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Estimates for the Bergman kernel and the multidimensional Suita conjecture

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Abstract. We study the lower bound for the Bergman kernel in terms of volume of sublevel sets of the pluricomplex Green function. We show that it implies a bound in terms of volume of the Azukawa indicatrix which can be treated as a multidimensional version of the Suita conjecture. We also prove that the corresponding upper bound holds for convex domains and discuss it in bigger detail on some convex complex ellipsoids.

Contents

1. Introduction and statement of main results

Let Ω be a pseudoconvex domain in $\mathbb{C}^n$. The following lower bound for the Bergman kernel in terms of the pluricomplex Green function was recently proved in [6] using methods of the $\bar\partial$-equation: for any $t \le 0$ and $w \in \Omega$ one has

$$K_\Omega(w) \ge \frac{1}{e^{-2nt}\lambda(\{G_{\Omega,w} < t\})}. \tag{1}$$

Here

$$K_\Omega(w) = \sup\left\{|f(w)|^2 : f \in \mathcal{O}(\Omega),\ \int_\Omega |f|^2 d\lambda \le 1\right\}$$

and

$$G_{\Omega,w} = \sup\{u \in PSH^-(\Omega) : u \le \log|\cdot - w| + C \text{ near } w\}.$$

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The constant in (1) is optimal for every t, for example we have the equality if Ω is a ball centered at w. The behaviour of the right-hand side of (1) as $t \to -\infty$ seems of particular interest. For example for $n = 1$ we easily have

$$\lim_{t\to-\infty} e^{-2t}\lambda(\{G_{\Omega,w} < t\}) = \frac{\pi}{(c_\Omega(w))^2}, \tag{2}$$

where

$$c_\Omega(w) = \exp \lim_{z\to w} \big(G_{\Omega,w}(z) - \log|z-w|\big)$$

is the logarithmic capacity of the complement of Ω with respect to w. This gave another proof in [6] of the Suita conjecture [17]

$$c_\Omega^2 \le \pi K_\Omega, \tag{3}$$

originally shown in [5].

Our first result is a counterpart of (2) in higher dimensions:

Theorem 1. *Assume that Ω is a bounded hyperconvex domain in $\mathbb{C}^n$. Then*

$$\lim_{t\to-\infty} e^{-2nt}\lambda(\{G_{\Omega,w} < t\}) = \lambda(I^A_\Omega(w)),$$

where

$$I^A_\Omega(w) = \{X \in \mathbb{C}^n : \overline{\lim_{\zeta\to 0}} \big(G_{\Omega,w}(w+\zeta X) - \log|\zeta|\big) < 0\}$$

is the Azukawa indicatrix of Ω at w.

It would be interesting to generalize this to a bigger class of domains. Combining (1) with Theorem 1 and approximating pseudoconvex domains by hyperconvex ones from inside we obtain the following multidimensional version of the Suita conjecture:

Theorem 2. *For a pseudoconvex domain Ω in $\mathbb{C}^n$ and $w \in \Omega$ we have*

$$K_\Omega(w) \ge \frac{1}{\lambda(I^A_\Omega(w))}. \tag{4}$$

Possible monotonicity of convergence in Theorem 1 is an interesting problem. We state the following:

Conjecture 1. *If Ω is pseudoconvex in $\mathbb{C}^n$ then the function*

$$t \longmapsto e^{-2nt}\lambda(\{G_{\Omega,w} < t\})$$

is nondecreasing on $(-\infty, 0]$.

We will show the following result:

Theorem 3. *Conjecture 1 is true for $n = 1$.*

The main tool will be the isoperimetric inequality. In fact, the proof of Theorem 3 will show that Conjecture 1 in arbitrary dimension is equivalent to the following *pluricomplex isoperimetric inequality*:

$$\int_{\partial\Omega} \frac{d\sigma}{|\nabla G_{\Omega,w}|} \ge 2n\lambda(\Omega)$$

for bounded strongly pseudoconvex Ω with smooth boundary (by [3] the left-hand side is then well-defined).

The following conjecture would easily give an affirmative answer to Conjecture 1:

Conjecture 2. *If Ω is pseudoconvex in $\mathbb{C}^n$ then the function*

$$t \longmapsto \log \lambda(\{G_{\Omega,w} < t\})$$

is convex on $(-\infty, 0]$.

Unfortunately, we do not know if it is true even for $n = 1$.

In [4] the question was raised whether for $n = 1$ a reverse inequality to (3)

$$K_\Omega \leq C c_\Omega^2$$

holds for some constant C. We answer it here in the negative:

Proposition 4. *Assume that $0 < r < 1$ and let $P_r = \{z \in \mathbb{C} : r < |z| < 1\}$. Then*

$$\frac{K_\Omega(\sqrt{r})}{(c_\Omega(\sqrt{r}))^2} \geq \frac{-2\log r}{\pi^3}. \tag{5}$$

It is nevertheless still plausible that there is an upper bound for the Bergman kernel in terms of logarithmic capacity which would give a quantitative version of the well-known result of Carleson [8] that for domains in $\mathbb{C}$ whose complement is a polar set the Bergman kernel vanishes. The opposite implication was also shown in [8] and the quantitative version of this is given by (3).

There is however a class of domains for which the upper bound does hold: a domain $\Omega \subset \mathbb{C}^n$ is called $\mathbb{C}$-*convex* if its intersection with every complex affine line is connected and simply connected (or empty).

Theorem 5. *For a $\mathbb{C}$-convex domain Ω in $\mathbb{C}^n$ and $w \in \Omega$ one has*

$$K_\Omega(w) \leq \frac{C^n}{\lambda(I_\Omega^A(w))}$$

with $C = 16$. If Ω is convex then the estimate holds with $C = 4$ and if it is in addition symmetric with respect to w then we can take $C = 16/\pi^2$.

By Theorems 2 and 5 for $\mathbb{C}$-convex domains the function

$$F_\Omega(w) := \big(K_\Omega(w)\lambda(I_\Omega^A(w))\big)^{1/n}$$

defined for $w \in \Omega$ with $K_\Omega(w) > 0$, satisfies

$$1 \leq F_\Omega \leq 16. \tag{6}$$

One can easily check that F_Ω is biholomorphically invariant. If Ω is pseudoconvex and balanced with respect to w (that is $w+z \in \Omega$ implies $w+\zeta z \in \Omega$ for $\zeta \in \bar{\Delta}$, where Δ is the unit disk) then $F_\Omega(w) = 1$. In fact a symmetrized bidisk

$$\mathbb{G}_2 = \{(\zeta_1 + \zeta_2, \zeta_1\zeta_2) : \zeta_1, \zeta_2 \in \Delta\},$$

is an example of a $\mathbb{C}$-convex domain (see [15]) with $F_\Omega \not\equiv 1$. By [9] we have $K_{\mathbb{G}_2}(0) = 2/\pi^2$ and by [1]

$$I^A_{\mathbb{G}_2}(0) = \{X \in \mathbb{C}^2 : |X_1| + 2|X_2| < 2\}.$$

Therefore $\lambda(I^A_{\mathbb{G}_2}(0)) = 2\pi^2/3$ and $F_{\mathbb{G}_2}(0) = 2/\sqrt{3} = 1.15470\dots$

Especially interesting is the class of convex domains. It is well-known that then the closure of the Azukawa indicatrix is equal to the Kobayashi indicatrix

$$I^K_\Omega(w) = \{\varphi'(0) : \varphi \in \mathcal{O}(\Delta, \Omega),\ \varphi(0) = w\}.$$

This follows from Lempert's results [14], see [12]. For such domains the inequality $F_\Omega \geq 1$ was proved in [6] and seems very accurate. It is in fact much more difficult than for $\mathbb{C}$-convex domains to compute an example where one does not have equality. This can be done for some convex complex ellipsoids:

Theorem 6. *For $n \geq 2$ and $m \geq 1/2$ define*

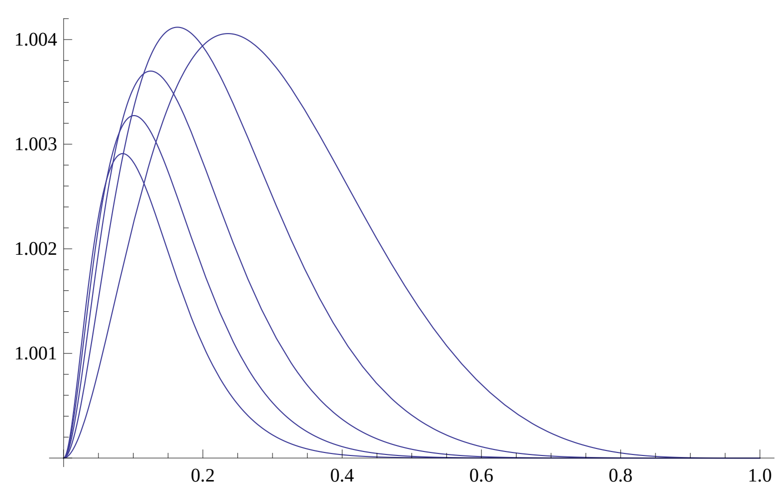
$$\Omega = \{z \in \mathbb{C}^n : |z_1| + |z_2|^{2m} + \dots + |z_n|^{2m} < 1\}. \tag{7}$$

Then for $w = (b, 0, \dots, 0)$, where $0 < b < 1$, one has

$$K_\Omega(w)\lambda(I^K_\Omega(w)) = 1 + (1-b)^a \frac{(1+b)^a - (1-b)^a - 2ab}{2ab(1+b)^a}, \tag{8}$$

where $a = (n-1)/m + 2$.

For example, Theorem 6 gives the following graphs of $F_\Omega(b, 0, \dots, 0)$ for $m = 1/2$ and $2 \leq n \leq 6$ [1]:



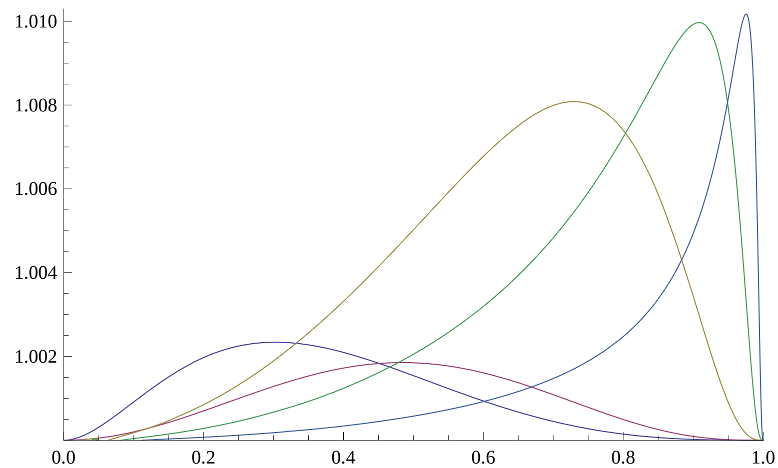
One can check numerically that the highest value of $F_\Omega(b, 0, \dots, 0)$ is attained for $m = 1/2$, $n = 3$ at $b = 0.163501\dots$, and is equal to $1.004178\dots$

[1]Figures were done using *Mathematica*.

Using [2] one can compute numerically $F_\Omega(b,0)$ for the ellipsoid

$$\Omega = \{z \in \mathbb{C}^2 : |z_1|^{2m} + |z_2|^2 < 1\},$$

where $m \geq 1/2$. This has an advantage compared to the ellipsoid given by (7) because using holomorphic automorphisms we can easily show that all values of F_Ω are attained at $(b,0)$, where $0 \leq b < 1$. Here is the graph of $F_\Omega(b,0)$ for m equal to 1/2, 2, 8, 32, and 128:



One can compute that the maximum converges to $1.010182\ldots$ as $m \to \infty$. This is the highest value of F_Ω for convex Ω we have been able to obtain so far. It would be interesting to find an optimal upper bound for F_Ω when Ω is convex, how close to 1 it really is. We suspect that it is attained for the ellipsoid

$$\{z \in \mathbb{C}^n : |z_1| + \cdots + |z_n| < 1\}$$

at a point of the form $w = (b, \ldots, b)$.

Conjecture 3. *Let Ω be convex and $w \in \Omega$ be such that $K_\Omega(w) > 0$. Then $F_\Omega(w) = 1$ if and only if there exists a balanced domain Ω' (not necessarily convex) and a biholomorphic mapping $H : \Omega \to \Omega'$ such that $H(w) = 0$.*

It was recently shown in [10] that the equality holds in (3) if and only if Ω is biholomorphic to $\Delta \setminus K$ for some closed polar subset K, this was also conjectured by Suita in [17].

The paper is organized as follows: in Section 2 we show Theorems 1 and 3. Upper bounds for the Bergman kernel are discussed in Section 3, we prove Proposition 4 and Theorem 5 there. Finally, in Section 4 the case of convex complex ellipsoids is treated.

2. Sublevel sets of the Green function

Proof of Theorem 1. Without loss of generality we may assume that $w = 0$. Write $G := G_{\Omega,0}$ and for $t \leq 0$ set

$$I_t := e^{-t}\{G < t\}.$$

We can find $R > 0$ such that $\Omega \subset B(0,R)$. Then $\log(|z|/R) \leq G$ and $I_t \subset B(0,R)$. In our case by [18] the function

$$A(X) = \overline{\lim_{\zeta\to 0}} \big(G(\zeta X) - \log|\zeta|\big)$$

is continuous on $\mathbb{C}^n$ and $\overline{\lim}$ is equal to lim. Therefore

$$A(X) = \lim_{t\to-\infty} \big(G(e^t X) - t\big)$$

and by the Lebesgue bounded convergence theorem

$$\lim_{t\to-\infty} \lambda(I_t) = \lambda(\{A < 0\}). \qquad \square$$

Proof of Theorem 3. Set

$$f(t) := \log \lambda(\{G < t\}) - 2t,$$

where $G = G_{\Omega,w}$. It is enough to show that if t is a regular value of G then $f'(t) \geq 0$. We have

$$f'(t) = \frac{\frac{d}{dt}\lambda(\{G < t\})}{\lambda(\{G < t\})} - 2.$$

The co-area formula gives

$$\lambda(\{G < t\}) = \int_{-\infty}^{t} \int_{\{G=s\}} \frac{d\sigma}{|\nabla G|} ds$$

and therefore

$$\frac{d}{dt}\lambda(\{G < t\}) = \int_{\{G=t\}} \frac{d\sigma}{|\nabla G|}.$$

By the Cauchy-Schwarz inequality

$$\frac{d}{dt}\lambda(\{G < t\}) \geq \frac{(\sigma(\{G=t\}))^2}{\displaystyle\int_{\{G=t\}} |\nabla G| d\sigma} = \frac{(\sigma(\{G=t\}))^2}{2\pi}.$$

The isoperimetric inequality gives

$$(\sigma(\{G=t\}))^2 \geq 4\pi\lambda(\{G < t\})$$

and we obtain $f'(t) \geq 0$. $\square$

3. Upper bound for the Bergman kernel

We first show that the reverse estimate to (4) is not true in general.

Proof of Proposition 4. Since z^j, $j \in \mathbb{Z}$, is an orthogonal system in $H^2(P_r)$ and

$$||z^j||^2 = \begin{cases} \dfrac{\pi}{j+1}\big(1 - r^{2j+2}\big), & j \neq -1, \\ -2\pi \log r, & j = -1, \end{cases}$$

we have

$$K_{P_r}(w) = \frac{1}{\pi|w|^2}\left(\frac{1}{-2\log r} + \sum_{j\in\mathbb{Z}} \frac{j|w|^{2j}}{1-r^{2j}}\right)$$

and

$$K_{P_r}(\sqrt{r}) \geq \frac{1}{-2\pi r\log r}. \tag{9}$$

To estimate c_{P_r} from above consider the mapping

$$p(\zeta) = \exp\left(\frac{\log r}{\pi i}\operatorname{Log}\left(i\frac{1+\zeta}{1-\zeta}\right)\right), \quad \zeta \in \Delta,$$

where Log is the principal branch of the logarithm defined on $\mathbb{C} \setminus (-\infty, 0]$. We have $p(0) = \sqrt{r}$ and $p'(0) = -2i\sqrt{r}\log r/\pi$. Also

$$G_{P_r}(p(\zeta), \sqrt{r}) \leq \log|\zeta|$$

and therefore

$$c_{P_r}(\sqrt{r}) \leq \frac{1}{|p'(0)|} = \frac{\pi}{-2\sqrt{r}\log r}.$$

Combining this with (9) we get (5). □

Next, we show the reverse inequality to (4) for $\mathbb{C}$-convex domains.

Proof of Theorem 5. Write $I = I^A_\Omega(w)$. We may assume that $w = 0$. We claim that it is enough to show that

$$I \subset \sqrt{C}\,\Omega. \tag{10}$$

Indeed, since I is balanced we would then have

$$K_\Omega(0) \leq K_{I/\sqrt{C}}(0) = \frac{1}{\lambda(I/\sqrt{C})} = \frac{C^n}{\lambda(I)}.$$

The proof of (10) will be similar to the proof of Proposition 1 in [16]. Choose $X \in I$ and by L denote the complex line generated by X. Let a be a point from $L \cap \partial\Omega$ with the smallest distance to the origin. We can find a hyperplane H in $\mathbb{C}^n$ such that $H \cap \Omega = \emptyset$ (cf. [11], Theorem 4.6.8). Let D be the set of those $\zeta \in \mathbb{C}$ such that ζX belongs to the projection of Ω on L along H. Then D is a simply connected domain (cf. [11], Proposition 4.6.7). Let φ be a biholomorphic mapping $\Delta \to D$ such that $\varphi(0) = 0$. We then have

$$0 > \overline{\lim}\left(G_{\Omega,0}(\zeta X) - \log|\zeta|\right) \geq \overline{\lim}\left(G_{D,0}(\zeta) - \log|\zeta|\right) = -\log|\varphi'(0)|.$$

By the Koebe quarter theorem $|\varphi'(0)| \leq 4r$, where r is the distance from the origin to ∂D. Since $r = |a|/|X|$, we obtain $|X| < 4|a|$. This gives (10) for $\mathbb{C}$-convex domains with $C = 16$. If Ω is convex then so is D and we may assume that it is a half-plane. Then $|\varphi'(0)| \leq 2r$ and we get (10) with $C = 4$. Finally, if Ω is symmetric then we may assume that D is a strip centered at the origin and we get $|\varphi'(0)| \leq 4r/\pi$. □

4. Complex ellipsoids

We first recall a general formula from [13] (it is in fact a consequence of Lempert's theory [14]) for geodesics in convex complex ellipsoids

$$\mathcal{E}(p) = \{z \in \mathbb{C}^n : |z_1|^{2p_1} + \cdots + |z_n|^{2p_n} < 1\},$$

where $p = (p_1, \dots, p_n)$, $p_j \geq 1/2$. For $A \subset \{1, \dots, n\}$ holomorphic mappings $\varphi : \Delta \to \mathcal{E}(p)$ of the form

$$\varphi_j(\zeta) = \begin{cases} a_j \dfrac{\zeta - \alpha_j}{1 - \bar{\alpha}_j \zeta} \left(\dfrac{1 - \bar{\alpha}_j \zeta}{1 - \bar{\alpha}_0 \zeta} \right)^{1/p_j}, & j \in A, \\ a_j \left(\dfrac{1 - \bar{\alpha}_j \zeta}{1 - \bar{\alpha}_0 \zeta} \right)^{1/p_j}, & j \notin A, \end{cases} \tag{11}$$

where $a_j \in \mathbb{C}_*$, $\alpha_j \in \Delta$ for $j \in A$, $\alpha_j \in \bar{\Delta}$ for $j \notin A$,

$$\alpha_0 = |a_1|^{2p_1} \alpha_1 + \cdots + |a_n|^{2p_n} \alpha_n,$$

and

$$1 + |\alpha_0|^2 = |a_1|^{2p_1}(1 + |\alpha_1|^2) + \cdots + |a_n|^{2p_n}(1 + |\alpha_n|^2),$$

form the set of almost all geodesics in Ω (possible exceptions form a lower-dimensional set). A component φ_j has a zero in Δ if and only if $j \in A$. We have

$$\varphi_j(0) = \begin{cases} -a_j \alpha_j, & j \in A, \\ a_j, & j \notin A, \end{cases}$$

and

$$\varphi_j'(0) = \begin{cases} a_j \left(1 + (\dfrac{1}{p_j} - 1)|\alpha_j|^2 - \dfrac{\alpha_j \bar{\alpha}_0}{p_j} \right), & j \in A, \\ a_j \dfrac{\bar{\alpha}_0 - \bar{\alpha}_j}{p_j}, & j \notin A. \end{cases}$$

For $w \in \mathcal{E}(p)$ the set of vectors $\varphi'(0)$ where $\varphi(0) = w$ forms a subset of $\partial I^K_{\mathcal{E}(p)}(w)$ of a full measure.

Now assume that $w = (b, 0, \dots, 0)$. There are two possibilities: either $A = \{1, \dots, n\}$ or $A = \{2, \dots, n\}$. Since $\varphi(0) = w$, it follows that $\alpha_2 = \cdots = \alpha_n = 0$, hence $\alpha_0 = |a_1|^{2p_1} \alpha_1$ and

$$1 + |a_1|^{4p_1} |\alpha_1|^2 = |a_1|^{2p_1}(1 + |\alpha_1|^2) + |a_2|^{2p_2} + \cdots + |a_n|^{2p_n}. \tag{12}$$

Moreover,

$$\begin{cases} a_1 \alpha_1 = -b, & 1 \in A, \\ a_1 = b, & 1 \notin A. \end{cases}$$

We will get vectors $X = \varphi'(0)$ from $\partial I^K_{\mathcal{E}(p)}(w)$, where

$$X_1 = \begin{cases} -\dfrac{b}{\alpha_1} \left(1 + (\dfrac{1}{p_1} - 1)|\alpha_1|^2 - \dfrac{b^{2p_1} |\alpha_1|^{2-2p_1}}{p_1} \right), & 1 \in A, \\ -\bar{\alpha}_1 \dfrac{b(1-b)}{p_1}, & 1 \notin A, \end{cases} \tag{13}$$

and $X_j = a_j$, $j = 2, \dots, n$. By (12) the parameters are related by

$$|a_2|^{2p_2} + \cdots + |a_n|^{2p_n} = \begin{cases} (1 - b^{2p_1}|\alpha_1|^{-2p_1})(1 - b^{2p_1}|\alpha_1|^{2-2p_1}), & 1 \in A, \\ (1 - b^{2p_1})(1 - b^{2p_1}|\alpha_1|^2), & 1 \notin A. \end{cases}$$

If now $p_1 = 1/2$ as in Theorem 6 then by (13)

$$|\alpha_1| = \begin{cases} \dfrac{2b^2 + |X_1| - \sqrt{(2b^2 + |X_1|)^2 - 4b^2}}{2b}, & 1 \in A, \\ \dfrac{|X_1|}{2b(1-b)}, & 1 \notin A. \end{cases}$$

After simple transformation we will obtain the following result:

Theorem 7. *Assume that $p_1 = 1/2$, $p_j \geq 1/2$ for $j \geq 2$, and $0 < b < 1$. Then*

$$I^K_{\mathcal{E}(p)}((b, 0, \dots, 0)) = \{X \in \mathbb{C}^n : |X_2|^{2p_2} + \cdots + |X_n|^{2p_n} \leq \gamma(|X_1|)\},$$

where

$$\gamma(r) = \begin{cases} 1 - b - \dfrac{r^2}{4b(1-b)}, & r \leq 2b(1-b), \\ 1 - b^2 - r, & r > 2b(1-b). \end{cases}$$

Proof of Theorem 6. Denoting

$$\omega = \lambda(\{z \in \mathbb{C}^{n-1} : |z_1|^{2m} + \cdots + |z_{n-1}|^{2m} < 1\}$$

we will get from Theorem 7

$$\begin{aligned} \lambda(I^K_\Omega((b, 0, \dots, 0))) &= 2\pi\omega \int_0^{1-b^2} r(\gamma(r))^{(n-1)/m} dr \\ &= 2\pi\omega(1-b)^a \frac{(1-b)^a + 2ab}{a(a-1)}. \end{aligned} \tag{14}$$

It remains to compute the Bergman kernel. By the deflation method from [7] we obtain

$$K_\Omega((b, 0, \dots, 0)) = \frac{\lambda(\mathcal{E}(1/2, m/(n-1)))}{\lambda(\Omega)} K_{\mathcal{E}(1/2, m/(n-1))}((b, 0)).$$

By Example 12.1.13 in [12] (see also formula (9) in [7])

$$K_{\mathcal{E}(1/2,1/p)}((b, 0)) = \frac{p+1}{4\pi^2 b}\big((1-b)^{-p-2} - (1+b)^{-p-2}\big).$$

We also have $\lambda(\mathcal{E}(1/2, 1/p) = 2\pi^2/((p+1)(p+2))$ and $\lambda(\Omega) = 2\pi\omega/(a(a-1))$. It follows that

$$K_\Omega((b, 0, \dots, 0)) = \frac{a-1}{4\pi\omega b}\big((1-b)^{-a} - (1+b)^{-a}\big)$$

and combining this with (14) gives (8). □

Added in proof. Professor J. E. Fornaess found an example (already in dimension one) showing that Conjecture 2 does not hold.

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