New York Journal of Mathematics

New York J. Math. **22** (2016) 277–291.

Realising the Toeplitz algebra of a higher-rank graph as a Cuntz–Krieger algebra

Yosafat E. P. Pangalela

Abstract. For a row-finite higher-rank graph Λ, we construct a higher-rank graph $T\Lambda$ such that the Toeplitz algebra of Λ is isomorphic to the Cuntz–Krieger algebra of $T\Lambda$. We then prove that the higher-rank graph $T\Lambda$ is always aperiodic and use this fact to give another proof of a uniqueness theorem for the Toeplitz algebras of higher-rank graphs.

Contents

1. Introduction

Higher-rank graphs and their Cuntz–Krieger algebras were introduced by Kumjian and Pask in [5] as a generalisation of the Cuntz–Krieger algebras of directed graphs. Kumjian and Pask proved an analogue of the Cuntz–Krieger uniqueness theorem for a family of *aperiodic* higher-rank graphs [5, Theorem 4.6]. Aperiodicity is a generalisation of Condition (L) for directed graphs and comes in several forms for different kinds of higher-rank graphs (see [1, 5, 6, 10, 11, 12, 13, 14]).

The Toeplitz algebra of a directed graph is an extension of the Cuntz–Krieger algebra in which the Cuntz–Krieger equations at vertices are replaced by inequalities. An analogous family of Toeplitz algebras for higher-rank graph was introduced and studied by Raeburn and Sims [9]. They

Received April 29, 2015.

2010 *Mathematics Subject Classification.* Primary 46L05.

Key words and phrases. Cuntz–Krieger algebra, Toeplitz algebra, higher-rank graph, the Cuntz–Krieger uniqueness theorem, the uniqueness theorem for Toeplitz algebras.

This research is part of the author's Ph.D. thesis, supervised by Professor Iain Raeburn and Dr. Lisa Orloff Clark.

ISSN 1076-9803/2016

proved a uniqueness theorem for Toeplitz algebras [9, Theorem 8.1], generalising a previous theorem for directed graphs [3, Theorem 4.1].

For a directed graph E, the Toeplitz algebra of E is canonically isomorphic to the Cuntz–Krieger algebra of a graph TE (see [7, Theorem 3.7] and [15, Lemma 3.5]). Here we provide an analogous construction for a row-finite higher-rank graph Λ. We build a higher-rank graph $T\Lambda$, and show that the Toeplitz algebra of Λ is canonically isomorphic to the Cuntz–Krieger algebra of $T\Lambda$ (Theorem 4.1). Our proof relies on the uniqueness theorem of [9]. However, it is interesting to observe that the higher-rank graph $T\Lambda$ is always aperiodic. Hence our isomorphism shows that the uniqueness theorem of [9] is a consequence of the general Cuntz–Krieger uniqueness theorem of [11] (see Remark 4.3).

2. Higher-rank graphs

Let k be a positive integer. We regard $\mathbb{N}^k$ as an additive semigroup with identity 0. For $m, n \in \mathbb{N}^k$, we write $m \vee n$ for their coordinate-wise maximum.

A *higher-rank graph* or *k-graph* is a pair (Λ, d) consisting of a countable small category Λ together with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *factorisation property*: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu v$ and $d(\mu) = m$, $d(\nu) = n$. We then write $\lambda(0, m)$ for μ and $\lambda(m, m+n)$ for ν. We regard elements of Λ^0 as *vertices* and elements of Λ as *paths*. For detailed explanation and examples, see [8, Chapter 10].

For $v \in \Lambda^0$ and $E \subseteq \Lambda$, we define $vE := \{\lambda \in E : r(\lambda) = v\}$ and $m \in \mathbb{N}^k$, we write $\Lambda^m := \{\lambda \in \Lambda : d(\lambda) = m\}$.We use term *edge* to denote a path $e \in \Lambda^{e_i}$ where $1 \leq i \leq k$, and write

$$\Lambda^1 := \bigcup_{1 \leq i \leq k} \Lambda^{e_i}$$

for the set of all edges. We say that Λ is *row-finite* if for every $v \in \Lambda^0$, the set $v\Lambda^{e_i}$ is finite for $1 \leq i \leq k$. Finally, we say $v \in \Lambda^0$ is a *source* if there exists $m \in \mathbb{N}^k$ such that $v\Lambda^m = \emptyset$.

For a row-finite k-graph Λ, we shall construct a k-graph $T\Lambda$ which is row-finite and always has sources. Our k-graph $T\Lambda$ is typically not *locally convex* in the sense of [10, Definition 3.9] (see Remark 3.3), so the appropriate definition of Cuntz–Krieger Λ-family is the one in [11]. For detailed discussion about row-finite k-graphs and their generalisations, see [16, Section 2].

From now on, we focus on a row-finite k-graph Λ. For $\lambda, \mu \in \Lambda$, we say that τ is a *minimal common extension* of λ and μ if

$$d(\tau) = d(\lambda) \vee d(\mu), \ \tau(0, d(\lambda)) = \lambda \text{ and } \tau(0, d(\mu)) = \mu.$$

Let $\mathrm{MCE}(\lambda, \mu)$ denote the collection of all minimal common extensions of λ and μ. Then we write

$$\Lambda^{\min}(\lambda, \mu) := \left\{ \left(\lambda', \mu'\right) \in \Lambda \times \Lambda : \lambda\lambda' = \mu\mu' \in \mathrm{MCE}(\lambda, \mu) \right\}.$$

A set $E \subseteq v\Lambda^1$ is *exhaustive* if for all $\lambda \in v\Lambda$, there exists $e \in E$ such that $\Lambda^{\min}(\lambda, e) \neq \emptyset$.

A *Toeplitz–Cuntz–Krieger* Λ*-family* is a collection $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries in a C^*-algebra B satisfying:

(TCK1) $\left\{t_v : v \in \Lambda^0\right\}$ is a collection of mutually orthogonal projections.
(TCK2) $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$.
(TCK3) $t_\lambda^* t_\mu = \sum_{(\lambda', \mu') \in \Lambda^{\min}(\lambda, \mu)} t_{\lambda'} t_{\mu'}^*$ for all $\lambda, \mu \in \Lambda$.

Remark 2.1. In [9, Lemma 9.2], Raeburn and Sims required also that "for all $m \in \mathbb{N}^k \setminus \{0\}$, $v \in \Lambda^0$, and every set $E \subseteq v\Lambda^m$, $t_v \geq \sum_{\lambda \in E} t_\lambda t_\lambda^*$". However, by [11, Lemma 2.7 (iii)], this follows from (TCK1)–(TCK3), and hence our definition is basically same as that of [9].

Meanwhile, based on [11, Proposition C.3], a *Cuntz–Krieger* Λ*-family* is a Toeplitz–Cuntz–Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ which satisfies

(CK) $\prod_{e \in E} (t_v - t_e t_e^*) = 0$ for all $v \in \Lambda^0$ and exhaustive $E \subseteq v\Lambda^1$.

Raeburn and Sims proved in [9, Section 4] that there is a C^*-algebra $TC^*(\Lambda)$ generated by a universal Toeplitz–Cuntz–Krieger Λ-family

$$\{t_\lambda : \lambda \in \Lambda\}.$$

If $\{T_\lambda : \lambda \in \Lambda\}$ is a Toeplitz–Cuntz–Krieger Λ-family in a C^*-algebra B, we write ϕ_T for the homomorphism of $TC^*(\Lambda)$ into B such that $\phi_T(t_\lambda) = T_\lambda$ for $\lambda \in \Lambda$. The quotient of $TC^*(\Lambda)$ by the ideal generated by

$$\left\{ \prod_{e \in E} (t_v - t_e t_e^*) : v \in \Lambda^0, E \subseteq v\Lambda^1 \text{ is exhaustive} \right\}$$

is generated by a universal family of the Cuntz–Krieger Λ-family

$$\{s_\lambda : \lambda \in \Lambda\},$$

and hence we can identify it with the C^*-algebra $C^*(\Lambda)$. For a Cuntz–Krieger Λ-family $\{S_\lambda : \lambda \in \Lambda\}$ in a C^*-algebra B, we write π_S for the homorphism of $C^*(\Lambda)$ into B such that $\pi_S(s_\lambda) = S_\lambda$ for $\lambda \in \Lambda$. Furthermore, we have $s_v \neq 0$ for $v \in \Lambda^0$ [11, Proposition 2.12].

As for directed graphs, we have uniqueness theorems for the Toeplitz algebra [9, Theorem 8.1] and the Cuntz–Krieger algebra [6, Theorem 4.7]. The former does not need any hypothesis on the k-graph as stated in the following theorem.

Theorem 2.2. *Let Λ be a row-finite k-graph. Let $\{T_\lambda : \lambda \in \Lambda\}$ be a Toeplitz–Cuntz–Krieger Λ-family in a C^*-algebra B. Suppose that for every $v \in \Lambda^0$,*

$$(*) \qquad \prod_{e \in v\Lambda^1} (T_v - T_e T_e^*) \neq 0$$

(where this includes $T_v \neq 0$ if $v\Lambda^1 = \emptyset$). Suppose that $\phi_T : TC^*(\Lambda) \to B$ is the homomorphism such that $\phi_T(t_\lambda) = T_\lambda$ for $\lambda \in \Lambda$. Then

$$\phi_T : TC^*(\Lambda) \to B$$

is injective.

Remark 2.3. Every k-graph Λ gives a product system of graphs over $\mathbb{N}^k$ and a Toeplitz–Cuntz–Krieger Λ-family gives a Toeplitz Λ-family of the product system [9, Lemma 9.2]. Lemma 9.3 of [9] shows that, if the Toeplitz–Cuntz–Krieger Λ-family satisfies $(*)$, then the Toeplitz Λ-family satisfies the hypothesis of [9, Theorem 8.1].

Remark 2.4. In the actual hypothesis, we need to verify whether

$$\prod_{1 \leq i \leq k} \left(T_v - \sum_{e \in G_i} T_e T_e^* \right) \neq 0$$

for every $v \in \Lambda^0$, $1 \leq i \leq k$, and finite set $G_i \subseteq v\Lambda^{e_i}$. However, since we only consider row-finite k-graphs, then for every $v \in \Lambda^0$ and $1 \leq i \leq k$, the set $v\Lambda^{e_i}$ is finite. Thus for a row finite k-graph, we can simplify Lemma 9.3 of [9] as Theorem 2.2.

On the other hand, Lewin and Sims in [6, Theorem 4.7] proved that the Cuntz–Krieger uniqueness theorem only holds for k-graphs which satisfy the following *aperiodicity* condition: for every pair of distinct paths $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$, there exists $\eta \in s(\lambda)\Lambda$ such that $\operatorname{MCE}(\lambda\eta, \mu\eta) = \emptyset$ [6, Definition 3.1]. (For discussion about the equivalence of various aperiodicity definitions, see [6, 12, 13, 14].) Now we state the uniqueness theorem as follows:

Theorem 2.5 ([6, Theorem 4.7])**.** *Suppose that Λ is an aperiodic row-finite k-graph and $\{S_\lambda : \lambda \in \Lambda\}$ is a Cuntz–Krieger Λ-family in a C^*-algebra B such that $S_v \neq 0$ for $v \in \Lambda^0$. Suppose that $\pi_S : C^*(\Lambda) \to B$ is the homomorphism such that $\pi_S(s_\lambda) = S_\lambda$ for $\lambda \in \Lambda$ for $\lambda \in \Lambda$. Then π_S is an injective homomorphism.*

3. The k-graph $T\Lambda$

Suppose that Λ is a row-finite k-graph. In this section, we define a k-graph $T\Lambda$; later we show that $TC^*(\Lambda) \cong C^*(T\Lambda)$ (Theorem 4.1). Interestingly, our k-graph $T\Lambda$ is always aperiodic (Proposition 3.5).

Proposition 3.1. *Let $\Lambda = (\Lambda, d, r, s)$ be a row-finite k-graph. Then define sets $T\Lambda^0$ and $T\Lambda$ as follows:*

$$T\Lambda^0 := \left\{\alpha(v) : v \in \Lambda^0\right\} \cup \left\{\beta(v) : v\Lambda^1 \neq \emptyset\right\};$$

$$T\Lambda := \left\{\alpha(\lambda) : \lambda \in \Lambda\right\} \cup \left\{\beta(\lambda) : \lambda \in \Lambda, s(\lambda)\Lambda^1 \neq \emptyset\right\}.$$

Define functions $r, s : T\Lambda \backslash T\Lambda^0 \to T\Lambda^0$ *by*

$$r(\alpha(\lambda)) = \alpha(r(\lambda)),\ s(\alpha(\lambda)) = \alpha(s(\lambda)),$$
$$r(\beta(\lambda)) = \alpha(r(\lambda)),\ s(\beta(\lambda)) = \beta(s(\lambda))$$

(r, s *are the identity on* $T\Lambda^0$*). We also define a partially defined product* $(\tau, \omega) \mapsto \tau\omega$ *from*

$$\{(\tau, \omega) \in T\Lambda \times T\Lambda : s(\tau) = r(\omega)\}$$

to $T\Lambda$*, where*

$$(\alpha(\lambda), \alpha(\mu)) \mapsto \alpha(\lambda\mu)$$
$$(\alpha(\lambda), \beta(\mu)) \mapsto \beta(\lambda\mu)$$

and a function $d : T\Lambda \to \mathbb{N}^k$ *where*

$$d(\alpha(\lambda)) = d(\beta(\lambda)) = d(\lambda).$$

Then $(T\Lambda, d)$ *is a* k*-graph.*

Proof. First we claim that $T\Lambda$ is a countable category. Note that $T\Lambda$ is countable since Λ is countable.

Now we show that for all paths η, τ, ω in $T\Lambda$ where $s(\eta) = r(\tau)$ and $s(\tau) = r(\omega)$, we have $s(\tau\omega) = s(\omega)$, $r(\tau\omega) = r(\tau)$, and $(\eta\tau)\omega = \eta(\tau\omega)$. If one of τ, ω is a vertex then we are done. So assume otherwise, and we have $\eta = \alpha(\lambda)$, $\tau = \alpha(\mu)$, and ω is either $\alpha(\nu)$ or $\beta(\nu)$ for some paths λ, μ, ν in Λ. In both cases, we always have $s(\lambda) = r(\mu)$, $s(\mu) = r(\nu)$, and $(\lambda\mu)\nu = \lambda(\mu\nu)$. If $\omega = \alpha(\nu)$, we have

$$\begin{aligned} s(\tau\omega) &= s(\alpha(\mu)\alpha(\nu)) = s(\alpha(\mu\nu)) \\ &= \alpha(s(\mu\nu)) = \alpha(s(\nu)) = s(\alpha(\nu)) = s(\omega), \end{aligned}$$

$$\begin{aligned} r(\tau\omega) &= r(\alpha(\mu)\alpha(\nu)) = r(\alpha(\mu\nu)) \\ &= \alpha(r(\mu\nu)) = \alpha(r(\mu)) = r(\alpha(\mu)) = r(\tau), \end{aligned}$$

and

$$\begin{aligned} (\eta\tau)\omega &= \big(\alpha(\lambda)\alpha(\mu)\big)\alpha(\nu) = \alpha(\lambda\mu)\alpha(\nu) = \alpha((\lambda\mu)\nu) \\ &= \alpha(\lambda(\mu\nu)) = \alpha(\lambda)\alpha(\mu\nu) = \alpha(\lambda)\big(\alpha(\mu)\alpha(\nu)\big) = \eta(\tau\omega). \end{aligned}$$

On the other hand, if $\omega = \beta(\nu)$, then

$$\begin{aligned} s(\tau\omega) &= s(\alpha(\mu)\beta(\nu)) = s(\beta(\mu\nu)) \\ &= \beta(s(\mu\nu)) = \beta(s(\nu)) = s(\beta(\nu)) = s(\omega), \end{aligned}$$

$$\begin{aligned} r(\tau\omega) &= r(\alpha(\mu)\beta(\nu)) = r(\beta(\mu\nu)) \\ &= \alpha(r(\mu\nu)) = \alpha(r(\mu)) = r(\alpha(\mu)) = r(\tau), \end{aligned}$$

and

$$\begin{aligned} (\eta\tau)\omega &= \big(\alpha(\lambda)\alpha(\mu)\big)\beta(\nu) = \alpha(\lambda\mu)\beta(\nu) = \beta((\lambda\mu)\nu) \\ &= \beta(\lambda(\mu\nu)) = \alpha(\lambda)\beta(\mu\nu) = \alpha(\lambda)\big(\alpha(\mu)\beta(\nu)\big) = \eta(\tau\omega). \end{aligned}$$

Thus, $T\Lambda$ is a countable category, as claimed.

Now we show that d is a functor. Note that both $T\Lambda$ and $\mathbb{N}^k$ are categories. First take object $x \in T\Lambda^0$, then $d(x) = 0$ is an object in category $\mathbb{N}^k$. Next take morphisms $\tau, \omega \in T\Lambda$ with $s(\tau) = r(\omega)$. Then by definition of d,

$$d(\tau\omega) = d(\tau) + d(\omega).$$

Hence, d is a functor.

To show that d satisfies the factorisation property, take $\omega \in T\Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\omega) = m + n$. By definition, ω is either $\alpha(\lambda)$ or $\beta(\lambda)$ for some path λ in Λ. In both cases, there exist paths μ, ν in Λ such that $\lambda = \mu\nu$, $d(\mu) = m$, and $d(\nu) = n$. Then, we have $d(\alpha(\mu)) = m$, $d(\alpha(\nu)) = d(\beta(\nu)) = n$, and ω is either equal to $\alpha(\mu)\alpha(\nu)$ or $\alpha(\mu)\beta(\nu)$. Therefore, the existence of factorisation is guaranteed.

Now we show that the factorisation is unique. First suppose

$$\omega = \alpha(\mu)\alpha(\nu) = \alpha(\mu')\alpha(\nu')$$

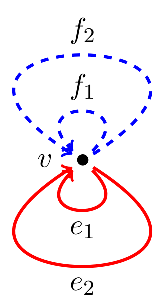
where $d(\alpha(\mu)) = d(\alpha(\mu'))$ and $d(\alpha(\nu)) = d(\alpha(\nu'))$. We consider paths $\lambda = \mu\nu$ and $\lambda' = \mu'\nu'$. Since $\alpha(\lambda) = \omega = \alpha(\lambda')$, then $\lambda = \lambda'$. This implies $\mu = \mu'$ and $\nu = \nu'$ based on the uniquness of factorisation in Λ. Then $\alpha(\mu) = \alpha(\mu')$ and $\alpha(\nu) = \alpha(\nu')$. For the case $\omega = \alpha(\mu)\beta(\nu)$, we get the same result by using the same argument. The conclusion follows. □

Remark 3.2. For a directed graph E (that is, for $k = 1$), the graph TE was constructed by Muhly and Tomforde [7, Definition 3.6] (denoted E_V), and by Sims [15, Section 3] (denoted $\widetilde{E}$). Our notation follows that of Sims because we want to distinguish between paths in $T\Lambda$ (denoted $\alpha(\lambda)$ and $\beta(\lambda)$) and those in Λ (denoted λ).

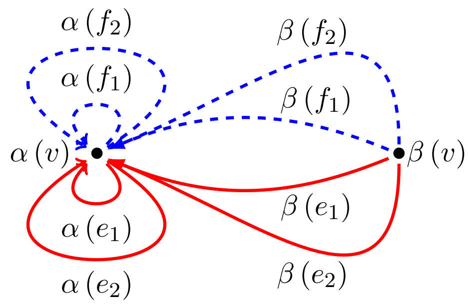
Remark 3.3. Every vertex $\beta(v)$ satisfies $\beta(v)T\Lambda^1 = \emptyset$. Then if Λ has a vertex v which receives edges e, f with $d(e) \neq d(f)$, then there is no edge $g \in \beta(s(e))T\Lambda^{d(f)}$ (or $g \in \alpha(s(e))T\Lambda^{d(f)}$ if $s(e)\Lambda = \emptyset$), and hence $T\Lambda$ is not locally convex.

To give an illustration how we construct the k-graph $T\Lambda$ from a k-graph Λ, we first recall coloured graphs of [4]. By choosing k-different colours $c_1, \ldots, c_k$, we can view paths in Λ^{e_i} as edges of colour c_i. For a k-graph Λ, we call its corresponding coloured graph the *skeleton* of Λ. For further discussion about k-graphs and their skeletons, see [4].

Example 3.4. Consider the 2-graph Λ which has skeleton



where $e_i f_j = f_i e_j$ for all $i, j \in \{1, 2\}$, the solid edges have degree $(1, 0)$ and the dashed edges have degree $(0, 1)$. Then the 2-graph $T\Lambda$ has skeleton



where $\alpha(e_i)\,\alpha(f_j) = \alpha(f_i)\,\alpha(e_j)$ and $\alpha(e_i)\,\beta(f_j) = \alpha(f_i)\,\beta(e_j)$ for all $i, j \in \{1, 2\}$, the solid edges have degree $(1, 0)$ and the dashed edges have degree $(0, 1)$.

The following lemma tells about properties of the k-graph $T\Lambda$.

Proposition 3.5. *Let Λ be a row-finite k-graph and $T\Lambda$ be the k-graph as in Proposition 3.1. Then,*

(a) *$T\Lambda$ is row-finite.*
(b) *$T\Lambda$ is aperiodic.*

Proof. To show part (a), take $x \in T\Lambda^0$. If $x = \beta(v)$ for some $v \in \Lambda^0$, then $xT\Lambda^1 = \emptyset$ by Remark 3.3. Suppose $x = \alpha(v)$ for some $v \in \Lambda^0$. If $v\Lambda^1 = \emptyset$, then $xT\Lambda^1 = \emptyset$. Otherwise, for $1 \le i \le k$ such that $v\Lambda^{e_i} \neq \emptyset$, we have

$$|xT\Lambda^{e_i}| \le 2\,|v\Lambda^{e_i}|\,,$$

which is finite.

For part (b), take $\tau, \omega \in T\Lambda$ such that $\tau \neq \omega$ and $s(\tau) = s(\omega)$. We have to show there exists $\eta \in s(\tau)\,T\Lambda$ such that $\mathrm{MCE}(\tau\eta, \omega\eta) = \emptyset$. If $s(\tau) = \beta(v)$ for some $v \in \Lambda^0$, then choose $\eta = \beta(v)$ and $\mathrm{MCE}(\tau\eta, \omega\eta) = \emptyset$. So suppose $s(\tau) = \alpha(v)$ for some $v \in \Lambda^0$. If $v\Lambda^1 = \emptyset$, then choose $\eta = \alpha(v)$ and $\mathrm{MCE}(\tau\eta, \omega\eta) = \emptyset$. Suppose $v\Lambda^1 \neq \emptyset$. Take $e \in v\Lambda^1$. If $s(e)\,\Lambda^1 = \emptyset$, then choose $\eta = \alpha(e)$ and $\mathrm{MCE}(\tau\eta, \omega\eta) = \emptyset$. Otherwise, we have $s(e)\,\Lambda^1 \neq \emptyset$. Then choose $\eta = \beta(e)$ and $\mathrm{MCE}(\tau\eta, \omega\eta) = \emptyset$. Hence, $T\Lambda$ is aperiodic. □

4. Realising $TC^*(\Lambda)$ as a Cuntz–Krieger algebra

Let Λ be a row-finite k-graph and $T\Lambda$ be the k-graph as in Proposition 3.1. In this section, we show that $TC^*(\Lambda)$ is isomorphic to $C^*(T\Lambda)$.

Theorem 4.1. *Let Λ be a row-finite k-graph and $T\Lambda$ be the k-graph as in Proposition 3.1. Let $\{t_\lambda : \lambda \in \Lambda\}$ be the universal Toeplitz–Cuntz–Krieger*

Λ-family and $\{s_\omega : \omega \in T\Lambda\}$ be the universal Cuntz–Krieger $T\Lambda$-family. For $\lambda \in \Lambda$, let

$$T_\lambda := \begin{cases} s_{\alpha(\lambda)} + s_{\beta(\lambda)} & \textit{if } s(\lambda)\Lambda^1 \neq \emptyset \\ s_{\alpha(\lambda)} & \textit{if } s(\lambda)\Lambda^1 = \emptyset. \end{cases}$$

Then there is an isomorphism $\phi_T : TC^*(\Lambda) \to C^*(T\Lambda)$ satisfying

$$\phi_T(t_\lambda) = T_\lambda$$

for every $\lambda \in \Lambda$.

Furthermore, $s_{\alpha(\lambda)} = \phi_T(t_\lambda)$ if $s(\lambda)\Lambda^1 = \emptyset$. Meanwhile, if $s(\lambda)\Lambda^1 \neq \emptyset$, we have

$$s_{\alpha(\lambda)} = \phi_T\left(t_\lambda - t_\lambda \prod_{e \in s(\lambda)\Lambda^1} (t_v - t_e t_e^*)\right),$$

$$s_{\beta(\lambda)} = \phi_T\left(t_\lambda \prod_{e \in s(\lambda)\Lambda^1} (t_v - t_e t_e^*)\right).$$

Proof that $\{T_\lambda : \lambda \in \Lambda\}$ is a Toeplitz–Cuntz–Krieger Λ-family. To avoid an argument by cases, for $\lambda \in \Lambda$ with $s(\lambda)\Lambda^1 = \emptyset$, we write

$$s_{\beta(\lambda)} := 0,$$

so that

$$T_\lambda = s_{\alpha(\lambda)} + s_{\beta(\lambda)}.$$

First, we want to show $\{T_\lambda : \lambda \in \Lambda\}$ is a Toeplitz–Cuntz–Krieger Λ-family in $C^*(T\Lambda)$. For (TCK1), take $v \in \Lambda^0$. Since $\left\{s_{\alpha(v)}\right\} \cup \left\{s_{\beta(v)}\right\}$ are mutually orthogonal projections, then T_v is a projection. Meanwhile, for $v, w \in \Lambda^0$ with $v \neq w$,

$$T_v T_w = s_{\alpha(v)} s_{\alpha(w)} + s_{\alpha(v)} s_{\beta(w)} + s_{\beta(v)} s_{\alpha(w)} + s_{\beta(v)} s_{\beta(w)} = 0.$$

Next we show (TCK2). Take $\mu, \nu \in \Lambda$ where $s(\mu) = r(\nu)$. Then

$$T_\mu T_\nu = s_{\alpha(\mu)} s_{\alpha(\nu)} + s_{\alpha(\mu)} s_{\beta(\nu)} + s_{\beta(\mu)} s_{\alpha(\nu)} + s_{\beta(\mu)} s_{\beta(\nu)}.$$

If ν is a vertex, the middle terms vanish and we get

$$T_\mu T_\nu = s_{\alpha(\mu)} + s_{\beta(\mu)} = T_\mu,$$

as required. Otherwise, the last two terms vanish and we get

$$T_\mu T_\nu = s_{\alpha(\mu)} s_{\alpha(\nu)} + s_{\alpha(\mu)} s_{\beta(\nu)} = s_{\alpha(\mu\nu)} + s_{\beta(\mu\nu)} = T_{\mu\nu},$$

which is (TCK2).

To show (TCK3), take $\lambda, \mu \in \Lambda$. Then

$$T_\lambda^* T_\mu = s_{\alpha(\lambda)}^* s_{\alpha(\mu)} + s_{\alpha(\lambda)}^* s_{\beta(\mu)} + s_{\beta(\lambda)}^* s_{\alpha(\mu)} + s_{\beta(\lambda)}^* s_{\beta(\mu)}. \tag{4.1}$$

We give separate arguments for $\Lambda^{\min}(\lambda,\mu)=\emptyset$ and $\Lambda^{\min}(\lambda,\mu)\neq\emptyset$. For case $\Lambda^{\min}(\lambda,\mu)=\emptyset$, we have

$$\begin{aligned}\emptyset &= T\Lambda^{\min}(\alpha(\lambda),\alpha(\mu)) = T\Lambda^{\min}(\alpha(\lambda),\beta(\mu))\\ &= T\Lambda^{\min}(\beta(\lambda),\alpha(\mu)) = T\Lambda^{\min}(\beta(\lambda),\beta(\mu)).\end{aligned}$$

Hence, $s^*_{\alpha(\lambda)}s_{\alpha(\mu)} = s^*_{\alpha(\lambda)}s_{\beta(\mu)} = s^*_{\beta(\lambda)}s_{\alpha(\mu)} = s^*_{\beta(\lambda)}s_{\beta(\mu)} = 0$ and then Equation (4.1) becomes

$$T^*_\lambda T_\mu = 0 = \sum_{(\lambda',\mu')\in\Lambda^{\min}(\lambda,\mu)} T_{\lambda'}T^*_{\mu'}.$$

Now suppose $\Lambda^{\min}(\lambda,\mu)\neq\emptyset$. Take $(a,b)\in\Lambda^{\min}(\lambda,\mu)$. We consider several cases: whether a equals $s(\lambda)$ and/or b equals $s(\mu)$. First suppose $a=s(\lambda)$ and $b=s(\mu)$. So $\lambda=\lambda s(\lambda)=\mu s(\mu)=\mu$. Because $\alpha(\lambda)$ and $\beta(\lambda)$ are paths with the same degree and different sources, then $T\Lambda^{\min}(\alpha(\lambda),\beta(\lambda))=\emptyset$. Thus,

$$s^*_{\beta(\lambda)}s_{\alpha(\lambda)} = 0 = s^*_{\alpha(\lambda)}s_{\beta(\lambda)}$$

and Equation (4.1) becomes

$$\begin{aligned}T^*_\lambda T_\lambda &= s^*_{\alpha(\lambda)}s_{\alpha(\lambda)} + s^*_{\beta(\lambda)}s_{\beta(\lambda)}\\ &= s_{s(\alpha(\lambda))} + s_{s(\beta(\lambda))} = s_{\alpha(s(\lambda))} + s_{\beta(s(\lambda))}\\ &= T_{s(\lambda)} = T_{s(\lambda)}T^*_{s(\lambda)}\\ &= \sum_{(\lambda',\mu')\in\Lambda^{\min}(\lambda,\lambda)} T_{\lambda'}T^*_{\mu'} \text{ (since } \Lambda^{\min}(\lambda,\lambda)=\{s(\lambda),s(\lambda)\}\text{).}\end{aligned}$$

Next suppose $a=s(\lambda)$ and $b\neq s(\mu)$. Then $\lambda=\mu b$ and

$$T\Lambda^{\min}(\alpha(\lambda),\beta(\mu)) = \emptyset = T\Lambda^{\min}(\beta(\lambda),\beta(\mu))$$

since $s(\beta(\mu))T\Lambda^1=\emptyset$. Hence

$$s^*_{\alpha(\lambda)}s_{\beta(\mu)} = 0 = s^*_{\beta(\lambda)}s_{\beta(\mu)}$$

and Equation (4.1) becomes

$$T^*_\lambda T_\mu = s^*_{\alpha(\lambda)}s_{\alpha(\mu)} + s^*_{\beta(\lambda)}s_{\alpha(\mu)}.$$

Every $(\alpha(s(\lambda)),\eta)\in T\Lambda^{\min}(\alpha(\lambda),\alpha(\mu))$ has $\eta=\alpha(\mu')$ with

$$(s(\lambda),\mu')\in\Lambda^{\min}(\lambda,\mu).$$

Similarly, every $(\beta(s(\lambda)), \eta) \in T\Lambda^{\min}(\beta(\lambda), \alpha(\mu))$ has $\eta = \beta(\mu')$ with $(s(\lambda), \mu') \in \Lambda^{\min}(\lambda, \mu)$. Thus, by using (TCK3) in $C^*(T\Lambda)$,

$$\begin{aligned}
&T_\lambda^* T_\mu \\
&= s_{\alpha(\lambda)}^* s_{\alpha(\mu)} + s_{\beta(\lambda)}^* s_{\alpha(\mu)} \\
&= \sum_{(\alpha(s(\lambda)),\eta) \in T\Lambda^{\min}(\alpha(\lambda),\alpha(\mu))} s_{\alpha(s(\lambda))} s_\eta^* + \sum_{(\beta(s(\lambda)),\eta) \in T\Lambda^{\min}(\beta(\lambda),\alpha(\mu))} s_{\beta(s(\lambda))} s_\eta^* \\
&= \sum_{(s(\lambda),\mu') \in \Lambda^{\min}(\lambda,\mu)} s_{\alpha(s(\lambda))} s_{\alpha(\mu')}^* + \sum_{(s(\lambda),\mu') \in \Lambda^{\min}(\lambda,\mu)} s_{\beta(s(\lambda))} s_{\beta(\mu')}^* \\
&= \sum_{(s(\lambda),\mu') \in \Lambda^{\min}(\lambda,\mu)} \left(s_{\alpha(s(\lambda))} s_{\alpha(\mu')}^* + s_{\beta(s(\lambda))} s_{\beta(\mu')}^* \right) \\
&= \sum_{(s(\lambda),\mu') \in \Lambda^{\min}(\lambda,\mu)} \left(s_{\alpha(s(\lambda))} + s_{\beta(s(\lambda))} \right) \left(s_{\alpha(\mu')}^* + s_{\beta(\mu')}^* \right) \\
&= \sum_{(s(\lambda),\mu') \in \Lambda^{\min}(\lambda,\mu)} T_{s(\lambda)} T_{\mu'}^* = \sum_{(\lambda',\mu') \in \Lambda^{\min}(\lambda,\mu)} T_{\lambda'} T_{\mu'}^*.
\end{aligned}$$

By taking adjoints, we deduce (TCK3) when $a \neq s(\lambda)$ and $b = s(\mu)$.

Now we consider the last case, which is $a \neq s(\lambda)$ and $b \neq s(\mu)$. This means we have neither $\lambda = \mu b$ nor $\mu = \lambda a$. Hence,

$$T\Lambda^{\min}(\alpha(\lambda), \beta(\mu)) = T\Lambda^{\min}(\beta(\lambda), \alpha(\mu)) = T\Lambda^{\min}(\beta(\lambda), \beta(\mu)) = \emptyset$$

since $s(\beta(\lambda)) T\Lambda^1 = \emptyset = s(\beta(\mu)) T\Lambda^1 = \emptyset$. Hence,

$$s_{\alpha(\lambda)}^* s_{\beta(\mu)} = s_{\beta(\lambda)}^* s_{\alpha(\mu)} = s_{\beta(\lambda)}^* s_{\beta(\mu)} = 0.$$

On the other hand, we have

$$\begin{aligned}
&T\Lambda^{\min}(\alpha(\lambda), \alpha(\mu)) \\
&\qquad = \left\{ \left(\alpha(\lambda'), \alpha(\mu') \right), \left(\beta(\lambda'), \beta(\mu') \right) : (\lambda', \mu') \in \Lambda^{\min}(\lambda, \mu) \right\}.
\end{aligned}$$

Therefore, Equation (4.1) becomes

$$\begin{aligned}
T_\lambda^* T_\mu = s_{\alpha(\lambda)}^* s_{\alpha(\mu)} &= \sum_{(\omega,\eta) \in T\Lambda^{\min}(\alpha(\lambda),\alpha(\mu))} s_\omega s_\eta^* \\
&= \sum_{(\lambda',\mu') \in \Lambda^{\min}(\lambda,\mu)} \left(s_{\alpha(\lambda')} s_{\alpha(\mu')}^* + s_{\beta(\lambda')} s_{\beta(\mu')}^* \right) \\
&= \sum_{(\lambda',\mu') \in \Lambda^{\min}(\lambda,\mu)} \left(s_{\alpha(\lambda')} + s_{\beta(\lambda')} \right) \left(s_{\alpha(\mu')}^* + s_{\beta(\mu')}^* \right) \\
&= \sum_{(\lambda',\mu') \in \Lambda^{\min}(\lambda,\mu)} T_{\lambda'} T_{\mu'}^*.
\end{aligned}$$

So for all cases, we have

$$T_\lambda^* T_\mu = \sum_{(\lambda',\mu') \in \Lambda^{\min}(\lambda,\mu)} T_{\lambda'} T_{\mu'}^*$$

and $\{T_\lambda : \lambda \in \Lambda\}$ satisfies (TCK3). □

Proof that ϕ_T is injective. Now the universal property of $TC^*(\Lambda)$ gives a homomorphism $\phi_T : TC^*(\Lambda) \to C^*(T\Lambda)$ satisfying $\phi_T(t_\lambda) = T_\lambda$ for every $\lambda \in \Lambda$.

We show the injectivity of ϕ_T by using Theorem 2.2. Take $v \in \Lambda^0$. We show

$$\prod_{e \in v\Lambda^1} (T_v - T_e T_e^*) \neq 0.$$

First suppose $v\Lambda^1 \neq \emptyset$. Take $1 \leq i \leq k$ such that $v\Lambda^{e_i} \neq \emptyset$. We claim

$$\prod_{e \in v\Lambda^{e_i}} (T_v - T_e T_e^*) \geq s_{\beta(v)}.$$

Since $v\Lambda^{e_i} \neq \emptyset$, then $\alpha(v) T\Lambda^{e_i} \neq \emptyset$ and by [11, Lemma 2.7 (iii)],

$$\begin{aligned}
s_{\alpha(v)} &\geq \sum_{g \in \alpha(v) T\Lambda^{e_i}} s_g s_g^* \\
&= \sum_{e \in v\Lambda^{e_i}} s_{\alpha(e)} s_{\alpha(e)}^* + \sum_{\substack{e \in v\Lambda^{e_i} \\ s(e)\Lambda^1 \neq \emptyset}} s_{\beta(e)} s_{\beta(e)}^* \\
&= \sum_{\substack{e \in v\Lambda^{e_i} \\ s(e)\Lambda^1 \neq \emptyset}} \left(s_{\alpha(e)} s_{\alpha(e)}^* + s_{\beta(e)} s_{\beta(e)}^* \right) + \sum_{\substack{e \in v\Lambda^{e_i} \\ s(e)\Lambda^1 = \emptyset}} s_{\alpha(e)} s_{\alpha(e)}^* \\
&= \sum_{\substack{e \in v\Lambda^{e_i} \\ s(e)\Lambda^1 \neq \emptyset}} T_e T_e^* + \sum_{\substack{e \in v\Lambda^{e_i} \\ s(e)\Lambda^1 = \emptyset}} T_e T_e^* \\
&= \sum_{e \in v\Lambda^{e_i}} T_e T_e^*.
\end{aligned}$$

Meanwhile, since every $e \in v\Lambda^{e_i}$ has the same degree,

$$\begin{aligned}
\prod_{e \in v\Lambda^{e_i}} (T_v - T_e T_e^*) &= T_v - \sum_{e \in v\Lambda^{e_i}} T_e T_e^* \\
&= \left(s_{\alpha(v)} + s_{\beta(v)} \right) - \sum_{e \in v\Lambda^{e_i}} T_e T_e^* \\
&= s_{\beta(v)} + \left(s_{\alpha(v)} - \sum_{e \in v\Lambda^{e_i}} T_e T_e^* \right) \\
&\geq s_{\beta(v)},
\end{aligned}$$

as claimed. This claim implies

$$\prod_{e \in v\Lambda^1} (T_v - T_e T_e^*) \geq \prod_{\{i : v\Lambda^{e_i} \neq \emptyset\}} s_{\beta(v)} = s_{\beta(v)} \neq 0$$

since $v\Lambda^1 \neq \emptyset$, as required.

Finally, for $v \in \Lambda^0$ with $v\Lambda^1 = \emptyset$, we have

$$T_v = s_{\alpha(v)} \neq 0.$$

Hence, by Theorem 2.2, ϕ_T is injective. □

Proof that ϕ_T is surjective. Now we show the surjectivity of ϕ_T. Since $C^*(T\Lambda)$ is generated by $\{s_\tau : \tau \in T\Lambda\}$, then it suffices to show that for every $\tau \in T\Lambda$, $s_\tau \in \operatorname{im}(\phi_T)$. Recall that for every $\tau \in T\Lambda$, s_τ is either $s_{\alpha(\lambda)}$ or $s_{\beta(\lambda)}$ for some $\lambda \in \Lambda$.

Take $v \in \Lambda^0$. First we show $s_{\alpha(v)}$ and $s_{\beta(v)}$ (if it exists) belong to $\operatorname{im}(\phi_T)$. If $v\Lambda^1 = \emptyset$, then

$$s_{\alpha(v)} = T_v \in \operatorname{im}(\phi_T).$$

Next suppose $v\Lambda^1 \neq \emptyset$. First we show that $s_{\beta(v)} = \prod_{e\in v\Lambda^1}(T_v - T_eT_e^*)$. Note that for every $f \in \alpha(v)T\Lambda^1$, the projection $s_{\alpha(v)} - s_fs_f^* \leq s_{\alpha(v)}$ is othogonal to $s_{\beta(v)}$. This implies

$$\begin{aligned}\prod_{f\in\alpha(v)T\Lambda^1}((s_{\alpha(v)} + s_{\beta(v)}) - s_fs_f^*) &= s_{\beta(v)} + \prod_{f\in\alpha(v)T\Lambda^1}(s_{\alpha(v)} - s_fs_f^*)\\ &= s_{\beta(v)},\end{aligned}$$

since $v\Lambda^1$ is an exhaustive set. Hence,

$$\begin{aligned}s_{\beta(v)} &= \prod_{f\in\alpha(v)T\Lambda^1}((s_{\alpha(v)} + s_{\beta(v)}) - s_fs_f^*)\\ &= \prod_{e\in v\Lambda^1}(T_v - s_{\alpha(e)}s_{\alpha(e)}^*)\prod_{\substack{e\in v\Lambda^1\\ s(e)\Lambda^1\neq\emptyset}}(T_v - s_{\beta(e)}s_{\beta(e)}^*)\\ &= \prod_{\substack{e\in v\Lambda^1\\ s(e)\Lambda^1=\emptyset}}(T_v - s_{\alpha(e)}s_{\alpha(e)}^*)\prod_{\substack{e\in v\Lambda^1\\ s(e)\Lambda^1\neq\emptyset}}(T_v - s_{\alpha(e)}s_{\alpha(e)}^*)(T_v - s_{\beta(e)}s_{\beta(e)}^*)\\ &= \prod_{\substack{e\in v\Lambda^1\\ s(e)\Lambda^1=\emptyset}}(T_v - s_{\alpha(e)}s_{\alpha(e)}^*)\prod_{\substack{e\in v\Lambda^1\\ s(e)\Lambda^1\neq\emptyset}}(T_v - (s_{\alpha(e)}s_{\alpha(e)}^* + s_{\beta(e)}s_{\beta(e)}^*))\\ &= \prod_{\substack{e\in v\Lambda^1\\ s(e)\Lambda^1=\emptyset}}(T_v - T_eT_e^*)\prod_{\substack{e\in v\Lambda^1\\ s(e)\Lambda^1\neq\emptyset}}(T_v - T_eT_e^*)\\ &= \prod_{e\in v\Lambda^1}(T_v - T_eT_e^*),\end{aligned}$$

as required, and $s_{\beta(v)}$ belongs to $\operatorname{im}(\phi_T)$. Furthermore,

$$s_{\alpha(v)} = T_v - s_{\beta(v)} = T_v - \prod_{e\in v\Lambda^1}(T_v - T_eT_e^*) \in \operatorname{im}(\phi_T),$$

as required.

Now take $\lambda \in \Lambda$. We have to show $s_{\alpha(\lambda)}$ and $s_{\beta(\lambda)}$ (if it exists) belong to $\mathrm{im}\,(\phi_T)$. If $s(\lambda)\Lambda^1 = \emptyset$, then

$$s_{\alpha(\lambda)} = s_{\alpha(\lambda)} s_{\alpha(s(\lambda))} = T_\lambda T_{s(\lambda)} = T_\lambda \in \mathrm{im}\,(\phi_T)\,.$$

Next suppose $s(\lambda)\Lambda^1 \neq \emptyset$. Then $s_{\beta(\lambda)} s_{\alpha(s(\lambda))} = 0$ and $s_{\alpha(\lambda)} s_{\beta(s(\lambda))} = 0$. Hence,

$$\begin{aligned} s_{\alpha(\lambda)} &= s_{\alpha(\lambda)} s_{\alpha(s(\lambda))} = \left(s_{\alpha(\lambda)} + s_{\beta(\lambda)}\right) s_{\alpha(s(\lambda))} \\ &= T_\lambda \left(T_{s(\lambda)} - \prod_{e \in s(\lambda)\Lambda^1} (T_{s(\lambda)} - T_e T_e^*) \right) \\ &= T_\lambda - T_\lambda \prod_{e \in s(\lambda)\Lambda^1} (T_{s(\lambda)} - T_e T_e^*) \in \mathrm{im}\,(\phi_T) \end{aligned}$$

and

$$\begin{aligned} s_{\beta(\lambda)} &= s_{\beta(\lambda)} s_{\beta(s(\lambda))} = \left(s_{\alpha(\lambda)} + s_{\beta(\lambda)}\right) s_{\beta(s(\lambda))} \\ &= T_\lambda \prod_{e \in s(\lambda)\Lambda^1} (T_{s(\lambda)} - T_e T_e^*) \in \mathrm{im}\,(\phi_T)\,. \end{aligned}$$

Therefore, ϕ_T is surjective and an isomorphism. □

Corollary 4.2. *Let Λ be a row-finite k-graph and $T\Lambda$ be the k-graph as in Proposition* 3.1*. Let $\{t_\lambda : \lambda \in \Lambda\}$ be the universal Toeplitz–Cuntz–Krieger Λ-family and $\{s_\omega : \omega \in T\Lambda\}$ be the universal Cuntz–Krieger $T\Lambda$-family. For $\tau \in T\Lambda$, define*

$$S_\tau := \begin{cases} t_\lambda & \text{if } \tau = \alpha(\lambda) \text{ with } s(\lambda)\Lambda^1 = \emptyset \\ t_\lambda - t_\lambda \prod_{e \in s(\lambda)\Lambda^1} (t_v - t_e t_e^*) & \text{if } \tau = \alpha(\lambda) \text{ with } s(\lambda)\Lambda^1 \neq \emptyset \\ t_\lambda \prod_{e \in s(\lambda)\Lambda^1} (t_v - t_e t_e^*) & \text{if } \tau = \beta(\lambda) \text{ with } s(\lambda)\Lambda^1 \neq \emptyset. \end{cases}$$

Suppose that $\phi_T : TC^*(\Lambda) \to C^*(T\Lambda)$ is the isomorphism as in Theorem 4.1 *and $\pi_S : C^*(T\Lambda) \to TC^*(\Lambda)$ is the homomorphism such that $\pi_S(s_\tau) = S_\tau$ for $\tau \in T\Lambda$. Then $\phi_T^{-1} = \pi_S$.*

Proof. Take $\lambda \in \Lambda$. By Theorem 4.1, we get $\phi_T^{-1}\left(s_{\alpha(\lambda)}\right) = t_\lambda$ if $s(\lambda)\Lambda^1 = \emptyset$. Meanwhile, if $s(\lambda)\Lambda^1 \neq \emptyset$, by Theorem 4.1, we have

$$\begin{aligned} \phi_T^{-1}\left(s_{\alpha(\lambda)}\right) &= t_\lambda - t_\lambda \prod_{e \in v\Lambda^1} (t_v - t_e t_e^*), \\ \phi_T^{-1}\left(s_{\beta(\lambda)}\right) &= t_\lambda \prod_{e \in v\Lambda^1} (t_v - t_e t_e^*). \end{aligned}$$

Hence, $\phi_T^{-1}(s_\tau) = S_\tau$ for $\tau \in T\Lambda$. This implies that $\{S_\tau : \tau \in T\Lambda\}$ is a Cuntz–Krieger $T\Lambda$-family, and then $\phi_T^{-1} = \pi_S$. □

Remark 4.3. Proposition 3.5 says that $T\Lambda$ is always aperiodic, and hence the Cuntz–Krieger uniqueness theorem always applies to $T\Lambda$. This helps explain why no hypothesis on Λ is required in the uniqueness theorem of [9, Theorem 8.1]. Indeed, we could have deduced that theorem by applying the Cuntz–Krieger uniqueness theorem to $T\Lambda$. With our current proof of Theorem 4.1, this argument would be circular, since we used [9, Theorem 8.1] in the proof of Theorem 4.1. However, we could prove Corollary 4.2 directly by showing that $\{S_\tau : \tau \in T\Lambda\}$ is a Cuntz–Krieger $T\Lambda$-family in $TC^*(\Lambda)$, hence gives a homomorphism $\pi_S : C^*(T\Lambda) \to TC^*(\Lambda)$, and using the Cuntz–Krieger uniqueness theorem to see that π_S is injective. Then we could deduce [9, Theorem 8.1] from Corollary 4.2, and this would be a legitimate proof. We worked out the details of this approach, but it seemed to require an extensive cases argument, and hence became substantially more complicated.

References

[1] Farthing, Cynthia; Muhly, Paul S.; Yeend, Trent. Higher-rank graph C^*-algebras: an inverse semigroup and groupoid approach. *Semigroup Forum* **71** (2005), no. 2, 159–187. MR2184052 (2006h:46052), Zbl 1099.46036, arXiv:math/0409505, doi: 10.1007/s00233-005-0512-2.

[2] Fowler, Neal J.; Laca, Marcelo; Raeburn, Iain. The C^*-algebras of infinite graphs. *Proc. Amer. Math. Soc.* **128** (2000), no. 8, 2319–2327. MR1670363 (2000k:46079), Zbl 0956.46035, doi: 10.1090/S0002-9939-99-05378-2.

[3] Fowler, Neal J.; Raeburn, Iain. The Toeplitz algebra of a Hilbert bimodule. *Indiana Univ. Math. J.* **48** (1999), no. 1, 155–181. MR1722197 (2001b:46093), Zbl 0938.47052, arXiv:math/9806093, doi: 10.1512/iumj.1999.48.1639.

[4] Hazlewood, Robert; Raeburn, Iain; Sims, Aidan; Webster, Samuel B. G. Remarks on some fundamental results about higher-rank graphs and their C^*-algebras. *Proc. Edinb. Math. Soc.* **56** (2013), no. 2, 575–597. MR3056660, Zbl 1264.05064, arXiv:1110.2269, doi: 10.1017/S0013091512000338.

[5] Kumjian, Alex; Pask, David. Higher rank graph C^*-algebras. *New York J. Math.* **6** (2000), 1–20. MR1745529 (2001b:46102), Zbl 0946.46044, arXiv:math/0007029.

[6] Lewin, Peter; Sims, Aidan. Aperiodicity and cofinality for finitely aligned higher-rank graphs. *Math. Proc. Cambridge Philos. Soc.* **149** (2010), no. 2, 333–350. MR2670219 (2012b:46106), Zbl 1213.46047, arXiv:0905.0735, doi: 10.1017/S0305004110000034.

[7] Muhly, Paul S.; Tomforde, Mark. Adding tails to C^*-correspondences. *Doc. Math.* **9** (2004), 79–106. MR2054981 (2005a:46117), Zbl 1049.46045, arXiv:math/0212277.

[8] Raeburn, Iain. Graph Algebras. CBMS Regional Conference Series in Math., 103. *Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI*, 2005. vi+113 pp. ISBN: 0-8218-3660-9. MR2135030 (2005k:46141), Zbl 1079.46002.

[9] Raeburn, Iain; Sims, Aidan. Product systems of graphs and the Toeplitz algebras of higher-rank graphs. *J. Operator Theory* **53** (2005), no. 2, 399–429. MR2153156 (2006d:46073), Zbl 1093.46032, arXiv:math/0305371.

[10] Raeburn, Iain; Sims, Aidan; Yeend, Trent. Higher-rank graphs and their C^*-algebras. *Proc. Edinb. Math. Soc.* **46** (2003), no. 1, 99–115. MR1961175 (2004f:46068), Zbl 1031.46061, arXiv:math/0107222, doi: 10.1017/S0013091501000645.

[11] Raeburn, Iain; Sims, Aidan; Yeend, Trent. The C^*-algebras of finitely aligned higher-rank graphs. *J. Funct. Anal.* **213** (2004), no. 1, 206–240. MR2069786 (2005e:46103), Zbl 1063.46041, arXiv:math/0305370, doi: 10.1016/j.jfa.2003.10.014.

[12] Robertson, David I.; Sims, Aidan. Simplicity of C^*-algebras associated to higher-rank graphs. *Bull. London Math. Soc.* **39** (2007), no. 2, 337–344. MR2323468 (2008g:46099), Zbl 1125.46045, arXiv:math/0602120, doi: 10.1112/blms/bdm006.

[13] Robertson, David; Sims, Aidan. Simplicity of C^*-algebras associated to row-finite locally convex higher-rank graphs. *Israel J. Math.* **172** (2009), 171–192. MR2534246 (2010f:46104), Zbl 1189.46047, arXiv:0708.0245, doi: 10.1007/s11856-009-0070-5.

[14] Shotwell, Jacob. Simplicity of finitely-aligned k-graph C^*-algebras. *J. Operator Theory* **67** (2012), no. 2, 335–347. MR2928319, Zbl 1265.46086, arXiv:0810.4567.

[15] Sims, Aidan. The co-universal C^*-algebra of a row-finite graph. *New York J. Math.* **16** (2010), 507–524. MR2740588 (2011j:46093), Zbl 1234.46044, arXiv:0809.2333.

[16] Webster, Samuel B. G. The path space of a higher-rank graph. *Studia Math.* **204** (2011), no. 2, 155–185. MR2805537 (2012e:46119), Zbl 1235.46049, arXiv:1102.1229, doi: 10.4064/sm204-2-4.

(Yosafat E. P. Pangalela) Department of Mathematics and Statistics, University of Otago, PO Box 56, Dunedin 9054, New Zealand
yosafat.pangalela@maths.otago.ac.nz

This paper is available via http://nyjm.albany.edu/j/2016/22-13.html.