a Mori module and $\{N_m\}$ is a strictly descending chain of divisorial submodule of M. Then, by Theorem 2.2, R is a Mori domain and $\{[N_m :_R M]\}$ is a strictly descending chain of divisorial ideals of R. So, by [Raill75, Theorem A.O], $\bigcap [N_m :_R M] = (0)$. Therefore

$$\cap N_m = \bigcap ([N_m :_R M])M = (0).$$

Conversely, let $\{N_m\}$ be a strictly descending chain of divisorial submodule of M such that $\bigcap N_m = (0)$. Then $\{[N_m :_R M]\}$ is a strictly descending chain of divisorial ideals of R such that $\bigcap [N_m :_R M] = (0)$. Hence, by [Raill75, Theorem A.O], R is a Mori domain and therefore by Theorem 2.2, M is a Mori module. □

Corollary 2.4. *Let R be an integral domain and M a faithful multiplication R-module. If M is a Mori module, then every divisorial submodule of M is contained in only a finite number of maximal divisorial submodules.*

Proof. Let M be a Mori module and N a divisorial submodule of M. Then by Theorem 2.2, R is a Mori domain and $[N :_R M]$ is a divisorial submodule of R. So, by [BG87], $[N :_R M]$ is contained in only a finite number of maximal divisorial ideals. Since M is faithful multiplication module, N is contained in only a finite number of maximal divisorial submodules of M. □

Note that if N is a divisorial submodule of R-module M, then N_S is a divisorial submodule of R_S-module M_S for each multiplicatively closed subset of R, because $N = N_\nu M$ and therefore $N_S = (N_\nu M)_S = (N_\nu)_S M_S$.

Theorem 2.5. *Let M be an Mori R-module. Then M_S is a Mori R_S-module for each multiplicatively closed subset of R.*

Proof. Let $\{\mathcal{N}_m\}$ be an ascending chain of divisorial submodules of M_S. Then $\{\mathcal{N}_m^c\}$ is an ascending chain of divisorial submodules of M. Thus there exits an integer $n \geq 1$ such that $\mathcal{N}_n^c = \mathcal{N}_m^c$ for each $m \geq n$. Therefore $\mathcal{N}_n = \mathcal{N}_n^{ce} = \mathcal{N}_m^{ce} = \mathcal{N}_m$ for each $m \geq n$. So M_S is a Mori module. □

Definition 2.6. A submodule N of M is said to be strong if $NN^{-1} = N$. N is strongly divisorial if it is both strong and divisorial.

Lemma 2.7. *Let R be an integral domain an M be a faithful multiplication R-module. Let I be an ideal of R and N ba a submodule of M. Then:*

(1) *N is strong (strong divisorial) submodule if and only if $[N :_R M]$ is strong (strong divisorial) ideal.*
(2) *I is strong (strong divisorial) ideal if and only if IM is strong (strong divisorial) submodule.*

Proof. It is obvious by [Ali09-a, Lemma 1]. □

Proposition 2.8. *Let R be an integral domain and M a faithful multiplication R-module. Let M be a Mori module and P be a prome submodule of M with $\mathrm{ht}(P) = 1$. Then P is a divisorial submodule of M. If $\mathrm{ht}(P) \geq 2$, then either $P^{-1} = R$ or P_ν is a strong divisorial submodule of M.*

Proof. Let M be a Mori module and P be a prome submodule of M with $\mathrm{ht}(P) = 1$. Then, by Theorem 2.2, R is a Mori domain and $[P :_R M]$ is a prime ideal of R such that $\mathrm{ht}([P :_R M]) = 1$. Therefore, by [Querr71, Proposition 1], $[P :_R M]$ is a divisorial ideal of R and so N is a divisorial submodule of M. If $\mathrm{ht}(P) \geq 2$, then $\mathrm{ht}([P :_R M]) \geq 2$. So, by [BG87], $[P :_R M]^{-1} = R$ or $[P :_R M]_\nu$ is a strong divisorial ideal of R. Therefore, by [Ali09-a, Lemma 1], $P^{-1} = R$ or P_ν is a strong divisorial submodule of M. □

Theorem 2.9. *Let R be an integral domain and M a faithful multiplication R-module. Then M is a Mori module if and only if for each nonzero submodule N of M, there is a finitely generated submodule $L \subset N$ such that $N^{-1} = L^{-1}$, equivalently, $N_\nu = L_\nu$.*

Proof. Let M be a Mori module and N be a nonzero submodule of M. Then, by Theorem 2.2, R is a Mori domain and $[N :_R M]$ is a nonzero ideal of R. Thus, by [Querr71, Theorem 1], there is a finitely generated ideal $J \subset [N :_R M] := I$ such that $J^{-1} = I^{-1}$. Hence there is a finitely generated submodule $L := JM \subset IM = N$ such that $N^{-1} = L^{-1}$ by [Ali09-a, Lemma 1].

Conversely, if for each nonzero submodule N of M, there is a finitely generated submodule $L \subset N$ such that $N^{-1} = L^{-1}$, then for each nonzero ideal $[N :_R M]$ of R, there is a finitely generated ideal $[L :_R M] \subset [N :_R M]$ such that $[N :_R M]^{-1} = [L :_R M]^{-1}$ by [Ali09-a, Lemma 1]. Thus, by [Querr71, Theorem 1], R is a Mori domain and so by Theorem 2.2, M is a Mori module. □

Corollary 2.10. *Let R be an integral domain and M a faithful multiplication R-module. If M is a Mori module, then every divisorial submodule of M is a divisorial submodule of finite type.*

3. ϕ-Mori modules

In this section, we define the concept of Φ-Mori module and give some results of this class of modules.

Definition 3.1. Let R be a ring and $M \in \mathbb{H}$ be an R-module. A nonnil submodule N of M is said to be a Φ-divisorial if $\Phi(N)$ is divisorial submodule of $\Phi(M)$. Also, N is called a Φ-divisorial of finite type of M if $\Phi(N)$ is a divisorial submodule of finite type of $\Phi(M)$.

Definition 3.2. Let R be a ring and $M \in \mathbb{H}$ be an R-module. Then M is said to be a Φ-Mori module if it satisfies the ascending chain condition on Φ-divisorial submodules.

Lemma 3.3. *Let $M \in \mathbb{H}$ be an R-module and N, L be nonnil submodules of M. Then $N = L$ if and only if $\Phi(N) = \Phi(L)$.*

Proof. It is clear that $N = L$ follows $\Phi(N) = \Phi(L)$. Conversely, since $\mathrm{Nil}(M)$ is a divided prime submodule of M and neither N nor L is contained in $\mathrm{Nil}(M)$, both poperly contain $\mathrm{Nil}(M)$. Thus both contain $\mathrm{Ker}(\Phi)$, by [MY, Proposition 2.1]. The result follows from standard module theory. □

Proposition 3.4 ([MY, Proposition 2.2])**.** *Let R be a ring and M a finitely generated faithful multiplication R-module with $M \in \mathbb{H}$. Then:*

(1) $\mathrm{Nil}(M_P) = \Phi(\mathrm{Nil}(M)) = \mathrm{Nil}(\Phi(M))$.
(2) $\mathrm{Nil}(\mathfrak{T}(M)) = \mathrm{Nil}(M)$.
(3) $\Phi(M) \in \mathbb{H}$.

Theorem 3.5. *Let $M \in \mathbb{H}$. Then M is a Φ-Mori module if and only if $\Phi(M)$ is a Mori module.*

Proof. Each submodule of $\Phi(M)$ is the image of a unique nonnil submodule of M and $\Phi(N)$ is a submodule of $\Phi(M)$ for each nonnil submodule N of M. Morover, by definition, if $L = \Phi(N)$, then L is a divisorial submodule of $\Phi(M)$ if and only if N is a Φ-divisorial submodule of M. Thus a chain of Φ-divisorial submodules of M stabilizes if and only if the corresponding chain of divisorial submodules of $\Phi(M)$ stabilizes. It follows that M is a Φ-Mori module if and only if $\Phi(M)$ is a Mori module. □

It is worthwhile to note that if R is a commutative ring and $M \in \mathbb{H}$ is an R-module, then $\frac{N}{\mathrm{Nil}(M)}$ is a divisorial submodule of $\frac{M}{\mathrm{Nil}(M)}$ if and only if $\frac{\Phi(N)}{\mathrm{Nil}(\Phi(M))}$ is a divisorial submodule of $\frac{\Phi(M)}{\mathrm{Nil}(\Phi(M))}$. For if $\frac{\Phi(N)}{\mathrm{Nil}(\Phi(M))}$ is not divisorial, then $\frac{\Phi(N)}{\mathrm{Nil}(\Phi(M))} \neq \frac{\Phi(N)_\nu}{\mathrm{Nil}(\Phi(M))} \frac{\Phi(M)}{\mathrm{Nil}(\Phi(M))}$. So $\Phi(N) \neq \Phi(N)_\nu \Phi(M) = \Phi(N_\nu M)$. Thus, by Lemma 3.3, $N \neq N_\nu M$. Therefore,

$$\frac{N}{\mathrm{Nil}(M)} \neq \frac{N_\nu M}{\mathrm{Nil}(M)} = \left(\frac{N}{\mathrm{Nil}(M)}\right)_\nu \frac{M}{\mathrm{Nil}(M)},$$

which is a contradiction.

Lemma 3.6. *Let $M \in \mathbb{H}$. For each nonnil submodule N of M, N is Φ-divisorial if and only if $\frac{N}{\mathrm{Nil}(M)}$ is a divisorial submodule of $\frac{M}{\mathrm{Nil}(M)}$. Moreover, $\Phi(N)$ is invertible if and only if $\frac{N}{\mathrm{Nil}(M)}$ is invertible.*

Proof. Let N is Φ-divisorial submodule of M. Then $\Phi(N)$ is divisorial and so $\Phi(N) = \Phi(N)_\nu \Phi(M)$. Thus $\frac{\Phi(N)}{\mathrm{Nil}(\Phi(M))} = \frac{\Phi(N)_\nu}{\mathrm{Nil}(\Phi(M))} \frac{\Phi(M)}{\mathrm{Nil}(\Phi(M))}$. Therefore $\frac{\Phi(N)}{\mathrm{Nil}(\Phi(M))}$ is a divisorial submodule of $\frac{\Phi(M)}{\mathrm{Nil}(\Phi(M))}$. Thus $\frac{N}{\mathrm{Nil}(M)}$ is a divisorial submodule of $\frac{M}{\mathrm{Nil}(M)}$. Conversely, is same. □

Theorem 3.7. *Let $M \in \mathbb{H}$. Then M is a Φ-Mori module if and only if $\frac{M}{\mathrm{Nil}(M)}$ is a Mori module.*

Proof. Suppose that M is a Φ-Mori module. Let $\{\frac{N_m}{\mathrm{Nil}(M)}\}$ be an ascending chain of divisorial submodules of $\frac{M}{\mathrm{Nil}(M)}$ where each N_m is a nonnil submodule of M. Hence $\{\Phi(N_m)\}$ is an ascending chain of divisorial submodules of $\Phi(M)$, by Lemma 3.6. Thus there exists an integer $n \geq 1$ such that $\Phi(N_n) = \Phi(N_m)$ for each $m \geq n$ and so $N_n = N_m$ by Lemma 3.3. It follows that $\frac{N_n}{\mathrm{Nil}(M)} = \frac{N_m}{\mathrm{Nil}(M)}$ as well.

Conversely, suppose that $\frac{M}{\mathrm{Nil}(M)}$ isa Mori module. Let $\{N_m\}$ be an ascending chain of nonnil Φ-divisorial submodules of M. Thus, by Lemma 3.6, $\{\frac{N_m}{\mathrm{Nil}(M)}\}$ is an ascending chain of divisorial submodules of $\frac{M}{\mathrm{Nil}(M)}$. Hence there exists an integer $n \geq 1$ such that $\frac{N_n}{\mathrm{Nil}(M)} = \frac{N_m}{\mathrm{Nil}(M)}$ for each $m \geq n$. As above, we have $N_n = N_m$ for each $m \geq n$. So M is a Φ-Mori module. □

Theorem 3.8. *Let R be a ring and M be a finitely generated faithful multiplication R-module. The following statements are equivalent:*

(1) *If $R \in \mathcal{H}$ is a ϕ-Mori ring, then M is a Φ-Mori module.*
(2) *If $M \in \mathbb{H}$ is a Φ-Mori module, then R is a ϕ-Mori ring.*

Proof. Since $\mathrm{Nil}(R) \subseteq \mathrm{Ann}(\frac{M}{\mathrm{Nil}(R)M}) = \mathrm{Ann}(\frac{M}{\mathrm{Nil}(M)})$, we have:

(1)⇒(2) Let $R \in \mathcal{H}$. Then, by [Yous, Proposition 3], $M \in \mathbb{H}$. If R is a ϕ-Mori ring, then by [BadaL06, Theorem 2.5], $\frac{R}{\mathrm{Nil}(R)}$ is a Mori domain. So, by Theorem 2.2, $\frac{M}{\mathrm{Nil}(M)}$ is a Mori module. Therefore, by Theorem 3.7, M is a Φ-Mori module.

(2)⇒(1) Let $M \in \mathbb{H}$. Then, by [Yous, Proposition 3], $R \in \mathcal{H}$. If M is a Φ-Mori module, then by Theorem 3.7, $\frac{M}{\mathrm{Nil}(M)}$ is a Mori module. So, by Theorem 2.2, $\frac{R}{\mathrm{Nil}(R)}$ is a Mori domain. Therefore, by [BadaL06, Theorem 2.5], R is a ϕ-Mori ring. □

Theorem 3.9 ([MY, Lemma 2.6])**.** *Let R be a ring and M a finitely generated faithful multiplication R-module with $M \in \mathbb{H}$. Then $\frac{M}{\mathrm{Nil}(M)}$ is isomorphic to $\frac{\Phi(M)}{\mathrm{Nil}(\Phi(M))}$ as R-module.*

Corollary 3.10. *Let R be a ring and M a finitely generated faithful multiplication R-module with $M \in \mathbb{H}$. Then M is a Φ-Mori module if and only if $\frac{\Phi(M)}{\mathrm{Nil}(\Phi(M))}$ is a Mori module.*

Lemma 3.11. *Let R be a ring and M a finitely generated faithful multiplication R-module with $M \in \mathbb{H}$. Suppose that a nonnil submodule N of M is a divisorial submodule of M. Then $\Phi(N)$ is a divisorial submodule of $\Phi(M)$, i.e., N is a Φ-divisorial submodule of M.*

Proof. We must show that $\Phi(N) = \Phi(N)_\nu \Phi(M)$. Since

$$[\Phi(N) :_R \Phi(M)] \subseteq [\Phi(N) :_R \Phi(M)]_\nu,$$

$[\Phi(N) :_R \Phi(M)]\Phi(M) \subseteq [\Phi(N) :_R \Phi(M)]_\nu \Phi(M)$. Hence

$$\Phi(N) \subseteq \Phi(N)_\nu \Phi(M)$$

by [Ali09-a, Lemma 1]. Now, let $y \in \Phi(N)_\nu \Phi(M)$. Then $y = \sum a_i m_i$ where $a_i \in \Phi(N)_\nu$ and $m_i = \Phi(m_i) \in \Phi(M)$. Since $\Phi(N)_\nu \subseteq R$, $a_i \in R$. If $x \in N^{-1}$ then $\Phi(x) \in \Phi(N)^{-1} = [\Phi(M) :_R \Phi(N)]$. Therefore

$$\begin{aligned} y\Phi(x) &= \Big(\sum a_i m_i\Big)\Phi(x) = \Big(\sum a_i \Phi(m_i)\Big)\Phi(x) = \sum a_i \Phi(m_i x) \\ &= \sum \Phi(a_i m_i x) = \Phi\Big(\sum a_i m_i x\Big). \end{aligned}$$

Since $\Phi(N)_\nu \Phi(N)^{-1} \subseteq \Phi(M)$, $y\Phi(x) = \Phi(\sum a_i m_i x) \in \Phi(M)$. Hence $(\sum a_i m_i)x \in M$. Since N is a divisorial submodule and $x \in N^{-1}$ is arbitrary, $\sum a_i m_i \in N$. Thus $\Phi(\sum a_i m_i) = \sum \Phi(a_i m_i) = \sum a_i \Phi(m_i) \in \Phi(N)$. Therefore $y = \sum a_i m_i \in \Phi(N)$ as well. □

Theorem 3.12. *Let R be a ring and M a finitely generated faithful multiplication R-module with $M \in \mathbb{H}$. If M is a Φ-Mori module, then M satiesfies the A.C.C on nonnil divisorial submodules of M. In particular M is a Mori module.*

Proof. Let N_m be an ascending chain of nonnil divisorial submodules of M. Hence, by Lemma 3.11, $\Phi(N_m)$ is an ascending chain of divisorial submodules of $\Phi(M)$. Since $\Phi(M)$ is a Mori module by Theorem 3.5, there exists an integer $n \geq 1$ such that $\Phi(N_n) = \Phi(N_m)$ for each $m \geq n$. Thus $N_n = N_m$ by Lemma 3.3. The "In particular" statement is now clear. □

Theorem 3.13. *Let $M \in \mathbb{H}$ be a Φ-Noetherian module. Then M is a Φ-Mori module.*

Proof. It is clear by [Yous, Theorem 10]. □

Theorem 3.14. *Let R be a ring and M a finitely generated faithful multiplication R-module with $M \in \mathbb{H}$. Let M be a Φ-Mori module and N be a Φ-divisorial submodule of M. Then N contains a power of its radical.*

Proof. Let M be a Φ-Mori module. Then, by Theorem 3.7, $\frac{M}{\mathrm{Nil}(M)}$ is a Mori module and so R is a Mori domain. Since N is a Φ-divisorial submodule of M, then $\frac{N}{\mathrm{Nil}(M)}$ is a divisorial submodule of $\frac{M}{\mathrm{Nil}(M)}$ by Lemma 3.6. Hence $[\frac{N}{\mathrm{Nil}(M)} :_R \frac{M}{\mathrm{Nil}(M)}]$ is a divisorial ideal of R and therefore contains a power of its radical by [Raill75, Theorem 5]. In other words, there exists an positive integer n such that

$$\left(\sqrt{\left[\frac{N}{\mathrm{Nil}(M)} :_R \frac{M}{\mathrm{Nil}(M)}\right]}\right)^n \subseteq \left[\frac{N}{\mathrm{Nil}(M)} :_R \frac{M}{\mathrm{Nil}(M)}\right].$$

Hence $\left(\sqrt{\frac{N}{\mathrm{Nil}(M)}}\right)^n \subseteq \frac{N}{\mathrm{Nil}(M)}$. Since $\mathrm{Nil}(M)$ is divided, N contains a power of its radical. □

We will extend concepts of definition 2.6 to the module in $\mathbb{H}$.

Definition 3.15. Let $M \in \mathbb{H}$ and N be a nonnil submodule of M. Then N is Φ-strong if $\Phi(N)$ is strong, i.e., $\Phi(N)\Phi(N)^{-1} = \Phi(N)$. Also, N is strongly Φ-divisorial if N is both Φ-strong and Φ-divisorial.

Obviously, N is Φ-strong (or strongly Φ-divisorial) if and only if $\Phi(N)$ is strong (or strongly divisorial).

Lemma 3.16. *Let $M \in \mathbb{H}$ be a Φ-Mori module and N be a nonnil submodule of M. Then the following hold:*

(1) *N is a Φ-strong submodule of M if and only if $\frac{N}{\mathrm{Nil}(M)}$ is a strong submodule of $\frac{M}{\mathrm{Nil}(M)}$.*
(2) *N is strongly Φ-divisorial if and only if $\frac{N}{\mathrm{Nil}(M)}$ is a strongly divisorial submodule of $\frac{M}{\mathrm{Nil}(M)}$.*

Proof. (1) N is a Φ-strong if and only if $\Phi(N)$ is strong if and only if $\Phi(N)\Phi(N)^{-1} = \Phi(N)$ if and only if $\frac{\Phi(N)}{\mathrm{Nil}(\Phi(M))}\frac{\Phi(N)^{-1}}{\mathrm{Nil}(\Phi(M))} = \frac{\Phi(N)}{\mathrm{Nil}(\Phi(M))}$ if and only if $\frac{\Phi(N)}{\mathrm{Nil}(\Phi(M))}$ is strong if and only if $\frac{N}{\mathrm{Nil}(M)}$ is strong.

(2) N is strongly Φ-divisorial if and only N is both Φ-strong and Φ-divisorial if and only if $\Phi(N)$ is both strong and divisorial if and only if $\frac{\Phi(N)}{\mathrm{Nil}(\Phi(M))}$ is both strong and divisorial if and only if $\frac{N}{\mathrm{Nil}(M)}$ is a strongly divisorial. □

Set $P := (\mathrm{Nil}(M) :_R M)$. Then P is a prime ideal of R and we have

$$\left(\frac{M}{\mathrm{Nil}(M)}\right)_P = \frac{M_P}{\mathrm{Nil}(M_P)},$$

[MY].

Theorem 3.17. *Let $M \in \mathbb{H}$ be a Φ-Mori module. Then M_P is a Φ-Mori module.*

Proof. Let M be a Φ-Mori module. Then, by Theorem 3.7, $\frac{M}{\mathrm{Nil}(M)}$ is a Mori module. Hence $(\frac{M}{\mathrm{Nil}(M)})_P = \frac{M_P}{\mathrm{Nil}(M_P)}$ is a Mori module by Theorem 2.5. Therefore, by Theorem 3.7, M_P is a Φ-Mori module. □

Theorem 3.18. *Let R be a ring and M a finitely generated faithful multiplication R-module with $M \in \mathbb{H}$. Let M be a Φ-Mori module and P be a nonnil prime submodule of M minimal over a nonnil principal submodule N of M. If P is finitely generated, then* $\mathrm{ht}(P) = 1$.

Proof. Let M be a Φ-Mori module. Then, by Theorem 3.7, $\frac{M}{\mathrm{Nil}(M)}$ is a Mori module and so R is a Mori domain. Also, by [MY, Theorem 2.8 and Corollary 2.9], we have $\frac{P}{\mathrm{Nil}(M)}$ is a minimal finitely generated prime submodule of $\frac{M}{\mathrm{Nil}(M)}$ over the principal submodule $\frac{N}{\mathrm{Nil}(M)}$ of $\frac{M}{\mathrm{Nil}(M)}$. Thus $[\frac{P}{\mathrm{Nil}(M)} :_R \frac{M}{\mathrm{Nil}(M)}]$ is a minimal finitely generated prime ideal of R over the principal ideal $[\frac{N}{\mathrm{Nil}(M)} :_R \frac{M}{\mathrm{Nil}(M)}]$ of R. Then, by [BAD87, Theorem 3.4], $\mathrm{ht}([\frac{P}{\mathrm{Nil}(M)} :_R \frac{M}{\mathrm{Nil}(M)}]) = 1$. Therefore $\mathrm{ht}(\frac{P}{\mathrm{Nil}(M)}) = 1$ and so $\mathrm{ht}(P) = 1$. □

Proposition 3.19. *Let R be an integral domain and M a faithful multiplication R-module with $M \in \mathbb{H}$. Let M be a Φ-Mori R-module and P be a nonnil prime submodule of M such that* $\mathrm{ht}(P) = 1$. *Then P is a Φ-divisorial submodule of M. If* $\mathrm{ht}(P) \geq 2$, *then either $P^{-1} = R$ or P_ν is a strong divisorial submodule of M.*

Proof. Let M be a Φ-Mori R-module and P be a nonnil prime submodule of M. Then, by Theorem 3.7, $\frac{M}{\mathrm{Nil}(M)}$ is a Mori module and $\frac{P}{\mathrm{Nil}(M)}$ is a prime submoduel of $\frac{M}{\mathrm{Nil}(M)}$ with $\mathrm{ht}(\frac{P}{\mathrm{Nil}(M)}) = 1$. Therefore, by Proposition 2.8, $\frac{P}{\mathrm{Nil}(M)}$ is a divisorial submodule of $\frac{M}{\mathrm{Nil}(M)}$ and so by Theorem 3.6, P is a Φ-divisorial submodule of M. Now, let $\mathrm{ht}(P) \geq 2$. Then $\mathrm{ht}(\frac{P}{\mathrm{Nil}(M)}) \geq 2$ and so by Proposition 2.8, $(\frac{P}{\mathrm{Nil}(M)})^{-1} = R$ or $(\frac{P}{\mathrm{Nil}(M)})_\nu$ is a strong divisorial submodule of M. Therefore, $P^{-1} = R$ or P_ν is a strong divisorial submodule of M. □

Theorem 3.20. *Let R be a ring and M a finitely generated faithful multiplication R-module with $M \in \mathbb{H}$. Then M is a Φ-Mori module if and only if for each nonnil submodule N of M, there exists a nonnil finitely generated submodule $L \subset N$ such that $\Phi(N)^{-1} = \Phi(L)^{-1}$, equivalently $\Phi(N)_\nu = \Phi(L)_\nu$.*

Proof. Suppose that M is a Φ-Mori module and N be a nonnil submodule of M. Since by Theorem 3.7, $\frac{M}{\mathrm{Nil}(M)}$ is a Mori module and $F := \frac{N}{\mathrm{Nil}(M)}$ is a nonzero submodule of $\frac{M}{\mathrm{Nil}(M)}$, there exists a finitely generated submodule $L \subset F$ such that $F^{-1} = L^{-1}$. Since $L = \frac{K}{\mathrm{Nil}(M)}$ for some nonnil finitely generated submodule $K$of M by [MY, Theorem 2.8], and $\mathfrak{T}(\frac{M}{\mathrm{Nil}(M)}) = \mathfrak{T}(\frac{\Phi(M)}{\mathrm{Nil}(\Phi(M))})$, we conclude that $\Phi(N)^{-1} = \Phi(L)^{-1}$.

Conversely, suppose that for each nonnil submodule N of M, there exists a nonnil finitely generated submodule $L \subset N$ such that $\Phi(N)^{-1} = \Phi(L)^{-1}$. Then for each nonzero submodule $F := \frac{N}{\mathrm{Nil}(M)}$ of $\frac{M}{\mathrm{Nil}(M)}$ there exists a finitely generated submodule $K \subset F$ such that $F^{-1} = K^{-1}$. Hence $\frac{M}{\mathrm{Nil}(M)}$ is a

Mori module by Theorem 2.9. Therefore, by Theorem 3.7, M is a Φ-Mori module. □

Corollary 3.21. *Let R be a ring and M a finitely generated faithful multiplication R-module with $M \in \mathbb{H}$. If M is a Φ-Mori module, then every Φ-divisorial submodule of M is a Φ-divisorial submodule of finite type.*

Proof. Let M be a Φ-Mori module and N be a Φ-divisorial submodule of M. Then, by Theorem 3.5, $\Phi(M)$ is a Mori module and $\Phi(N)$ is a divisorial submodule of $\Phi(M)$. Thus, by Theorem 2.9, there is a finitely generated submodule $\Phi(L) \subseteq \Phi(N)$ such that $\Phi(N)_\nu = \Phi(L)_\nu$. Since $\Phi(N)$ is divisorial, $\Phi(N) = \Phi(L)_\nu$. Therefore N is a Φ-divisorial submodule of finite type. □

Theorem 3.22. *Let R be a ring and M a finitely generated faithful multiplication R-module with $M \in \mathbb{H}$. Then the following statements are equivalent:*

(1) *M is a Φ-Mori module.*
(2) *R is a ϕ-Mori ring.*
(3) *$\Phi(M)$ is a Mori module.*
(4) *$\frac{M}{\mathrm{Nil}(M)}$ is a Mori module.*
(5) *$\frac{\Phi(M)}{\mathrm{Nil}(\Phi(M))}$ is a Mori module.*
(6) *For each nonnil submodule N of M, there exists a nonnil finitely generated submodule $L \subset N$ such that $\Phi(N)^{-1} = \Phi(L)^{-1}$.*
(7) *For each nonnil submodule N of M, there exists a nonnil finitely generated submodule $L \subset N$ such that $\Phi(N)_\nu = \Phi(L)_\nu$.*

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