

Differential K -theory as equivalence classes of maps to Grassmannians and unitary groups

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ABSTRACT. We construct a model of differential K -theory whose cocycles are certain equivalence classes of maps into the Grassmannians and unitary groups. In particular, we produce the circle integration maps for these models by using certain differential forms that witness the incompatibility between the even and odd universal Chern forms. By the uniqueness theorem of Bunke and Schick, this model agrees with the spectrum based models in the literature whose nongeometrically defined Chern cocycles are compatible with the delooping maps of the spectrum. These constructions favor geometry over homotopy theory.

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1. Introduction

Differential cohomology theories provide a refinement of cohomology theories by incorporating additional geometric data. A historically important example is *differential ordinary cohomology*, in which ordinary cohomology classes are promoted to richer cocycles. For instance, an integral cohomology class of degree one, which can be thought of as a homotopy class of a continuous map from the manifold to the circle, is enhanced to an actual smooth map from the manifold to the circle. Similarly, an integral cohomology class of degree two, which determines an isomorphism class of a continuous complex line bundle, is enriched to an isomorphism class of a smooth line bundle with a connection.

Constructions of these sorts have their roots in the original work of Cheeger and Simons' differential characters [CheeS], Harvey and Lawson's spark complexes [HL], and Deligne cohomology defined originally in the holomorphic setting [D].

Hopkins and Singer in [HS] showed that every generalized cohomology theory has a differential refinement and gave a construction of the differential cohomology theory as a homotopy fiber product of two ingredients, a cohomology theory (homotopy invariant datum) and differential forms (non-homotopy-invariant datum).

In [BS2], Bunke and Schick gave axioms for differential cohomology theories and showed that a differential extension, together with an S^1 -integration map, often determine the theory up to a unique natural isomorphism. We apply their characterization to our model of differential K -theory. We refer the reader to [BS3] for a survey of applications of differential cohomologies in mathematics and physics.

In [BS], Bunke and Schick construct a geometric model of differential K -theory, based on ideas of index theory of Dirac operators, with an obvious advantage in its immediate connection to geometry. Subsequently, Simons and Sullivan [SS] constructed a differential extension of the even degree part of K -theory as the Grothendieck group of appropriate equivalence classes of vector bundles *with connection*. A particularly nice feature of this model is that it does not require the additional data of differential forms, as required in previous models. That is to say, the even degree cocycles are represented by vector bundles with connection and no additional form data is required, as opposed to the case in [FL], where an index theorem was established. More recently, [BNV] proposed that differential cohomology theories can be viewed in terms of a natural decomposition of general sheaves of spectra on the site of manifolds into a homotopy invariant and a non-homotopy-invariant part. In [Schr], differential cohomology theories are explained in the context of cohesive topoi.

In Section 2 we review the definitions needed in this paper. For a fixed differential cohomology theory, it follows from the axioms that the underlying functor is well-defined on a nontrivial quotient of the category of smooth

manifolds. Here, smooth maps are identified using an equivalence relation determined by the fiber integration of “Chern forms” over homotopies (see Remark 2.6). This provides a refinement of the homotopy category that is sensitive to the differential theory. For differential K -theory, this category is denoted by $\text{Smooth}_{\hat{K}}$, and the relation is that two smooth maps $f_0, f_1 : M \rightarrow N$ are equivalent if there is a smooth homotopy f_t such that $\int_I f_t^* X$ is a Chern form on M , whenever X is a Chern form on N .

In Section 3 we construct models for differential extensions of complex K -theory in odd and even degrees. The odd case is similar to the authors’ work in [TWZ2], using *only* the mapping space of the stable unitary group U , though the space U is modified slightly to better accommodate the S^1 -integration maps. The construction of a differential extension of even K -theory defined here is entirely new, and is given by certain equivalence classes of maps into a specific model of $BU \times \mathbb{Z}$. That the additional data of differential forms is not required as input in our cocycles is due to the nontrivial fact that the geometric classifying space used here contains all Chern forms as well as all of their transgressions, see Theorem 3.17.

The work of [SS] suggested to us that one might hope for such a model since one knows that, up to isomorphism, every connection is a pullback of the universal connection. Yet, there are many seemingly unrelated maps giving rise to the same isomorphism class of a bundle and connection, which makes the problem of identifying our model in even degree with the Simons–Sullivan model somewhat nontrivial.

To make the construction complete, several technical difficulties regarding a suitable model of $BU \times \mathbb{Z}$, with explicit requisite structure (*e.g.*, the monoid structure) had to be surmounted. In one part, we produce explicit homotopies for commutativity and the additive unit on which the Chern–Simons forms vanish. It would be interesting to know if there are abstract reasons why such homotopies exist. Regardless, one can envision possible uses for the explicit homotopies (*cf.* Lemma 3.9, Lemma 3.23, Lemma 3.24). For instance, the classical Chern–Simons formulae often used with great success in physics are a result of making specific choices of homotopies, *i.e.*, the straight line segment between connections.

A nice consequence of the model given here is that the set of cocycles is a rather small and familiar objects from geometry. Moreover, there is a natural map from the even part of our model to the Simons–Sullivan model of differential K -theory which, by a strengthening of the Narasimhan–Ramanan theorem, is an isomorphism. The even and odd models are both expected to have interesting connections with higher degree gerbes and field theories.

From the onset, in addition to using the explicit classifying spaces, we insist on using the geometrically defined Chern forms, as opposed to abstract cocycle representatives used in models constructed by means of homotopy theory. Unfortunately, in using these classifying spaces, it is not clear how

to express the multiplicative structure even at the level of non-differential K -theory. In Section 4 we do construct an explicit S^1 -integration map for the theory. In particular, by uniqueness results in [BS], this implies that the definition given in [TWZ2] does correctly define the odd degree part of differential K -theory. This was previously shown by Hekmati, Murray, Schlegel, and Vozzo, in [HMSV], using different methods.

To build the S^1 -integration map, we construct explicit differential forms which measure the discrepancy between the Chern forms in one parity and the S^1 -integration of the Chern forms in the opposite parity, via explicit homotopy equivalences $BU \times \mathbb{Z} \rightarrow \Omega U$ and $\Omega(BU \times \mathbb{Z}) \rightarrow U$. The existence of such cochains in both parities, that agree on the nose with one another through S^1 -integration, can be shown using abstract homotopy theoretic arguments [HS]. Those cocycles, however, are not geometric, and in so far as they make subsequent constructions easy, they deviate from natural geometric objects and constructions.

We close with some comments on how our results relate to the possible *representability* of differential K -theory. Of course, one could not possibly represent this functor by a finite dimensional manifold, and on the other hand, any functor extends to a representable functor on some category (*e.g.*, the functor category, by Yoneda's Lemma). One could ask for representability in some geometric category such as diffeological spaces.

The results here produce *by definition* set bijections

$$\hat{K}^0(M) = \text{Hom}(M, BU \times \mathbb{Z}) \quad \hat{K}^{-1}(M) = \text{Hom}(M, U)$$

where the Hom sets are defined as CS-equivalence classes of maps into the respective target: two maps f_0 and f_1 are equivalent if and only if there is a smooth homotopy f_t such that $\int_I f_t^* \text{Ch}$ is a Chern form on M , where Ch denotes *the* Chern form on $BU \times \mathbb{Z}$ or U defined below, respectively. One might hope that the equivalence relation used above to define $\text{Hom}(M, N)$ in $\text{Smooth}_{\hat{K}}$, restricted to the cases $N = BU \times \mathbb{Z}$ and $N = U$, would give differential K -theory as well, but it turns out that this set is too large, even for the point, $M = pt$. We are resolved then that differential K -theory is equal to equivalence classes of maps determined by fiber integration of *the* Chern form on targets $BU \times \mathbb{Z}$ and U , and is functorial with respect to equivalence classes of maps determined by fiber integration of *any* Chern form on the target N .

2. Differential extensions of K -theory

In this section, we recall the definition of a differential extension of K -theory with S^1 -integration map from Bunke and Schick [BS2] and [BS3], as well as a uniqueness theorem for differential K -theory, [BS3, Theorem 3.3], which will be used to identify the construction given here with differential K -theory. Denote by $K^*(M)$ the complex K -theory of a manifold M

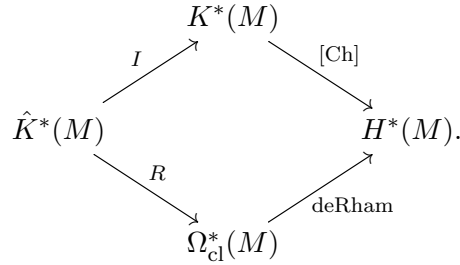
(possibly with corners). Recall that the Chern character Ch induces a (\mathbb{Z}_2 -graded) natural transformation $[\text{Ch}] : K^*(M) \rightarrow H^*(M)$. (For an explicit construction of these, see the next section.)

Definition 2.1 (Definition 2.1, [BS3]). A *differential extension of K -theory* is a quadruple (\hat{K}, R, I, a) , where \hat{K} is a contravariant functor from the category of compact smooth manifolds to the category of \mathbb{Z}_2 -graded abelian groups, and R, I, a are natural transformations

- (1) $R : \hat{K}^*(M) \rightarrow \Omega_{\text{cl}}^*(M; \mathbb{R})$,
- (2) $I : \hat{K}^*(M) \rightarrow K^*(M)$,
- (3) $a : \Omega^*(M; \mathbb{R}) / \text{Im}(d) \rightarrow \hat{K}^{*+1}(M)$,

such that:

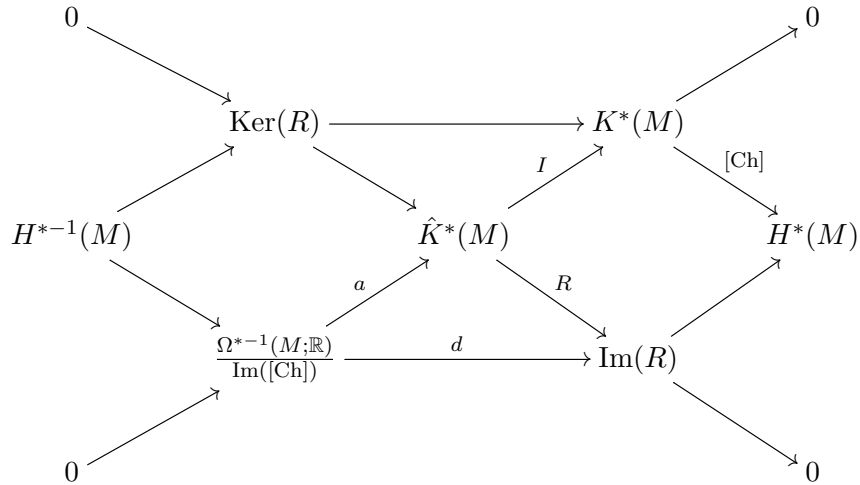
- (4) The following diagram commutes



- (5) $R \circ a = d$.
- (6) The following sequence is exact

$$K^{*-1}(M) \xrightarrow{[\text{Ch}]} \Omega^{*-1}(M; \mathbb{R}) / \text{Im}(d) \xrightarrow{a} \hat{K}^*(M) \xrightarrow{I} K^*(M) \xrightarrow{0} 0.$$

Remark 2.2. The diagram in condition (4) fits into the following *character diagram* for differential K -theory, where the top and bottom sequences that connect $H^{*-1}(M)$ with $H^*(M)$ form long exact sequences.



The group $\text{Ker}(R)$ has been considered by Karoubi [K], and subsequently by Lott [L], as \mathbb{R}/\mathbb{Z} -valued K -theory. Bunke and Schick have shown that $\text{Ker}(R)$, the so-called flat-theory, is a homotopy invariant, and in fact extends to a cohomology theory (Theorem 7.11 [BS2]).

The following homotopy lemma is given in [BS2, Lemma 5.1].

Lemma 2.3. *Suppose $f_t : M \times I \rightarrow N$ is a smooth homotopy. Then for any $x \in \hat{K}^*(N)$ we have*

$$f_1^*x - f_0^*x = a \left(\int_I f_t^*R(x) \right).$$

Definition 2.4. For any differential extension \hat{K} of K , there is a category $\text{Smooth}_{\hat{K}}$ with objects smooth manifolds, and morphisms given by equivalence classes of smooth maps, where two smooth maps $f_0, f_1 : M \rightarrow N$ are equivalent if there is a smooth homotopy $f_t : M \times I \rightarrow N$, such that

$$\int_I f_t^*X \in \text{Im}(R) \quad \text{whenever} \quad X \in \text{Im}(R).$$

Let us check this indeed defines a category. The relation is easily seen to be an equivalence relation, since integration along the fiber is additive, and image of R is a subgroup. The only remaining item to check is that composition of morphisms is well-defined. Suppose $f_0 \sim f_1$, via some smooth homotopy $f_t : M \times I \rightarrow N$, and $g_0 \sim g_1$, via some smooth homotopy $g_t : N \times I \rightarrow P$. Then $h_t : M \times I \rightarrow P$ defined by $h_t = (g_t \circ f_0) * (g_1 \circ f_t)$ is a homotopy from $g_0 \circ f_0$ to $g_1 \circ f_1$, and, if $X \in \text{Im}(R)$ then

$$\int_I h_t^*X = f_0^* \int_I g_t^*X + \int_I f_t^*(g_1^*X).$$

For the first summand on the right hand side, $\int_I g_t^*X$ is in the image of R by assumption on g_t , and therefore so is $f_0^* \int_I g_t^*X$ by naturality of \hat{K} . Also, $X \in \text{Im}(R)$ implies $g_1^*X \in \text{Im}(R)$ by naturality of \hat{K} , so $\int_I f_t^*(g_1^*X)$ is in the image of R by assumption on f_t . This shows $g_0 \circ f_0$ is equivalent to $g_1 \circ f_1$, and we are done.

Corollary 2.5. *For any differential extension \hat{K} of K , the underlying functor \hat{K} is well-defined on $\text{Smooth}_{\hat{K}}$.*

Proof. It suffices to show that if two smooth maps $f_0, f_1 : M \rightarrow N$ are equivalent, then $f_0^* = f_1^*$. Suppose $f_t : M \times I \rightarrow N$ is a smooth homotopy such that

$$\int_I f_t^*X \in \text{Im}(R)$$

whenever $X \in \text{Im}(R)$. Then, by the homotopy lemma 2.3, for any $x \in \hat{K}(N)$ we have

$$f_1^*x - f_0^*x = a \left(\int_I f_t^*R(x) \right).$$

By commutative diagram of \hat{K}^* we have that $\text{Im}([\text{Ch}])$ is equal to $\text{Im}(R)$ mod exact, and by the exact sequence we have that $\text{Ker}(a) = \text{Im}([\text{Ch}])$, so the right hand side vanishes, and therefore $f_0^* = f_1^*$. \square

Remark 2.6. One can similarly define a differential extension \hat{E} of any cohomology theory E , [BS2]. Just as in Definition 2.4 above, one obtains a quotient of the category Smooth of smooth manifolds, by declaring two smooth maps $f_0, f_1 : M \rightarrow N$ to be equivalent if there is a smooth homotopy $f_t : M \times I \rightarrow N$, such that

$$\int_I f_t^* X \in \text{Im}(R) \quad \text{whenever} \quad X \in \text{Im}(R).$$

where $R : \hat{E} \rightarrow \Omega$ is given in the differential extension. It follows just as in Corollary 2.5, using only the axioms of a differential extension, that \hat{E} is well-defined on this category.

2.1. S^1 -integration. We next recall that the S^1 -integration map in K -theory, and then the notion of an S^1 -integration for a differential extension of K -theory.

Definition 2.7. The inclusion $j : M \rightarrow M \times S^1$ via the basepoint and the projection $p : M \times S^1 \rightarrow M$ induce a direct sum decomposition

$$K^*(M \times S^1) \cong \text{Im}(p^*) \oplus \text{Ker}(j^*),$$

where the map is given by $\alpha \mapsto (p^*j^*\alpha, \alpha - p^*j^*\alpha)$. The map

$$q : M \times S^1 \rightarrow \Sigma M_+$$

induces an isomorphism $q^* : \tilde{K}^*(\Sigma M_+) \rightarrow \text{Ker}(j^*)$, which is composed with the suspension isomorphism $\sigma : K^{*-1}(M) \rightarrow \tilde{K}^*(\Sigma M_+)$ in the following way to define the S^1 -integration map in K -theory

$$\int_{S^1} : K^*(M \times S^1) \xrightarrow{\text{pr}} \text{Ker}(j^*) \xrightarrow{(q^*)^{-1}} \tilde{K}^*(\Sigma M_+) \xrightarrow{\sigma^{-1}} K^{*-1}(M).$$

Definition 2.8 (Definition 1.3, [BS3]). Let (\hat{K}, R, I, a) be a differential extension of K -theory. An S^1 integration map is by definition a natural transformation of functors $\mathcal{I} : \hat{K}^{*+1}(- \times S^1) \rightarrow \hat{K}^*(-)$ satisfying the following three properties:

- (1) $\mathcal{I} \circ (id \times r)^* = -\mathcal{I}$ where $r : S^1 \rightarrow S^1$ is given by $r(z) = \bar{z}$.
- (2) $\mathcal{I} \circ p^* = 0$ where $p : M \times S^1 \rightarrow M$ is projection.

(3) The following diagram commutes for all manifolds M

$$\begin{array}{ccccc}
 & & & \xrightarrow{R} & \\
 \Omega^*(M \times S^1)/\text{Im}(d) & \xrightarrow{a} & \hat{K}^{*+1}(M \times S^1) & \xrightarrow{I} & K^{*+1}(M \times S^1) & \xrightarrow{R} & \Omega_{\text{cl}}^*(M \times S^1) \\
 \downarrow \int_{S^1} & & \downarrow \mathcal{I} & & \downarrow \int_{S^1} & & \downarrow \int_{S^1} \\
 \Omega^{*-1}(M)/\text{Im}(d) & \xrightarrow{a} & \hat{K}^*(M) & \xrightarrow{I} & K^*(M) & \xrightarrow{R} & \Omega_{\text{cl}}^{*-1}(M)
 \end{array}$$

where the maps \int_{S^1} on differential forms are integration over the fiber S^1 and the map $\int_{S^1} : K^{*+1}(M \times S^1) \rightarrow K^*(M)$ is the S^1 -integration map in K -theory.

Bunke and Schick have shown these structure uniquely determine differential K theory. The following theorem follows from [BS2], but was succinctly stated as such in Theorem 3.3 of [BS3].

Theorem 2.9 ([BS2], [BS3]). *Let $(\hat{K}, R, I, a, \mathcal{I})$ and $(\hat{K}', R', I', a', \mathcal{I}')$ be two differential extensions of complex K -theory with S^1 -integrations. Then there is a unique natural isomorphism $\hat{K} \rightarrow \hat{K}'$, compatible with all the given structures.*

3. An explicit differential extension of K -Theory

In this section we give an explicit differential extension of K -theory by taking certain equivalence classes of maps into the infinite unitary group U or $BU \times \mathbb{Z}$. The odd part of the differential extension given here is essentially the one developed in [TWZ2], whereas the even part is to our knowledge new. We emphasize that this approach does not require the additional data of differential forms as input. We rely on specific models for U and $BU \times \mathbb{Z}$, with explicit monoid structures, universal Chern forms, and a Bott periodicity map.

3.1. Model for U and the odd differential extension. We first recall some specific constructions for the unitary group U .

Let $\mathbb{C}_{-\infty}^{\infty} = \text{span}(\{e_i\}_{i \in \mathbb{Z}})$ be the complex vector space given by the span of vectors $\{e_i\}_{i \in \mathbb{Z}}$. We will also denote this by $\mathbb{C}_{-\infty}^{\infty} = \mathbb{C}_{\text{cpt}}^{\mathbb{Z}}$, the space of maps from \mathbb{Z} to \mathbb{C} with compact support. It will be useful to adopt the following notation

$$\begin{aligned}
 \mathbb{C}_p^q &= \text{span}\{e_i \mid p \leq i < q\}, \\
 \mathbb{C}_{-\infty}^q &= \text{span}\{e_i \mid i < q\}, \\
 \mathbb{C}_p^{\infty} &= \text{span}\{e_i \mid p \leq i\}.
 \end{aligned}$$

There is an inner product on \mathbb{C}_p^q given by $\langle e_i, e_j \rangle = \delta_{i,j}$. Note that we have the inclusions $\mathbb{C}_p^q \subset \mathbb{C}_p^{q+1}$ and $\mathbb{C}_p^q \subset \mathbb{C}_{p-1}^q$.

Definition 3.1. Let U_p^q be the Lie group of unitary operators A on \mathbb{C}_p^q , *i.e.*, the space of linear maps $A : \mathbb{C}_p^q \rightarrow \mathbb{C}_p^q$ such that $\langle A(x), A(y) \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{C}_p^q$. We have the inclusions $U_0^0 \subset U_{-1}^1 \subset U_{-2}^2 \subset \dots$ given by

$$A \in U_{-p}^p \quad \mapsto \quad \text{Id}_{\mathbb{C}} \oplus A \oplus \text{Id}_{\mathbb{C}} \in U_{-(p+1)}^{p+1}.$$

Let

$$U = U(\mathbb{C}_{-\infty}^{\infty}) = \bigcup_{p \geq 0} U_{-p}^p$$

be the stable infinite unitary group on $\mathbb{C}_{-\infty}^{\infty} = \mathbb{C}_{\text{cpt}}^{\mathbb{Z}}$. Equivalently, U is the group of unitary operators on $\mathbb{C}_{-\infty}^{\infty}$ whose difference from the identity Id of $\mathbb{C}_{-\infty}^{\infty}$ has finite rank. We put the final topology on U , that is, a subset $V \subset U$ is open if and only if the space $V \cap U_{-p}^p$ is open in U_{-p}^p for all p .

We remark that this definition of the stable unitary group is isomorphic to the group $U(\mathbb{C}_0^{\infty})$ of unitary operators on $\mathbb{C}_0^{\infty} = \mathbb{C}_{\text{cpt}}^{\mathbb{N}_0}$ whose difference from the identity on $\mathbb{C}_{\text{cpt}}^{\mathbb{N}_0}$ has finite rank. In fact, any isomorphism $\rho : \mathbb{C}_{\text{cpt}}^{\mathbb{N}_0} \rightarrow \mathbb{C}_{\text{cpt}}^{\mathbb{Z}}$ permuting the ordered basis $\{e_i\}_i$ induces an isomorphism of the unitary groups

$$\tilde{\rho} : U(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}}) \rightarrow U(\mathbb{C}_{\text{cpt}}^{\mathbb{N}_0}), \quad \tilde{\rho}(A) = \rho^{-1} \circ A \circ \rho.$$

In [TWZ2] the authors construct a differential extension of odd K -theory using the group $U(\mathbb{C}_0^{\infty})$. These constructions work equally well with the group $U = U(\mathbb{C}_{-\infty}^{\infty})$ defined above, but the model using $\mathbb{C}_{-\infty}^{\infty}$ fits better with our discussion of $BU \times \mathbb{Z}$ below, so we will review it here for completeness.

In our discussion below, we will need to consider the smooth structures and deRham forms on U and $BU \times \mathbb{Z}$. Since U and $BU \times \mathbb{Z}$ are not finite dimensional manifolds, we use the more general notion of plots and differential forms given by plots, see, *e.g.*, [Chen, Definitions 1.2.1 and 1.2.2]. Since U and $BU \times \mathbb{Z}$ are filtered by finite dimensional smooth manifolds, it is sufficient (and in fact equivalent) to consider differential forms on each finite manifold that are compatible with the filtration, thus justifying the following definition.

A k -form α on U is given by a sequence of forms $\{\alpha_p \in \Omega^k(U_{-p}^p)\}_{p \geq 0}$ such that, for all $p \geq 0$, we have

$$\text{incl}_p^*(\alpha_{p+1}) = \alpha_p,$$

where $\text{incl}_p : U_{-p}^p \subset U_{-(p+1)}^{p+1}$ is the inclusion.

Definition 3.2. We define the universal odd Chern form $\text{Ch} \in \Omega_{\text{cl}}^{\text{odd}}(U)$ by

$$(3.1) \quad \text{Ch} := \text{Tr} \sum_{n \geq 0} \frac{(-1)^n}{(2\pi i)^{n+1}} \frac{n!}{(2n+1)!} \omega^{2n+1},$$

Here $\omega = \{\omega_p\}$ where $\omega_p \in \Omega^1(U_{-p}^p)$ is the left invariant Lie algebra valued 1-form on the unitary groups U_{-p}^p . Note that ω is well-defined since the inclusions $\text{incl}_p : U_{-p}^p \rightarrow U_{-(p+1)}^{p+1}$ are group homomorphism and thus

$\text{incl}_p^*(\omega_{p+1}) = \omega_p$. In particular, for any smooth map $g : M \rightarrow U$ we have an odd degree closed form

$$\text{Ch}(g) := g^*(\text{Ch}) \in \Omega_{\text{cl}}^{\text{odd}}(M).$$

Definition 3.3. The odd Chern form induces an associated transgression form $\text{CS} \in \Omega^{\text{even}}(PU)$ on the path space PU of U , defined

$$(3.2) \quad \text{CS} := \int_{t \in I} \text{ev}_t^*(\text{Ch})$$

where $\text{ev}_t : PU \rightarrow U$ is evaluation of a path at time t .

By Stokes' theorem we have for $\gamma \in PU$ that

$$(3.3) \quad d\text{CS}_\gamma = \text{ev}_1^*(\text{Ch}) - \text{ev}_0^*(\text{Ch}).$$

In particular, for a map $g_t : M \times I \rightarrow U$, i.e., $g_t : M \rightarrow PU$, we have

$$\text{CS}(g_t) := g_t^*(\text{CS}) = \int_{t \in I} \text{Ch}(g_t)$$

satisfies $d\text{CS}(g_t) = \text{Ch}(g_1) - \text{Ch}(g_0)$. Note that this implies that the form $\text{CS} \in \Omega^{\text{even}}(\Omega U)$, obtained by restriction to the based loop space ΩU of U , is closed.

Definition 3.4. Two maps $g_0, g_1 : M \rightarrow U$ are CS-equivalent if there is a smooth homotopy $g_t : M \times I \rightarrow U$ such that $\text{CS}(g_t) \in \Omega^{\text{even}}(M)$ is *exact*.

Remark 3.5. It follows from Theorem 3.17 below that the previously introduced equivalence relation is the same as the equivalence relation: two maps $g_0, g_1 : M \rightarrow U$ are CS-equivalent if there is a smooth homotopy $g_t : M \times I \rightarrow U$ such that $\text{CS}(g_t) \in \Omega^{\text{even}}(M)$ is an even degree *Chern form* on M (see Definition 3.15).

We now wish to introduce a monoid structure on U , making the set of CS-equivalence class of maps $g : M \rightarrow U$ into an abelian group. This was achieved in [TWZ2] using the elementary block sum operation \oplus on $U(\mathbb{C}_0^\infty)$ defined below, which works perfectly well for constructing both the odd K -theory group of M , and the differential extension in [TWZ2]. But, since this operation is discontinuous on U , it is cumbersome to use this operation to build S^1 -integration maps that are group homomorphisms induced by Bott Periodicity maps between $BU \times \mathbb{Z}$ and ΩU .

Instead, we will introduce another monoid operation on U , denoted by \boxplus , which is continuous and in fact smooth on plots. As we will show, these two operations will induce the same operations on the CS-equivalence classes, cf. Lemma 3.9. First let us recall the elementary block sum.

Remark 3.6. In [TWZ2, Section 2] the block sum operation on $U(\mathbb{C}_{\text{cpt}}^{\mathbb{N}_0})$ is defined as follows. If $A, B \in U(\mathbb{C}_{\text{cpt}}^{\mathbb{N}_0})$ are given by $A = A_0^k \oplus \text{Id}_{k+1}^\infty$ and $B = B_0^\ell \oplus \text{Id}_{\ell+1}^\infty$, then $A \oplus B = A_0^k \oplus B_0^\ell \oplus \text{Id}_{k+\ell+1}^\infty$. Note that this definition depends on the chosen integer k , and there is no consistent choice for k to

make this block sum into a continuous operation on all of U . However, we may remedy this below by a shuffle sum operation \boxplus , which does not depend on any choice.

We now define a shuffle sum operation \boxplus which is continuous on U but not associative.

Definition 3.7. Consider the inclusion

$$U(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}}) \times U(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}}) \hookrightarrow U(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}} \oplus \mathbb{C}_{\text{cpt}}^{\mathbb{Z}}) = U(\mathbb{C}_{\text{cpt}}^{\mathbb{Z} \sqcup \mathbb{Z}}).$$

For any isomorphism $\rho : \mathbb{C}_{\text{cpt}}^{\mathbb{Z}} \rightarrow \mathbb{C}_{\text{cpt}}^{\mathbb{Z} \sqcup \mathbb{Z}}$ given by relabeling the basis elements e_i of $\mathbb{C}_{\text{cpt}}^{\mathbb{Z}}$ and $\mathbb{C}_{\text{cpt}}^{\mathbb{Z} \sqcup \mathbb{Z}}$, there is an induced isomorphism of the unitary groups

$$\tilde{\rho} : U(\mathbb{C}_{\text{cpt}}^{\mathbb{Z} \sqcup \mathbb{Z}}) \rightarrow U(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}}) = U, \quad \tilde{\rho}(A) = \rho^{-1} \circ A \circ \rho.$$

We choose ρ to be the shuffle map $\rho_{\text{sh}} : \mathbb{C}_{\text{cpt}}^{\mathbb{Z}} \rightarrow \mathbb{C}_{\text{cpt}}^{\mathbb{Z} \sqcup \mathbb{Z}}$, which maps a basis element e_i with an even index to $\rho_{\text{sh}}(e_{2k}) = e_k$ into the first \mathbb{Z} component, and a basis element with an odd index $\rho_{\text{sh}}(e_{2k+1}) = e'_k$ into the second \mathbb{Z} component. With this, we define the (shuffle) block sum as

$$\boxplus : U(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}}) \times U(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}}) \xrightarrow{\text{incl}} U(\mathbb{C}_{\text{cpt}}^{\mathbb{Z} \sqcup \mathbb{Z}}) \xrightarrow{\tilde{\rho}_{\text{sh}}} U(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}}).$$

Note, that \boxplus is a composition of two continuous maps, and thus continuous.

Remark 3.8. With respect to the two operations above, we have

$$\text{Ch}(f \boxplus g) = \text{Ch}(f) + \text{Ch}(g) \quad \text{and} \quad \text{CS}(f_t \boxplus g_t) = \text{CS}(f_t) + \text{CS}(g_t),$$

because in both cases the trace is additive. Moreover,

$$\text{CS}(f_t \circ g_t) = \text{CS}(f_t) + \text{CS}(g_t).$$

It follows that the set of CS-equivalence classes of maps $M \rightarrow U$ inherits two binary operations, and is a monoid under the operation induced by \boxplus . The following lemma shows that these two operations on CS-equivalence classes are in fact equal.

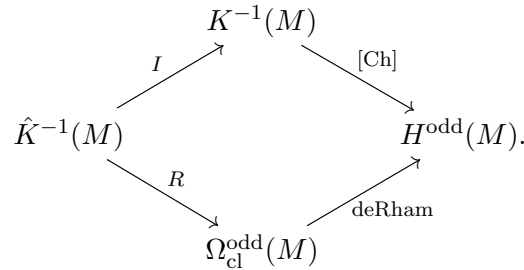
Lemma 3.9. *Let $f, g : M \rightarrow U$ be smooth. There is a smooth homotopy $h_t : M \times I \rightarrow U$ satisfying $h_0 = f \oplus g$, $h_1 = f \boxplus g$ and $\text{CS}(h_t) = 0$. In particular, $f \oplus g$ is CS-equivalent to $f \boxplus g$, so that $[f \oplus g] = [f \boxplus g]$.*

Proof. We may assume $f, g : M \rightarrow U_{-n}^n$ for some $n \in \mathbb{N}$, so that $f \oplus g$ and $f \boxplus g$ are maps from M to U_{-2n}^{2n} . The maps $f \oplus g$ and $f \boxplus g$ differ only by an automorphism of U_{-2n}^{2n} induced by a permutation of coordinates, and since an arbitrary permutation of the coordinates of \mathbb{C}_{-2n}^{2n} may be obtained as a composition of transpositions of adjacent coordinates, it suffices to show there is a path S_t from $A \oplus B$ to $B \oplus A$, for maps $A, B : M \rightarrow U$ such that $\text{CS}(S_t) = 0$. This is proved in Lemma 3.6 of [TWZ2]. \square

In [TWZ2] it is shown that \oplus in fact induces an abelian group structure on the set of CS-equivalence classes of maps $g : M \rightarrow U$, with the inverse of $[g]$ given by $[g^{-1}]$. So, by the above Lemma, there is also the same abelian group structure induced by \boxplus .

This defines a contravariant functor from compact manifolds to abelian groups, which we denote by $M \mapsto \hat{K}^{-1}(M)$. In [TWZ2] the remaining data of a differential extension of K^{-1} are defined. To conclude this section, we review these here as they'll be used later for the construction of S^1 -integration.

Definition 3.10 ([TWZ2]). We define $R = \text{Ch} : \hat{K}^{-1}(M) \rightarrow \Omega_{\text{cl}}^{\text{odd}}(M)$ and $I : \hat{K}^{-1}(M) \rightarrow K^{-1}(M)$ to be the forgetful map, sending a CS-equivalence class of a map $g : M \rightarrow U$ to its underlying homotopy class. This yields a commutative diagram



The remaining data is given by a map $a : \Omega^{\text{even}}(M; \mathbb{R}) / \text{Im}(d) \rightarrow \hat{K}^{-1}(M)$, constructed as follows. To define the map a we first construct an isomorphism

$$\widehat{\text{CS}} : \text{Ker}(I) \rightarrow (\Omega^{\text{even}}(M) / \text{Im}(d)) / \text{Im}([\text{Ch}])$$

where

$$\text{Ker}(I) = \{[g] \mid \text{there is a path } g_t \text{ such that } g_1 = g \text{ and } g_0 = 1\}.$$

The map $\widehat{\text{CS}}$ is defined for $[g] \in \text{Ker}(I) \subset \hat{K}^{-1}(M)$ by choosing a (non-unique) map $g_t : M \times I \rightarrow U$ such that $g_1 = g$ and $g_0 = 1$ is the constant map $M \rightarrow U$ to the identity of U , and letting

$$\widehat{\text{CS}}([g]) = \text{CS}(g_t) \in (\Omega^{\text{even}}(M) / \text{Im}(d)) / \text{Im}([\text{Ch}]).$$

According to [TWZ2], this map is well-defined independently of choice of representative since, modulo exact forms, every even degree CS-form on M of a loop $M \rightarrow \Omega(U)$ can be written as an even degree Chern form of some connection; see [TWZ2, Theorem 3.5]. Moreover this map is surjective since, modulo exact forms, every even form on M is a the CS-form of some path; see [TWZ2, Corollary 5.8].

Finally, the map $a = \widehat{\text{CS}}^{-1} \circ \pi$ is defined to be the composition of the projection π with $\widehat{\text{CS}}^{-1}$,

$$\begin{array}{ccc} \Omega^{\text{even}}(M)/\text{Im}(d) & \xrightarrow{\pi} & (\Omega^{\text{even}}(M)/\text{Im}(d))/\text{Im}([\text{Ch}]) \\ & \searrow a & \downarrow \widehat{\text{CS}}^{-1} \\ & & \text{Ker}(I) \subset \hat{K}^{-1}(M) \end{array}$$

and this map satisfies $\text{Ch} \circ a = d$, $\text{Ker}(a) = \text{Im}([\text{Ch}])$, and $\text{Im}(a) = \text{Ker}(I)$, yielding the exact sequence

$$K^{*-1}(M) \xrightarrow{[\text{Ch}]} \Omega^{*-1}(M; \mathbb{R})/\text{Im}(d) \xrightarrow{a} \hat{K}^*(M) \xrightarrow{I} K^*(M) \xrightarrow{0} 0.$$

3.2. Model for $BU \times \mathbb{Z}$ and the even differential extension. We now recall the construction of the model of $BU \times \mathbb{Z}$ from McDuff [MacD]. We will use this model of $BU \times \mathbb{Z}$ to define a differential extension \hat{K}^0 of K^0 .

Let U be the unitary group of operators on $\mathbb{C}_{-\infty}^{\infty}$, as in Definition 3.1. We denote by $I_k : \mathbb{C}_{-\infty}^{\infty} \rightarrow \mathbb{C}_{-\infty}^{\infty}$ the orthogonal projection onto $\mathbb{C}_{-\infty}^k$. Of particular interest to us will be the orthogonal projection I_0 onto $\mathbb{C}_{-\infty}^0$.

Definition 3.11. A Hermitian operator on \mathbb{C}_p^q is a linear map $h : \mathbb{C}_p^q \rightarrow \mathbb{C}_p^q$ such that $\langle h(x), y \rangle = \langle x, h(y) \rangle$. We denote by H_p^q the space of Hermitian operators on \mathbb{C}_p^q with eigenvalues in $[0, 1]$. Note, that there are inclusions $H_0^0 \subset H_{-1}^1 \subset H_{-2}^2 \subset \dots$ given by

$$h \in H_{-p}^p \quad \mapsto \quad \text{Id}_{\mathbb{C}} \oplus h \oplus 0 \in H_{-(p+1)}^{p+1}.$$

Let

$$H = \bigcup_{p \geq 0} H_{-p}^p$$

be the union of the spaces under the inclusions. In other words, H is the set of Hermitian operators h on $\mathbb{C}_{-\infty}^{\infty}$ with eigenvalues in the interval $[0, 1]$, such that $h - I_0$ has finite rank. Again, we put the final topology on the space H ; *i.e.*, $V \subset H$ is open iff $V \cap H_{-p}^p \subset H_{-p}^p$ is open for all p .

There is an exponential map

$$\exp : H \rightarrow U, \quad \exp(h) := e^{2\pi i h} = \sum_{n \geq 0} \frac{(2\pi i h)^n}{n!}.$$

The fiber $\exp^{-1}(\text{Id})$ of the identity $\text{Id} = \text{Id}_{\mathbb{C}_{-\infty}^{\infty}} \in U$ is the subset of H of operators with eigenvalues in $\{0, 1\}$,

$$\begin{aligned} \exp^{-1}(\text{Id}) &= \{P \in \text{End}(\mathbb{C}_{-\infty}^{\infty}) \mid \\ &P \text{ is Hermitian, } \text{spec}(P) \subset \{0, 1\}, \text{ and } \text{rank}(P - I_0) < \infty\}. \end{aligned}$$

The space H is contractible with contracting homotopy

$$h(t) = th + (1 - t)I_0,$$

providing a path which connects any $h \in H$ to $I_0 \in H$. Therefore, there is an induced map E to the based loop space ΩU of U . According to [D], [AP], [B], [B2] this map E is a homotopy equivalence

Proposition 3.12 ([MacD]). *The map $E : \exp^{-1}(\text{Id}) \rightarrow \Omega U$ given by*

$$E(P)(t) = e^{2\pi i(tP+(1-t)I_0)}$$

is a homotopy equivalence.

This justifies the following definition, which will be the model of $BU \times \mathbb{Z}$ used throughout this paper.

Definition 3.13. We will denote by $BU \times \mathbb{Z}$ the space

$$BU \times \mathbb{Z} := \exp^{-1}(\text{Id}) = \{P \in \text{End}(\mathbb{C}_{-\infty}^{\infty}) \mid \\ P \text{ is Hermitian, } \text{spec}(P) \subset \{0, 1\}, \text{ and } \text{rank}(P - I_0) < \infty\}.$$

Note, that there is a one-to-one correspondence between $BU \times \mathbb{Z}$ and the space of linear subspaces of $\mathbb{C}_{-\infty}^{\infty}$ which contain some $\mathbb{C}_{-\infty}^p$ and which are contained in some $\mathbb{C}_{-\infty}^q$,

$$\exp^{-1}(\text{Id}) \cong \{V \subset \mathbb{C}_{-\infty}^{\infty} \mid \mathbb{C}_{-\infty}^p \subset V \subset \mathbb{C}_{-\infty}^q \text{ for some } p, q \in \mathbb{Z}\}.$$

The equivalence is given by $P \mapsto V = \text{Im}(P)$ with inverse $V \mapsto \text{proj}_V$, where proj_V denotes the orthogonal projection to a subspace $V \subset \mathbb{C}_{-\infty}^{\infty}$.

Next, we show how to recover the integer \mathbb{Z} factor of $BU \times \mathbb{Z}$. We define the rank $: BU \times \mathbb{Z} \rightarrow \mathbb{Z}$ as follows. Let $P \in BU \times \mathbb{Z}$ and let $V = \text{Im}(P)$ be the image of P , so that $\mathbb{C}_{-\infty}^p \subset V \subset \mathbb{C}_{-\infty}^q$ for some $p, q \in \mathbb{Z}$. With this, the rank of P is defined to be

$$(3.4) \quad \text{rank}(P) := p + \dim(V/\mathbb{C}_{-\infty}^p).$$

Notice that the rank(P) is a well-defined integer independent of the choice of p and q .

The topology that $BU \times \mathbb{Z}$ inherits as a closed subspace of H has the property that for any compact subset $K \subset BU \times \mathbb{Z}$, there exist integers p and q such that $K \subset (BU \times \mathbb{Z})_p^q$, where

$$(3.5) \quad (BU \times \mathbb{Z})_p^q := \{P \in BU \times \mathbb{Z} \mid \mathbb{C}_{-\infty}^p \subset \text{Im}(P) \subset \mathbb{C}_{-\infty}^q\}.$$

Let M be a smooth compact manifold. A map $P : M \rightarrow BU \times \mathbb{Z}$ determines a vector bundle with connection over M , which is well-defined up to the addition of the trivial line bundle with the trivial connection d , and a realization of this bundle as a sub-bundle of a trivial bundle, in the following way.

Definition 3.14. Let $P : M \rightarrow BU \times \mathbb{Z}$ be smooth. Choose $p, q \in \mathbb{Z}$ such that $\mathbb{C}_{-\infty}^p \subset \text{Im}(P) \subset \mathbb{C}_{-\infty}^q$, i.e.,

$$\text{Im}(P(x)) = \mathbb{C}_{-\infty}^p \oplus V(x), \text{ with } V(x) \subset \mathbb{C}_p^q$$

for all $x \in M$. Let $E_P = \sqcup_{x \in M} V(x)$, which is a sub-bundle of the trivial bundle $M \times \mathbb{C}^q$. This vector bundle inherits a fiber-wise metric by restriction of the metric on H . The projection operator onto $V(x)$ defines a connection on E_P , given by $\nabla_P(s) = P \circ d(s)$, which is compatible with the metric. Note that changing the integer p , say by subtracting one, adds on the trivial line bundle with the trivial connection. So we have an assignment

$$P \mapsto (E_P, \nabla_P)$$

where the right hand side is well-defined up to addition of (\mathbb{C}^n, d) , for some $n \in \mathbb{N}$.

The curvature of the connection ∇_P is given by $R = P(dP)^2 = (dP)^2P$. This justifies the following definition, in which we use the notion of plots and forms given by plots, where $BU \times \mathbb{Z}$ is filtered by the spaces $(BU \times \mathbb{Z})_{-p}^p$ from Equation (3.5). In particular, a k -form α on $BU \times \mathbb{Z}$ is given by a sequence of forms $\{\alpha_p \in \Omega^k((BU \times \mathbb{Z})_{-p}^p)\}_{p \geq 0}$ such that

$$\text{incl}_p^*(\alpha_{p+1}) = \alpha_p,$$

where $\text{incl}_p : (BU \times \mathbb{Z})_{-p}^p \subset (BU \times \mathbb{Z})_{-(p+1)}^{p+1}$, $P \mapsto \text{Id}_{\mathbb{C}} \oplus P \oplus 0$, is the inclusion.

Definition 3.15. The universal even Chern form $\text{Ch} \in \Omega_{\text{cl}}^{\text{even}}(BU \times \mathbb{Z})$ is defined by

$$(3.6) \quad \text{Ch}(P) := \text{Tr} \sum_{n \geq 0} \frac{1}{(2\pi i)^n} \frac{1}{n!} P(dP)^{2n}$$

where, by definition, $\text{Tr}(P) = \text{rank}(P)$, cf. Equation (3.4).

Note this is well-defined since PdP^2 is invariant under pullback along the maps incl_p , since $d(\text{Id}_{\mathbb{C}} \oplus P \oplus 0) = 0 \oplus dP \oplus 0$. We also have the associated Chern–Simons form.

Definition 3.16. Denote by $P(BU \times \mathbb{Z})$ the path space of $BU \times \mathbb{Z}$. The universal Chern–Simons form $\text{CS} \in \Omega^{\text{odd}}(P(BU \times \mathbb{Z}))$ is given using the evaluation map at time t , $\text{ev}_t : P(BU \times \mathbb{Z}) \rightarrow BU \times \mathbb{Z}$, by

$$(3.7) \quad \text{CS} := \int_{t \in I} \text{ev}_t^*(\text{Ch}).$$

It is straightforward to check that for a map $P_t : M \times I \rightarrow BU \times \mathbb{Z}$, i.e., $P_t : M \rightarrow P(BU \times \mathbb{Z})$, we have that the pullback $P_t^*(\text{CS}) \in \Omega^{\text{odd}}(M)$ is given by (cf. [TWZ2, (2.2)])

$$(3.8) \quad \begin{aligned} &\text{CS}(P_t) \\ &:= P_t^*(\text{CS}) \\ &= \int_0^1 \sum_{n \geq 0} \frac{1}{(2\pi i)^{n+1}} \frac{1}{n!} \text{Tr} \left((\text{Id} - 2P_t) \left(\frac{\partial}{\partial t} P_t \right) \overbrace{dP_t \wedge \cdots \wedge dP_t}^{2n+1 \text{ factors}} \right) dt \end{aligned}$$

where $R_t = P_t(dP_t)^2$ is the curvature of P_t . Moreover, by Stokes' theorem we have that

$$(3.9) \quad d\text{CS}_\gamma = \text{ev}_1^*(\text{Ch}) - \text{ev}_0^*(\text{Ch}).$$

Let $K^0(M) = [M, BU \times \mathbb{Z}]$ denote the homotopy classes of maps from M to $BU \times \mathbb{Z}$. It follows from (3.9) that there is an induced Chern homomorphism $[\text{Ch}] : K^0(M) \rightarrow H^{\text{even}}(M)$, given by $P \mapsto [\text{Ch}(P)]$.

We now state some fundamental results concerning Chern and Chern–Simons forms on a manifold.

Theorem 3.17. *Let M be a smooth compact manifold and consider the maps*

$$\begin{aligned} \text{Ch} : \text{Map}^0(M, U) &\rightarrow \Omega_{\text{cl}}^{\text{odd}}(M; \mathbb{R}) \\ \text{Ch} : \text{Map}^0(M, BU \times \mathbb{Z}) &\rightarrow \Omega_{\text{cl}}^{\text{even}}(M; \mathbb{R}) \end{aligned}$$

given by the pullback of forms (3.1) and (3.6), where Map^0 denotes the connected component of the constant map. Furthermore, consider the maps

$$\begin{aligned} \text{CS} : \text{Map}_*(M \times I, U) &\rightarrow \Omega^{\text{even}}(M; \mathbb{R}) \\ \text{CS} : \text{Map}_*(M \times I, BU \times \mathbb{Z}) &\rightarrow \Omega^{\text{odd}}(M; \mathbb{R}) \end{aligned}$$

given by the pullback of forms (3.3) and (3.7), where Map_ indicates based maps whose time zero value is constant at $1 \in U$, respectively $I_0 \in BU \times \mathbb{Z}$. Finally, consider the induced maps*

$$\begin{aligned} \text{CS}_\Omega : \text{Map}(M, \Omega U) &\rightarrow \Omega_{\text{cl}}^{\text{even}}(M; \mathbb{R}) \\ \text{CS}_\Omega : \text{Map}(M, \Omega(BU \times \mathbb{Z})) &\rightarrow \Omega_{\text{cl}}^{\text{odd}}(M; \mathbb{R}) \end{aligned}$$

obtained by restricting the maps CS above to those based paths that are based loops. Then, for both even and odd degree cases, we have the following statements.

- (1) $\text{Im}(d) \subset \text{Im}(\text{Ch})$, where d is the DeRham differential.
- (2) CS is onto.
- (3) $\text{Im}(\text{Ch}) \equiv \text{Im}(\text{CS}_\Omega)$ modulo exact forms.

Proof. The first statement (1) for odd forms is given by Corollary 2.7 of [TWZ2]. For positive degree even forms we know from Proposition 2.1 of [PiT] such exact even real forms can be written as the Chern form of connection on a trivial bundle, so the result follows from Narasimhan–Ramanan [NR] and the fact that trivial bundles are represented by nullhomotopic maps.

The second statement (2) for odd forms was proved in Corollary 2.2 of [PiT] (sharpening Proposition 2.10 of [SS]), and the second statement for even forms is proved as follows. Corollary 5.8 of [TWZ2] states that the map CS is onto $\Omega^{\text{even}}(M)/\text{Im}(d)$. By the previous case, we may write an exact error as an even Chern form, which by Theorem 3.5 of [TWZ2] may be written as the even CS-form of a based loop. By taking block sum of (or concatenating) the path with the loop, we obtain the desired result.

For the third statement (3), first note that the map CS_Ω indeed lands in the space of closed forms, due to Equations (3.3) and (3.9).

For the case of even forms, statement (3) is Theorem 3.5 of [TWZ2]. (Alternatively this also follows from Lemma 4.1 below using the map E , by an argument very similar to the one that we give next for the case of odd forms.) For the case of odd forms, we use Lemma 4.15 below, which shows that the map $h : \Omega(BU \times \mathbb{Z}) \rightarrow U$ induced by holonomy satisfies $\text{CS}_\Omega(f) \equiv \text{Ch}(h \circ f) \in \Omega_{\text{cl}}^{\text{odd}}(M; \mathbb{R})$ modulo exact forms. But the map h is a homotopy equivalence, so by choosing any homotopy inverse we see that the other containment holds modulo exact forms as well. \square

Definition 3.18. Two maps $P_0, P_1 : M \rightarrow BU \times \mathbb{Z}$ are CS-equivalent if there is a smooth homotopy $P_t : M \times I \rightarrow BU \times \mathbb{Z}$ from P_0 to P_1 such that $P_t^*(\text{CS}) = \text{CS}(P_t) \in \Omega^{\text{odd}}(M)$ is exact.

Remark 3.19. It follows from Theorem 3.17 that the previously introduced equivalence relation is the same as the equivalence relation: two maps $P_0, P_1 : M \rightarrow BU \times \mathbb{Z}$ are CS-equivalent if there is a smooth homotopy $P_t : M \times I \rightarrow BU \times \mathbb{Z}$ from P_0 to P_1 such that $P_t^*(\text{CS}) = \text{CS}(P_t) \in \Omega^{\text{odd}}(M)$ is an odd degree *Chern form* on M .

We denote the set of CS-equivalence classes of maps from M to $BU \times \mathbb{Z}$ by the suggestive notation $\hat{K}^0(M)$. This is a contravariant functor from the category of smooth manifolds and maps to sets. We wish to introduce an abelian group structure and show this is indeed a differential extension of K -theory. The naive definition of sum is given as follows, but suffers the same issue of not being continuous as does Definition 3.6, since it depends on noncanonical choices. Nevertheless, on plots it is well-defined and associative.

Definition 3.20. We define $\oplus : (BU \times \mathbb{Z}) \times (BU \times \mathbb{Z}) \rightarrow BU \times \mathbb{Z}$ as follows. For $P, Q \in BU \times \mathbb{Z}$ choose maximal $p, m \in \mathbb{Z}$, and minimal $q, n \in \mathbb{Z}$, such that $\text{Im}(P) = \mathbb{C}_{-\infty}^p \oplus V$ and $\text{Im}(Q) = \mathbb{C}_{-\infty}^m \oplus W$, where V and W are finite dimensional subspaces satisfying $V \subset \mathbb{C}_p^q$ and $W \subset \mathbb{C}_m^n$. Then, define

$$P \oplus Q = \text{proj}_{\mathbb{C}_{-\infty}^{p+m} \oplus s_m(V) \oplus s_q(W)}$$

where proj_Z denotes the orthogonal projection to a subspace $Z \subset \mathbb{C}_{-\infty}^\infty$, and s_k denotes the operator given by $s_k(e_i) = e_{i+k}$. Note that the above definition depends on the integers p, q, m, n , which, in general, may vary discontinuously.

We now define a second sum operation, which is a continuous operation on $BU \times \mathbb{Z}$, but is not associative.

Definition 3.21. We define $\boxplus : (BU \times \mathbb{Z}) \times (BU \times \mathbb{Z}) \rightarrow BU \times \mathbb{Z}$ by defining a shuffle block sum $\boxplus : H \times H \rightarrow H$ and then by showing that this factors through $BU \times \mathbb{Z} \subset H$. Recall that $H = H(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}})$ is the set of

hermitian operators h on $\mathbb{C}_{\text{cpt}}^{\mathbb{Z}}$ with eigenvalues in $[0, 1]$ such that $h - I_0$ has finite rank, and, similarly, denote by $H(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}} \oplus \mathbb{C}_{\text{cpt}}^{\mathbb{Z}})$ the set of hermitian operators \tilde{h} on $\mathbb{C}_{\text{cpt}}^{\mathbb{Z}} \oplus \mathbb{C}_{\text{cpt}}^{\mathbb{Z}}$ with eigenvalues in $[0, 1]$, such that $\tilde{h} - (I_0 \oplus I_0)$ has finite rank. In analogy with Definition 3.7, we define the shuffle block sum to be the composition

$$\boxplus : H(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}}) \times H(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}}) \xrightarrow{\text{incl}} H(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}} \oplus \mathbb{C}_{\text{cpt}}^{\mathbb{Z}}) = H(\mathbb{C}_{\text{cpt}}^{\mathbb{Z} \sqcup \mathbb{Z}}) \xrightarrow{\tilde{\rho}_{\text{sh}}} H(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}}),$$

where the first map is the inclusion, and the second map is the map induced by the shuffle map $\rho_{\text{sh}} : \mathbb{C}_{\text{cpt}}^{\mathbb{Z}} \rightarrow \mathbb{C}_{\text{cpt}}^{\mathbb{Z} \sqcup \mathbb{Z}}$ from Definition 3.7 via

$$\tilde{\rho}_{\text{sh}} : H(\mathbb{C}_c^{\mathbb{Z} \sqcup \mathbb{Z}}) \rightarrow H(\mathbb{C}_c^{\mathbb{Z}}),$$

$\tilde{\rho}_{\text{sh}}(\tilde{h}) = \rho_{\text{sh}}^{-1} \circ \tilde{h} \circ \rho_{\text{sh}}$. Note that $\tilde{\rho}_{\text{sh}}$ indeed lands in $H(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}})$. Further, since the eigenvalues are preserved by *incl* and $\tilde{\rho}_{\text{sh}}$, we see that $BU \times \mathbb{Z}$ is preserved by \boxplus , inducing the desired map

$$\boxplus : (BU \times \mathbb{Z}) \times (BU \times \mathbb{Z}) \rightarrow BU \times \mathbb{Z}.$$

Remark 3.22. With respect to the two operations we have

$$\text{Ch}(f \oplus g) = \text{Ch}(f) + \text{Ch}(g) = \text{Ch}(f \boxplus g)$$

and

$$\text{CS}(f_t \oplus g_t) = \text{CS}(f_t) + \text{CS}(g_t) = \text{CS}(f_t \boxplus g_t),$$

since in either case trace is additive. Moreover, for the path composition $g_t * h_t$, we have $\text{CS}(f_t * g_t) = \text{CS}(f_t) + \text{CS}(g_t)$.

We will first show in Lemma 3.23 that restricted to any plot, the operation \oplus is abelian up to a path that has vanishing CS-form, and also that each plot $P : M \rightarrow BU \times \mathbb{Z}$ has an additive inverse $P^\perp : M \rightarrow BU \times \mathbb{Z}$, again up to a path that has vanishing CS-form. Then, in Lemma 3.24, we use similar techniques to show the operations \oplus and \boxplus are in fact homotopic, and even more, CS-equivalent, and so introduce the same operation on $\hat{K}^0(M)$. It will be useful for calculations and propositions that follows to have both operations.

Lemma 3.23. *For any $P, Q \in \text{Map}(M, BU \times \mathbb{Z})$, there exists a path $\Gamma_t \in \text{Map}(M \times I, BU \times \mathbb{Z})$ such that*

$$\Gamma_0 = P \oplus Q, \quad \Gamma_1 = Q \oplus P, \quad \text{and} \quad \text{CS}(\Gamma_t) = 0.$$

Also, for any $P \in \text{Map}(M, BU \times \mathbb{Z})$, there exists a path

$$\Gamma_t \in \text{Map}(M \times I, BU \times \mathbb{Z})$$

such that

$$\Gamma_0 = P \oplus P^\perp, \quad \Gamma_1 = I_0, \quad \text{and} \quad \text{CS}(\Gamma_t) = 0.$$

Here P^\perp is defined by

$$P^\perp = \text{proj}_{\mathbb{C}_{-\infty}^{-q} \oplus s_{-p-q}(V^\perp)}$$

where $\text{Im}(P) = \mathbb{C}_{-\infty}^p \oplus V$ with $V \subset \mathbb{C}_p^q$, and $V^\perp \subset \mathbb{C}_p^q$ is the orthogonal complement of V in \mathbb{C}_p^q , and s_ℓ denotes a “shift” by ℓ as in Definition 3.20.

Proof. For the first statement, we assume that for some integers $m, n, r, s \in \mathbb{Z}$, the maps P and Q have images $\text{Im}(P) = \mathbb{C}_{-\infty}^m \oplus V$, where $V \subset \mathbb{C}_m^n$, and $\text{Im}(Q) = \mathbb{C}_{-\infty}^r \oplus W$, where $W \subset \mathbb{C}_r^s$. By picking $p > 0$ large enough, we may assume without loss of generality, that $m = r = -p$ and $n = s = p$. We construct a path of the form $\Gamma_t = id_{\mathbb{C}_{-\infty}^{-2p}} \oplus S_t \oplus 0|_{\mathbb{C}_{2p}^\infty}$, such that Γ_t has vanishing CS-form and the path Γ_t equals $P \oplus Q$ and $Q \oplus P$ at the two endpoints.

To simplify the notation we'll define our paths on the interval $[0, \pi/2]$, which can always be reparametrized to be a path on $I = [0, 1]$, with the same properties as stated above. We will use $X(t) : \mathbb{C}_{-2p}^{2p} \rightarrow \mathbb{C}_{-2p}^{2p}$, defined by

$$X(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix},$$

where this block matrix acts on $\mathbb{C}_{-2p}^{2p} = \mathbb{C}_{-2p}^0 \oplus \mathbb{C}_0^{2p}$. Furthermore, we denote by A the $2p \times 2p$ -matrix representing the map $P|_{\mathbb{C}_{-p}^p} : \mathbb{C}_{-p}^p \rightarrow \mathbb{C}_{-p}^p$, and by B the $2p \times 2p$ -matrix representing $Q|_{\mathbb{C}_{-p}^p} : \mathbb{C}_{-p}^p \rightarrow \mathbb{C}_{-p}^p$, and we use these to define the map $F : \mathbb{C}_{-2p}^{2p} \rightarrow \mathbb{C}_{-2p}^{2p}$,

$$F = s_{-p}(P|_{\mathbb{C}_{-p}^p}) \oplus s_p(Q|_{\mathbb{C}_{-p}^p}) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

With this notation, consider the path $S_t \in \text{Map}(M \times [0, \pi/2], \text{End}(\mathbb{C}_{-2p}^{2p}))$,

$$S_t = X(t)FX(t)^{-1}$$

so that $S_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $S_{\pi/2} = \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix}$. Using the fact that

$$\frac{\partial}{\partial t}(X(t)^{-1}) = -X(t)^{-1}X'(t)X(t)^{-1},$$

we get

$$S'_t = X'(t)FX(t)^{-1} - X(t)FX(t)^{-1}X'(t)X(t)^{-1}$$

and we have that

$$(dS_t)^{2n+1} = (X(t)dFX(t)^{-1})^{2n+1} = X(t)(dF)^{2n+1}X(t)^{-1}$$

Using the explicit formula for $\text{CS}(\Gamma_t) = \text{CS}(S_t)$ from (3.8), it suffices to show

$$\text{Tr}\left((\text{Id} - 2S_t)S'_t(dS_t)^{2n+1}\right) = 0.$$

First, we have

$$\begin{aligned} & \operatorname{Tr}\left(S'_t(dS_t)^{2n+1}\right) \\ &= \operatorname{Tr}\left(X(t)'F(dF)^{2n+1}X(t)^{-1} - X(t)FX(t)^{-1}X'(t)(dF)^{2n+1}X(t)^{-1}\right) \\ &= \operatorname{Tr}\left(X(t)^{-1}X(t)'F(dF)^{2n+1} - X(t)^{-1}X'(t)F^\perp(dF)^{2n+1}\right), \end{aligned}$$

where we used $F(dF)^2 = (dF)F^\perp(dF) = (dF)^2F$ and the fact that trace is cyclic. Next, we compute

$$\begin{aligned} & \operatorname{Tr}\left((S_t)S'_t(dS_t)^{2n+1}\right) \\ &= \operatorname{Tr}\left(FX(t)^{-1}X'(t)F(dF)^{2n+1} - FX(t)^{-1}X'(t)(dF)^{2n+1}\right) \\ &= \operatorname{Tr}\left(X(t)^{-1}X'(t)(-F^\perp)(dF)^{2n+1}\right) \end{aligned}$$

since

$$\operatorname{Tr}\left(FX(t)^{-1}X'(t)F(dF)^{2n+1}\right) = \operatorname{Tr}\left(X(t)^{-1}X'(t)FF^\perp(dF)^{2n+1}\right) = 0.$$

Putting this together and using $F - F^\perp - 2(-F^\perp) = \operatorname{Id}$ we have

$$\operatorname{Tr}\left((\operatorname{Id} - 2S_t)S'_t(dS_t)^{2n+1}\right) = \operatorname{Tr}\left(X(t)^{-1}X'(t)(dF)^{2n+1}\right) = 0,$$

since $X(t)^{-1}X'(t) = \begin{bmatrix} 0 & \operatorname{Id} \\ -\operatorname{Id} & 0 \end{bmatrix}$, and $(dF)^{2n+1}$ is block diagonal.

For the second statement, write $\operatorname{Im}(P) = \mathbb{C}_{-\infty}^p \oplus V$, where $V \subset \mathbb{C}_p^q$. Denote by $V^\perp \subset \mathbb{C}_p^q$ the orthogonal projection of V in \mathbb{C}_p^q , and let P^\perp be the orthogonal projection onto $\mathbb{C}_{-\infty}^{-q} \oplus s_{-p-q}(V^\perp)$. By Definition 3.20, $P \oplus P^\perp$ is the projection onto $\mathbb{C}_{-\infty}^{p-q} \oplus s_{-q}(V) \oplus s_{-p}(V^\perp)$. Notice that $s_{-q}(V) \subset \mathbb{C}_{p-q}^0$ and $s_{-p}(V^\perp) \subset \mathbb{C}_0^{q-p}$. Similarly to the previous case, let $X(t) : \mathbb{C}_{p-q}^{q-p} \rightarrow \mathbb{C}_{p-q}^{q-p}$ be the map

$$X(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

regarded as an endomorphism of $\mathbb{C}_{p-q}^0 \oplus \mathbb{C}_0^{q-p}$. We denote by A the $(q-p) \times (q-p)$ -matrix representing the operation $P|_{\mathbb{C}_p^q} : \mathbb{C}_p^q \rightarrow \mathbb{C}_p^q$, and by A^\perp the $(q-p) \times (q-p)$ -matrix representing $P^\perp|_{\mathbb{C}_{-q}^{-p}} : \mathbb{C}_{-q}^{-p} \rightarrow \mathbb{C}_{-q}^{-p}$. Denote by $G : \mathbb{C}_{p-q}^{q-p} \rightarrow \mathbb{C}_{p-q}^{q-p}$ and $H : \mathbb{C}_{p-q}^{q-p} \rightarrow \mathbb{C}_{p-q}^{q-p}$ the maps

$$\begin{aligned} G &= s_{-q}(P|_{\mathbb{C}_p^q}) \oplus \operatorname{Id}|_{\mathbb{C}_0^{q-p}} = \begin{bmatrix} A & 0 \\ 0 & \operatorname{Id} \end{bmatrix}, \\ H &= \operatorname{Id}|_{\mathbb{C}_{p-q}^0} \oplus s_q(P^\perp|_{\mathbb{C}_{-q}^{-p}}) = \begin{bmatrix} \operatorname{Id} & 0 \\ 0 & A^\perp \end{bmatrix}. \end{aligned}$$

Consider the path $S_t \in \text{Map}(M \times [0, \pi/2], \text{End}(\mathbb{C}_{p-q}^{q-p}))$,

$$S_t = X(t)GX(t)^{-1}H = \begin{bmatrix} A + \sin^2(t)A^\perp & \cos(t)\sin(t)A^\perp \\ \cos(t)\sin(t)A^\perp & \cos^2(t)A^\perp \end{bmatrix}.$$

Then $S_0 = \begin{bmatrix} A & 0 \\ 0 & A^\perp \end{bmatrix}$, and $S_{\pi/2} = \begin{bmatrix} \text{Id} & 0 \\ 0 & 0 \end{bmatrix}$. Also, S_t is symmetric and $S_t^2 = S_t$, for each t , so $\Gamma_t := id_{\mathbb{C}_{-\infty}^{-p}} \oplus S_t \oplus 0$ satisfies $\Gamma_t : M \times [0, \pi/2] \rightarrow BU \times \mathbb{Z}$, and $\Gamma_0 = P \oplus P^\perp$ and $\Gamma_1 = I_0$. We then calculate

$$\begin{aligned} dS_t &= \begin{bmatrix} \cos^2(t) & -\cos(t)\sin(t) \\ -\cos(t)\sin(t) & -\cos^2(t) \end{bmatrix} dA, \\ (dS_t)^2 &= \begin{bmatrix} \cos^2(t) & 0 \\ 0 & \cos^2(t) \end{bmatrix} (dA)^2, \\ \frac{\partial}{\partial t} S_t &= \begin{bmatrix} 2\cos(t)\sin(t)A^\perp & (\cos^2(t) - \sin^2(t))A^\perp \\ (\cos^2(t) - \sin^2(t))A^\perp & -2\cos(t)\sin(t)A^\perp \end{bmatrix}, \end{aligned}$$

and with this,

$$\begin{aligned} &(\text{Id} - 2S_t)S'_t(dS_t)^{2n+1} \\ &= (\text{Id} - 2S_t)S'_t dS_t \cdot (dS_t)^{2n} \\ &= \begin{bmatrix} -\sin(t)\cos(t) \cdot A^\perp & -\cos^2(t) \cdot A^\perp \\ -\cos^2(t) \cdot A^\perp & \sin(t)\cos(t) \cdot A^\perp \end{bmatrix} \cdot \cos^{2n}(t)(dA)^{2n+1}. \end{aligned}$$

Thus, we see that we have a vanishing trace,

$$\text{Tr}((\text{Id} - 2S_t)S'_t(dS_t)^{2n+1}) = 0.$$

It therefore follows from Equation (3.8) that $\text{CS}(\Gamma_t) = \text{CS}(S_t) = 0$, which is the claim. \square

We now show the operations \oplus and \boxplus are CS-equivalent.

Lemma 3.24. *Let M be a smooth compact manifold.*

- (1) *Let $k \neq \ell \in \mathbb{Z}$ and let $\tilde{\tau} : BU \times \mathbb{Z} \rightarrow BU \times \mathbb{Z}$ be the map induced by the isomorphism $\tau : \mathbb{C}_{-\infty}^\infty \rightarrow \mathbb{C}_{-\infty}^\infty$ which exchanges the k^{th} and the ℓ^{th} basis vectors of $\mathbb{C}_{-\infty}^\infty$. Then there is a map*

$$\Gamma_t : BU \times \mathbb{Z} \times I \rightarrow BU \times \mathbb{Z},$$

i.e., $\Gamma : BU \times \mathbb{Z} \rightarrow P(BU \times \mathbb{Z})$, such that $\Gamma(P)_0 = P$, $\Gamma(P)_1 = \tilde{\tau} \circ P$, and $\text{CS}(\Gamma(P)_t) = 0$.

In particular, any plot $g : M \rightarrow BU \times \mathbb{Z}$ is CS-equivalent to $\tau \circ g$.

- (2) *Let $P, Q : M \rightarrow BU \times \mathbb{Z}$. There is a smooth homotopy*

$$\Gamma_t : M \times I \rightarrow BU \times \mathbb{Z}$$

satisfying $\Gamma_0 = P \oplus Q$, $\Gamma_1 = P \boxplus Q$ and $\text{CS}(\Gamma_t) = 0$.

In particular, $P \oplus Q$ is CS-equivalent to $P \boxplus Q$, so that

$$[P \oplus Q] = [P \boxplus Q] \in \hat{K}^0(M).$$

Proof. First the first statement we may assume $\ell = k + 1$ since any permutation can be obtained by a composition of such transposition and CS is additive with respect to compositions, cf. Remark 3.22. For this case we use the rotation family $X(t)$ given by $\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$ on \mathbb{C}_k^{k+2} and by the identity everywhere else (compare this with the proof of the first part of the previous Lemma 3.23). Setting $\Gamma(P)_t = X(t)PX(t)^{-1}$, the same argument and calculation as in the first part of the previous Lemma 3.23 shows that $\Gamma(P)_t$ swaps the k^{th} and $(k+1)^{\text{st}}$ basis vectors and has a vanishing CS-form.

Now, for the second claim, for fixed plots P, Q , we may find numbers p, q, m, n , such that $\text{Im}(P(x)) = \mathbb{C}_{-\infty}^p \oplus V(x)$ with $V(x) \subset \mathbb{C}_p^q$ for all $x \in M$, and $\text{Im}(Q(x)) = \mathbb{C}_{-\infty}^m \oplus W(x)$ with $W(x) \subset \mathbb{C}_m^n$ for all $x \in M$. Without loss of generality, we may assume that $p = m$ (by taking the minimum of p and m) and $q = n$ (by taking the maximum of q and n). Then

$$P(x) \oplus Q(x) = \text{proj}_{\mathbb{C}_{-\infty}^{2p}} + \text{proj}_{s_p(V(x))} + \text{proj}_{s_q(W(x))}$$

has an image $\text{Im}(P(x) \oplus Q(x)) = \mathbb{C}_{-\infty}^{2p} \oplus s_p(V(x)) \oplus s_q(W(x))$. Thus, the direct sum \oplus (for these plots) can be expressed as the composition

$$\oplus : H(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}}) \times H(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}}) \xrightarrow{\text{incl}} H(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}} \oplus \mathbb{C}_{\text{cpt}}^{\mathbb{Z}}) = H(\mathbb{C}_{\text{cpt}}^{\mathbb{Z} \sqcup \mathbb{Z}}) \xrightarrow{\tilde{\rho}} H(\mathbb{C}_{\text{cpt}}^{\mathbb{Z}}),$$

where $\rho : \mathbb{C}_{\text{cpt}}^{\mathbb{Z}} \rightarrow \mathbb{C}_{\text{cpt}}^{\mathbb{Z} \sqcup \mathbb{Z}}$ is given by

$$\rho(e_k) = \begin{cases} \rho_{\text{sh}}(e_k), & \text{for } k < 2p, \\ e_{k-p}, \text{ in the first } \mathbb{Z} \text{ component} & \text{for } 2p \leq k < p+q, \\ e'_{k-(p+q)}, \text{ in the second } \mathbb{Z} \text{ component} & \text{for } p+q \leq k < 2q, \\ \rho_{\text{sh}}(e_k), & \text{for } 2q \leq k. \end{cases}$$

Therefore, $P(x) \oplus Q(x)$ and $P(x) \boxplus Q(x) \in H(\mathbb{C}_{2p}^{\mathbb{Z}})$ differ only on \mathbb{C}_{2p}^{2q} , and there $P(x) \oplus Q(x) = \tilde{\alpha}(P(x) \boxplus Q(x))$ for some isomorphism

$$\alpha : \mathbb{C}_{2p}^{2q} \rightarrow \mathbb{C}_{2p}^{2q}$$

which relabels the basis elements of \mathbb{C}_{2p}^{2q} . Each such isomorphism is a composition of transpositions as in part (1), so that the result follows from the claim in (1). \square

Corollary 3.25. *The set of CS-equivalence classes of maps of M into $BU \times \mathbb{Z}$ has an abelian group structure induced by \oplus or \boxplus , with identity $[I_0]$, and so $M \rightarrow \hat{K}^0(M)$ defines a contravariant functor from compact manifolds to abelian groups.*

We now show that the functor \hat{K}^0 admits the data of a differential extension. By Theorem 2.9, i.e., [BS3, Theorem 3.3] of Bunke and Schick, it then follows that this model is isomorphic to any other model of even differential K -theory, via a unique natural isomorphism.

Definition 3.26. Let $I : \hat{K}^0(M) \rightarrow K^0(M)$ denote the forgetful map which sends a CS-equivalence class of maps to its homotopy class. By Equation (3.9) we have a well-defined map $R = \text{Ch} : \hat{K}^0(M) \rightarrow \Omega_{\text{cl}}^{\text{even}}(M)$, and by definition of the Chern character (3.6) we have a commutative diagram

$$\begin{array}{ccc}
 & K^0(M) & \\
 I \nearrow & & \searrow [\text{Ch}] \\
 \hat{K}^0(M) & & H^{\text{even}}(M) \\
 \text{Ch} \searrow & & \nearrow \text{deRham} \\
 & \Omega_{\text{cl}}^{\text{even}}(M) &
 \end{array}$$

The remaining data is given by a map $a : \Omega^{\text{odd}}(M; \mathbb{R}) / \text{Im}(d) \rightarrow \hat{K}^0(M)$, constructed as using the ideas from [SS]. To define the map a we first construct an isomorphism

$$\widehat{\text{CS}} : \text{Ker}(I) \rightarrow \left(\Omega^{\text{odd}}(M) / \text{Im}(d) \right) / \text{Im}([\text{Ch}])$$

where

$$\begin{aligned}
 \text{Ker}(I) = \{ [P] \mid & \text{there is a path } P_t : M \times I \rightarrow BU \times \mathbb{Z} \text{ such that } P_1 = P \\
 & \text{and } P_0 = I_0 : M \rightarrow BU \times \mathbb{Z} \text{ is the constant map to } I_0 \}.
 \end{aligned}$$

The map $\widehat{\text{CS}}$ is defined for $[P] \in \text{Ker}(I) \subset \hat{K}^0(M)$ by choosing a (nonunique) $P_t : M \times I \rightarrow U$ with $P_1 = P$ and $P_0 = I_0$, and letting

$$\widehat{\text{CS}}([P]) = \text{CS}(P_t) \in \left(\Omega^{\text{odd}}(M) / \text{Im}(d) \right) / \text{Im}([\text{Ch}]).$$

We first show this map is well-defined. For two different choices P_t and Q_t satisfying $P_1 = Q_1 = P$ and $P_0 = Q_0 = I_0$, consider the composition

$$F_t = Q_{-t} * P_t : M \rightarrow \Omega(BU \times \mathbb{Z}).$$

But $\text{CS}(P_t) - \text{CS}(Q_t) = \text{CS}(F_t) \in \text{Im}(\text{Ch})$ mod exact forms by Theorem 3.17(3), and $\text{Im}(d) \subset \text{Im}(\text{Ch})$ by Theorem 3.17(1). Thus, $\text{CS}(P_t)$ and $\text{CS}(Q_t)$ coincide modulo $\text{Im}(\text{Ch})$, which shows that $\widehat{\text{CS}}$ is well-defined. Next, the map $\widehat{\text{CS}}$ is onto, since CS is onto by Theorem 3.17(2).

Finally, we show the map $\widehat{\text{CS}}$ is one to one. Suppose $[P] \in \text{Ker}(I)$, so that $\widehat{\text{CS}}([P]) = 0$, i.e., that $\text{CS}(P_t) \in \text{Im}(\text{Ch})$ for some choice of P_t as above. Then $\text{CS}(P_t) = \text{Ch}(g)$ for some $g : M \rightarrow U$. By Theorem 3.17(3) we have $\text{Ch}(g) = \text{CS}(Q_t)$, modulo exact, for some $Q_t : M \rightarrow \Omega BU \times \mathbb{Z}$. Let $K_t = Q_{1-t} * P_t$ be the concatenation, which is a new homotopy from P to the constant I_0 , and satisfies $\text{CS}(K_t) = \text{CS}(P_t) - \text{CS}(Q_t) = \text{Ch}(g) - \text{Ch}(g) = 0$ modulo exact. Thus, K_t is a CS-exact path from P to I_0 , so that $[P] = [I_0] = 0$.

Using the map $\widehat{\text{CS}}$, we define the map a as the composition of the projection π with $\widehat{\text{CS}}^{-1}$,

$$\begin{array}{ccc} \Omega^{\text{odd}}(M)/\text{Im}(d) & \xrightarrow{\pi} & (\Omega^{\text{odd}}(M)/\text{Im}(d)) / \text{Im}([\text{Ch}]) \\ & \searrow a & \downarrow \widehat{\text{CS}}^{-1} \\ & & \text{Ker}(I) \subset \hat{K}^0(M) \end{array}$$

yielding the exact sequence

$$K^{-1}(M) \xrightarrow{[\text{Ch}]} \Omega^{\text{odd}}(M; \mathbb{R})/\text{Im}(d) \xrightarrow{a} \hat{K}^0(M) \xrightarrow{I} K^0(M) \xrightarrow{0} 0.$$

It remains to show this map satisfies $\text{Ch} \circ a = d$. This equation follows, since $d \circ \widehat{\text{CS}} = \text{Ch}$ as maps $\text{Ker}(I) \rightarrow \Omega_{\text{cl}}^{\text{even}}(M)$, which can be seen calculating for any $[P] \in \text{Ker}(I)$ and P_t connecting $P_1 = P$ with $P_0 = I_0$,

$$\begin{aligned} d(\widehat{\text{CS}}([P])) &= d(\text{CS}(P_t)) \stackrel{(3.9)}{=} \text{Ch}(P_1) - \text{Ch}(P_0) \\ &= \text{Ch}(P_1) = \text{Ch}(P) = \text{Ch}([P]). \end{aligned}$$

Remark 3.27. Note that the uniqueness of even differential K -theory does not require an S^1 -integration, see [BS3, Theorem 3.3]. Since the definition of $\hat{K}^0(M)$ given here satisfies the axioms in Definition 2.1, this is already isomorphic to any other model of even differential K -theory. For example, $\hat{K}^0(M)$ coincides with Simons and Sullivan’s model $\hat{K}_{SS}^0(M)$ in [SS], given by the Grothendieck group of structured vector bundles (i.e., isomorphism classes of CS-equivalence classes of bundles with connection). There is a natural map between these two models, given by the “pullback”: for

$$P : M \rightarrow BU \times \mathbb{Z}$$

consider the following difference of structured vector bundles

$$P^* = (E_P, \nabla_P) - (\mathbb{C}^k, d)$$

where the choice (E_P, ∇_P) is made as in Definition 3.14, and

$$k = \text{rank}(E_P) - \text{rank}(P).$$

Note this is well-defined since (E_P, ∇_P) is well-defined up to adding a trivial line bundle with trivial connection. In fact, this map induces an isomorphism, as we now explain.

Given a smooth map $f_t : M \times I \rightarrow BU \times Z$ such that $\text{CS}(f_t)$ is exact, consider a choice for the total pullback bundle $P^* = (E, \nabla) - (\mathbb{C}^k, d)$ over $M \times I$ whose slice at time $t \in I$ is $(E_t, \nabla_t) - (\mathbb{C}^k, d)$. Parallel transport $P_t : E_0 \rightarrow E_t$ provides an isomorphism between (E, ∇) and the product bundle $E_0 \times I$, and a path of connections $P_t^* \nabla_t$ on E_0 such that $\text{CS}(P_t^* \nabla_t) = \text{CS}(f_t)$ is exact. So, (E_0, ∇_0) and $(E_0, P_1^* \nabla_1) \cong (E_1, \nabla_1)$ are CS-equivalent, and the integer k is time independent, which shows the pullback map is well-defined.

Since any element of $\hat{K}_{SS}^0(M)$ has a representative of the form

$$\left((E, \nabla), (\mathbb{C}^k, d) \right)$$

for some k (*cf.* [SS] (3.1)), the map is a surjective by the the Narasimhan–Ramanan theorem [NR], and a group homomorphism by Lemma 3.24.

Finally, the map is one to one, as we now sketch. Given two maps $f_0, f_1 : M \rightarrow BU \times \mathbb{Z}$ whose pullbacks are equal in $\hat{K}_{SS}^0(M)$, we may as well assume the second (trivial) summands \mathbb{C}^k of the pullbacks are equal, and even that they are zero, since $f_0 \cong f_0 \boxplus \mathbb{C}^k \boxplus (\mathbb{C}^k)^\perp$, and similarly for f_1 . Then it suffices to consider (E_0, ∇_0) and (E_1, ∇_1) that represent the same isomorphism class of CS-equivalence class of bundles with connection. Using the isomorphism to transport both connections to E_0 , we have two connections, say ∇_0 and ∇_1 on E_0 , and a path of connections ∇_t from ∇_0 to ∇_1 such that $\text{CS}(\nabla_t)$ is exact. We can regard this as a bundle with connection on $M \times I$, and represent this as a map $G_t : M \times I \rightarrow BU \times \mathbb{Z}$ using the Narasimhan–Ramanan theorem [NR], so that $\text{CS}(G_t)$ is exact, by assumption. The pullback bundles via G_0 and G_1 may not equal E_0 and E_1 with their connections (respectively), but since they are isomorphic, there are homotopies $F_t : M \times I \rightarrow BU \times \mathbb{Z}$ from f_0 to G_0 , and $H_t : M \times I \rightarrow BU \times \mathbb{Z}$ from G_1 to f_1 defined explicitly by families of rotations of coordinates, by the main theorem of [Schl]. By an argument similar to Lemma 3.23 above, for such homotopies given by rotations we have $\text{CS}(F_t) = \text{CS}(H_t) = 0$. Therefore $F_t * G_t * H_t$ is a homotopy from E_0 to E_1 and

$$\text{CS}(F_t * G_t * H_t) = \text{CS}(F_t) + \text{CS}(G_t) + \text{CS}(H_t) = \text{CS}(G_t)$$

is exact, as desired. The details of this argument will be exploited in [TWZ4], to provide a means for classifying structured vector bundles.

4. Constructing S^1 -integration

Our goal in this section is to construct an S^1 -integration map (Definition 2.8) for the differential extension of K -theory defined in Section 3. By Theorem 2.9 (*i.e.*, [BS3, Theorem 3.3]) this shows we have defined a model for differential K -theory. In Section 4.1 we discuss the even to odd part, while in Section 4.2 we discuss the odd to even part of the S^1 -integration map.

Constructing the S^1 integration in these models is quite nontrivial, so we begin with a motivating discussion, focusing for concreteness on the integration map from the even to the odd part. Suppose that, for all compact manifolds with corners M , we had a natural map

$$i : \text{Map}(M \times S^1, BU \times \mathbb{Z}) \rightarrow \text{Map}(M, U)$$

that induces the S^1 -integration map $\int_{S^1} : K^0(M \times S^1) \rightarrow K^{-1}(M)$ in K -theory and makes the following diagram commute

$$\begin{array}{ccc} \text{Map}(M \times S^1, BU \times \mathbb{Z}) & \xrightarrow{\text{Ch}} & \Omega^{\text{even}}(M \times S^1) \\ \downarrow i & & \downarrow \int_{S^1} \\ \text{Map}(M, U) & \xrightarrow{\text{Ch}} & \Omega^{\text{odd}}(M). \end{array}$$

By applying this diagram to the manifold $M \times I$ and integrating out the interval I we obtain the commutative diagram

$$\begin{array}{ccccc} \text{Map}(M \times I \times S^1, BU \times \mathbb{Z}) & \xrightarrow{\text{Ch}} & \Omega^{\text{even}}(M \times I \times S^1) & \xrightarrow{\int_I} & \Omega^{\text{odd}}(M \times S^1) \\ \downarrow i & & \downarrow \int_{S^1} & & \downarrow \int_{S^1} \\ \text{Map}(M \times I, U) & \xrightarrow{\text{Ch}} & \Omega^{\text{odd}}(M \times I) & \xrightarrow{\int_I} & \Omega^{\text{even}}(M). \end{array}$$

The composition of the top and bottom horizontal rows define the even and odd Chern–Simons forms, respectively. The integration over S^1 is a chain map, since S^1 is closed. So, if the Chern–Simons form for an element of $\text{Map}(M \times I \times S^1, BU \times \mathbb{Z})$ is exact, then so is the Chern–Simons form for the element of $\text{Map}(M \times I, U)$ induced by i . This shows there is an induced map $i : \hat{K}^0(M \times S^1) \rightarrow \hat{K}^{-1}(M)$ making the following diagram commute

$$\begin{array}{ccc} \hat{K}^0(M \times S^1) & \xrightarrow{\text{Ch}} & \Omega^{\text{even}}(M \times S^1) \\ \downarrow i & & \downarrow \int_{S^1} \\ \hat{K}^{-1}(M) & \xrightarrow{\text{Ch}} & \Omega^{\text{odd}}(M). \end{array}$$

The maps $a : \Omega^*(M)/\text{Im}(d) \rightarrow \hat{K}^{*+1}(M)$ induce maps

$$\Omega^*(M)/\text{Im}(\text{Ch}) \rightarrow \text{Ker}(I) \subset \hat{K}^{*+1}(M)$$

which are isomorphisms. In fact, in both the even and odd parts of the differential extensions given here, the map a is defined to be the inverse of the map $\widehat{\text{CS}} : \text{Ker}(I) \rightarrow \Omega^*(M)/\text{Im}(\text{Ch})$ which is simply given by taking the Chern–Simons form. By the remarks above, the map induced by i commutes with the Chern–Simons forms, so it follows that the diagram

$$\begin{array}{ccc} \Omega^{*+1}(M \times S^1) & \xrightarrow{a} & \hat{K}^*(M \times S^1) \\ \downarrow \int_{S^1} & & \downarrow i \\ \Omega^*(M) & \xrightarrow{a} & \hat{K}^{*+1}(M) \end{array}$$

commutes, and so we would have produced an S^1 -integration map according to Definition 2.8.

Unfortunately, it is difficult to give natural maps which commute on the nose with these geometric representative for the Chern Character. The

natural candidates for these two maps from Bott periodicity and homotopy theory do not make the diagrams commute on the nose, but rather commute only up to an exact differential form, compare [FL]. This exact error makes the previous argument fail at the first step, since integration over the interval is not a chain map.

The resolution is to study this exact error and use it to define a new integration map. We will show using methods of homotopies and associated transgression forms how to correct such a situation, in the end yielding a bona fide S^1 -integration map.

4.1. Odd to even. Recall from Section 3.2 that H is the subspace of hermitian operators h on \mathbb{C}^∞_∞ with eigenvalues in $[0, 1]$ such that $h - I_0$ has finite rank (see page 539). Recall furthermore from page 539, that we have the exponential map $\exp : H \rightarrow U$, $\exp(h) = e^{2\pi ih} = \sum_{n \geq 0} \frac{(2\pi ih)^n}{n!}$, and from Proposition 3.12, that we denote by $E : BU \times \mathbb{Z} \rightarrow \Omega U$ the exponential of the straight line to I_0 ,

$$E(P) : S^1 \rightarrow U \quad \text{given by} \quad E(P)(t) = e^{2\pi i(tP+(1-t)I_0)}.$$

Recall from (3.6) that $\text{Ch} \in \Omega^{\text{even}}(BU \times \mathbb{Z})$, and from (3.3) that $\text{CS} \in \Omega^{\text{even}}(PU)$, which we may restrict from the path space on U to the based loop space of U , also denoted by $\text{CS} \in \Omega^{\text{even}}(\Omega U)$. The next lemma shows that $E^*\text{CS}$ equals Ch on $BU \times \mathbb{Z}$ modulo exact.

Lemma 4.1. *There exists a form $\beta \in \Omega^{\text{odd}}(BU \times \mathbb{Z})$ such that*

$$(4.1) \quad d\beta = E^*\text{CS} - \text{Ch}.$$

In other words, the following diagram commutes modulo $d\beta$ for all compact manifolds with corners.

$$\begin{array}{ccc} \text{Map}(M \times S^1, U) & \xrightarrow{\text{Ch}} & \Omega^{\text{odd}}(M \times S^1) \\ E^\sharp \uparrow & & \downarrow \int_{S^1} \\ \text{Map}(M, BU \times \mathbb{Z}) & \xrightarrow{\text{Ch}} & \Omega^{\text{even}}(M). \end{array}$$

Proof. For $k \in \mathbb{Z}$, consider the maps $\gamma_k : BU \times \mathbb{Z} \rightarrow \text{Map}([0, 1], H)$ given by taking the straight line path $\gamma_k(P)(t) = tP + (1 - t)I_k$ from I_k to P , where I_k is the projection to \mathbb{C}^k_∞ as before. Also denote by

$$\rho_k : BU \times \mathbb{Z} \rightarrow \text{Map}([0, 1], H)$$

the constant map to the straight line path $\rho_k(P)(t) = tI_k + (1 - t)I_{k-1}$ from I_{k-1} to I_k . Furthermore, there is a map $h_k : BU \times \mathbb{Z} \rightarrow \text{Map}([0, 1] \times [0, 1], H)$ given by $h_k(P)(s, t) = tP + (1 - t)(sI_k + (1 - s)I_{k-1})$ such that the boundary consists of $\partial h_k(P) = \gamma_k(P) - \gamma_{k-1}(P) - \text{const}_P + \rho_k(P)$. Note that γ_k, ρ_k , and h_k can be composed with $\exp : H \rightarrow U$, giving rise (by slight abuse of notation) to maps $\exp \circ \gamma_k : BU \times \mathbb{Z} \rightarrow \text{Map}([0, 1], U)$, etc.

According to [TWZ2, Propositions 3.4 and 3.2], one can construct a form $H \in \Omega^{\text{odd}}(\text{Map}([0, 1] \times [0, 1], U))$ such that

$$dH = \partial_1^* \text{CS} - \partial_2^* \text{CS} - \partial_3^* \text{CS} + \partial_4^* \text{CS},$$

where $\partial_i : \text{Map}([0, 1] \times [0, 1], U) \rightarrow \text{Map}([0, 1], U)$ is induced by the i th boundary component of $[0, 1] \times [0, 1]$, and $\text{CS} \in \Omega^{\text{even}}(\text{Map}([0, 1], U))$ is the universal Chern–Simons form. The form H is determined by its pullback under any map $g : M \rightarrow \text{Maps}([0, 1] \times [0, 1], U)$ to a manifold with corners M , and given by

(4.2)

$$g^*(H) = \text{Tr} \sum_{n \geq 1} \sum_{i \neq j} c_{n,i,j} \times \int_0^1 \int_0^1 \overbrace{(g^{-1} dg) \dots (g^{-1} \frac{\partial}{\partial t} g) \dots (g^{-1} \frac{\partial}{\partial s} g) \dots (g^{-1} dg)}^{2n+1 \text{ terms}} dt ds,$$

$\underbrace{\hspace{10em}}_{i^{\text{th}}}$ $\underbrace{\hspace{10em}}_{j^{\text{th}}}$

where $c_{n,i,j}$ are some constants. Note, that in the above situation, we have that

$$\begin{aligned} (\exp \circ h_k)^*(dH) &= (\exp \circ h_k)^*(\partial_1^* \text{CS} - \partial_2^* \text{CS} - \partial_3^* \text{CS} + \partial_4^* \text{CS}) \\ &= (\exp \circ \gamma_k)^* \text{CS} - (\exp \circ \gamma_{k-1})^* \text{CS} + (\exp \circ \rho_k)^* \text{CS}. \end{aligned}$$

We then define $\beta \in \Omega^{\text{odd}}(BU \times \mathbb{Z})$ by setting

$$\beta = \sum_{k \leq 0} (\exp \circ h_k)^* H = (\exp \circ h_0)^* H + (\exp \circ h_{-1})^* H + (\exp \circ h_{-2})^* H + \dots$$

We claim that β is a well-defined odd form on $BU \times \mathbb{Z}$, that is, the infinite sum reduces to a finite sum whenever applied to tangent vectors v_1, \dots, v_ℓ at some $P \in BU \times \mathbb{Z}$. Indeed, if we represent v_1, \dots, v_ℓ by a map

$$f : B(0) \rightarrow BU \times \mathbb{Z}$$

from a compact ball $B(0) \subset \mathbb{R}^\ell$ centered at 0 with $f(0) = P$ and $f_*(\frac{\partial}{\partial x^j}) = v_j$, then the image of f is contained in some subspace

$$\{P \in BU \times \mathbb{Z} \mid P|_{\mathbb{C}^r_\infty} = \text{Id}_{\mathbb{C}^r_\infty}\}$$

for some r (see page 540). Now, for any $k < r$, we claim that

$$(4.3) \quad (\exp \circ h_k \circ f)^* H = 0,$$

so that $\beta(v_1, \dots, v_\ell) = f^* \left(\sum_{k \leq 0} (\exp \circ h_k)^* H \right) \left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^\ell} \right)$ becomes a finite sum. To see (4.3), let $g = \exp \circ h_k \circ f : B(0) \rightarrow \text{Map}([0, 1] \times [0, 1], U)$,

which is given by

$$\begin{aligned} g(x)(s, t) &= e^{2\pi i(tf(x)+(1-t)(sI_k+(1-s)I_{k-1}))} \\ &= e^{2\pi itf(x)} \cdot e^{2\pi i(1-t)(sI_k+(1-s)I_{k-1})}, \end{aligned}$$

where the last equality follows, since $f(x)$ commutes with $(sI_k + (1-s)I_{k-1})$ for $k < r$, as f is the identity on \mathbb{C}^r_∞ . If we decompose $\mathbb{C}^\infty_\infty = \mathbb{C}^k_\infty \oplus \mathbb{C}^\infty_{k+1}$, then we can see that $g(x)(s, t)$ preserves these subspaces for each $x \in B(0)$ and $s, t \in [0, 1]$; that is, $g(x)(s, t)$ maps \mathbb{C}^k_∞ to \mathbb{C}^k_∞ and \mathbb{C}^∞_{k+1} to \mathbb{C}^∞_{k+1} . Thus, g^{-1} , dg , $\frac{\partial}{\partial t}g$, and $\frac{\partial}{\partial s}g$ also preserve this decomposition. Finally note, that $g^{-1} \frac{\partial}{\partial s}g = 2\pi i(1-t)(I_k - I_{k-1})$ which vanishes on \mathbb{C}^∞_{k+1} , while

$$g^{-1}dg = e^{-2\pi itf(x)}d(e^{2\pi itf(x)})$$

which vanishes on \mathbb{C}^k_∞ . Thus, by (4.2), we see that $g^*(H) = 0$, which is the claim of (4.3).

It remains to check Equation (4.1). We calculate $d\beta$ by evaluating the first k terms in the expansion of β as follows,

$$\begin{aligned} d\beta &= d((\exp \circ h_0)^*H + (\exp \circ h_{-1})^*H + (\exp \circ h_{-2})^*H + \dots) \\ &= (\exp \circ h_0)^*dH + (\exp \circ h_{-1})^*dH + (\exp \circ h_{-2})^*dH + \dots \\ &= (\exp \circ \gamma_0)^*CS - (\exp \circ \gamma_{-1})^*CS + (\exp \circ \rho_0)^*CS \\ &\quad + (\exp \circ \gamma_{-1})^*CS - (\exp \circ \gamma_{-2})^*CS + (\exp \circ \rho_{-1})^*CS \\ &\quad + (\exp \circ \gamma_{-2})^*CS - (\exp \circ \gamma_{-3})^*CS + (\exp \circ \rho_{-2})^*CS + \dots \\ &\quad + (\exp \circ \gamma_{-(k-1)})^*CS - (\exp \circ \gamma_{-k})^*CS + (\exp \circ \rho_{-(k-1)})^*CS \\ &\quad + \sum_{j \geq k} (\exp \circ h_{-j})^*dH. \end{aligned}$$

Now, since $(\exp \circ \rho_j)(P)(t) = e^{2\pi i(tI_j+(1-t)I_{j-1})}$ is a constant map (independent of P) with winding number 1, we see that $(\exp \circ \rho_j)^*CS = 1$. Next, note that $\exp \circ \gamma_0(P)(t) = e^{2\pi i(tP+(1-t)I_0)} = E(P)(t)$ so that the first term is $(\exp \circ \gamma_0)^*CS = E^*CS$, while the terms $(\exp \circ \gamma_j)^*CS$ for $j < 0$ cancel pairwise. Thus, for any $k \geq 0$, we have

$$d\beta = E^*CS - (\exp \circ \gamma_{-k})^*CS + k + \sum_{j \geq k} (\exp \circ h_{-j})^*dH.$$

Now, the calculation from [TWZ2, Theorem 3.5] shows that for $k \geq 0$ sufficiently large (*i.e.*, when P and I_{-k} commute), we have for all $j \geq k$ that $(\exp \circ \gamma_{-j})^*CS = \text{Ch} + j$, where j is a constant function. Thus, the terms $(\exp \circ h_{-j})^*dH = (\exp \circ \gamma_{-j})^*CS - (\exp \circ \gamma_{-(j+1)})^*CS + (\exp \circ \rho_{-j})^*CS$ vanish for all $j \geq k$. Using this together with $-(\exp \circ \gamma_{-k})^*CS + k = -\text{Ch}$, we obtain that $d\beta = E^*CS - \text{Ch}$, which is Equation (4.1). \square

We recall from [TWZ2, Lemma 3.6] that we may associate to each $g : M \rightarrow U^n_{-n}$ a map $\gamma_g : M \times I \rightarrow U^{2n}_{-2n}$, such that $\gamma_g(0) = g \oplus g^{-1}$ and

$\gamma_g(1) = id$, and $CS(\gamma_g) = 0$. The path γ_g is essentially given as in Lemma 3.23 using a sin/cos-matrix, but with U instead of $BU \times \mathbb{Z}$, but we repeat it here for completeness. For $g : M \rightarrow U_{-n}^n$ let $\gamma_g(t) : M \times I \rightarrow U_{-2n}^{2n}$ be given by

$$\gamma_g(t) = GX(t)HX(t)^{-1},$$

where (cf. Lemma 3.23)

$$G = \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 \\ 0 & g^{-1} \end{bmatrix} \quad X(t) = \begin{bmatrix} \cos(\pi t/2) & \sin(\pi t/2) \\ -\sin(\pi t/2) & \cos(\pi t/2) \end{bmatrix}$$

so that $\gamma_g(0) = g \oplus g^{-1}$ and $\gamma_g(1) = id$. It is straightforward to check that for all $g, h : M \rightarrow U_{-n}^n$ mapping into the same components U_{-n}^n of U , we obtain:

$$(4.4) \quad \gamma_{(g \boxplus h)} = \gamma_g \boxplus \gamma_h,$$

where $g \boxplus h$ is given by a map $g \boxplus h : M \rightarrow U_{-2n}^{2n}$.

The obvious map from the free loop space LU to the based loop space ΩU , given by left U -action at the base point, does not preserve the CS form. The following map, defined using \boxplus , does.

Definition 4.2. For any $g_t : M \rightarrow LU_{-n}^n \subset LU$ we define

$$*g_t : M \rightarrow \Omega U_{-2n}^{2n} \subset \Omega U$$

by conjugating $g_t \oplus g_0^{-1}$ (under path composition “ $*$ ”) with the path $\gamma_{g_0}(t)$, i.e.,

$$*g_t = \gamma_{g_0}(t) * (g_t \oplus g_0^{-1}) * \overline{\gamma_{g_0}(t)},$$

where $\bar{\gamma}$ denotes reversal of a path γ . The notation $*g_t$ is meant to denote the *based map* associated to the free loop g_t .

We give some useful properties of this map. In particular, it is a monoid morphism $LU \rightarrow \Omega U$ that preserves the CS-forms.

Lemma 4.3. *The based loop map induces a well-defined map $LU \rightarrow \Omega U$, denoted by $g_t \mapsto *g_t$ as above, and satisfies*

$$CS(*g_t) = CS(g_t) \quad \text{and} \quad CS(*g_t \oplus *h_t) = CS(*(g_t \oplus h_t)).$$

Furthermore, the map $g_t \mapsto *g_t$ is a homomorphism with respect to the shuffle block sum, i.e., for $g_t, h_t : M \rightarrow LU_{-n}^n$, we have

$$*(g_t \boxplus h_t) = *g_t \boxplus *h_t.$$

Proof. According to Lemma 3.6 of [TWZ2] we have $CS(\gamma_g(t)) = 0$ so the first result follows since $CS(g_t \oplus g_0^{-1}) = CS(g_t) + CS(g_0^{-1}) = CS(g_t)$. The second statement now follows formally since

$$\begin{aligned} CS(*g_t \oplus *h_t) &= CS(*g_t) + CS(*h_t) = CS(g_t) + CS(h_t) \\ &= CS(g_t \oplus h_t) = CS(*(g_t \oplus h_t)). \end{aligned}$$

The last statement follows using two facts. First, the operations \boxplus and \oplus satisfy the interchange law for maps $g_t, h_t, k_t, l_t : M \rightarrow LU_{-n}^n$, *i.e.*,

$$(g_t \oplus h_t) \boxplus (k_t \oplus l_t) = (g_t \boxplus k_t) \oplus (h_t \boxplus l_t)$$

and, secondly, since the shuffle sum preserves unitary matrices, and the inverse of a unitary matrix is the conjugate transpose, we have

$$(g \boxplus h)^{-1} = (g \boxplus h)^* = g^* \boxplus h^* = g^{-1} \boxplus h^{-1}.$$

Therefore, using (4.4),

$$\begin{aligned} & *(g_t \boxplus h_t) \\ &= \gamma_{(g \boxplus h)_0}(t) * ((g_t \boxplus h_t) \oplus ((g \boxplus h)_0)^{-1}) * \overline{\gamma_{(g \boxplus h)_0}(t)} \\ &= \left(\gamma_{g_0}(t) \boxplus \gamma_{h_0}(t) \right) * \left((g_t \oplus g_0^{-1}) \boxplus (h_t \oplus h_0^{-1}) \right) * \left(\overline{\gamma_{g_0}(t)} \boxplus \overline{\gamma_{h_0}(t)} \right) \\ &= *_g t \boxplus *_h t. \end{aligned} \quad \square$$

We are now ready to define the S^1 -integration map. We fix once and for all a (continuous) homotopy inverse to the Bott Periodicity map E , *i.e.*, a map $i : \Omega U \rightarrow BU \times \mathbb{Z}$, and a homotopy $F_r : \Omega U \times I \rightarrow \Omega U$ satisfying $F_0 = E \circ i$ and $F_1 = id$, and a homotopy $H_r : BU \times \mathbb{Z} \times I \rightarrow BU \times \mathbb{Z}$ such that $H_0 = i \circ E$ and $H_1 = id$.

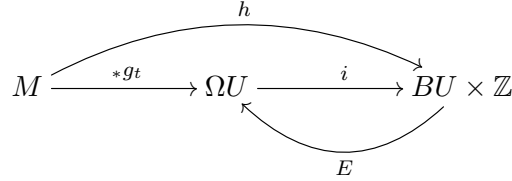
It is worth noting that if we could choose i , F_r and H_r to be smooth, then the presentation here could be simplified considerably.¹ Such a smooth choice is unknown to us at the time of this writing, nevertheless we'll overcome this by replacing our maps by certain smooth maps from M , and then showing our constructions are independent of the choices.

Definition 4.4. We now define the map $\mathcal{I} : \hat{K}^{-1}(M \times S^1) \rightarrow \hat{K}^0(M)$. Let $g_t : M \times S^1 \rightarrow U$ be a representative for an element $[g_t] \in K^{-1}(M \times S^1)$, and denote by $*_g t : M \rightarrow \Omega U$ the based map from Definition 4.2. Composing this with the fixed (continuous) homotopy inverse $i : \Omega U \rightarrow BU \times \mathbb{Z}$ to the Bott Periodicity map $E : BU \times \mathbb{Z} \rightarrow \Omega U$ gives a (continuous) map $i \circ *_g t : M \rightarrow BU \times \mathbb{Z}$.

To define \mathcal{I} , we make two choices. First, choose a smooth map $h : M \rightarrow BU \times \mathbb{Z}$ which is homotopic to $i \circ *_g t$, via some continuous homotopy $h_r : M \times I \rightarrow BU \times \mathbb{Z}$, so that $h_0 = h$ and $h_1 = i \circ *_g t$. Note that $(E \circ h_r) * (F_r \circ *_g t)$ is then a continuous homotopy from $E \circ h$ to $*_g t$.

¹There is a geometric map from ΩU to the space of Fredholm operators on a separable Hilbert space, extensively studied in [PrS]. It would be interesting and useful if this map can be used in the way mentioned above.

Second, choose a smooth homotopy $k_r : M \times I \rightarrow \Omega U$ between the smooth maps $k_0 = E \circ h$ and $k_1 = *g_t$



With these choices, denote by $\eta \in \Omega^{\text{odd}}(M)$ the form

$$\eta = \int_r k_r^* \text{CS} + h^* \beta$$

and define $\mathcal{I}(g_t) \in \hat{K}^0(M)$ by

$$\mathcal{I}(g_t) = [h] + a(\eta),$$

where $[h]$ indicates the CS-equivalence class of the smooth map $h : M \rightarrow BU \times \mathbb{Z}$, and $a : \Omega^{\text{odd}}(M)/\text{Im}(d) \rightarrow \hat{K}^0(M)$ is the map from Definition 3.26.

Remark 4.5. We note that the choice of homotopy h_r is not part of the data in the above definition. In fact, for any chosen h homotopic to $i \circ *g_t$, and any smooth homotopy $k_r : M \times I \rightarrow \Omega U$ between $k_0 = E \circ h$ and $k_1 = *g_t$, there always exists a homotopy h_r from $h_0 = h$ to $h_1 = i \circ *g_t$ such that k_r is homotopic to $(E \circ h_r) * (F_r \circ *g_t)$ relative to the endpoints $E \circ h$ and $*g_t$. To find the homotopy h_r from h to $i \circ *g_t$ with the desired property, we first choose any homotopy h'_r from $h'_0 = h$ to $h'_1 = i \circ *g_t$. Then, for any homotopy k_r from $k_0 = E \circ h$ to $k_1 = *g_t$, the path composition $k_r * (F_{1-r} \circ *g_t) * (E \circ h'_{1-r})$ is a homotopy from $E \circ h$ to itself, and it thus defines an element in the homotopy group $\pi_1((\Omega U)^M, E \circ h)$. Since the induced map $E^M : (BU \times \mathbb{Z})^M \rightarrow (\Omega U)^M$ is a homotopy equivalence, there is an element $h''_r \in \pi_1((BU \times \mathbb{Z})^M, h)$ such that $E^M(h''_r) = E \circ h''_r$ is homotopic to $k_r * (F_{1-r} \circ *g_t) * (E \circ h'_{1-r})$ relative to the endpoints.

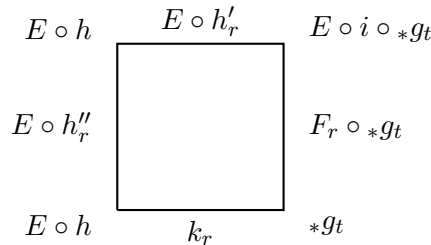


FIGURE 1. The homotopy from k_r to $(E \circ h_r) * (F_r \circ *g_t)$

But this means in turn that k_r is homotopic to $(E \circ h_r'') * (E \circ h_r') * (F_r \circ *g_t)$ relative to the endpoints. Choosing $h_r = h_r'' * h_r'$, we get the desired map from $h_0 = h$ to $h_1 = i \circ *g_t$, such that k_r is homotopic to $(E \circ h_r) * (F_r \circ *g_t)$ relative to the endpoints.

In the following Lemmas 4.6–4.12 we will show that \mathcal{I} is well-defined, and satisfies all the properties of an S^1 -integration from Definition 2.8.

Lemma 4.6. *The map $\mathcal{I} : \hat{K}^{-1}(M \times S^1) \rightarrow \hat{K}^0(M)$ is well-defined.*

Proof. We first show that $\mathcal{I}(g_t) = [h] + a(\eta)$ is independent of the choices of h and the homotopy k_r that are used to define $\mathcal{I}(g_t)$. Given

$$g_t : M \times S^1 \rightarrow U,$$

suppose we have two choices of smooth maps, h^0 and h^1 , which are both continuously homotopic to i_*g_t , via some homotopies h_r^0 and h_r^1 , respectively. Then h^0 and h^1 are smooth and continuously homotopic via the continuous homotopy $h_r^0 * h_{1-r}^1$. So we can choose a smooth homotopy $s_r : M \times I \rightarrow BU \times \mathbb{Z}$ between h^0 and h^1 , which is a deformation of the continuous homotopy $h_r^0 * h_{1-r}^1$. This means the triangle in Figure 2 can be filled in by a smooth homotopy T_1 , and so $E \circ T_1$ is a homotopy between $E \circ s_r$ and $(E \circ h_r^0) * (E \circ h_{1-r}^1)$, as indicated in Figure 2.

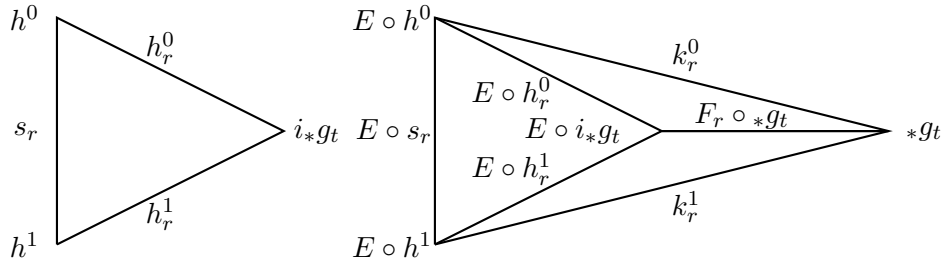


FIGURE 2. Homotopies between two possible choices. Triangle T_1 with vertices h^0, h^1 , and i_*g_t , and triangle T_2 with vertices $E \circ h^0, E \circ h^1$, and $*g_t$.

The homotopy $(E \circ h_r^0) * (F_r \circ g_t)$ is a continuous homotopy from $E \circ h^0$ to $*g_t$. We can deform this to a smooth homotopy k_r^0 , and similarly we can deform $(E \circ h_r^1) * (F_r \circ g_t)$ to a smooth homotopy k_r^1 . This shows the second triangle T_2 in Figure 2 can be filled in, and we may as well assume it is filled in by a smooth map, which we also denote by T_2 .

For $i = 0, 1$, let

$$\eta^i = \int_r (k_r^i)^* \text{CS} + (h^i)^* \beta.$$

By definition of the “ a ” map (see Definition 3.26), there is a path

$$\gamma_r : M \times I \rightarrow BU \times \mathbb{Z}$$

such that $a(\eta^0) = [\gamma_1]$, $\text{CS}(\gamma_r) = \eta^0$, and γ_0 is the constant map to the identity. Similarly, there is a path

$$\rho_r : M \times I \rightarrow BU \times \mathbb{Z}$$

such that $a(\eta^1) = [\rho_1]$, $\text{CS}(\rho_r) = \eta^1$, and ρ_0 is the constant map to the identity.

Then $s_r \boxplus (\rho_r \boxplus \gamma_{1-r})$ is a path from a representative of

$$[h^0 \boxplus (\rho_0 \boxplus \gamma_1)] = [h^0] + a(\eta^0)$$

to a representative of $[h^1 \boxplus (\rho_1 \boxplus \gamma_0)] = [h^1] + a(\eta^1)$, and it suffices to show that the CS-form of this path is exact. Note that the CS form is given by

$$\begin{aligned} \text{CS}(s_r \boxplus (\rho_r \boxplus \gamma_{1-r})) &= \text{CS}(s_r) + \text{CS}(\rho_r) - \text{CS}(\gamma_r) \\ &= \int_r s_r^* \text{Ch} + \eta^1 - \eta^0 \\ &= \int_r s_r^* \text{Ch} + \int_r (k_r^1)^* \text{CS} + (h^1)^* \beta \\ &\quad - \int_r (k_r^0)^* \text{CS} - (h^0)^* \beta. \end{aligned}$$

Using the fact that $d\beta = E^* \text{CS} - \text{Ch}$ we have

$$\int_r s_r^* \text{Ch} = \int_r s_r^* E^* \text{CS} - \int_r s_r^* d\beta = \int_r (E \circ s_r)^* \text{CS} + d \int_r s_r^* \beta - (h^1)^* \beta + (h^0)^* \beta.$$

So that, modulo exact forms,

$$\begin{aligned} \text{CS}(s_r \boxplus (\rho_r \boxplus \gamma_{1-r})) &= \int_r (E \circ s_r)^* \text{CS} + \int_r (k_r^1)^* \text{CS} - \int_r (k_r^0)^* \text{CS} \\ &= d \iint T_2^* \text{CS} \end{aligned}$$

where T_2 is the smooth homotopy filling the second triangle, as in Figure 2.

This shows that \mathcal{I} is independent of the choices made.

It remains to show that if $g_{t,0}$ and $g_{t,1}$ are CS-equivalent then $\mathcal{I}(g_{t,0}) = \mathcal{I}(g_{t,1})$. Let $g_{t,r} : M \times S^1 \times I \rightarrow U$ for $t \in S^1$, $r \in I$ be a smooth homotopy from $g_{t,0}$ to $g_{t,1}$ such that $\text{CS}(g_{t,r})$ is exact. By definition of the map \mathcal{I} we must first choose smooth maps h^0 and h^1 which are homotopic to $i \circ *_g g_{t,0}$ and $i \circ *_g g_{t,1}$, respectively. Since $g_{t,0}$ and $g_{t,1}$ are homotopic via the smooth homotopy $g_{t,r}$ above, it follows that $i \circ *_g g_{t,0}$ and $i \circ *_g g_{t,1}$ are (continuously) homotopic via the homotopy $i \circ *_g g_{t,r}$. Since we have already shown that $\mathcal{I}(g_t)$ is independent of the chosen representative h , we may as well choose the same map h for both $g_{t,0}$ and $g_{t,1}$. Next, we must choose a smooth homotopy $k_{r,0} : M \times I \rightarrow \Omega U$ from $E \circ h$ to $*g_{t,0}$, and a smooth homotopy $k_{r,1} : M \times I \rightarrow \Omega U$ from $E \circ h$ to $*g_{t,1}$. If we choose any homotopy $k_{r,0}$

from $E \circ h$ to $*g_{t,0}$, then we may pick $k_{r,1}$ to be the path composition $k_{r,1} = k_{r,0} * (*g_{t,r})$, since we have already shown that $\mathcal{I}(g_{t,1})$ is independent of the choice for $k_{r,1}$.

Now, to show that $\mathcal{I}(g_{t,0}) = [h] + a(\eta^0)$ and $\mathcal{I}(g_{t,1}) = [h] + a(\eta^1)$ are equal, it suffices to show $\eta^0 = \eta^1 \pmod{\text{exact}}$, since the map a vanishes on exact forms (cf. Definition 3.26). But for the chosen $k_{r,1}$, we get

$$\begin{aligned} \eta^1 &= \int_r (k_{r,1})^* \text{CS} + h^* \beta = \int_r (k_{r,0})^* \text{CS} + \int_r (*g_{t,r})^* \text{CS} + h^* \beta \\ &= \eta^0 + \int_r (*g_{t,r})^* \text{CS}. \end{aligned}$$

By Lemma 4.3 we have

$$\begin{aligned} \int_r (*g_{t,r})^* \text{CS} &= \int_r g_{t,r}^* \text{CS} = \int_{r \in I} \int_{t \in S^1} g_{t,r}^* \text{Ch} \\ &= \int_{t \in S^1} \int_{r \in I} g_{t,r}^* \text{Ch} = \int_{t \in S^1} \text{CS}(g_{t,r}). \end{aligned}$$

But $\text{CS}(g_{t,r})$ is exact by assumption and d commutes with the integral over S^1 since S^1 is closed, showing that η^1 and η^0 only differ by an exact form.

This completes the proof that \mathcal{I} is well-defined. \square

Lemma 4.7. *The map $\mathcal{I} : \hat{K}^{-1}(M \times S^1) \rightarrow \hat{K}^0(M)$ is a group homomorphism.*

Proof. Recall that the group structure “+” on $\hat{K}^*(M)$ is induced on representatives by “ \boxplus ” from Definitions 3.7 and 3.21. Let g_t and f_t be representatives for elements in $\hat{K}^{-1}(M \times S^1)$, and denote with the induced based maps $*g_t, *f_t : M \rightarrow \Omega U$. We must show that

$$\mathcal{I}(g_t \boxplus f_t) = [u] + a(\eta^{l_r})$$

is CS-equivalent to

$$\mathcal{I}(g_t) + \mathcal{I}(f_t) = [s] + [t] + a(\eta^{k_r}) + a(\eta^{h_r}).$$

where u is smooth and homotopic to $i \circ (*g_t \boxplus *f_t)$ via some homotopy u_r , s is smooth and homotopic to $i \circ *g_t$ via some homotopy s_r , and t is smooth and homotopic to $i \circ *f_t$ via some homotopy t_r . Recall that

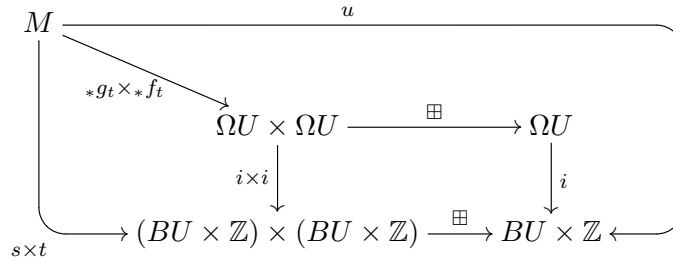
$$\begin{aligned} \eta^{l_r} &= \int_r (l_r)^* \text{CS} + u^* \beta \\ \eta^{k_r} &= \int_r (k_r)^* \text{CS} + s^* \beta \\ \eta^{h_r} &= \int_r (h_r)^* \text{CS} + t^* \beta \end{aligned}$$

where l_r, k_r, h_r are smooth and, by Remark 4.5, we may assume are homotopic to $(E \circ u_r) * (F_r \circ (*g_t \boxplus *f_t))$, $(E \circ s_r) * (F_r \circ *g_t)$, and $(E \circ t_r) * (F_r \circ *f_t)$, respectively.

We choose representatives for $a(\eta^{lr})$, $a(\eta^{kr})$ and $a(\eta^{hr}) \in \hat{K}^0(M)$ and denote these by γ_1 , κ_1 , and $\rho_1 : M \rightarrow BU \times \mathbb{Z}$, respectively. By definition of the “ a ” map (as the inverse of the CS-form obtained by a path to the identity), there are maps $\gamma_r, \kappa_r, \rho_r : M \times I \rightarrow BU \times \mathbb{Z}$ satisfying $\gamma_0 = \kappa_0 = \rho_0 = id$ and $CS(\gamma_r) = \eta^{lr}$, $CS(\kappa_r) = \eta^{kr}$, and $CS(\rho_r) = \eta^{hr}$.

The proof will be an adaptation of an argument by Upmeyer in [U].

Consider the following diagram



There is a homotopy $K : \Omega U \times \Omega U \times I \rightarrow BU \times \mathbb{Z}$ given by

$$K(r) = \begin{cases} i(F_r \boxplus F_r), & \text{if } r \in [0, 1/2] \\ H_r(i \boxplus i), & \text{if } r \in [1/2, 1] \end{cases}$$

where $F_r : \Omega U \times [0, 1/2] \rightarrow \Omega U$ is as before (suitably reparametrized for better readability), satisfying $F_0 = id$ and $F_{1/2} = E \circ i$, and

$$H : BU \times \mathbb{Z} \times [1/2, 1] \rightarrow BU \times \mathbb{Z}$$

satisfies $H_{1/2} = i \circ E$ and $H_1 = id$ so that $K(0) = i \circ \boxplus$ and $K(1) = i \boxplus i$. Note this is well-defined at $r = 1/2$ since $(A - cI_0) \boxplus (B - cI_0) = (A \boxplus B) - cI_0$ and $\exp(A \boxplus B) = \exp(A) \boxplus \exp(B)$, so that $E(P \boxplus Q) = E(P) \boxplus E(Q)$.

This implies there is a smooth homotopy between u and $s \boxplus t$, since u and $s \boxplus t$ are homotopic via the continuous homotopy

$$C(r) = u_r * (K(r) \circ (*gt \times *ft)) * (s_{1-r} \boxplus t_{1-r}).$$

For a choice of a smooth homotopy α_r from u to $s \boxplus t$, we must show that the smooth homotopy $\alpha_r \boxplus (\gamma_{1-r} \boxplus (\kappa_r \boxplus \rho_r))$, from the representative $u \boxplus (\gamma_1 \boxplus id)$ of $[u] + a(\eta^{lr})$ to the representative $(s \boxplus t) \boxplus (id \boxplus (\kappa_1 \boxplus \rho_1))$ of $[s] + [t] + a(\eta^{kr}) + a(\eta^{hr})$, has an exact CS-form. This CS-form is in fact equal to

$$\begin{aligned} CS(\alpha_r) - \eta^{lr} + \eta^{kr} + \eta^{hr} \\ = CS(\alpha_r) - \int_r l_r^* CS - u^* \beta + \int_r k_r^* CS + s^* \beta + \int_r h_r^* CS + t^* \beta \end{aligned}$$

In order to show that this form is exact, it is enough to show that the form is exact thought of as a cocycle, via the deRham Theorem.

Choose a nice cochain model with a representative for integration along the interval I . For example we may take the cubical singular model, and then for a cochain c we can represent integration along the interval by the

slant product $c \mapsto c \setminus [I]$, where $[I]$ is the fundamental chain of the interval. Notice that the deRham map $\omega \mapsto dR(\omega) = \int \omega$ commutes with the slant product and integration along the interval I , i.e., $dR(\int_I \omega) = dR(\omega) \setminus [I]$.

Now, for any choice of smooth α_r continuously homotopic to $C(r)$ (relative to the boundary smooth maps u and $s \boxplus t$) we have that the cochains $dR(\text{CS}(\alpha_r))$ and $C(r)^*(dR(\text{CS}))$ differ by a coboundary. In fact, if $\alpha_{r,s}$ is a continuous relative homotopy from α_r to $C(r)$ fixing the maps at the boundary, then integrating over $s \in I$ we have

$$\begin{aligned} \delta(\alpha_{r,s}^* dR(\text{CS}) \setminus [I]) &= \alpha_{r,1}^* dR(\text{CS}) - \alpha_{r,0}^* dR(\text{CS}) \\ &= \alpha_r^* dR(\text{CS}) - C(r)^* dR(\text{CS}) \end{aligned}$$

since $\delta dR(\text{CS}) = 0$. So, working modulo exact cochains, we can replace $\text{CS}(\alpha_r)$ by

$$\begin{aligned} C(r)^*(dR(\text{CS})) &= dR(\text{CS}(u_r)) + (K(r) \circ (*g_t \times *f_t))^*(dR(\text{CS})) \\ &\quad - dR(\text{CS}(s_r \boxplus t_r)) \end{aligned}$$

and it suffices to show that

$$\begin{aligned} X &= u_r^*(dR(\text{CS})) + (K(r) \circ (*g_t \times *f_t))^*(dR(\text{CS})) \\ &\quad - s_r^*(dR(\text{CS})) - t_r^*(dR(\text{CS})) \\ &\quad + dR\left(-\int_r l_r^* \text{CS} - u^* \beta + \int_r k_r^* \text{CS} + s^* \beta + \int_r h_r^* \text{CS} + t^* \beta\right) \end{aligned}$$

is an exact cochain, where we used that

$$\text{CS}(s_r \boxplus t_r) = \text{CS}(s_r) + \text{CS}(t_r).$$

Now, adding to X the exact cochains

$$\begin{aligned} -\delta\left((u_r^* dR(\beta)) \setminus [I]\right) &= -(i \circ (*g_t \boxplus *f_t))^* dR(\beta) + u^* dR(\beta) \\ &\quad + u_r^* dR(E^* \text{CS} - \text{Ch}) \setminus [I] \\ \delta\left((s_r^* dR(\beta)) \setminus [I]\right) &= (i \circ *g_t)^* dR(\beta) - s^* dR(\beta) - s_r^* dR(E^* \text{CS} - \text{Ch}) \setminus [I] \\ \delta\left((t_r^* dR(\beta)) \setminus [I]\right) &= (i \circ *f_t)^* dR(\beta) - t^* dR(\beta) - t_r^* dR(E^* \text{CS} - \text{Ch}) \setminus [I] \end{aligned}$$

and using the relations

$$\begin{aligned} -dR\left(\int_r l_r^* \text{CS}\right) &= -\left((E \circ u_r)^* dR(\text{CS})\right) \setminus [I] \\ &\quad - \left((F_r \circ (*g_t \boxplus *f_t))^* dR(\text{CS})\right) \setminus [I] \\ dR\left(\int_r k_r^* \text{CS}\right) &= \left((E \circ s_r)^* dR(\text{CS})\right) \setminus [I] + \left((F_r \circ *g_t)^* dR(\text{CS}) \setminus [I]\right) \\ dR\left(\int_r h_r^* \text{CS}\right) &= \left((E \circ t_r)^* dR(\text{CS})\right) \setminus [I] + \left((F_r \circ *f_t)^* dR(\text{CS}) \setminus [I]\right) \end{aligned}$$

which follow from our assumptions on l_r, k_r and h_r , we obtain, modulo exact, that

$$\begin{aligned} X &= (K(r) \circ (*g_t \times *f_t))^*(dR(\text{CS})) \\ &\quad - (i \circ (*g_t \boxplus *f_t))^*dR(\beta) + (i \circ *g_t)^*dR(\beta) + (i \circ *f_t)^*dR(\beta) \\ &\quad - \left((F_r \circ (*g_t \boxplus *f_t))^*dR(\text{CS}) \right) \setminus [I] \\ &\quad + \left((F_r \circ *g_t)^*dR(\text{CS}) \setminus [I] \right) + \left((F_r \circ *f_t)^*dR(\text{CS}) \right) \setminus [I]. \end{aligned}$$

Notice that this can be written as

$$X = (*g_t \times *f_t)^*(Y) \quad \text{for } Y \in C^{\text{odd}}(\Omega U \times \Omega U)$$

defined to be

$$\begin{aligned} Y &= K(r)^*(dR(\text{CS})) - (i \circ \boxplus)^*dR(\beta) + (i \circ pr_1)^*dR(\beta) \\ &\quad + (i \circ pr_2)^*dR(\beta) - \left((F_r \circ \boxplus)^*dR(\text{CS}) \right) \setminus [I] \\ &\quad + \left((F_r \circ pr_1)^*dR(\text{CS}) \right) \setminus [I] + \left((F_r \circ pr_2)^*dR(\text{CS}) \right) \setminus [I], \end{aligned}$$

where $pr_1, pr_2 : \Omega U \times \Omega U \rightarrow \Omega U$ are the projections onto the first and second factors, respectively. To check that X is exact it suffices to show that Y is exact, since the pullback $(*g_t \times *f_t)^*$ preserves exactness. Now, since $H^{\text{odd}}(\Omega U \times \Omega U) = 0$, it suffices to show that Y is closed. We check this using the facts that $\delta dR(\beta) = E^*dR(\text{CS}) - dR(\text{Ch})$, F_r is a homotopy from $F_0 = E \circ i$ to $F_1 = id$, CS is closed, and $K(r)$ is as above, so that

$$\begin{aligned} \delta \left(K(r)^*dR(\text{CS}) \right) &= \delta \left((K(r)^*dR(\text{Ch})) \setminus [I] \right) \\ &= K(1)^*dR(\text{Ch}) - K(0)^*dR(\text{Ch}) \\ &= (i \times i)^* \circ \boxplus^*(dR(\text{Ch})) - \boxplus^* \circ i^*(dR(\text{Ch})). \end{aligned}$$

With this we obtain

$$\begin{aligned} \delta Y &= (i \times i)^* \circ \boxplus^*(dR(\text{Ch})) - \boxplus^* \circ i^*(dR(\text{Ch})) \\ &\quad - (i \circ \boxplus)^*(E^*dR(\text{CS}) - dR(\text{Ch})) + (i \circ pr_1)^*(E^*dR(\text{CS}) - dR(\text{Ch})) \\ &\quad + (i \circ pr_2)^*(E^*dR(\text{CS}) - dR(\text{Ch})) - (\boxplus - E \circ i \circ \boxplus)^*dR(\text{CS}) \\ &\quad + (pr_1 - E \circ i \circ pr_1)^*dR(\text{CS}) + (pr_2 - E \circ i \circ pr_2)^*dR(\text{CS}) \\ &= (\boxplus \circ (i \times i) - i \circ pr_1 - i \circ pr_2)^*dR(\text{Ch}) - (\boxplus - pr_1 - pr_2)^*dR(\text{CS}) \\ &= (i \times i)^* \circ (\boxplus - pr_1 - pr_2)^*dR(\text{Ch}) - (\boxplus - pr_1 - pr_2)^*dR(\text{CS}) \\ &= 0, \end{aligned}$$

where in the last equality we used that, in general, $\text{Ch}(x \boxplus y) = \text{Ch}(x) + \text{Ch}(y)$ and $\text{CS}(x_t \boxplus y_t) = \text{CS}(x_t) + \text{CS}(y_t)$.

This completes the proof that \mathcal{I} is a group homomorphism. \square

Lemma 4.8. *The map \mathcal{I} satisfies $\mathcal{I} \circ (id \times r)^* = -\mathcal{I}$ where $r : S^1 \rightarrow S^1$ is given by $r(z) = \bar{z} = z^{-1}$.*

Proof. Note that for a map $g_t : M \times S^1 \rightarrow U$ we have that $(id \times r)^*g_t = \bar{g}_t$, where \bar{g}_t is the reversed loop. The desired condition holds if and only if $\mathcal{I}(g_t \boxplus \bar{g}_t) = 0$ since

$$\mathcal{I}(g_t) + \mathcal{I}(\bar{g}_t) = \mathcal{I}(g_t \boxplus \bar{g}_t).$$

Let $\phi : \Omega U \rightarrow \Omega U$ be given by $\phi(k_t) = k_t \boxplus \bar{k}_t$ and let $f_r : \Omega U \times I \rightarrow \Omega U$ be any (continuous) homotopy satisfying

$$f_0(k_t) = 1, \quad \text{and} \quad f_1(k_t) = \phi(k_t).$$

For example we may choose the composition of homotopies whose value on some k_t is given by

$$\begin{aligned} k_t \boxplus \bar{k}_t &= (k_t \boxplus 1) \cdot (1 \boxplus \bar{k}_t) \sim (k_t \boxplus 1) \cdot (\bar{k}_t \boxplus 1) \\ &= (k_t \cdot \bar{k}_t \boxplus 1) \sim (k_t \circ \bar{k}_t \boxplus 1) \sim (1 \boxplus 1). \end{aligned}$$

For later use, we note that $\phi^*(CS) = 0$, since for any plot $k_t : M \rightarrow \Omega U$, we have

$$\begin{aligned} k_t^*(\phi^*(CS)) &= CS(\phi(k_t)) = CS(k_t \boxplus \bar{k}_t) \\ &= CS(k_t) + CS(\bar{k}_t) = \int_{S^1} \text{Ch}(k_t) + \int_{S^1} \text{Ch}(\bar{k}_t) = 0. \end{aligned}$$

Since ϕ is homotopic via f_r to the constant map, $i(*g_t \boxplus *\bar{g}_t) = i(\phi(*g_t)) : M \rightarrow BU \times \mathbb{Z}$ is also homotopic to constant map. So, we may choose the map h as in the definition of \mathcal{I} to be the constant map, which is smooth. Let $k_r : M \times I \rightarrow \Omega U$ be given by $k_r = f_r \circ (*g_t \times id) : M \times I \xrightarrow{*g_t \times id} \Omega U \times I \xrightarrow{f_r} \Omega U$, so that k_r is a homotopy from $k_0 = 1$ to $k_1 = \phi(*g_t)$. This map may not be smooth, but we can choose relative homotopy to a map $k_r^s : M \times I \times I \rightarrow \Omega U$ which extends the map $k_r^0 = k_r$, fixing the smooth maps at the endpoints $k_0^0 = k_0^s = 1$, and $k_1^0 = k_1^s = \phi(*g_t)$, so that the map k_r^1 is smooth.

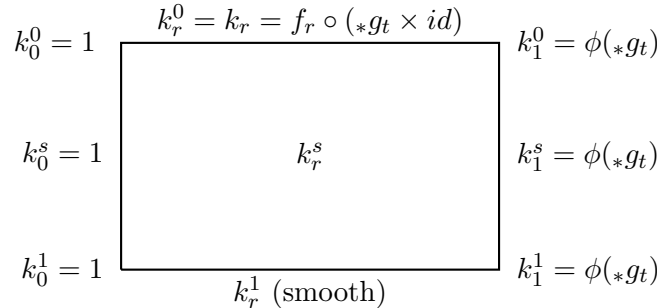


FIGURE 3. The homotopy $k_r^s : M \times I \times I \rightarrow \Omega U$

Then we let

$$(4.5) \quad \eta = \int_r (k_r^1)^* \text{CS} + h^* \beta = \int_r (k_r^1)^* \text{CS} \quad \in \Omega^{\text{odd}}(M)$$

and by definition we have $\mathcal{I}(g_t \boxplus \bar{g}_t) = [h] + a(\eta) = a(\eta)$, where the last equality follows since h is constant. In order to show that $\mathcal{I}(g_t \boxplus \bar{g}_t) = 0$, it now suffices to show that η is an exact form, since a vanished on exact forms. (Note that η is closed, since $d\eta = (k_1^1)^* \text{CS} - (k_0^1)^* \text{CS} = \text{CS}(\phi(*g_t)) - \text{CS}(\text{const}) = \text{CS}(\phi(*g_t)) = 0$.) We will show that η is indeed exact by showing that the deRham form η , thought of as a cocycle via the deRham Theorem, is an exact cocycle.

As in the previous lemma, we choose a nice cochain model with a representative for integration along the interval I , so that for a cochain c we can represent integration along the interval by the slant product $c \mapsto c \setminus [I]$, where $[I]$ is the fundamental chain of the interval. Recall that the deRham map $\omega \mapsto dR(\omega) = \int \omega$ commutes with the slant product and integration along the interval I , i.e., $dR(\int_I \omega) = dR(\omega) \setminus [I]$.

Consider the cochain

$$X = ((k_r^s)^* dR(\text{CS})) \setminus [I \times I]$$

where $\setminus [I \times I] = \setminus [I] \setminus [I]$. Then

$$\begin{aligned} \delta X &= ((k_r^1)^* dR(\text{CS})) \setminus [I] - ((k_r^0)^* dR(\text{CS})) \setminus [I] \\ &\quad - ((k_1^s)^* dR(\text{CS})) \setminus [I] - ((k_0^s)^* dR(\text{CS})) \setminus [I] \end{aligned}$$

Since $k_0^s = 1$, the last term vanishes. Also, the second to last term vanishes since $k_1^s = \phi(*g_t)$ and $\text{CS}(\phi(*g_t)) = 0$, as shown above. This shows that

$$\begin{aligned} \delta X &= ((k_r^1)^* dR(\text{CS})) \setminus [I] - ((k_r^0)^* dR(\text{CS})) \setminus [I] \\ &= dR\left(\int_r (k_r^1)^* \text{CS}\right) - ((f_r \circ (*g_t \times id))^* dR(\text{CS})) \setminus [I], \end{aligned}$$

since k_r^1 is smooth, and for cochains in the image of the deRham map, the slant product equals the integration along I . By Equation (4.5), it thus suffices to show that the last term is exact. But

$$\begin{aligned} ((f_r \circ (*g_t \times id))^* dR(\text{CS})) \setminus [I] &= ((*g_t \times id)^* \circ f_r^* dR(\text{CS})) \setminus [I] \\ &= (*g_t)^* (f_r^* dR(\text{CS})) \setminus [I] \end{aligned}$$

and $f_r^* dR(\text{CS}) \setminus [I]$ is a closed odd degree cochain on ΩU , since

$$\begin{aligned} \delta (f_r^* dR(\text{CS}) \setminus [I]) &= f_1^* dR(\text{CS}) - f_0^* dR(\text{CS}) \\ &= dR(\phi^* \text{CS}) - dR(\text{const}^* \text{CS}) = 0. \end{aligned}$$

Since $H^{\text{odd}}(\Omega U) = 0$, the term $f_r^* dR(\text{CS}) \setminus [I]$ must also be exact, showing that $(*g_t)^* (f_r^* dR(\text{CS}) \setminus [I])$ is exact as well, and so we are done. \square

Lemma 4.9. *The integration map $\mathcal{I} : \hat{K}^{-1}(M \times S^1) \rightarrow \hat{K}^0(M)$ satisfies $\mathcal{I} \circ p^* = 0$, where $p : M \times S^1 \rightarrow M$ is projection.*

Proof. Let $g : M \rightarrow U$ be a representative for $[g] \in \hat{K}^{-1}(M)$. Then $g_t : M \times S^1 \rightarrow U$ defined by $g_t = p^*g$ satisfies $g_t = g$ for all $t \in S^1$. Recall from page 556, that $\gamma_g : M \times I \rightarrow U$ is a path with $\gamma_g(0) = g \oplus g^{-1}$ and $\gamma_g(1) = id$. It follows that the induced based map $*g_t : M \rightarrow \Omega U$ defined by

$$*g_t = \gamma_g(t) * (g_t \oplus g_0^{-1}) * \overline{\gamma_g(t)}$$

is the path from id to $g \oplus g^{-1}$ and back to id (cf. Lemma 4.3). We must show $\mathcal{I}(g_t) = 0$.

Denote by $\gamma_g^s(t) = \gamma_g(1 - s(1 - t))$ the path from $\gamma_g(1 - s)$ to $\gamma_g(1) = id$, and consider the homotopy $h_{t,s} : M \times I \rightarrow \Omega U$ which contracts $*g_t$ to the identity, which is defined by concatenating the following paths,

$$h_{t,s} = \gamma_g^s * \text{const}_{\gamma_g(1-s)} * \overline{\gamma_g^s}, \quad \text{for } s \in I.$$

Then $h_{t,1} = \gamma_g * (g \oplus g^{-1}) * \overline{\gamma_g} = *g_t$, and $h_{t,0} : M \rightarrow \Omega U$ is the constant map to the identity in U .

Since $*g_t$ is homotopic to a constant, we can choose (in the definition of \mathcal{I}) $h : M \rightarrow BU \times \mathbb{Z}$ to be the constant map. Then $E \circ h$ is also constant and we can let $k_s = h_{t,s}$ be the chosen homotopy from $k_1 = *g_t$ to $k_0 = E \circ h$, the constant map. Then

$$\begin{aligned} \eta &= \int_s (k_s)^* \text{CS} + h^* \beta = \int_s (h_{t,s})^* \text{CS} \\ &= \text{CS}(\gamma_g^s) + \text{CS}(\text{const}) + \text{CS}(\overline{\gamma_g^s}) = \text{CS}(\gamma_g^s) - \text{CS}(\gamma_g^s) = 0. \end{aligned}$$

So that

$$\mathcal{I}(g_t) = [h] + a(\eta) = 0 \quad \square$$

Lemma 4.10. *The map \mathcal{I} makes the following diagram commute*

$$\begin{array}{ccc} \hat{K}^{-1}(M \times S^1) & \xrightarrow{\text{Ch}} & \Omega^{\text{odd}}(M \times S^1) \\ \downarrow \mathcal{I} & & \downarrow \int_{S^1} \\ \hat{K}^0(M) & \xrightarrow{\text{Ch}} & \Omega^{\text{even}}(M). \end{array}$$

Proof. This follows by a direct computation. As in the definition of \mathcal{I} , choose h homotopic to $i \circ *g_t$ and choose a homotopy k_r such that $k_0 = E \circ h$ and $k_1 = *g_t$. Define $\eta = \int_r k_r^* \text{CS} + h^* \beta$, then by Lemma 4.1 we have

$$\begin{aligned} d\eta &= k_1^* \text{CS} - k_0^* \text{CS} + h^*(E^* \text{CS} - \text{Ch}) \\ &= *g_t^* \text{CS} - (E \circ h)^* \text{CS} + h^*(E^* \text{CS} - \text{Ch}) = g_t^* \text{CS} - h^* \text{Ch} \end{aligned}$$

so that

$$\begin{aligned} \text{Ch}(\mathcal{I}(g_t)) &= \text{Ch}([h]) + \text{Ch}(a(\eta)) = \text{Ch}([h]) + d\eta \\ &= \text{Ch}([h]) + g_t^* \text{CS} - h^* \text{Ch} = g_t^* \text{CS} = \int_t g_t^* \text{Ch}. \quad \square \end{aligned}$$

Lemma 4.11. *The following diagram commutes.*

$$\begin{CD} \hat{K}^{-1}(M \times S^1) @>I>> K^{-1}(M \times S^1) \\ @VV\mathcal{I}V @VV\int_{S^1}V \\ \hat{K}^0(M) @>I>> K^0(M) \end{CD}$$

where the right vertical map is the S^1 -integration map in K -theory.

Proof. As in the definition of \mathcal{I} , choose h homotopic to $i \circ *_g t$ and choose a homotopy k_r such that $k_0 = E \circ h$ and $k_1 = *_g t$. Define $\eta = \int_r k_r^* CS + h^* \beta$. Since $\mathcal{I}(g_t) = [h] + a(\eta)$, where h is homotopic to $i \circ *_g t$ and $\text{Im}(a) = \text{Ker}(I)$, it suffices to show for any representative g_t that

$$I(i(*_g t)) = \int_{S^1} I(g_t).$$

Consider the following diagram, where the bottom row defines the S^1 -integration in K -theory as defined in Definition 2.7:

$$\begin{CD} [M, LU] @>\pi>> [M, \Omega U] @>i_*>> [M, BU \times \mathbb{Z}] \\ @VV=V @VV=V @VV=V \\ K^{-1}(M \times S^1) @>\text{pr}>> \text{Ker}(j^*) @>(q^*)^{-1}>> \tilde{K}^{-1}(\Sigma M_+) = K^{-1}(M \wedge S^1) @>\sigma^{-1}>> K^0(M) \end{CD}$$

The map π is defined as follows. The adjoint of a map $M \times S^1 \rightarrow U$ is a map $M \rightarrow LU$. Using the transport to the identity map, $g_t \mapsto g_0^{-1} g_t$, we have $LU \cong U \times \Omega U$, and we let π denote projection onto the ΩU factor. Note that since $g_0^{-1} g_t$ is homotopic to $*_g t$, we have that the induced map on homotopy classes $\pi : [M, LU] \rightarrow [M, \Omega U]$ is the same as the map $g_t \mapsto *_g t$. This shows that on homotopy classes we have

$$(i_* \circ \pi)(g_t) = i(*_g t),$$

so the top row of the diagram represents $I(i(*_g t))$.

The left square and middle triangle commute. So, it suffices to show the rightmost quadrilateral commutes. But i_* is an isomorphism with inverse induced by the Bott periodicity map $E : BU \times \mathbb{Z} \rightarrow \Omega U$, which induces the suspension isomorphism σ . \square

Lemma 4.12. *The following diagram commutes for all manifolds M :*

$$\begin{CD} \Omega^{\text{even}}(M \times S^1) / \text{Im}(d) @>a>> \hat{K}^{-1}(M \times S^1) \\ @VV\int_{S^1}V @VV\mathcal{I}V \\ \Omega^{\text{odd}}(M) / \text{Im}(d) @>a>> \hat{K}^0(M). \end{CD}$$

Proof. By definition of the maps $a : \Omega^{*+1}(M)/\text{Im}(d) \rightarrow \hat{K}^*(M)$ we can write the diagram of interest as

$$\begin{CD} \Omega^{\text{even}}(M \times S^1)/\text{Im}(d) @>\text{pr}>> \Omega^{\text{even}}(M \times S^1)/\text{Im}(\text{Ch}) @>\text{CS}^{-1}>> \text{Ker}(I) \subset \hat{K}^{-1}(M \times S^1) \\ @Vf_{S^1}VV @Vf_{S^1}VV @VV\mathcal{I}V \\ \Omega^{\text{odd}}(M)/\text{Im}(d) @>\text{pr}>> \Omega^{\text{odd}}(M)/\text{Im}(\text{Ch}) @>\text{CS}^{-1}>> \text{Ker}(I) \subset \hat{K}^0(M) \end{CD}$$

where pr is the projection, which is well-defined since $\text{Im}(d) \subset \text{Im}(\text{Ch})$ by Theorem 3.17(1), and the middle vertical map is well-defined since

$$\text{Im} \left(\int_{S^1} \circ \text{Ch} \right) \subset \text{Im}(\text{Ch}),$$

by Lemma 4.15 (which will be proven independently from this lemma) and Theorem 3.17(1). The left square clearly commutes, so it suffices to show that the right square commutes. Note, that the maps CS^{-1} are isomorphisms onto $\text{Ker}(I)$.

For given $g_t \in \text{Ker}(I) \subset \hat{K}^{-1}(M \times S^1)$ there is a path $g_{t,s} : (M \times S^1) \times I \rightarrow U$ such that $g_{t,0} = id$ is constant for all $t \in S^1$, and $g_{t,1} = g_t : M \times S^1 \rightarrow U$. Note that the parameter for this path $g_{t,s}$ is $s \in I$, so that

$$\text{CS}(g_{t,s}) = \int_{s \in I} \text{Ch}(g_{t,s}).$$

It suffices to show that

$$\text{CS}(\mathcal{I}(g_t)) = \int_{t \in S^1} \text{CS}(g_{t,s}) \quad \text{mod exact},$$

which is the same as showing

$$\text{CS}([h]) + \eta = \int_{t \in S^1} \int_{s \in I} \text{Ch}(g_{t,s}) \quad \text{mod exact}.$$

for some choices (as in the definition of \mathcal{I}) of a smooth map $h : M \rightarrow BU \times \mathbb{Z}$ that is homotopic to $i \circ *g_t$, and a smooth map $k_s : M \times I \rightarrow \Omega U$ such that $k_0 = E \circ h$ and $k_1 = *g_t$, where, as always

$$\eta = \int_s k_s^* \text{CS} + h^* \beta$$

Since the map g_t is nullhomotopic via $g_{t,s}$ from above, the map $i \circ *g_t$ is also nullhomotopic via $i \circ *g_{t,s}$, so we may as well choose h to be the constant map, which is smooth. Then $E \circ h$ is also the constant map (without loss of generality, to the identity). So, we may choose the homotopy $k_s = *g_{t,s}$ and then $k_0 = E \circ h$ is constant, and $k_1 = *g_t$. Then $\text{CS}([h]) = 0$ and using

Lemma 4.3

$$\begin{aligned} \text{CS}([h]) + \eta = \eta &= \int_s k_s^* \text{CS} = \int_s (*g_{t,s})^* \text{CS} \\ &= \int_s \text{CS}(*g_{t,s}) = \int_s \text{CS}(g_{t,s}) = \int_{t \in S^1} \int_{s \in I} \text{Ch}(g_{t,s}), \end{aligned}$$

so we are done. \square

4.2. Even to odd. We now define an integration map

$$\mathcal{I} : \hat{K}^0(M \times S^1) \rightarrow \hat{K}^{-1}(M)$$

and check the axioms for it.

In the spirit of the last subsection, there is at least morally a “wrong way” map

$$H : \text{Map}(M, U) \rightarrow \text{Map}(M \times S^1, BU \times \mathbb{Z})$$

such that the following diagram

$$\begin{array}{ccc} \text{Map}(M \times S^1, BU \times \mathbb{Z}) & \xrightarrow{\text{Ch}} & \Omega^{\text{even}}(M \times S^1) \\ H \uparrow & & \downarrow \int_{S^1} \\ \text{Map}(M, U) & \xrightarrow{\text{Ch}} & \Omega^{\text{odd}}(M) \end{array}$$

commutes for all compact manifolds with corners. Geometrically, for $g : M \rightarrow U(n) = U(\mathbb{C}^n)$, consider the \mathbb{C}^n -bundle with connection ∇ over $M \times S^1$ constructed by gluing the \mathbb{C}^n -bundle over $M \times I$ with connection $tg^{-1}dg$ for $t \in [0, 1]$ along the endpoints 0 and $1 \in [0, 1]$. Then the diagram commutes in the sense that (*cf.* [TWZ2, Equation (2.4)])

$$(4.6) \quad \int_{S^1} \text{Ch}(\nabla) = \text{CS}(tg^{-1}dg) = \text{Ch}(g).$$

We use this idea as a guide in constructing an integration map in the correct direction.

Recall from Definition 3.14 that a map $P_t : M \times S^1 \rightarrow BU \times \mathbb{Z}$ induces a projection operator P_t on a trivial \mathbb{C}_p^q bundle over $M \times S^1$ (for some integers p, q), and an induced sub-bundle $E_{P_t} \subset (M \times S^1) \times \mathbb{C}_p^q$, with connection $\nabla_{P_t} = P_t dP_t$, where E_{P_t} is well-defined up to adding a trivial bundle with trivial connection. To simplify the notation in this subsection, we will use the notation of denoting the Chern character $\text{Ch}(P_t)$ by

$$\text{Ch}(\nabla_{P_t}) = \text{Ch}(P_t).$$

There is a map

$$h_* : \text{Map}(M \times S^1, BU \times \mathbb{Z}) \rightarrow \text{Map}(M, U)$$

given by

$$h_*(P_t) = \text{hol}_{S^1}(\nabla_{P_t}) \oplus id$$

where $\text{hol}_{S^1}(\nabla_{P_t}) \in \text{End}(E_{P_t}|_{M \times \{0\}})$ is the holonomy of ∇_{P_t} along the S^1 factor of $M \times S^1$, regarded as an endomorphism of the fiber of

$$E_{P_t}|_{M \times \{0\}} \subset (M \times \{0\}) \times \mathbb{C}_p^q$$

over $M \times \{0\} \subset M \times S^1$, and id is the identity endomorphism on the orthogonal complement of $E_{P_t}|_{M \times \{0\}}$ in $(M \times \{0\}) \times \mathbb{C}_p^q$. Note that

$$h_*(P_t) : M \rightarrow U$$

is unchanged under adding a trivial bundle with trivial connection, since the holonomy of the trivial connection is the identity.

The following diagram commutes in degree one, but does *not* commute in general for higher degrees.

$$(4.7) \quad \begin{array}{ccc} \text{Map}(M \times S^1, BU \times \mathbb{Z}) & \xrightarrow{\text{Ch}} & \Omega^{\text{even}}(M \times S^1) \\ \downarrow h_* & & \downarrow \int_{S^1} \\ \text{Map}(M, U) & \xrightarrow{\text{Ch}} & \Omega^{\text{odd}}(M). \end{array}$$

In fact, here is an explicit example showing the diagram does not commute in higher degrees, compare [FL]. We show below (Lemma 4.15) that the error in the diagram is always given by an exact form on M .

Example 4.13. By the Narasimhan–Ramanan Theorem, it suffices to produce an example of a bundle with connection for which the diagram does not commute. Let $M = S^1 \times S^1 \times S^1$ and consider the trivial line bundle over $M \times S^1$ with connection $\nabla = i(A + Bdt)$, using coordinates $(p, q, s, t) \in M \times S^1$. We verify that the diagram does not commute in degree three if $A = f(p)dq$ and $B = g(s)$ where f and g are such that $\frac{\partial f}{\partial p} \frac{\partial g}{\partial s}$ is not identically zero. For example, we may let $f(p) = \cos(p)$ and $g(s) = \sin(s)$.

First, the degree three component of $\text{Ch}(\text{hol}_{S^1}(\nabla))$ is zero for line bundles since if $g = \text{hol}_{S^1}(\nabla) = \exp(2\pi i \int Bdt)$, then $(g^{-1}dg)^3 = 0$. On the other hand, $-R = -\nabla^2 = d_M A + d_M Bdt$, so that $R^2 = 2d_M A \wedge d_M Bdt$, and in degree three of $\int_{S^1} \text{Ch}(\nabla)$ we have

$$\int_{S^1} R^2 = \int_{S^1} 2d_M A \wedge d_M Bdt = 2 \frac{\partial f}{\partial p} \frac{\partial g}{\partial s} dpdqds.$$

For example, $f(p) = \cos(p)$ and $g(s) = \sin(s)$, then in degree three we have $\int_{S^1} \text{Ch}(\nabla) = -2 \sin(p) \cos(s) dpdqds$. Note this is exact on M , which is the case in general, by Lemma 4.15 below.

Now, let (E, ∇) be the bundle with connection induced by

$$P_t \in \text{Map}(M \times S^1, BU \times \mathbb{Z}).$$

Let ∇_t be the connection on the slice of E_t over $M \times \{t\}$ and let $g_t \in U(\mathbb{C}_p^q)$ be the parallel transport from E_0 to E_t . Then $g_0 = id$ and $g_1 = \text{hol}_{S^1}(\nabla)$. Note that $\nabla_0 = \nabla_1$. We can pullback the bundle $E \rightarrow M \times S^1$ along the

projection $M \times I \rightarrow M \times S^1$ via the map $I \mapsto S^1$ which identifies the endpoints of I . By abuse of notation, we also denote by (E, ∇) the pullback bundle $E \rightarrow M \times I$ from $E \rightarrow M \times S^1$. Then we have,

$$\int_{S^1} \text{Ch}(\nabla) = \int_I \text{Ch}(\nabla).$$

The latter term can be expressed equivalently using the following gauge transformation induced by parallel transport. There is a bundle isomorphism $g_t : E_0 \times I \rightarrow E$ given by $(v, t) \mapsto g_t(v)$, so the bundle $E_0 \times I$ over $M \times I$, with connection $g_t^* \nabla$, is isomorphic to the bundle $E \rightarrow M \times I$ with connection ∇ . Therefore

$$\int_{S^1} \text{Ch}(\nabla) = \int_I \text{Ch}(\nabla) = \int_I \text{Ch}(g_t^* \nabla).$$

Finally, since the connection $g_t^* \nabla$ vanishes on $\partial/\partial t$, we can regard this as a path of connections on the bundle $E_0 \rightarrow M$, and so

$$(4.8) \quad \int_{S^1} \text{Ch}(\nabla) = \int_I \text{Ch}(\nabla) = \int_I \text{Ch}(g_t^* \nabla) = \text{CS}(g_t^* \nabla).$$

To see that diagram (4.7) commutes up to an exact form, it now suffices to show that $\text{CS}(g_t^* \nabla_t)$ and $\text{Ch}(g_1)$ differ by an exact form that depends naturally on the map $P_t : M \times S^1 \rightarrow BU \times \mathbb{Z}$ that determines ∇ . This is the form η we now define.

Definition 4.14. For a map $P_t : M \times S^1 \rightarrow BU \times \mathbb{Z}$, let ∇ be the induced connection on the sub-bundle $E \subset (M \times S^1) \times \mathbb{C}_p^q$, and let ∇^\perp be the induced connection on the complementary bundle E^\perp . Define $\eta \in \Omega^{\text{even}}(M)$ by

$$\begin{aligned} \eta &= \eta_{P_t} \\ &= \int \int_{(r,t) \in I \times I} \text{Ch} \left((1-r)\nabla_0 + r g_t^* \nabla_t \right) \\ &\quad + \int \int_{(r,t) \in I \times I} \text{Ch} \left((1-r) \left(t\nabla_0 \oplus t\nabla_0^\perp \right) + r(g_1 \oplus id)^* \left(t\nabla_0 \oplus t\nabla_0^\perp \right) \right). \end{aligned}$$

Note, that η is given by integrating the Chern forms of two two-parameter families of connections over a square; see Figure 4. Note further, that since η_{P_t} is determined by the connection $\nabla = P_t dP_t$ defined by the projection operator P_t , it follows that η_{P_t} is natural in the sense that for a map $f : N \rightarrow M$, we have that $f^*(\eta_{P_t}) = \eta_{P_t \circ (f \times id)} \in \Omega^{\text{even}}(N)$.

The definition of the η form and some of its important properties are illuminated by the following lemma and its proof.

Lemma 4.15. For a map $P_t : M \times S^1 \rightarrow BU \times \mathbb{Z}$ we have

$$d(\eta_{P_t}) = \int_{S^1} \text{Ch}(P_t) - \text{Ch}(h_*(P_t)).$$

This shows that h_* makes the diagram in (4.7) commute modulo exact forms.

Proof. Let ∇ be the connection induced by P_t . The two terms in η_{P_t} are induced by squares of connections as in Figure 4. We obtain $d(\eta_{P_t})$ from the sum of the boundaries of the two squares.

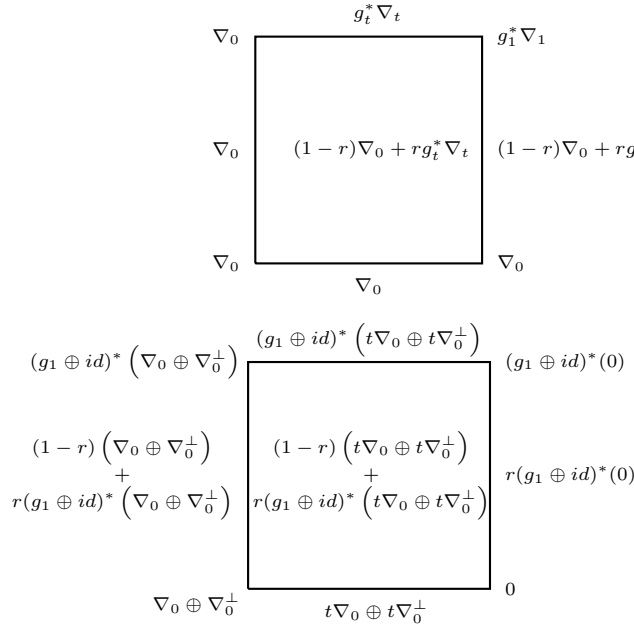


FIGURE 4. The two parameter squares of connections defining η .

For the first term we have

$$d \left(\int \int_{(r,t) \in I \times I} \text{Ch}((1-r)\nabla_0 + rg_t^*\nabla_t) \right) = \text{CS}(g_t^*\nabla_t) - \text{CS}((1-r)\nabla_0 + rg_1^*\nabla_1)$$

since the other two boundary terms are constant paths of connections, so the Chern–Simons form vanishes. By Equation (4.8) above we have

$$\text{CS}(g_t^*\nabla_t) = \int_I \text{Ch}(g_t^*\nabla_t) = \int_{S^1} \text{Ch}(\nabla).$$

Calculating the boundary for the second square, the top and bottom edges $(g_1 \oplus id)^*(t\nabla_0 \oplus t\nabla_0^\perp)$ and $t\nabla_0 \oplus t\nabla_0^\perp$ are gauge equivalent, so that they have equal Chern–Simons forms, which cancel, since they appear with opposite

signs. Therefore,

$$\begin{aligned} & d \left(\int \int_{(r,t) \in I \times I} \text{Ch} \left((1-r) \left(t \nabla_0 \oplus t \nabla_0^\perp \right) + r(g_1 \oplus id)^* \left(t \nabla_0 \oplus t \nabla_0^\perp \right) \right) \right) \\ &= \text{CS} \left((1-r) \left(\nabla_0 \oplus \nabla_0^\perp \right) + r(g_1 \oplus id)^* \left(\nabla_0 \oplus \nabla_0^\perp \right) \right) - \text{CS} \left(r(g_1 \oplus id)^*(0) \right). \end{aligned}$$

Note, that the last term does not vanish, since $g_1^*(0) = g_1^{-1}dg_1$, but it is

$$\begin{aligned} \text{CS}(r(g_1 \oplus id)^*(0)) &= \text{CS}(r(g_1^{-1}dg_1 \oplus 0)) \\ &= \text{CS}(rg_1^{-1}dg_1) \stackrel{(4.6)}{=} \text{Ch}(g_1) = \text{Ch}(h_*(P_t)). \end{aligned}$$

The remaining two CS terms cancel, because

$$(1-r) \left(\nabla_0 \oplus \nabla_0^\perp \right) + r(g_1 \oplus id)^* \left(\nabla_0 \oplus \nabla_0^\perp \right) = \left((1-r)\nabla_0 + rg_1^*\nabla_0 \right) \oplus \nabla_0^\perp,$$

and since the right summand ∇_0^\perp is time independent, it does not contribute to the CS form, so that using $\nabla_0 = \nabla_1$, we have that

$$\text{CS} \left((1-r) \left(\nabla_0 \oplus \nabla_0^\perp \right) + r(g_1 \oplus id)^* \left(\nabla_0 \oplus \nabla_0^\perp \right) \right) = \text{CS} \left((1-r)\nabla_0 + rg_1^*\nabla_1 \right).$$

This shows that $d\eta = \int_{S^1} \text{Ch}(\nabla) - \text{Ch}(h_*(P_t))$, which is the claim of the lemma. \square

We define the S^1 -integration map \mathcal{I} using this error of making diagram (4.7) commute, cf. Definition 4.4.

Definition 4.16. We define $\mathcal{I} : \hat{K}^0(M \times S^1) \rightarrow \hat{K}^1(M)$ for a representative $P_t : M \times S^1 \rightarrow BU \times \mathbb{Z}$ by setting

$$\mathcal{I}(P_t) = [h_*(P_t)] + a(\eta_{P_t})$$

where $[h_*(P_t)] \in \hat{K}^{-1}(M)$ is the equivalence class of $h_*(P_t)$.

In the following Lemmas 4.17-4.23, we will show that \mathcal{I} is well-defined, and satisfies the axioms for the integration map from Definition 2.8.

Lemma 4.17. *The map $\mathcal{I} : \hat{K}^0(M \times S^1) \rightarrow \hat{K}^{-1}(M)$ is well-defined.*

Proof. Let $P_{t,0}, P_{t,1} : M \times S^1 \rightarrow BU \times \mathbb{Z}$ be given and CS-equivalent, so there is a path $P_{t,s} : M \times S^1 \times I \rightarrow BU \times \mathbb{Z}$ such that

$$\text{CS}(P_{t,s}) = \int_{s \in I} \text{Ch}(P_{t,s}) \in \Omega^{\text{odd}}(M \times S^1)$$

is exact. Note this assumption implies that

$$\int_{t \in S^1} \int_{s \in I} \text{Ch}(P_{t,s}) = \int_{s \in I} \int_{t \in S^1} \text{Ch}(P_{t,s})$$

is also exact.

We show that

$$[h_*(P_{t,1})] - [h_*(P_{t,0})] = a(\eta_{P_{t,0}}) - a(\eta_{P_{t,1}}).$$

Since $h_*(P_{t,0})$ and $h_*(P_{t,1})$ are homotopic via $s \mapsto h_*(P_{t,s})$, the class

$$[h_*(P_{t,1})] - [h_*(P_{t,0})]$$

is in the kernel of I . So it suffices to show that the path $s \mapsto h_*(P_{t,s})$ satisfies

$$\text{CS}(h_*(P_{t,s})) = \eta_{P_{t,1}} - \eta_{P_{t,0}} \quad \text{mod exact forms.}$$

and that

$$\text{CS}([h_*(P_{t,1})] - [h_*(P_{t,0})]) = \text{CS}(h_*(P_{t,s})) \quad \text{mod Im(Ch).}$$

The idea for showing the first equality is to consider the data in Figure 4 varying smoothly with parameter $s \in [0, 1]$, and construct a differential form whose exterior derivative is the difference $\text{CS}(h_*(P_{t,s})) + \eta_{P_{t,1}} - \eta_{P_{t,0}}$. Let

$$\omega = \int_{s \in I} \eta_{P_{t,s}}.$$

Then, with $d\eta_{P_{t,s}} = \int_{t \in S^1} \text{Ch}(P_{t,s}) - \text{Ch}(h_*(P_{t,s}))$, we have

$$\begin{aligned} d\omega &= \eta_{P_{t,1}} - \eta_{P_{t,0}} - \int_{s \in I} d\eta_{P_{t,s}} \\ &= \eta_{P_{t,1}} - \eta_{P_{t,0}} - \int_{s \in I} \left(\int_{t \in S^1} \text{Ch}(P_{t,s}) - \text{Ch}(h_*(P_{t,s})) \right). \end{aligned}$$

But $\int_{t \in S^1} \text{Ch}(h_*(P_{t,s})) = \text{CS}(h_*(P_{t,s}))$ and $\int_{s \in I} \int_{t \in S^1} \text{Ch}(P_{t,s})$ is exact by assumption.

It remains to show $\text{CS}([h_*(P_{t,1})] - [h_*(P_{t,0})]) = \text{CS}(h_*(P_{t,s})) \text{ mod Im(Ch)}$. There is a path γ_s from $h_*(P_{t,0}) \oplus h_*(P_{t,0})^{-1}$ to 0 such that $\text{CS}(\gamma_s) = 0$ (by the proof of the existence of inverses). Composing this with the path $h_*(P_{t,s}) \oplus id$, from $h_*(P_{t,1}) \oplus h_*(P_{t,0})^{-1}$ to $h_*(P_{t,0}) \oplus h_*(P_{t,0})^{-1}$, we obtain a path $(h_*(P_{t,s}) \oplus id) * \gamma_s$ from $h_*(P_{t,1}) - h_*(P_{t,0}) = h_*(P_{t,1}) \oplus h_*(P_{t,0})^{-1}$ to 0 whose CS-form equals $\text{CS}(h_*(P_{t,s})) + \text{CS}(\gamma_s) = \text{CS}(h_*(P_{t,s}))$. This shows that

$$\text{CS}(h_*(P_{t,s})) = \text{CS}([h_*(P_{t,1})] - [h_*(P_{t,0})]) \quad \text{mod Im(Ch).}$$

This completes the proof of the lemma. □

Lemma 4.18. *The map $\mathcal{I} : \hat{K}^0(M \times S^1) \rightarrow \hat{K}^{-1}(M)$ is a group homomorphism.*

Proof. Let $P_{t,1}, P_{t,2} : M \times S^1 \rightarrow BU \times \mathbb{Z}$. By re-ordering we have

$$h_*(P_{t,1} \oplus P_{t,2}) = \text{hol}_{S^1}(P_{t,1} \oplus P_{t,2}) \oplus id$$

is CS-equivalent to

$$h_*(P_{t,1}) \oplus h(P_{t,2}) = \text{hol}_{S^1}(P_{t,1}) \oplus id \oplus \text{hol}_{S^1}(P_{t,2}) \oplus id$$

so $[h_*(P_{t,1} \oplus P_{t,2})] = [h_*(P_{t,1})] + [h_*(P_{t,2})]$. Also,

$$\eta_{P_{t,1} \oplus P_{t,2}} = \eta_{P_{t,1}} + \eta_{P_{t,2}}$$

since the induced connections' parallel transport, and the Chern Character, respect block sum. So, we conclude that \mathcal{I} is a group homomorphism. □

Lemma 4.19. *The map $\mathcal{I} : \hat{K}^0(M \times S^1) \rightarrow \hat{K}^{-1}(M)$ satisfies*

$$\mathcal{I} \circ (id \times r)^* = -\mathcal{I}$$

where $r : S^1 \rightarrow S^1$ is given by $r(z) = \bar{z} = z^{-1}$.

Proof. Since holonomy along the reverse loop gives the inverse group element, we have, for $P_t : M \times S^1 \rightarrow BU \times \mathbb{Z}$,

$$[h_*((id \times r)^* P_t)] = [(h_*(P_t))^{-1}] = -[(h_*(P_t))].$$

Secondly, since a is linear, it suffices to show that $\eta_{(id \times r)^* P_t} = -\eta_{P_t}$. Recall

$$\begin{aligned} \eta_{P_t} &= \int \int_{(s,t) \in I \times I} \text{Ch}((1-s)\nabla_0 + s g_t^* \nabla_t) \\ &\quad + \int \int_{(s,t) \in I \times I} \text{Ch}\left((1-s)\left(t\nabla_0 \oplus t\nabla_0^\perp\right) + s(g_1 \oplus id)^*\left(t\nabla_0 \oplus t\nabla_0^\perp\right)\right), \end{aligned}$$

where for the purposes of the integral the variable $t \in I$ can also be regarded as $t \in S^1$. In this way, the map $r : S^1 \rightarrow S^1$ corresponds to the substitution $t \mapsto 1 - t$ for $t \in I$. Making this substitution in the equation above yields $-\eta_{(id \times r)^* P_t}$, which shows the claim. \square

Lemma 4.20. *The map $\mathcal{I} : \hat{K}^0(M \times S^1) \rightarrow \hat{K}^{-1}(M)$ satisfies $\mathcal{I} \circ p^* = 0$, where $p : M \times S^1 \rightarrow M$ is the projection.*

Proof. Let $P : M \times S^1 \rightarrow BU \times \mathbb{Z}$ be a representative for $[P] \in \hat{K}^0(M)$. Then $P_t : M \times S^1 \rightarrow BU \times \mathbb{Z}$ defined by $P_t = p^*(P)$ satisfies $P_t = P$ for all $t \in S^1$. Therefore the connection ∇ on $E \rightarrow M \times S^1$ induced by P_t is zero in the S^1 direction, i.e., $\nabla_{\partial/\partial t} = 0$, and that $\nabla_t = \nabla_0$ for all t . It follows that the parallel transport in the S^1 direction satisfies $g_t = id$ for all t , and in particular, the holonomy along the S^1 direction satisfies $h_*(p^*(P)) = id$. Also, the two squares of connections defining $\eta_{p^*(P)}$, as in Figure 4, do not depend on r , since

$$(1-r)\nabla_0 + r g_t^* \nabla_t = \nabla_0$$

and

$$(1-r)\left(t\nabla_0 \oplus t\nabla_0^\perp\right) + r(g_1 \oplus id)^*\left(t\nabla_0 \oplus t\nabla_0^\perp\right) = t\nabla_0 \oplus t\nabla_0^\perp$$

for all r . Thus, $\eta_{p^*(P)} = 0$. It follows that $a(\eta_{p^*(P)}) = 0$ and so

$$\mathcal{I}(p^*(P)) = [h_*(p^*(P))] + \eta_{p^*(P)} = 0.$$

This completes the proof of the lemma. \square

Lemma 4.21. *The map $\mathcal{I} : \hat{K}^0(M \times S^1) \rightarrow \hat{K}^{-1}(M)$ makes the following diagram commute*

$$\begin{array}{ccc} \hat{K}^0(M \times S^1) & \xrightarrow{\text{Ch}} & \Omega^{\text{even}}(M \times S^1) \\ \downarrow \mathcal{I} & & \downarrow f_{S^1} \\ \hat{K}^{-1}(M) & \xrightarrow{\text{Ch}} & \Omega^{\text{odd}}(M). \end{array}$$

Proof. Given a representative $P_t : M \times S^1 \rightarrow BU \times \mathbb{Z}$ for $[P_t] \in \hat{K}^0(M \times S^1)$, we have, using $\text{Ch} \circ a = d$, that

$$\begin{aligned} \text{Ch}(\mathcal{I}(P_t)) &= \text{Ch}(h_*(P_t)) + \text{Ch}(a(\eta_{P_t})) \\ &= \left(\int_{S^1} \text{Ch}(P_t) - d\eta_{P_t} \right) + d\eta_{P_t} = \int_{S^1} \text{Ch}(P_t). \end{aligned}$$

This is the claim of the lemma. □

Lemma 4.22. *The map $\mathcal{I} : \hat{K}^0(M \times S^1) \rightarrow \hat{K}^{-1}(M)$ makes the following diagram commute*

$$\begin{array}{ccc} \hat{K}^0(M \times S^1) & \xrightarrow{I} & K^0(M \times S^1) \\ \downarrow \mathcal{I} & & \downarrow \int_{S^1} \\ \hat{K}^{-1}(M) & \xrightarrow{I} & K^{-1}(M). \end{array}$$

Proof. Since $\text{Im}(a) \subset \text{Ker}(I)$, it suffices to show that if $P_t : M \times S^1 \rightarrow BU \times \mathbb{Z}$, then

$$(4.9) \quad \int_{S^1} I([P_t]) = I(h_*(P_t)).$$

Recall that the S^1 -integration map in ordinary K -theory is defined by the lower row in the following diagram,

$$\begin{array}{ccc} [M \times S^1, BU \times \mathbb{Z}] & \xrightarrow{h_*} & [M, U] \\ \downarrow = & & \downarrow = \\ K^0(M \times S^1) & \xrightarrow{\text{pr}} \text{Ker}(j^*) \xrightarrow{(q^*)^{-1}} \tilde{K}^0(\Sigma M_+) = K^0(M \wedge S^1) \xrightarrow{\sigma^{-1}} & K^{-1}(M) \end{array}$$

where $h_*(P_t) = \text{hol}_{S^1}(P_t) \oplus id$ is the map induced by holonomy in the S^1 direction, and $I(h_*(P_t)) : M \rightarrow U$ is the homotopy class of the holonomy. The lower row can be interpreted geometrically as follows. If E is the bundle over $M \times S^1$ determined by P_t , then $pr(E) = E \oplus p^*j^*E^\perp$ is the bundle over $M \times S^1$ obtained by adding to E the bundle $p^*E_0^\perp$ over $M \times S^1$, where $p : M \times S^1 \rightarrow M$ is projection, and j^*E is the restriction of E to $M \times \{0\}$. This bundle $pr(E)$ is trivial over $M \times S^1$, and the induced bundle over ΣM_+ is determined by (a homotopy class of) a map $M \rightarrow U$, given by the clutching map, which is determined by trivializing the bundle over each half of the suspension. But, this clutching map is given up to homotopy by $h_*(P_t) = \text{hol}_{S^1}(P_t) \oplus id$ since parallel transport of the connection trivializes the bundle over each cone. This shows that the above diagram commutes up to homotopy, and thus we have Equation (4.9). This completes the proof. □

Lemma 4.23. *The map $\mathcal{I} : \hat{K}^0(M \times S^1) \rightarrow \hat{K}^{-1}(M)$ makes the following diagram commute*

$$\begin{CD} \Omega^{\text{odd}}(M \times S^1)/\text{Im}(d) @>a>> \hat{K}^0(M \times S^1) \\ @Vf_{S^1}VV @VV\mathcal{I}V \\ \Omega^{\text{even}}(M)/\text{Im}(d) @>a>> \hat{K}^{-1}(M). \end{CD}$$

Proof. By definition of the maps $a : \Omega^{*+1}(M)/\text{Im}(d) \rightarrow \hat{K}^*(M)$, we can write the diagram in the statement of the lemma as

$$\begin{CD} \Omega^{\text{odd}}(M \times S^1)/\text{Im}(d) @>\text{pr}>> \Omega^{\text{odd}}(M \times S^1)/\text{Im}(\text{Ch}) @>\text{CS}^{-1}>> \text{Ker}(I) \subset \hat{K}^0(M \times S^1) \\ @Vf_{S^1}VV @Vf_{S^1}VV @VV\mathcal{I}V \\ \Omega^{\text{even}}(M)/\text{Im}(d) @>\text{pr}>> \Omega^{\text{even}}(M)/\text{Im}(\text{Ch}) @>\text{CS}^{-1}>> \text{Ker}(I) \subset \hat{K}^{-1}(M) \end{CD}$$

where pr is projection, which is well-defined since $\text{Im}(d) \subset \text{Im}(\text{Ch})$ by Theorem 3.17(1), and the middle vertical map is well-defined since

$$\text{Im} \left(\int_{S^1} \circ \text{Ch} \right) \subset \text{Im}(\text{Ch})$$

by Lemma 4.10. The left square clearly commutes so it suffices to show the right square commutes. The maps CS^{-1} are isomorphisms onto $\text{Ker}(I)$.

Given $P_t : M \times S^1 \rightarrow BU \times \mathbb{Z}$ such that $P_t \in \text{Ker}(I)$ we can choose $P_{t,s} : M \times S^1 \times I \rightarrow BU \times \mathbb{Z}$ such that $P_{t,1} = P_t$ and $P_{t,0}$ is constant. We need to show that

$$\text{CS}(\mathcal{I}(P_t)) = \int_{t \in S^1} \text{CS}(P_{t,s}) \quad \text{mod exact forms.}$$

The class $[h_*(P_{t,1})]$ is also in the kernel of I , and $h_*(P_{t,s})$ is a path from $h_*(f_{P,1})$ to the constant $h_*(P_{t,0})$. So it suffices so to show

$$\text{CS}(h_*(P_{t,s})) + \eta_{P_{t,1}} = \int_{t \in S^1} \text{CS}(P_{t,s}) \quad \text{mod exact forms.}$$

As in Lemma 4.17, let

$$\omega = \int_{s \in I} \eta_{P_{t,s}}.$$

Using $d\eta_{P_{t,s}} = \int_{t \in S^1} \text{Ch}(P_{t,s}) - \text{Ch}(h_*(P_{t,s}))$ we have

$$d\omega = \eta_{P_{t,1}} - \eta_{P_{t,0}} - \int_{s \in I} \int_{t \in S^1} \text{Ch}(P_{t,s}) + \int_{s \in I} \text{Ch}(h_*(P_{t,s})).$$

Now $\eta_{P_{t,0}} = 0$ since $P_{t,0}$ is constant, $\int_{s \in I} \int_{t \in S^1} \text{Ch}(P_{t,s}) = \int_{t \in S^1} \text{CS}(P_{t,s})$, and $\int_{t \in S^1} \text{Ch}(h_*(P_{t,s})) = \text{CS}(h_*(P_{t,s}))$, which shows that

$$d\omega = \eta_{P_{t,1}} - \int_{t \in S^1} \text{CS}(P_{t,s}) + \text{CS}(h_*(P_{t,s})).$$

This completes the proof of the lemma. □

The collection of lemmas in subsections 4.1 and 4.2 prove the following corollary.

Corollary 4.24. *The map $\mathcal{I} : \hat{K}^{*+1}(M \times S^1) \rightarrow \hat{K}^*(M)$ defines an S^1 -integration map.*

Finally, by the uniqueness theorem of differential K -theory (Theorem 3.3 of [BS3]) we have

Theorem 4.25. *Differential K -theory is represented by*

$$\hat{K}^0(M) = \text{Hom}(M, BU \times \mathbb{Z}) \quad \hat{K}^{-1}(M) = \text{Hom}(M, U)$$

as functors on $\text{Smooth}_{\hat{K}}$ (see Definition 2.4) where Hom means CS-equivalence classes of maps (see Definitions 3.18 and 3.4).

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